Rook sums in the symmetric group algebra

Darij Grinberg (Drexel University)

Howard University, DC, 2024-04-07

slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/dc2024.pdf

paper (draft): https://www.cip.ifi.lmu.de/~grinberg/algebra/rooksn.pdf

- **Definition.** Fix a commutative ring $\mathbf{k}$. (The main examples are $\mathbb{Z}$ and $\mathbb{Q}$.)
  For each $n \in \mathbb{N}$, let S_n be the n-th symmetric group, and $\mathbf{k}[S_n]$ its group algebra over $\mathbf{k}$. So

$$\mathbf{k}[S_n] = \left\{ \text{formal linear combinations } \sum_{w \in S_n} \alpha_w w \text{ with } \alpha_w \in \mathbf{k} \right\}.$$

  Also, let $[n] := \{1, 2, \ldots, n\}$ for each $n \in \mathbb{N}$.

- **Definition.** For any two subsets A and B of $[n]$, we define the elements

$$\nabla_{B,A} := \sum_{\substack{w \in S_n; \\ w(A)=B}} w \qquad \text{and} \qquad \widetilde{\nabla}_{B,A} := \sum_{\substack{w \in S_n; \\ w(A) \subseteq B}} w$$

of $\mathbf{k}[S_n]$. We shall refer to these elements as **rectangular rook sums**.

Rook-to-rook sums: definition

- **Definition.** For any two subsets A and B of $[n]$, we define the elements

$$\nabla_{B,A} := \sum_{\substack{w \in S_n; \\ w(A) = B}} w \qquad \text{and} \qquad \widetilde{\nabla}_{B,A} := \sum_{\substack{w \in S_n; \\ w(A) \subseteq B}} w$$

  of $\mathbf{k}[S_n]$. We shall refer to these elements as **rectangular rook sums**.

- **Examples.**

$$\nabla_{\varnothing,\varnothing} = \nabla_{[n],[n]} = (\text{sum of all } w \in S_n);$$
$$\nabla_{\{2\},\{1\}} = (\text{sum of all } w \in S_n \text{ sending } 1 \text{ to } 2);$$
$$\widetilde{\nabla}_{\{2,3\},\{1\}} = (\text{sum of all } w \in S_n \text{ sending } 1 \text{ to } 2 \text{ or } 3).$$

Rook-to-rook sums: simple properties

- **Proposition.** Let A and B be two subsets of $[n]$. Then:
  (a) We have $\nabla_{B,A} = 0$ if $|A| \neq |B|$.
  (b) We have $\widetilde{\nabla}_{B,A} = 0$ if $|A| > |B|$.

- **Proposition.** Let A and B be two subsets of $[n]$. Then:
  - **(a)** We have $\nabla_{B,A} = 0$ if $|A| \neq |B|$.
  - **(b)** We have $\widetilde{\nabla}_{B,A} = 0$ if $|A| > |B|$.
  - **(c)** We have $\widetilde{\nabla}_{B,A} = \sum_{\substack{V \subseteq B; \\ |V| = |A|}} \nabla_{V,A}$.

Rook-to-rook sums: simple properties

- **Proposition.** Let A and B be two subsets of $[n]$. Then:
  (a) We have $\nabla_{B,A} = 0$ if $|A| \neq |B|$.
  (b) We have $\widetilde{\nabla}_{B,A} = 0$ if $|A| > |B|$.
  (c) We have $\widetilde{\nabla}_{B,A} = \sum_{\substack{V \subseteq B; \\ |V| = |A|}} \nabla_{V,A}$.
  (d) We have $\nabla_{B,A} = \nabla_{[n] \setminus B,\ [n] \setminus A}$.
  (e) If $|A| = |B|$, then $\nabla_{B,A} = \widetilde{\nabla}_{B,A}$.

- **Proposition.** Let A and B be two subsets of $[n]$. Then:
  (a) We have $\nabla_{B,A} = 0$ if $|A| \neq |B|$.
  (b) We have $\widetilde{\nabla}_{B,A} = 0$ if $|A| > |B|$.
  (c) We have $\widetilde{\nabla}_{B,A} = \sum_{\substack{V \subseteq B; \\ |V| = |A|}} \nabla_{V,A}$.
  (d) We have $\nabla_{B,A} = \nabla_{[n] \setminus B,\ [n] \setminus A}$.
  (e) If $|A| = |B|$, then $\nabla_{B,A} = \widetilde{\nabla}_{B,A}$.

  Next, let $S : \mathbf{k}[S_n] \to \mathbf{k}[S_n]$ be the **antipode** of $\mathbf{k}[S_n]$; this is the $\mathbf{k}$-linear map sending each permutation $w \in S_n$ to w^{-1}. Then:
  (f) We have $S(\nabla_{B,A}) = \nabla_{A,B}$.
  (g) We have $S\left(\widetilde{\nabla}_{B,A}\right) = \widetilde{\nabla}_{[n] \setminus A,\ [n] \setminus B}$.

- The simplest rectangular rook sum is

$$\nabla_{\varnothing,\varnothing} = (\text{sum of all } w \in S_n)\,.$$

Easily, $\nabla^2_{\varnothing,\varnothing} = n!\nabla_{\varnothing,\varnothing}$, so that

$$P\left(\nabla_{\varnothing,\varnothing}\right) = 0 \qquad \text{for the polynomial } P\left(x\right) = x\left(x - n!\right).$$

Minimal polynomials: a question

- The simplest rectangular rook sum is

$$\nabla_{\varnothing,\varnothing} = (\text{sum of all } w \in S_n) \, .$$

Easily, $\nabla_{\varnothing,\varnothing}^2 = n! \nabla_{\varnothing,\varnothing}$, so that

$$P\left(\nabla_{\varnothing,\varnothing}\right) = 0 \qquad \text{for the polynomial } P\left(x\right) = x\left(x - n!\right).$$

- **Question:** What polynomials P satisfy $P\left(\nabla_{B,A}\right) = 0$ or $P\left(\widetilde{\nabla}_{B,A}\right) = 0$ for arbitrary A, B ?
  In particular, what is the minimal polynomial of $\widetilde{\nabla}_{B,A}$? (The only interesting $\nabla_{B,A}$'s are those for $|A| = |B|$, and they agree with $\widetilde{\nabla}_{B,A}$, so that we need not study them separately.)

- **Example.** The minimal polynomial of $\widetilde{\nabla}_{\{2,4,5,6\},\ \{1,2\}}$ for $n = 6$ is $(x - 288)x(x + 12)(x + 36)$.

- **Example.** The minimal polynomial of $\widetilde{\nabla}_{\{2,4,5,6\},\ \{1,2\}}$ for $n = 6$ is $(x - 288)x(x + 12)(x + 36)$.
- **Example.** The minimal polynomial of $\widetilde{\nabla}_{\{1,2,5,6\},\ \{1,2,3\}}$ for $n = 6$ is $(x - 144)(x + 16)x^2$.

Minimal polynomials: experimental data

- **Example.** The minimal polynomial of $\widetilde{\nabla}_{\{2,4,5,6\},\ \{1,2\}}$ for $n = 6$ is $(x - 288)x(x + 12)(x + 36)$.
- **Example.** The minimal polynomial of $\widetilde{\nabla}_{\{1,2,5,6\},\ \{1,2,3\}}$ for $n = 6$ is $(x - 144)(x + 16)x^2$.
- Looks like the minimal polynomial always splits over $\mathbb{Z}$ (i.e., factors into linear factors)!

- **Example.** The minimal polynomial of $\widetilde{\nabla}_{\{2,4,5,6\},\ \{1,2\}}$ for $n = 6$ is $(x - 288)x(x + 12)(x + 36)$.
- **Example.** The minimal polynomial of $\widetilde{\nabla}_{\{1,2,5,6\},\ \{1,2,3\}}$ for $n = 6$ is $(x - 144)(x + 16)x^2$.
- Looks like the minimal polynomial always splits over $\mathbb{Z}$ (i.e., factors into linear factors)!
- How can we prove this?

A product rule

- A crucial step in the proof is a product rule for ∇s:
- **Theorem (product rule).** Let A, B, C, D be four subsets of $[n]$ such that $|A| = |B|$ and $|C| = |D|$. Then,

$$\nabla_{D,C} \nabla_{B,A} = \omega_{B,C} \sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U| = |V|}} (-1)^{|U| - |B \cap C|} \binom{|U|}{|B \cap C|} \nabla_{U,V}.$$

Here, for any two subsets B and C of $[n]$, we set

$$\omega_{B,C} := |B \cap C|! \cdot |B \setminus C|! \cdot |C \setminus B|! \cdot |[n] \setminus (B \cup C)|! \in \mathbb{Z}.$$

A product rule

- A crucial step in the proof is a product rule for ∇s:
- **Theorem (product rule).** Let A, B, C, D be four subsets of $[n]$ such that $|A| = |B|$ and $|C| = |D|$. Then,

$$\nabla_{D,C} \nabla_{B,A} = \omega_{B,C} \sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U| = |V|}} (-1)^{|U| - |B \cap C|} \binom{|U|}{|B \cap C|} \nabla_{U,V}.$$

Here, for any two subsets B and C of $[n]$, we set

$$\omega_{B,C} := |B \cap C|! \cdot |B \setminus C|! \cdot |C \setminus B|! \cdot |[n] \setminus (B \cup C)|! \in \mathbb{Z}.$$

- **Proof.** Nice exercise in enumeration! First step is to show that

$$\nabla_{D,C} \nabla_{B,A} = \omega_{B,C} \sum_{\substack{w \in S_n; \\ |w(A) \cap D| = |B \cap C|}} w.$$

- Recall that $\widetilde{\nabla}_{B,A}$ is the sum of all $\nabla_{V,A}$'s for $V \subseteq B$ satisfying $|V| = |A|$. Thus, the product rule rewrites as follows:
- **Theorem (product rule, rewritten).** Let A, B, C, D be four subsets of $[n]$ such that $|A| = |B|$ and $|C| = |D|$. Then,

$$\nabla_{D,C} \nabla_{B,A} = \omega_{B,C} \sum_{V \subseteq A} (-1)^{|V| - |B \cap C|} \binom{|V|}{|B \cap C|} \widetilde{\nabla}_{D,V}.$$

An incomplete filtration

- Now, fix a subset D of $[n]$. Define

$$\mathcal{F}_k := \operatorname{span}\left\{ \widetilde{\nabla}_{D,C} \ \mid\ C \subseteq [n] \text{ with } |C| \leq k \right\}$$

for each $k \in \mathbb{Z}$.

An incomplete filtration

- Now, fix a subset D of $[n]$. Define

$$\mathcal{F}_k := \operatorname{span}\left\{\widetilde{\nabla}_{D,C} \ \mid \ C \subseteq [n] \text{ with } |C| \leq k\right\}$$

for each $k \in \mathbb{Z}$. Of course,

$$\mathcal{F}_n \supseteq \mathcal{F}_{n-1} \supseteq \cdots \supseteq \mathcal{F}_0 \supseteq \mathcal{F}_{-1} = 0.$$

It is easy to see that $\mathcal{F}_0$ is spanned by $\widetilde{\nabla}_{D,\varnothing} = \nabla_{\varnothing,\varnothing} = \sum\limits_{w \in S_n} w$.

An incomplete filtration

- Now, fix a subset D of $[n]$. Define

$$\mathcal{F}_k := \operatorname{span}\left\{\widetilde{\nabla}_{D,C} \ \mid \ C \subseteq [n] \text{ with } |C| \leq k\right\}$$

for each $k \in \mathbb{Z}$. Of course,

$$\mathcal{F}_n \supseteq \mathcal{F}_{n-1} \supseteq \cdots \supseteq \mathcal{F}_0 \supseteq \mathcal{F}_{-1} = 0.$$

- For any subset $C \subseteq [n]$ and any $k \in \mathbb{N}$, we define the integer

$$\delta_{D,C,k} := \sum_{\substack{B \subseteq D; \\ |B|=k}} \omega_{B,C} \left(-1\right)^{k-|B \cap C|} \binom{k}{|B \cap C|} \in \mathbb{Z}.$$

- Now, fix a subset D of $[n]$. Define
$$\mathcal{F}_k := \operatorname{span}\left\{ \widetilde{\nabla}_{D,C} \ \mid \ C \subseteq [n] \text{ with } |C| \leq k \right\}$$
for each $k \in \mathbb{Z}$. Of course,
$$\mathcal{F}_n \supseteq \mathcal{F}_{n-1} \supseteq \cdots \supseteq \mathcal{F}_0 \supseteq \mathcal{F}_{-1} = 0.$$

- For any subset $C \subseteq [n]$ and any $k \in \mathbb{N}$, we define the integer
$$\delta_{D,C,k} := \sum_{\substack{B \subseteq D; \\ |B|=k}} \omega_{B,C} \left(-1\right)^{k-|B\cap C|} \binom{k}{|B \cap C|} \in \mathbb{Z}.$$

- **Proposition.** Let $C \subseteq [n]$ satisfy $|C| = |D|$. Let $k \in \mathbb{N}$. Then,
$$\left(\nabla_{D,C} - \delta_{D,C,k}\right) \mathcal{F}_k \subseteq \mathcal{F}_{k-1}.$$

- Now, fix a subset D of $[n]$. Define

$$\mathcal{F}_k := \operatorname{span}\left\{ \widetilde{\nabla}_{D,C} \ \mid \ C \subseteq [n] \text{ with } |C| \leq k \right\}$$

for each $k \in \mathbb{Z}$. Of course,

$$\mathcal{F}_n \supseteq \mathcal{F}_{n-1} \supseteq \cdots \supseteq \mathcal{F}_0 \supseteq \mathcal{F}_{-1} = 0.$$

- For any subset $C \subseteq [n]$ and any $k \in \mathbb{N}$, we define the integer

$$\delta_{D,C,k} := \sum_{\substack{B \subseteq D; \\ |B| = k}} \omega_{B,C} \left(-1\right)^{k - |B \cap C|} \binom{k}{|B \cap C|} \in \mathbb{Z}.$$

- **Proposition.** Let $C \subseteq [n]$ satisfy $|C| = |D|$. Let $k \in \mathbb{N}$. Then,

$$\left(\nabla_{D,C} - \delta_{D,C,k}\right) \mathcal{F}_k \subseteq \mathcal{F}_{k-1}.$$

- **Proof.** Follows from the rewritten product rule.

- So we have proved $(\nabla_{D,C} - \delta_{D,C,k})\,\mathcal{F}_k \subseteq \mathcal{F}_{k-1}$ whenever $|C| = |D|$ and $k \in \mathbb{N}$.

Annihilating polynomials, 1

- So we have proved $(\nabla_{D,C} - \delta_{D,C,k})\,\mathcal{F}_k \subseteq \mathcal{F}_{k-1}$ whenever $|C| = |D|$ and $k \in \mathbb{N}$.
  Since $\nabla_{D,C} \in \mathcal{F}_n$ and $\mathcal{F}_{-1} = 0$, this entails

$$\left(\prod_{k=0}^{|D|} (\nabla_{D,C} - \delta_{D,C,k})\right) \nabla_{D,C} = 0.$$

- So we have proved $(\nabla_{D,C} - \delta_{D,C,k}) \mathcal{F}_k \subseteq \mathcal{F}_{k-1}$ whenever $|C| = |D|$ and $k \in \mathbb{N}$.
  Since $\nabla_{D,C} \in \mathcal{F}_n$ and $\mathcal{F}_{-1} = 0$, this entails

$$\left(\prod_{k=0}^{|D|} (\nabla_{D,C} - \delta_{D,C,k}) \right) \nabla_{D,C} = 0.$$

- However, the $\mathcal{F}_k$ depend only on D, not on C, so that we can apply the same reasoning to any linear combination

$$\nabla_{D,\alpha} := \sum_{\substack{C \subseteq [n]; \\ |C|=|D|}} \alpha_C \nabla_{D,C}$$

  of $\nabla_{D,C}$'s instead of a single $\nabla_{D,C}$.

- So we have proved $(\nabla_{D,C} - \delta_{D,C,k}) \mathcal{F}_k \subseteq \mathcal{F}_{k-1}$ whenever $|C| = |D|$ and $k \in \mathbb{N}$.
  Since $\nabla_{D,C} \in \mathcal{F}_n$ and $\mathcal{F}_{-1} = 0$, this entails

$$\left(\prod_{k=0}^{|D|} (\nabla_{D,C} - \delta_{D,C,k}) \right) \nabla_{D,C} = 0.$$

- However, the $\mathcal{F}_k$ depend only on D, not on C, so that we can apply the same reasoning to any linear combination

$$\nabla_{D,\alpha} := \sum_{\substack{C \subseteq [n]; \\ |C| = |D|}} \alpha_C \nabla_{D,C}$$

  of $\nabla_{D,C}$'s instead of a single $\nabla_{D,C}$.
- Thus we find:

- **Theorem.** Let $D \subseteq [n]$. Let $\alpha = (\alpha_C)_{C \subseteq [n];\ |C|=|D|}$ be a family of scalars in $\mathbf{k}$ indexed by the $|D|$-element subsets of $[n]$. Then,

$$\left(\prod_{k=0}^{|D|} \left(\nabla_{D,\alpha} - \delta_{D,\alpha,k} \right) \right) \nabla_{D,\alpha} = 0,$$

where

$$\nabla_{D,\alpha} := \sum_{\substack{C \subseteq [n]; \\ |C|=|D|}} \alpha_C \nabla_{D,C} \in \mathbf{k}[S_n] \qquad \text{and}$$

$$\delta_{D,\alpha,k} := \sum_{\substack{C \subseteq [n]; \\ |C|=|D|}} \alpha_C \delta_{D,C,k} \in \mathbf{k}.$$

- **Theorem.** Let $D \subseteq [n]$. Let $\alpha = (\alpha_C)_{C \subseteq [n];\ |C| = |D|}$ be a family of scalars in $\mathbf{k}$ indexed by the $|D|$-element subsets of $[n]$. Then,

$$\left(\prod_{k=0}^{|D|} \left(\nabla_{D,\alpha} - \delta_{D,\alpha,k} \right) \right) \nabla_{D,\alpha} = 0,$$

where

$$\nabla_{D,\alpha} := \sum_{\substack{C \subseteq [n]; \\ |C| = |D|}} \alpha_C \nabla_{D,C} \in \mathbf{k}\left[S_n\right] \qquad \text{and}$$

$$\delta_{D,\alpha,k} := \sum_{\substack{C \subseteq [n]; \\ |C| = |D|}} \alpha_C \delta_{D,C,k} \in \mathbf{k}.$$

- Thus, the minimal polynomial of $\nabla_{D,\alpha}$ splits over $\mathbf{k}$.

Annihilating polynomials, 2

- **Theorem.** Let $D \subseteq [n]$. Let $\alpha = (\alpha_C)_{C \subseteq [n];\ |C|=|D|}$ be a family of scalars in $\mathbf{k}$ indexed by the $|D|$-element subsets of $[n]$. Then,

$$\left(\prod_{k=0}^{|D|} \left(\nabla_{D,\alpha} - \delta_{D,\alpha,k} \right) \right) \nabla_{D,\alpha} = 0,$$

  where

$$\nabla_{D,\alpha} := \sum_{\substack{C \subseteq [n]; \\ |C|=|D|}} \alpha_C \nabla_{D,C} \in \mathbf{k}\left[S_n\right] \qquad \text{and}$$

$$\delta_{D,\alpha,k} := \sum_{\substack{C \subseteq [n]; \\ |C|=|D|}} \alpha_C \delta_{D,C,k} \in \mathbf{k}.$$

- Thus, the minimal polynomial of $\nabla_{D,\alpha}$ splits over $\mathbf{k}$.
- In particular, the minimal polynomial of $\widetilde{\nabla}_{D,C}$ splits over $\mathbb{Z}$ (since $\widetilde{\nabla}_{D,C} = \nabla_{D,\alpha}$ for an appropriate α).

- The product rule for the ∇'s suggests another question.

- The product rule for the ∇'s suggests another question.
- The ∇'s are not linearly independent (e.g., we have $\nabla_{B,A} = \nabla_{[n]\backslash B,\ [n]\backslash A}$).
  What happens if we create linearly independent "abstract ∇'s" (call them Δ's) and define their product using the product rule?

- The product rule for the ∇'s suggests another question.
- The ∇'s are not linearly independent (e.g., we have $\nabla_{B,A} = \nabla_{[n]\setminus B,\ [n]\setminus A}$).
  What happens if we create linearly independent "abstract ∇'s" (call them Δ's) and define their product using the product rule?
- **Definition.** For any two subsets A and B of $[n]$ satisfying $|A| = |B|$, introduce a formal symbol $\Delta_{B,A}$. Let $\mathcal{D}$ be the free $\mathbf{k}$-module with basis $(\Delta_{B,A})_{A,B\subseteq[n] \text{ with } |A|=|B|}$. Define a multiplication on $\mathcal{D}$ by

$$\Delta_{D,C}\Delta_{B,A} := \omega_{B,C} \sum_{\substack{U\subseteq D, \\ V\subseteq A; \\ |U|=|V|}} (-1)^{|U|-|B\cap C|} \binom{|U|}{|B\cap C|} \Delta_{U,V}.$$

- The product rule for the ∇'s suggests another question.
- The ∇'s are not linearly independent (e.g., we have $\nabla_{B,A} = \nabla_{[n]\setminus B,\ [n]\setminus A}$).
  What happens if we create linearly independent "abstract ∇'s" (call them Δ's) and define their product using the product rule?
- **Definition.** For any two subsets A and B of $[n]$ satisfying $|A| = |B|$, introduce a formal symbol $\Delta_{B,A}$. Let $\mathcal{D}$ be the free **k**-module with basis $(\Delta_{B,A})_{A,B\subseteq[n] \text{ with } |A|=|B|}$. Define a multiplication on $\mathcal{D}$ by

$$\Delta_{D,C}\Delta_{B,A} := \omega_{B,C} \sum_{\substack{U\subseteq D,\\ V\subseteq A;\\ |U|=|V|}} (-1)^{|U|-|B\cap C|} \binom{|U|}{|B\cap C|} \Delta_{U,V}.$$

- **Theorem.** This makes $\mathcal{D}$ into a nonunital **k**-algebra.

- The product rule for the ∇'s suggests another question.
- The ∇'s are not linearly independent (e.g., we have $\nabla_{B,A} = \nabla_{[n]\setminus B,\ [n]\setminus A}$).
  What happens if we create linearly independent "abstract ∇'s" (call them Δ's) and define their product using the product rule?
- **Definition.** For any two subsets A and B of $[n]$ satisfying $|A| = |B|$, introduce a formal symbol $\Delta_{B,A}$. Let $\mathcal{D}$ be the free $\mathbf{k}$-module with basis $(\Delta_{B,A})_{A,B\subseteq[n] \text{ with } |A|=|B|}$. Define a multiplication on $\mathcal{D}$ by

$$\Delta_{D,C}\Delta_{B,A} := \omega_{B,C} \sum_{\substack{U\subseteq D,\\ V\subseteq A;\\ |U|=|V|}} (-1)^{|U|-|B\cap C|} \binom{|U|}{|B\cap C|} \Delta_{U,V}.$$

- **Theorem.** This makes $\mathcal{D}$ into a nonunital $\mathbf{k}$-algebra.
- **Conjecture.** If $n!$ is invertible in $\mathbf{k}$, then this algebra $\mathcal{D}$ has a unity.

- **Example.** For $n = 1$, the nonunital algebra $\mathcal{D}$ has basis (u, v) with $u = \Delta_{\varnothing,\varnothing}$ and $v = \Delta_{\{1\},\{1\}}$, and multiplication

$$uu = uv = vu = u, \qquad vv = v.$$

It is just $\mathbf{k} \times \mathbf{k}$.

- **Example.** For $n = 1$, the nonunital algebra $\mathcal{D}$ has basis (u, v) with $u = \Delta_{\varnothing,\varnothing}$ and $v = \Delta_{\{1\},\{1\}}$, and multiplication

$$uu = uv = vu = u, \qquad vv = v.$$

  It is just $\mathbf{k} \times \mathbf{k}$.

- **Example.** For $n = 2$, the nonunital algebra $\mathcal{D}$ has basis $(u, v_{11}, v_{12}, v_{21}, v_{22}, w)$ with $u = \Delta_{\varnothing,\varnothing}$ and $v_{ij} = \Delta_{\{i\},\{j\}}$ and $w = \Delta_{[2],[2]}$. The multiplication on $\mathcal{D}$ is

$$uu = uw = wu = 2u, \qquad uv_{ij} = v_{ij}u = u,$$
$$v_{dc}v_{ba} = u - v_{da} \qquad \text{if } b \neq c;$$
$$v_{dc}v_{ba} = v_{da} \qquad \text{if } b = c,$$
$$v_{ij}w = v_{i1} + v_{i2}, \qquad wv_{ij} = v_{1j} + v_{2j},$$
$$ww = 2w.$$

  This nonunital $\mathbf{k}$-algebra $\mathcal{D}$ has a unity if and only if 2 is invertible in $\mathbf{k}$. This unity is $\frac{1}{4}(v_{11} + v_{22} - v_{12} - v_{21} + 2w)$.

- **Question.** Is $\mathcal{D}$ a known object? Since $\mathcal{D}$ is a free $\mathbf{k}$-module of rank $\dbinom{2n}{n}$, could $\mathcal{D}$ be a nonunital $\mathbb{Z}$-form of the planar rook algebra (which is known to be $\cong \prod_{k=0}^{n} \mathbf{k}^{\binom{n}{k} \times \binom{n}{k}}$)?

- **Question.** Is $\mathcal{D}$ a known object? Since $\mathcal{D}$ is a free $\mathbf{k}$-module of rank $\binom{2n}{n}$, could $\mathcal{D}$ be a nonunital $\mathbb{Z}$-form of the planar rook algebra (which is known to be $\cong \prod_{k=0}^{n} \mathbf{k}^{\binom{n}{k} \times \binom{n}{k}}$)?
- **Question.** Barring that, is there a nice proof of the above theorem?

Row-to-row sums

- Let us generalize the $\nabla_{B,A}$.

- Let us generalize the $\nabla_{B,A}$.
- **Definition.** A **set composition** of $[n]$ is a tuple $\mathbf{U} = (U_1, U_2, \ldots, U_k)$ of disjoint nonempty subsets of $[n]$ such that $U_1 \cup U_2 \cup \cdots \cup U_k = [n]$. We set $\ell(\mathbf{U}) = k$ and call k the **length** of $\mathbf{U}$.

- Let us generalize the $\nabla_{B,A}$.
- **Definition.** A **set composition** of $[n]$ is a tuple $\mathbf{U} = (U_1, U_2, \ldots, U_k)$ of disjoint nonempty subsets of $[n]$ such that $U_1 \cup U_2 \cup \cdots \cup U_k = [n]$. We set $\ell(\mathbf{U}) = k$ and call k the **length** of $\mathbf{U}$.
- **Definition.** Let $\operatorname{SC}(n)$ be the set of all set compositions of $[n]$.

- Let us generalize the $\nabla_{B,A}$.
- **Definition.** A **set composition** of $[n]$ is a tuple $\mathbf{U} = (U_1, U_2, \ldots, U_k)$ of disjoint nonempty subsets of $[n]$ such that $U_1 \cup U_2 \cup \cdots \cup U_k = [n]$. We set $\ell(\mathbf{U}) = k$ and call k the **length** of $\mathbf{U}$.
- **Definition.** Let $\operatorname{SC}(n)$ be the set of all set compositions of $[n]$.
- **Definition.** If $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_k)$ are two set compositions of $[n]$ having the same length, then we define the **row-to-row sum**

$$\nabla_{\mathbf{B},\mathbf{A}} := \sum_{\substack{w \in S_n; \\ w(A_i) = B_i \text{ for all } i}} w \qquad \text{in } \mathbf{k}[S_n].$$

- Let us generalize the $\nabla_{B,A}$.
- **Definition.** A **set composition** of $[n]$ is a tuple $\mathbf{U} = (U_1, U_2, \ldots, U_k)$ of disjoint nonempty subsets of $[n]$ such that $U_1 \cup U_2 \cup \cdots \cup U_k = [n]$. We set $\ell(\mathbf{U}) = k$ and call k the **length** of $\mathbf{U}$.
- **Definition.** Let $\operatorname{SC}(n)$ be the set of all set compositions of $[n]$.
- **Definition.** If $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_k)$ are two set compositions of $[n]$ having the same length, then we define the **row-to-row sum**

$$\nabla_{\mathbf{B},\mathbf{A}} := \sum_{\substack{w \in S_n; \\ w(A_i) = B_i \text{ for all } i}} w \qquad \text{in } \mathbf{k}[S_n].$$

- **Example.** We have

$$\nabla_{B,A} = \nabla_{\mathbf{B},\mathbf{A}} \qquad \text{for } \mathbf{B} = (B,\ [n] \setminus B) \text{ and } \mathbf{A} = (A,\ [n] \setminus A).$$

- **Proposition.** Let $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_k)$.
  - **(a)** We have $\nabla_{\mathbf{B},\mathbf{A}} = 0$ unless $|A_i| = |B_i|$ for all i.
  - **(b)** We have $\nabla_{\mathbf{B},\mathbf{A}} = \nabla_{\mathbf{B}\sigma,\mathbf{A}\sigma}$ for any $\sigma \in S_k$ (acting on set compositions by permuting the blocks).
  - **(c)** We have $S(\nabla_{\mathbf{B},\mathbf{A}}) = \nabla_{\mathbf{A},\mathbf{B}}$, where $S(w) = w^{-1}$ for all $w \in S_n$ as before.

- **Proposition.** Let $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_k)$.
  - **(a)** We have $\nabla_{\mathbf{B},\mathbf{A}} = 0$ unless $|A_i| = |B_i|$ for all i.
  - **(b)** We have $\nabla_{\mathbf{B},\mathbf{A}} = \nabla_{\mathbf{B}\sigma,\mathbf{A}\sigma}$ for any $\sigma \in S_k$ (acting on set compositions by permuting the blocks).
  - **(c)** We have $S(\nabla_{\mathbf{B},\mathbf{A}}) = \nabla_{\mathbf{A},\mathbf{B}}$, where $S(w) = w^{-1}$ for all $w \in S_n$ as before.
- The minimal polynomial of $\nabla_{\mathbf{B},\mathbf{A}}$ does not always split over $\mathbb{Z}$ unless $\ell(\mathbf{A}) \leq 2$.

- **Proposition.** Let $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_k)$.
  - **(a)** We have $\nabla_{\mathbf{B},\mathbf{A}} = 0$ unless $|A_i| = |B_i|$ for all i.
  - **(b)** We have $\nabla_{\mathbf{B},\mathbf{A}} = \nabla_{\mathbf{B}\sigma,\mathbf{A}\sigma}$ for any $\sigma \in S_k$ (acting on set compositions by permuting the blocks).
  - **(c)** We have $S(\nabla_{\mathbf{B},\mathbf{A}}) = \nabla_{\mathbf{A},\mathbf{B}}$, where $S(w) = w^{-1}$ for all $w \in S_n$ as before.
- The minimal polynomial of $\nabla_{\mathbf{B},\mathbf{A}}$ does not always split over $\mathbb{Z}$ unless $\ell(\mathbf{A}) \leq 2$.
- The $\nabla_{\mathbf{B},\mathbf{A}}$ are not entirely new:
  The **Murphy basis** of $\mathbf{k}[S_n]$ consists of the elements $\nabla_{\mathbf{B},\mathbf{A}}$ for the **standard** set compositions $\mathbf{A}$ and $\mathbf{B}$ of $[n]$. Here, "standard" means that the blocks are the rows of a standard Young tableau (in particular, they must be of partition shape).
  See G. E. Murphy, *On the Representation Theory of the Symmetric Groups and Associated Hecke Algebras*, 1991.

- **Theorem.** Let $\mathcal{A} = \mathbf{k}[S_n]$. Let $k \in \mathbb{N}$. We define two $\mathbf{k}$-submodules $\mathcal{I}_k$ and $\mathcal{J}_k$ of $\mathcal{A}$ by

$$\mathcal{I}_k := \operatorname{span}\{\nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{A}, \mathbf{B} \in \operatorname{SC}(n) \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k\}$$

and

$$\mathcal{J}_k := \mathcal{A} \cdot \operatorname{span}\left\{\boldsymbol{\alpha}_U^- \mid U \subseteq [n] \text{ of size } k+1\right\} \cdot \mathcal{A},$$

where

$$\boldsymbol{\alpha}_U^- := \sum_{\sigma \in S_U} (-1)^\sigma \sigma \in \mathbf{k}[S_n].$$

Then:

- **Theorem.** Let $\mathcal{A} = \mathbf{k}[S_n]$. Let $k \in \mathbb{N}$. We define two $\mathbf{k}$-submodules $\mathcal{I}_k$ and $\mathcal{J}_k$ of $\mathcal{A}$ by

  $$\mathcal{I}_k := \operatorname{span}\{\nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{A}, \mathbf{B} \in \mathrm{SC}(n) \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k\}$$

  and

  $$\mathcal{J}_k := \mathcal{A} \cdot \operatorname{span}\left\{\boldsymbol{\alpha}_U^- \mid U \subseteq [n] \text{ of size } k+1\right\} \cdot \mathcal{A},$$

  where

  $$\boldsymbol{\alpha}_U^- := \sum_{\sigma \in S_U} (-1)^\sigma \sigma \in \mathbf{k}[S_n].$$

  Then:

  (a) Both $\mathcal{I}_k$ and $\mathcal{J}_k$ are ideals of $\mathcal{A}$, and are preserved under S.

- **Theorem (cont'd).**

  (b) We have

$$\mathcal{I}_k = \mathcal{J}_k^{\perp} = \operatorname{LAnn}\mathcal{J}_k = \operatorname{RAnn}\mathcal{J}_k \qquad \text{and}$$
$$\mathcal{J}_k = \mathcal{I}_k^{\perp} = \operatorname{LAnn}\mathcal{I}_k = \operatorname{RAnn}\mathcal{I}_k.$$

  Here, $\mathcal{U}^{\perp}$ means orthogonal complement wrt the standard bilinear form on $\mathcal{A}$, whereas LAnn and RAnn mean left and right annihilators.

- **Theorem (cont'd).**

  (b) We have

$$\mathcal{I}_k = \mathcal{J}_k^{\perp} = \operatorname{LAnn} \mathcal{J}_k = \operatorname{RAnn} \mathcal{J}_k \qquad \text{and}$$
$$\mathcal{J}_k = \mathcal{I}_k^{\perp} = \operatorname{LAnn} \mathcal{I}_k = \operatorname{RAnn} \mathcal{I}_k.$$

  Here, $\mathcal{U}^{\perp}$ means orthogonal complement wrt the standard bilinear form on $\mathcal{A}$, whereas LAnn and RAnn mean left and right annihilators.

  (c) The **k**-module $\mathcal{I}_k$ is free of rank $= \#$ of $(1, 2, \ldots, k+1)$-avoiding permutations in S_n.

  (d) The **k**-module $\mathcal{J}_k$ is free of rank $= \#$ of $(1, 2, \ldots, k+1)$-nonavoiding permutations in S_n.

The span of the generalized ∇'s, 2

- **Theorem (cont'd).**

  (b) We have

$$\mathcal{I}_k = \mathcal{J}_k^{\perp} = \operatorname{LAnn} \mathcal{J}_k = \operatorname{RAnn} \mathcal{J}_k \qquad \text{and}$$
$$\mathcal{J}_k = \mathcal{I}_k^{\perp} = \operatorname{LAnn} \mathcal{I}_k = \operatorname{RAnn} \mathcal{I}_k .$$

  Here, $\mathcal{U}^{\perp}$ means orthogonal complement wrt the standard bilinear form on $\mathcal{A}$, whereas LAnn and RAnn mean left and right annihilators.

  (c) The $\mathbf{k}$-module $\mathcal{I}_k$ is free of rank $= \#$ of $(1, 2, \ldots, k+1)$-avoiding permutations in S_n.

  (d) The $\mathbf{k}$-module $\mathcal{J}_k$ is free of rank $= \#$ of $(1, 2, \ldots, k+1)$-nonavoiding permutations in S_n.

  (e) The quotients $\mathcal{A}/\mathcal{J}_k$ and $\mathcal{A}/\mathcal{I}_k$ are also free, with the same ranks as $\mathcal{I}_k$ and $\mathcal{J}_k$ (respectively), and with bases consisting of (residue classes of) the relevant permutations.

- **Theorem (cont'd).**
  (f) If $n!$ is invertible in $\mathbf{k}$, then $\mathcal{A} = \mathcal{I}_k \oplus \mathcal{J}_k$ (internal direct sum) as $\mathbf{k}$-modules, and $\mathcal{A} \cong \mathcal{I}_k \times \mathcal{J}_k$ as $\mathbf{k}$-algebras.

- **Theorem (cont'd).**
  (f) If $n!$ is invertible in $\mathbf{k}$, then $\mathcal{A} = \mathcal{I}_k \oplus \mathcal{J}_k$ (internal direct sum) as $\mathbf{k}$-modules, and $\mathcal{A} \cong \mathcal{I}_k \times \mathcal{J}_k$ as $\mathbf{k}$-algebras.
- **Proof.** When $\mathbf{k}$ is a char-0 field, this can be done using representations (note that $\nabla_{\mathbf{B},\mathbf{A}}$ vanishes on each Specht module S^λ with $\ell(\lambda) > \ell(\mathbf{A})$). In particular, $\mathcal{A} \cong \mathcal{I}_k \times \mathcal{J}_k$ is (up to iso? morally?) a coarsening of the Artin–Wedderburn decomposition of $\mathcal{A}$.

- **Theorem (cont'd).**
  (f) If $n!$ is invertible in $\mathbf{k}$, then $\mathcal{A} = \mathcal{I}_k \oplus \mathcal{J}_k$ (internal direct sum) as $\mathbf{k}$-modules, and $\mathcal{A} \cong \mathcal{I}_k \times \mathcal{J}_k$ as $\mathbf{k}$-algebras.
- **Proof.** When $\mathbf{k}$ is a char-0 field, this can be done using representations (note that $\nabla_{\mathbf{B},\mathbf{A}}$ vanishes on each Specht module S^λ with $\ell(\lambda) > \ell(\mathbf{A})$). In particular, $\mathcal{A} \cong \mathcal{I}_k \times \mathcal{J}_k$ is (up to iso? morally?) a coarsening of the Artin–Wedderburn decomposition of $\mathcal{A}$.
  The case of general $\mathbf{k}$ is harder and has to be done from scratch.

- **Theorem (cont'd).**
  (f) If $n!$ is invertible in $\mathbf{k}$, then $\mathcal{A} = \mathcal{I}_k \oplus \mathcal{J}_k$ (internal direct sum) as $\mathbf{k}$-modules, and $\mathcal{A} \cong \mathcal{I}_k \times \mathcal{J}_k$ as $\mathbf{k}$-algebras.
- **Proof.** When $\mathbf{k}$ is a char-0 field, this can be done using representations (note that $\nabla_{\mathbf{B},\mathbf{A}}$ vanishes on each Specht module S^λ with $\ell(\lambda) > \ell(\mathbf{A})$). In particular, $\mathcal{A} \cong \mathcal{I}_k \times \mathcal{J}_k$ is (up to iso? morally?) a coarsening of the Artin–Wedderburn decomposition of $\mathcal{A}$.
  The case of general $\mathbf{k}$ is harder and has to be done from scratch.
- **Question.** Is there a product rule for the $\nabla_{\mathbf{B},\mathbf{A}}$'s?
- **Question.** How much of the representation theory of S_n can be developed using the $\nabla_{\mathbf{B},\mathbf{A}}$'s? (e.g., I think you can prove $\sum\limits_{\lambda \vdash n} \left(f^\lambda\right)^2 = n!$ using the Murphy basis and the Garnir relations.)

Unrelated(?): A commuting family, 1

- Here is something rather different.

Unrelated(?): A commuting family, 1

- Here is something rather different.
- The following is joint work with Theo Douvropoulos, inspired by the work of Mukhin/Tarasov/Varchenko on the Gaudin Bethe ansatz.

- Here is something rather different.
- The following is joint work with Theo Douvropoulos, inspired by the work of Mukhin/Tarasov/Varchenko on the Gaudin Bethe ansatz.
- **Definition.** Let $\sigma \in S_n$ be a permutation. Then, we define

$$\operatorname{exc} \sigma := (\# \text{ of } i \in [n] \text{ such that } \sigma(i) > i) \qquad \text{and}$$
$$\operatorname{anxc} \sigma := (\# \text{ of } i \in [n] \text{ such that } \sigma(i) < i)$$

  (the "**excedance number**" and the "**anti-excedance number**" of σ).

Unrelated(?): A commuting family, 1

- Here is something rather different.
- The following is joint work with Theo Douvropoulos, inspired by the work of Mukhin/Tarasov/Varchenko on the Gaudin Bethe ansatz.
- **Definition.** Let $\sigma \in S_n$ be a permutation. Then, we define

$$\operatorname{exc} \sigma := (\# \text{ of } i \in [n] \text{ such that } \sigma(i) > i) \qquad \text{and}$$
$$\operatorname{anxc} \sigma := (\# \text{ of } i \in [n] \text{ such that } \sigma(i) < i)$$

  (the "**excedance number**" and the "**anti-excedance number**" of σ).
- For any $a, b \in \mathbb{N}$, define

$$\mathbf{X}_{a,b} := \sum_{\substack{\sigma \in S_n; \\ \operatorname{exc} \sigma = a; \\ \operatorname{anxc} \sigma = b}} \sigma \in \mathbf{k}[S_n].$$

- **Conjecture.** The elements $\mathbf{X}_{a,b}$ for all $a, b \in \mathbb{N}$ commute (for fixed n).
- Checked for all $n \leq 7$ using SageMath.

- The antipode plays well with these elements:

$$S(\mathbf{X}_{a,b}) = \mathbf{X}_{b,a}.$$

- **Question.** What can be said about the $\mathbf{k}$-subalgebra $\mathbf{k}[\mathbf{X}_{a,b} \mid a, b \in \{0, 1, \ldots, n\}]$ of $\mathbf{k}[S_n]$? Note:

n	1	2	3	4	5	6
$\dim(\mathbb{Q}[\mathbf{X}_{a,b}])$	1	2	4	10	26	76

.

So far, this looks like the # of involutions in S_n, which is exactly the dimension of the Gelfand–Zetlin subalgebra (generated by the Young–Jucys–Murphy elements)!

- What is the exact relation?

Thank you,

- **Per Alexandersson** and **Theo Douvropoulos** for conversations in 2023 that motivated this project.
- **Nadia Lafrenière, Jon Novak, Vic Reiner, Richard P. Stanley** for helpful comments.
- **the organizers** for the invitation.
- **you** for your patience.