Abstract. A number of combinatorial identities are concerned with certain classes of subsets of a finite set (e.g., matchings of a graph); they can be viewed as saying (roughly speaking) that equal numbers of these subsets have even size and odd size. In this talk, I will discuss a few such identities – some of them new – and their topological meaning. As a common theme, the "parity bias" (or lack thereof) is the Euler characteristic of a simplicial complex, and thus any expression for it is potentially the tip of a topological iceberg. Underneath are questions of homology, homotopy or even discrete Morse theory. Aside from the specific complexes in question, I hope to provide one more pair of "simplex glasses" through which combinatorial identities appear in a new light.

Preprint:
1. Introduction

1.1. Alternating sums

- Enumerative combinatorics is full of alternating sums. Some examples:

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \quad \text{for integers } n > 0;
\]

\[
\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m} \quad \text{for } m \geq 0;
\]

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} = (1 \text{ or } 0 \text{ or } -1);
\]

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{ak+b}{c} = 0 \quad \text{for } c, n \in \mathbb{N} \text{ with } c < n;
\]

\[
\sum_{i=0}^{m} (-1)^i \text{sur}(m, i) = (-1)^m,
\]

where \( \text{sur}(m, i) = (\# \text{ of surjections from } \{1, 2, \ldots, m\} \text{ to } \{1, 2, \ldots, i\}) \).

- These alternating sums are among the most helpful tools in proving identities. (They often play a similar role as the formula \( 1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1} = 0 \) for \( \zeta \) being a nontrivial \( n \)-th root of unity plays in the discrete Fourier transform.)

- An alternating sum identity generally looks like this:

\[
\sum_{\text{(some finite set)}} (-1)^{\text{(something)}} \ (\text{something}) = \ (\text{something typically simpler}).
\]

- In this talk, I shall

  - present some alternating sum identities and their combinatorial proofs by “toggling” or “sign-reversing involutions”;
  - discuss how a few of these identities can be lifted to topological statements about simplicial complexes,
  - and how these topological statements can be lifted to combinatorial statements again using discrete Morse theory.

- This is not a theory talk; you’ll hear my personal favorites, not the most general or most important results.

- There will be various open questions.
2. Toggling

2.1. All subsets

• We start with the first identity listed above:

**Theorem.** Let \( n \) be a positive integer. Then,

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.
\]

• There are many ways to prove this:
e.g., expand \((1 - 1)^n\) using the binomial theorem.

• Here is a **combinatorial proof**:
  Set \([n] = \{1, 2, \ldots, n\}\). Then,

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \sum_{I \subseteq [n]} (-1)^{|I|}.
\]

**Claim:** In the sum on the RHS, all the addends cancel out.

**Proof.** For each subset \( I \) of \([n]\), we can

1. insert 1 into \( I \) if \( 1 \notin I \), or
2. remove 1 from \( I \) if \( 1 \in I \).

This gives us a new subset of \([n]\), which we denote by \( I \triangle \{1\} \).

Easy to see: The map

\[
\{ \text{subsets of } [n] \} \to \{ \text{subsets of } [n] \},
I \mapsto I \triangle \{1\}
\]

is an involution (i.e., applying it twice gives the identity), and it flips the
sign (meaning \((-1)^{|I\triangle\{1\}|} = - (-1)^{|I|}\) for any subset \( I \) of \([n]\)).

Hence, all addends in the sum \( \sum_{I \subseteq [n]} (-1)^{|I|} \) cancel out (the \( I \)-addend can-
celling the \( I \triangle \{1\} \)-addend). Thus, the sum is 0, qed.

• Our notation \( I \triangle \{1\} \) is a particular case of the notation

\[
I \triangle J = (I \cup J) \setminus (I \cap J)
= (I \setminus J) \cup (J \setminus I)
= \{ \text{all elements that belong to exactly one of } I \text{ and } J \}
\]

for any two sets \( I \) and \( J \).
• If \( a \) is any element, then the operation of replacing a set \( I \) by \( I \triangle \{a\} \) (that is, inserting \( a \) into \( I \) if \( a \not\in I \), and removing \( a \) from \( I \) otherwise) is called **toggling \( a \) in \( I \)**. This is always an involution: \((I \triangle \{a\}) \triangle \{a\} = I\) for any \( I \) and \( a \).

• **Remark:** It was actually sufficient for our proof that the map \( I \mapsto I \triangle \{1\} \) is a bijection, not necessarily an involution. But all such maps we will encounter are involutions.

### 2.2. All subsets not too large

• Let us try the second identity:

**Theorem.** Let \( n \) be any number (e.g., a real), and let \( m \) be a nonnegative integer. Then,

\[
\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}.
\]

• **Proof.** First of all, we are proving a polynomial identity in \( n \), so we WLOG assume that \( n \) is a positive integer (since two polynomials over a field are equal if they agree on sufficiently many points).

We have

\[
\sum_{k=0}^{m} (-1)^k \binom{n}{k} = \sum_{I \subseteq [n]; |I| \leq m} (-1)^{|I|}.
\]

Now, we try the involution from the previous proof:

\[
\{\text{subsets of } [n]\} \rightarrow \{\text{subsets of } [n]\},
I \mapsto I \triangle \{1\}.
\]

Unfortunately, applying it to a set \( I \) might break the \( |I| \leq m \) restriction. But it restricts to an involution

\[
\mathcal{A} \rightarrow \mathcal{A},
I \mapsto I \triangle \{1\},
\]

where

\[
\mathcal{A} = \{\text{subsets } I \text{ of } [n] \text{ with } |I \setminus \{1\}| < m\}.
\]

Thus, all addends in the sum \( \sum_{I \subseteq [n]; |I| \leq m} (-1)^{|I|} \) cancel except for those with \( |I \setminus \{1\}| = m \). We get

\[
\sum_{I \subseteq [n]; |I| \leq m} (-1)^{|I|} = \sum_{I \subseteq [n]; |I| \leq m; 1 \not\in I; |I \setminus \{1\}| = m} (-1)^{|I|} = (-1)^m \binom{n-1}{m}.
\]
since there are exactly $\binom{n-1}{m}$ many subsets $I$ of $[n]$ satisfying $1 \not\in I$ and $|I| = m$. This completes our proof.

2.3. Lacunar subsets

• Now to the third identity:

**Theorem.** Let $n$ be a nonnegative integer. Then,

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} = \begin{cases} 1, & \text{if } n \% 6 \in \{0, 1\}; \\ 0, & \text{if } n \% 6 \in \{2, 5\}; \\ -1, & \text{if } n \% 6 \in \{3, 4\}, \end{cases}$$

where $n \% 6$ means the remainder of $n$ divided by 6.

• To prove this combinatorially, we need to find out what $\binom{n-k}{k}$ counts.

• **Convention.** We shall write $[m]$ for $\{1, 2, \ldots, m\}$ whenever $m \in \mathbb{Z}$.

• **Definition.** A set $I$ of integers is said to be lacunar if it contains no two consecutive integers (i.e., there is no $i \in I$ such that $i + 1 \in I$).

• For example, $\{1, 3, 6\}$ is lacunar, but $\{1, 3, 4\}$ is not. Empty and one-element sets are always lacunar.

• Note that any lacunar subset of $[n-1]$ has size $\leq \lfloor n/2 \rfloor$.

• **Proposition.** For any $n \geq k \geq 0$, the number of lacunar $k$-element subsets of $[n-1]$ is $\binom{n-k}{k}$.

• **Proof.** Write “elt” for “element”, and “subs” for “subsets”.

There is a bijection

\[
\{\text{lacunar } k\text{-elt subs of } [n-1]\} \rightarrow \{k\text{-elt subs of } \{0,1,\ldots,n-k-1\}\}, \\
\{i_1 < i_2 < \cdots < i_k\} \mapsto \{i_1 - 1 < i_2 - 2 < \cdots < i_k - k\}.
\]

• Thus, we can start a combinatorial proof of our theorem as follows:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} = \sum_{I \subseteq [n-1]; I \text{ is lacunar}} (-1)^{|I|}.$$
So let us try to construct a sign-reversing involution on the set
\[
\{\text{lacunar subsets of } [n-1]\}
\]
except for possibly one element.
Let \(I\) be a lacunar subset of \([n-1]\).

- We try to toggle 1 in \(I\), but we only do this if the result is lacunar.
  If we succeed (i.e., if the result is lacunar), then we are done.

**Examples:**
- If \(I = \{1, 3, 7\}\), then we toggle 1, and obtain the set \(\{3, 7\}\). Thus, in this case, we succeed and have found the image of \(I\) under our involution.
- If \(I = \{3, 7\}\), then we toggle 1, and obtain the set \(\{1, 3, 7\}\). Thus, in this case, we succeed and have found the image of \(I\) under our involution.
- If \(I = \{2, 7\}\), then we cannot toggle 1, since this would produce the non-lacunar set \(\{1, 2, 7\}\). Thus, in this case, we don’t succeed and move on to the next step.

- If we have not succeeded in the previous step, then 2 \(\in\) \(I\) and thus 3 \(\notin\) \(I\).
  Thus we try to toggle 4 in \(I\), but we only do this if the result is lacunar.
  If we succeed, then we are done.

**Examples:**
- If \(I = \{2, 4, 9\}\), then we toggle 4, and obtain the set \(\{2, 9\}\). Thus, in this case, we succeed and have found the image of \(I\) under our involution.
- If \(I = \{2, 9\}\), then we toggle 4, and obtain the set \(\{2, 4, 9\}\). Thus, in this case, we succeed and have found the image of \(I\) under our involution.
- If \(I = \{2, 5, 8\}\), then we cannot toggle 4, since this would produce the non-lacunar set \(\{2, 4, 5, 8\}\). Thus, in this case, we don’t succeed and move on to the next step.
- If \(I = \{1, 3, 7\}\), then we do not get to this step in the first place, since the first step has already succeeded (turning \(I\) into \(\{3, 7\}\)).

- If we have not succeeded in the previous step, then 5 \(\in\) \(I\) and thus 6 \(\notin\) \(I\).
  Thus we try to toggle 7 in \(I\), but we only do this if the result is lacunar.
  If we succeed, then we are done.
– And so on.
This operation goes on until we have run out of elements of \([n - 1]\) to toggle. The only case in which we fail to toggle anything is if \(n \not\equiv 2 \text{ mod } 3\). Let \(I = \{2, 5, 8, \ldots\} \cap [n - 1]\).

Thus we have found a sign-reversing involution on the set \{lacunar subsets of \([n - 1]\)\} with the exception of a single lacunar subset if \(n \not\equiv 2 \text{ mod } 3\) (and with no exceptions if \(n \equiv 2 \text{ mod } 3\)). The conclusion easily follows.

• This proof is in [BenQui08] (where it is worded using domino tilings instead of lacunar subsets).

2.4. Independent sets of a graph

• Let us generalize this.

• **Definition.** Let \(\Gamma = (V, E)\) be an (undirected) graph. An independent set of \(\Gamma\) means a subset \(I\) of \(V\) such that no two vertices in \(I\) are adjacent (i.e., no edge of \(\Gamma\) connects two vertices in \(I\)).

• **Example.** For the following graph:

![Graph Image]

the independent sets are
\[
\{x, y\}, \{y, z\}, \{z, x\}, \{u, x\}, \{v, y\}, \{w, z\}, \{x, y, z\}
\]
as well as all 1-element sets and the empty set.

• For any \(m \geq 0\), let the \(m\)-path be the graph

\[
1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow m
\]

(that is, the graph with vertices 1, 2, \ldots, \(m\) and edges \(\{i, i + 1\}\) for each \(0 < i < m\)).

Then, the lacunar subsets of \([m]\) are the independent sets of the \(m\)-path.
• Now we can generalize our previous theorem as follows:

**Question:** For what graphs $\Gamma$ do we have

$$\sum_{I \text{ is an independent set of } \Gamma} (-1)^{|I|} \in \{1, 0, -1\} \ ?$$

• Certainly not for all graphs $\Gamma$ (e.g., the 3-cycle is a counterexample).
• But we know it’s true for path graphs. For what other graphs?
• We can try to construct a sign-reversing involution again, and see where we fail.
• What order do we try to toggle the vertices in?
• Well, we can always pick some order at random.
• Unfortunately, toggling a vertex might be blocked by several vertices.
• Trying to solve the resulting conflicts often fails (e.g., for a 4-cycle, even though the sum is $-1$ for a 4-cycle).
• Our above proof can be adapted when $\Gamma$ is a tree.
• However, a much more general result holds:

**Theorem (conjectured by Kalai and Meshulam, 1990s, proved by Chudnovsky, Scott, Seymour, Spirkl, 2018 ([CSSS18]))**: Let $\Gamma$ be a simple loop-less undirected graph that has no induced cycle of length divisible by 3. Then,

$$\sum_{I \text{ is an independent set of } \Gamma} (-1)^{|I|} \in \{1, 0, -1\} .$$

**Question:** Is there any proof under 10 pages length?

2.5. Dominating sets of a graph

• **Definition.** Let $\Gamma = (V, E)$ be an (undirected) graph. A **dominating set** of $\Gamma$ means a subset $I$ of $V$ such that each vertex of $\Gamma$ belongs to $I$ or has a neighbor in $I$. 
• **Example.** For the following graph:

```
  z
 /|
 / |
/  |
  y----v
      |
      |
  w
```

the *non-dominating* sets are

\[
\{u, y, z\}, \ \{v, z, x\}, \ \{w, x, y\}
\]

as well as all their subsets.

• **Theorem (e.g., Brouwer 2009 ([Brouwe09], [BrCsSc09])):** The number of dominating sets of a graph $\Gamma$ is always odd.

• **Theorem (Heinrich, Tittmann, 2017 ([HeiTit17], [Grinbe17, Theorem 3.2.2])):** The number of dominating sets of a graph $\Gamma = (V, E)$ is

\[
2^{|V|} - 1 + \sum_{\text{pairs } (A, B) \text{ of disjoint nonempty subsets of } V; \{a, b\} \not\in E \text{ for all } a \in A \text{ and } b \in B; |A| \equiv |B| \mod 2} (-1)^{|A|}.
\]

This is even for symmetry reasons (for any $(A, B)$, there is a $(B, A)$).

• What about the alternating sum

\[
\sum_{I \text{ is a dominating set of } \Gamma} (-1)^{|I|} ?
\]

Is it $\pm 1$ ?

• No; for example:

**Theorem (Alikhani, 2012 ([Alikha12, Lemma 1])):** If $\Gamma$ is an $n$-cycle (for $n > 0$), then this alternating sum is

\[
\begin{cases}
3, & \text{if } n \equiv 0 \mod 4; \\
-1, & \text{otherwise}.
\end{cases}
\]

**Exercise:** Prove this! (Is there a nice proof without too much casework?)
Theorem (Ehrenborg, Hetyei, 2005 ([EhrHet06, §7])): The alternating sum is $\pm 1$ whenever $\Gamma$ is a forest.

3. Simplicial complexes

3.1. Basic definitions

- The sums we have been discussing so far didn’t range over some random collections of sets. Most of them had a commonality: If a set $I$ appeared in the sum, then so did any subset of $I$.
- Such collections of sets are called **simplicial complexes**.
- Formally:

  **Definition.** A simplicial complex means a pair $(S, \Delta)$, where $S$ is a finite set and $\Delta$ is a collection (= set) of subsets of $S$ such that any $I \in \Delta$ and $J \subseteq I$ satisfy $J \in \Delta$.

- We often just write $\Delta$ for a simplicial complex $(S, \Delta)$.
- A **face** of a simplicial complex $\Delta$ means a set $I \in \Delta$.
- Note that $\{\}$ and $\{\emptyset\}$ are two different simplicial complexes on any set $S$.
- **Examples** of simplicial complexes:
  - $\{\text{all subsets of } S\}$ for a given finite set $S$.
  - $\{\text{all lacunar subsets of } [m]\}$ for a given $m \in \mathbb{N}$.
  - $\{\text{all independent sets of } \Gamma\}$ for a given graph $\Gamma$.
  - $\textbf{not} \{\text{all dominating sets of } \Gamma\}$ for a given graph $\Gamma$.
  - $\{\text{all non-dominating sets of } \Gamma\}$ and $\{\text{all complements of dominating sets of } \Gamma\}$ for a given graph $\Gamma$.

(Here the ground set is the set of vertices of $\Gamma$.)

3.2. Geometric realizations

- Each simplicial complex $(S, \Delta)$ has a geometric realization $|\Delta|$, which is a topological space glued out of (geometric) simplices. The easiest way to define it is by assuming (WLOG) that $S = [n]$ for some $n \in \mathbb{N}$, and setting

$$|\Delta| = \{(t_1, t_2, \ldots, t_n) \in \mathbb{R}_{\geq 0}^n \mid t_1 + t_2 + \cdots + t_n = 1 \quad \text{and} \quad \{i \mid t_i > 0\} \in \Delta\}.$$
• Normally we don’t draw the literal \(|\Delta|\) (since \(\mathbb{R}^n\) has too high dimension) but just something homeomorphic to it (usually in a smaller space).

• Some examples:
  – The complex \{all independent sets of \(\Gamma\)\} of the graph \(\Gamma\) on the left is the simplicial complex drawn on the right:

  ![Graph and Simplicial Complex 1]

  – The complex \{all non-dominating sets of \(\Gamma\)\} of the graph \(\Gamma\) on the left is the simplicial complex drawn on the right:

  ![Graph and Simplicial Complex 2]

3.3. Homotopy and homology

• A lot of features come for free with the geometric realization:

The homotopy type, the homology and the reduced Euler characteristic of a simplicial complex \(\Delta\) are defined to be the homotopy type, the homology and the reduced Euler characteristic of its geometric realization.
• Explicitly, the Euler characteristic of a complex $\Delta$ is simply

$$\sum_{I \in \Delta} (-1)^{|I|-1}.$$  

(The “$-1$” in the exponent just negates the whole sum.)

• Thus, the alternating sums we have been computing are actually Euler characteristics in disguise.

• Homology is a stronger invariant than Euler characteristic, and homotopy type is an even stronger invariant than homology:

$$(\text{homotopy type}) \rightarrow (\text{homology over } \mathbb{Z}) \rightarrow (\text{homology over } \mathbb{Q}) \rightarrow (\text{Euler characteristic}).$$

Our results above are all about Euler characteristics; can we lift them to those stronger invariants?

• Note that homology can be easily redefined combinatorially in terms of $\Delta$. (Homotopy type, too, but less easily; see [Kozlov20, Proposition 9.28].)

### 3.4. Examples of homotopy types

• Our first theorem said that the reduced Euler characteristic of the simplicial complex

$$\{\text{all subsets of } E\}$$

is 0 for any nonempty finite set $E$. This lifts all the way up to homotopy level:

**Proposition.** This simplicial complex is contractible (i.e., homotopy-equivalent to a point).

Geometrically, this is clear: Its geometric realization is a simplex, hence homeomorphic to an $(n-1)$-ball, where $n = |E|$.

• Our second theorem was

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}.$$  

This corresponds to the simplicial complex

$$\{\text{all subsets of } [n] \text{ having size } \leq m\}.$$  

This is called the $(m-1)$-skeleton of the $(n-1)$-ball. By classical algebraic topology, it is homotopy-equivalent to a bouquet of $\binom{n-1}{m}$ many $(m-1)$-spheres, which again explains the Euler characteristic.
• Now, recall the independent sets of graphs.

**Theorem (Kalai, Meshulam, Engström, Chudnovsky, Scott, Seymour, Spirkl, Zhang, Wu, Kim, 2021 ([ZhaWu20], [Kim21])):** Let $\Gamma$ be a simple loopless undirected graph that has no induced cycle of length divisible by 3. Then, the simplicial complex

\[
\{\text{independent sets of } \Gamma\}
\]

is either contractible or homotopy-equivalent to a sphere (whence its reduced Euler characteristic is in \{1, 0, −1\}).

• As we recall, the dominating sets of a graph do not form a simplicial complex, but their complements do, and so do the non-dominating sets. As far as the alternating sum $\sum_I (−1)^{|I|}$ is concerned, these are just as good (switching between dominating and non-dominating sets or between the sets and their complements changes the sum by a factor of ±1).

**Theorem (Ehrenborg, Hetyei, 2005 ([EhrHet06, §7])):** Let $\Gamma$ be a forest. Then, both simplicial complexes

\[
\{\text{non-dominating sets of } \Gamma\} \quad \text{and} \quad \{\text{complements of dominating sets of } \Gamma\}
\]

are either contractible or homotopy-equivalent to a sphere.

• **Question:** What can be said about the case when $\Gamma$ is an $n$-cycle?

3.5. Discrete Morse theory

• Thus we have two approaches to proving formulas for alternating sums:

\[
\sum_{I \in \Delta} (−1)^{|I|}
\]

• Could these two approaches be combined? I.e., is there a technique that gets us both homotopy information and a sign-reversing involution in one
(possibly harder) swoop?

\[ \sum_{I \in \Delta} (-1)^{|I|} \]

- **Discrete Morse theory** is an answer. We will use some of its very basics.

- **Definition.** For two sets \( I \) and \( J \), we write \( I \prec J \) if \( J = I \cup \{ \text{a single element} \} \) (that is, if \( I \subseteq J \) and \( |J \setminus I| = 1 \)). Equivalently, we write \( J \succ I \) for this.

- **Definition.** Let \((S, \Delta)\) be a simplicial complex. A **partial matching** on \( \Delta \) shall mean an involution \( \mu : \Delta \to \Delta \) such that

  \[ \mu(I) = I \text{ or } \mu(I) \prec I \text{ or } \mu(I) \succ I \text{ for each } I \in \Delta. \]

  In other words, \( \mu(I) \) is either \( I \) itself or is obtained from \( I \) by removing or inserting a single element.

- **Definition.** If \( \mu \) is a partial matching on \( \Delta \), then the sets \( I \in \Delta \) satisfying \( \mu(I) = I \) will be called **unmatched** (by \( \mu \)).

  - Thus, if \( \mu \) is a partial matching on \( \Delta \), then

    \[ \sum_{I \in \Delta} (-1)^{|I|} = \sum_{I \in \Delta \text{ is unmatched}} (-1)^{|I|} \]

    (by cancellation).

- Thus, partial matchings are just our partial sign-reversing involutions rewritten (instead of taking some sets out of our complex, we are now leaving them fixed).

- What about the homotopy information? We cannot in general “cancel” matched faces from a simplicial complex and hope that the homotopy information is preserved.

- However, we can restrict our matchings in a way that will make them homotopy-friendly! This is one of the main contributions of Forman that became discrete Morse theory ([Forman02 §3, §6], [Kozlov20]):
• **Definition.** Let $(S, \Delta)$ be a simplicial complex. A partial matching $\mu$ on $\Delta$ is said to be **acyclic** (or a **Morse matching** if there exists no “cycle” of the form

$$I_1 \succ \mu(I_1) \prec I_2 \succ \mu(I_2) \prec \cdots \prec I_n \succ \mu(I_n) \prec I_1$$

with $n \geq 2$ and with $I_1, I_2, \ldots, I_n$ distinct.

• **Intuition:** The easiest way to ensure this is by making sure that when $\mu$ adds an element to a face $I$, then it does so in an “optimal” way (i.e., among all ways to add an element to $I$ and still obtain a face of $\Delta$, it picks the “best” one in some sense). This way, in the above “cycle”, the faces $I_1, I_2, \ldots, I_n, I_1$ become “better and better”, so the cycle cannot exist. There is freedom in defining what “optimal”/“best” is (it means specifying some partial order on the faces of any given size).

This is why Forman calls acyclic matchings “gradient vector fields” in [Forman02].

• **Empiric fact(?):** Sign-reversing involutions in combinatorics tend to be acyclic partial matchings.

• **Question:** Really? Check some of the more complicated ones!

• **Theorem (Forman, I believe).** Let $(S, \Delta)$ be a simplicial complex, and $\mu$ an acyclic partial matching on $\Delta$. For each $k \in \mathbb{N}$, let $c_k$ be the number of unmatched size-$k$ faces of $\Delta$.

Then, there is a CW-complex homotopy-equivalent to $\Delta$ that has exactly $c_k$ faces of dimension $k - 1$ for each $k \in \mathbb{N}$.

• **Corollary.** (a) If a simplicial complex $(S, \Delta)$ has an acyclic partial matching that leaves no face unmatched, then it is contractible.

(b) If a simplicial complex $(S, \Delta)$ has an acyclic partial matching that leaves exactly one face unmatched, then it is homotopy-equivalent to a sphere.

• As a consequence, having a good Morse matching gets us good (if not 100% complete) information both about the homotopy type and about the combinatorics of a simplicial complex.

• For example, all the sign-reversing involutions we used in our proofs above are Morse matchings.

4. **Elser’s “pandemic” complex**

• A remarkable alternating sum identity appeared in a 1984 paper by Elser on mathematical physics (percolation theory) [Elser84]. I shall restate it in a slightly simpler language.
• Fix a (finite undirected multi)graph $\Gamma$ with vertex set $V$ and edge set $E$.
  Fix a vertex $v \in V$.

• If $F \subseteq E$, then an $F$-path shall mean a path of $\Gamma$ such that all edges of the
  path belong to $F$.

• If $e \in E$ is any edge and $F \subseteq E$ is any subset, then we say that $F$ infects $e$
  if there exists an $F$-path from $v$ to some endpoint of $e$.
  (My go-to mental model: A virus starts out in $v$ and spreads along any
  $F$-edge it can get to. Then, $F$ infects $e$ if the virus will eventually reach an
  endpoint of $e$. Note that $F$ always infects any edge through $v$.)

• A subset $F \subseteq E$ is said to be pandemic if it infects each edge $e \in E$.

• Example: Let $\Gamma$ be

  ![](graph.png)

  Then:
  
  – The set $\{1, 2\} \subseteq E$ infects edges 1, 2, 3, 6, 8 (but no others), since the
    virus gets to the vertices $v, p, q$.
  – The set $\{1, 2, 5\}$ infects the same edges.
  – The set $\{1, 2, 3\}$ infects every edge other than 5.
  – The set $\{1, 2, 3, 4\}$ infects each edge, and thus is pandemic (even
    though the virus never gets to vertex $w$).

• Theorem (Elser, 1984 ([Elser84, Lemma 1], [Grinbe20, Theorem 1.2])):
  Assume that $E \neq \emptyset$. Then,

  $$\sum_{\substack{F \subseteq E \text{ is} \\ \text{pandemic}}} (-1)^{|F|} = 0.$$ 

• Remark: A version of pandemicity in which $F$ has to infect all vertices
  (rather than all edges) would fail to produce such a theorem.
4.1. More generally

- If \( F \) is a subset of \( E \), then we define a subset \( \text{Shade} \, F \) of \( E \) by

\[
\text{Shade} \, F = \{ e \in E \mid F \text{ infects } e \}.
\]

- **Example:** Let \( \Gamma \) be

\[\begin{align*}
p & \quad 2 \quad q \\
1 & \quad 8 \quad 7 \quad 4 \\
v & \quad 6 \\
w & \quad 5 \\
t & \quad 3
\end{align*}\]

Then, \( \text{Shade} \, \{1, 2\} = \{1, 2, 3, 6, 8\} \) and \( \text{Shade} \, \{1\} = \{1, 2, 6\} \) and \( \text{Shade} \, \{8\} = \{1, 6\} \).

- **Theorem ([Grinbe20, Theorem 2.5], generalizing Elser’s theorem):** Let \( G \) be any subset of \( E \). Assume that \( E \neq \emptyset \). Then,

\[
\sum_{\substack{F \subseteq E; \\
G \subseteq \text{Shade} \, F}} (-1)^{|F|} = 0.
\]

- **Theorem ([Grinbe20 Theorem 2.6], equivalent restatement of previous theorem):** Let \( G \) be any subset of \( E \). Then,

\[
\sum_{\substack{F \subseteq E; \\
G \not\subseteq \text{Shade} \, F}} (-1)^{|F|} = 0.
\]

- This restatement looks useful since it gets rid of the \( E \neq \emptyset \) assumption. That’s a good sign!

4.2. Proof idea

- Let’s prove this latter restatement. Here is it again:

**Theorem ([Grinbe20 Theorem 2.6], equivalent restatement of previous theorem):** Let \( G \) be any subset of \( E \). Then,

\[
\sum_{\substack{F \subseteq E; \\
G \not\subseteq \text{Shade} \, F}} (-1)^{|F|} = 0.
\]
• **Proof.** Let

\[ A = \{ F \subseteq E \mid G \nsubseteq \text{Shade } F \} \]

Equip the set \( E \) with a total order. If \( F \in A \), then let \( \varepsilon (F) \) be the **smallest** edge \( e \in G \setminus \text{Shade } F \).

Define a sign-reversing involution

\[
\mathcal{A} \rightarrow \mathcal{A}, \\
F \mapsto F \triangle \{ \varepsilon (F) \}.
\]

Check that this works! (The key observation: \( \text{Shade } F \) does not change when we toggle \( \varepsilon (F) \) in \( F \).)

### 4.3. Variants

• We cannot replace “infects all edges” by “infects all vertices” as long as we work with sets of edges.

• However, we can work with sets of vertices instead (mutatis mutandis).

• In detail:

  • If \( F \subseteq V \), then an **F-vertex-path** shall mean a path of \( \Gamma \) such that all vertices of the path except (possibly) for its two endpoints belong to \( F \). (Thus, if a path has only one edge or none, then it automatically is an F-vertex-path.)

  • If \( w \in V \setminus \{ v \} \) is any vertex and \( F \subseteq V \setminus \{ v \} \) is any subset, then we say that \( F \) vertex-infects \( w \) if there exists an \( F \)-vertex-path from \( v \) to \( w \). (This is always true when \( w \) is \( v \) or a neighbor of \( v \).)

  • A subset \( F \subseteq V \setminus \{ v \} \) is said to be **vertex-pandemic** if it vertex-infects each vertex \( w \in V \setminus \{ v \} \).

• **Theorem ([Grinbe20, Theorem 3.2])**. Assume that \( V \setminus \{ v \} \neq \emptyset \). Then,

\[
\sum_{\substack{F \subseteq V \setminus \{ v \} \text{ is vertex-pandemic}}} (-1)^{|F|} = 0.
\]

• Generalizations similar to the one above also hold.

### 4.4. A hammer in search of nails

• The proofs of the original Elser’s theorem and of its vertex variant are suspiciously similar.
• Even worse, they use barely any graph theory. All we needed is that $E$ is a finite set, and that $\text{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ (where $\mathcal{P}(E) = \{ \text{all subsets of } E \}$) is a map with the property that

$$\text{Shade}(F \triangle \{u\}) = \text{Shade} F$$

for any $F \subseteq E$ and $u \in E \setminus \text{Shade} F$.

I call such a map Shade a *shade map*. Our above argument then shows that

$$\sum_{F \subseteq E;\ G \not\subseteq \text{Shade} F} (-1)^{|F|} = 0 \quad \text{for any } G \subseteq E.$$  

• **Question.** Have you seen other maps satisfying this property in the wild?

• **Answer 1.** Let $A$ be an affine space over $\mathbb{R}$. Fix a finite subset $E$ of $A$. For any $F \subseteq E$, we define

$$\text{Shade} F = \{ e \in E \mid \text{e is not a nontrivial convex combination of } F \}.$$

(A convex combination is said to be *nontrivial* if all coefficients are $< 1$.) Then, this map $\text{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ is a shade map.

• Other answers? Can you get shade maps from matroids? spanning trees? closure operators? lattices?

### 4.5. The topological viewpoint

• Now let us return to the case of a graph $\Gamma = (V,E)$. Fix a subset $G$ of $E$, and let

$$\mathcal{A} = \{ F \subseteq E \mid G \not\subseteq \text{Shade} F \} = \{ F \subseteq E \mid \text{not every edge in } G \text{ is infected by } F \}$$

as in the proof above.

• This $\mathcal{A}$ is clearly a simplicial complex on ground set $E$.

• **Theorem (G., 2020 ([Grinbe20, Theorem 5.5]))**. This simplicial complex has a Morse matching (i.e., an acyclic partial matching) with no unmatched faces. Thus, it is contractible.

• **Proof idea.** Argue that the sign-reversing involution above is a Morse matching.
4.6. The Alexander dual

- The complex \( \mathcal{A} = \{ F \subseteq E \mid G \not\subseteq \text{Shade} \} \)

is not the only simplicial complex we can obtain from our setup. There is also
\[
\mathcal{A}^* = \{ F \subseteq E \mid G \subseteq \text{Shade} (E \setminus F) \}.
\]

- More generally, if \((S, \Delta)\) is any simplicial complex, then we can define a new simplicial complex \((S, \Delta^*)\), where
\[
\Delta^* := \{ I \subseteq S \mid S \setminus I \notin \Delta \} = \{ \text{the complements of the non-faces of } \Delta \}.
\]

This \((S, \Delta^*)\) is called the Alexander dual of \((S, \Delta)\).

- The homologies of \((S, \Delta^*)\) and \((S, \Delta)\) are isomorphic (folklore – see, e.g., [BjoTan09]); thus the Euler characteristics agree up to sign.

But the homotopy types are not in general equivalent! Nor is the existence of a Morse matching with good properties.

- Thus, for any homotopy type question we can answer, we can state an analogous one for its dual.

- **Question.** What is the homotopy type of the \(\mathcal{A}^*\) above?

4.7. Multi-shades?

- I can’t help spreading yet another open question that essentially comes from Dorpalen-Barry et al. [DHLetc19, Conjecture 9.1].

- Return to the setup of a graph \(\Gamma = (V, E)\), but don’t fix the vertex \(v\) this time.

- Rename Shade \(F\) as Shade\(_v\) \(F\) to stress its dependence on \(v\).

- For any subset \(U \subseteq V\), define the simplicial complex
\[
\mathcal{A}_U := \{ F \subseteq E \mid G \not\subseteq \text{Shade}_v F \text{ for some } v \in U \}.
\]

- **Question:** What can we say about the homotopy and discrete Morse theory of \(\mathcal{A}_U\)? What about its Alexander dual?

- An optimistic yet reasonable expectation would be: a Morse matching whose unmatched faces all have the same size. (Thus, \(\mathcal{A}_U\) should be homotopy-equivalent to a bouquet of spheres.)
5. Bonus: Path-free and path-missing complexes

- This is joint work with Lukas Katthän and Joel Brewster Lewis \[\text{GrKaLe21}\].
- Fix a **directed** graph \(G = (V, E)\) and two vertices \(s\) and \(t\). We define the two simplicial complexes
  \[
P F (G) = \{ F \subseteq E \mid \text{there is no } F\text{-path from } s \text{ to } t \}\]
  (the "path-free" complex of \(G\))

  and

  \[
P M (G) = \{ F \subseteq E \mid \text{there is an } (E \setminus F)\text{-path from } s \text{ to } t \}\]
  (the "path-missing" complex of \(G\)).

(These are Alexander duals of each other.)

- **Example:** Let \(G\) be the following directed graph:

![Graph](image)

Then:

- The faces of the simplicial complex \(P F(G)\) are the sets
  \[
  \{b, c, e, f, g\}, \{a, c, e, f, g\}, \{b, c, d, g\}, \{a, c, d, f, g\}, \{a, b, e, f\}, \{a, b, d, f, g\}
  \]
  as well as all their subsets.

- The faces of the simplicial complex \(P M(G)\) are the sets
  \[
  \{d, e, f, g\}, \{c, d, f\}, \{a, b, c, f, g\}, \{a, e, g\}
  \]
  as well as all their subsets.

- **Theorem (G., Katthän, Lewis, 2021 \([\text{GrKaLe21}]\)).** Assume that \(s \neq t\)
  and \(E \neq \emptyset\) (the other cases are trivial). Then, both complexes \(P F (G)\)
  and \(P M (G)\) are contractible or homotopy-equivalent to spheres. The dimensions of
  the spheres can be determined explicitly. The complexes are contractible if and only if
  \(G\) has a useless edge (i.e., an edge that appears in no path from \(s\) to \(t\)) or a (directed) cycle.
• Theorem (G., Katthän, Lewis, 2021+ ([GrKaLe21, future version])). Both complexes $\mathcal{P}_F(G)$ and $\mathcal{P}_M(G)$ have Morse matchings with at most one unmatched face.

• The proofs use (fairly intricate) deletion/contraction arguments.

• **Question.** Is there a good combinatorial description of these Morse matchings?

**Acknowledgments**

Thanks to

• **Lukas Katthän**, who showed me how simple discrete Morse theory is;

• **Anders Björner, Galen Dorpalen-Barry, Dmitry Feichtner-Kozlov, Patricia Hersh, Vic Reiner, Tom Roby** and **Richard Stanley** for insightful conversations;

• **an anonymous referee** for one of the most useful reports I have ever gotten (within just 2 weeks), greatly simplifying one of my proofs;

• the **Mathematisches Forschungsinstitut Oberwolfach 2020** and specifically the programme “Oberwolfach Leibniz Fellows” for its hospitality (in 2020, of all times);

• **you** for your patience!

**References**


