Counterexample for MathOverflow #84345

Darij Grinberg

version 19 May 2012

This is not a paper, but just a computation that I am writing up in order to ensure there are no errors. (As if this could be ensured this way.)

The purpose of the computation is to negatively answer the Concrete Question in [3].

§1. The largest involutive quotient of a Hopf algebra

First a very easy positive result.

Definition 1. Let H be a Hopf algebra over a field k. We say that the Hopf algebra H is *involutive* if $S^2 = \mathrm{id}$, where S denotes the antipode of H.

Theorem 2. Let H be a Hopf algebra over a field k. Let $\operatorname{Inv} H$ denote the k-submodule $H \cdot ((S^2 - \mathrm{id})(H)) \cdot H$ of H (where S denotes the antipode of H). Then:

(a) The k-submodule $\operatorname{Inv} H$ is a Hopf ideal of H.

(b) The quotient Hopf algebra $H \diagup (\operatorname{Inv} H)$ is an involutive Hopf algebra.

(c) Whenever G is an involutive Hopf algebra and $f : H \to G$ is a Hopf algebra homomorphism, we have $f(\operatorname{Inv} H) = 0$, so that the homomorphism f factors through $H \diagup (\operatorname{Inv} H)$.

(d) If H is a graded Hopf algebra, then $\operatorname{Inv} H$ is a homogeneous ideal, so that $H \diagup (\operatorname{Inv} H)$ is a graded Hopf algebra canonically.

(e) If H is a connected graded Hopf algebra, then $H \diagup (\operatorname{Inv} H)$ is a connected graded Hopf algebra.

Theorem 2 shows that we can consider $H \diagup (\operatorname{Inv} H)$ as the largest involutive quotient Hopf algebra of H. While it is well-known that every commutative and every cocommutative Hopf algebra is involutive, it is definitely not true that every involutive Hopf algebra is either commutative or cocommutative; and Theorem 2 shows how to construct involutive Hopf algebras which are neither commutative nor cocommutative from "generic" Hopf algebras.

There is also a different way to construct involutive Hopf algebras: namely, by taking the tensor product of a commutative with a cocommutative Hopf algebra. However, these are not really the most general possible case; in particular, they don't help me answering the Concrete Question in [3].

In order to prove the main part of Theorem 2 **(a)**, we abstract from the antipode and prove something more general:

Theorem 3. Let H be a bialgebra over a field k. Let $t : H \to H$ be a coalgebra homomorphism. Let $\operatorname{Inv}_t H$ denote the k-submodule $H \cdot ((t - \mathrm{id})(H)) \cdot H$ of H. Then, the k-submodule $\operatorname{Inv}_t H$ is a biideal of H.

In the following proofs of Theorem 2 and Theorem 3, we will use the standard notations for bialgebras: namely, we will denote by Δ the comultiplication of H, and by ε the counit of H.

Proof of Theorem 3. In the following, id will always denote id_H, whereas the identity map of $H \otimes H$ will always be written $\mathrm{id}_{H\otimes H}$.

First of all,

$$H \cdot \underbrace{(\operatorname{Inv}_t H)}_{=H\cdot((t-\mathrm{id})(H))\cdot H} = \underbrace{H \cdot H}_{\subseteq H} \cdot ((t-\mathrm{id})(H)) \cdot H \subseteq H \cdot ((t-\mathrm{id})(H)) \cdot H = \operatorname{Inv}_t H,$$

so that $\operatorname{Inv}_t H$ is a left ideal of H. Similarly, $\operatorname{Inv}_t H$ is a right ideal of H. Thus, $\operatorname{Inv}_t H$ is both a left ideal and a right ideal of H. In other words, $\operatorname{Inv}_t H$ is an ideal of H.

Note that

$$\operatorname{Inv}_t H = \underbrace{H}_{\ni 1} \cdot ((t-\mathrm{id})(H)) \cdot \underbrace{H}_{\ni 1} \supseteq 1 \cdot ((t-\mathrm{id})(H)) \cdot 1 = (t-\mathrm{id})(H).$$

Furthermore, $t \otimes t = (t \otimes \mathrm{id}) \circ (\mathrm{id} \otimes t)$, where id denotes id_H [1]. Also notice that $\Delta \circ t = (t \otimes t) \circ \Delta$ (since t is a coalgebra homomorphism). Thus,

$$\Delta \circ t = \underbrace{(t \otimes t)}_{=(t\otimes\mathrm{id})\circ(\mathrm{id}\otimes t)} \circ \Delta = (t \otimes \mathrm{id}) \circ (\mathrm{id} \otimes t) \circ \Delta.$$

Hence,

$$\begin{aligned}\Delta \circ (t-\mathrm{id}) &= \underbrace{\Delta \circ t}_{=(t\otimes\mathrm{id})\circ(\mathrm{id}\otimes t)\circ\Delta} - \underbrace{\Delta \circ \mathrm{id}}_{=\Delta=\mathrm{id}_{H\otimes H}\circ\Delta} \qquad \text{(since composition of } k\text{-linear maps is } k\text{-bilinear)}\\ &= (t \otimes \mathrm{id}) \circ (\mathrm{id} \otimes t) \circ \Delta - \mathrm{id}_{H\otimes H} \circ \Delta = ((t \otimes \mathrm{id}) \circ (\mathrm{id} \otimes t) - \mathrm{id}_{H\otimes H}) \circ \Delta.\end{aligned}$$

[1] *Proof.* Every $(x, y) \in H \otimes H$ satisfies

$$\begin{aligned}((t \otimes \mathrm{id}) \circ (\mathrm{id} \otimes t))(x \otimes y) &= (t \otimes \mathrm{id}) \left(\underbrace{(\mathrm{id} \otimes t)(x \otimes y)}_{=\mathrm{id}(x)\otimes t(y)} \right) = (t \otimes \mathrm{id})(\mathrm{id}(x) \otimes t(y))\\ &= t\left(\underbrace{\mathrm{id}(x)}_{=x}\right) \otimes \underbrace{\mathrm{id}(t(y))}_{=t(y)} = t(x) \otimes t(y) = (t \otimes t)(x \otimes y).\end{aligned}$$

In other words, the two maps $(t \otimes \mathrm{id}) \circ (\mathrm{id} \otimes t)$ and $t \otimes t$ are equal to each other on every pure tensor. Since these two maps are k-linear, this yields that these two maps are identic (because two k-linear maps from a tensor product must be identic if they are equal to each other on every pure tensor). In other words, $t \otimes t = (t \otimes \mathrm{id}) \circ (\mathrm{id} \otimes t)$, qed.

Since

$$\begin{aligned}
&(t\otimes \mathrm{id})\circ(\mathrm{id}\otimes t)-\mathrm{id}_{H\otimes H}\\
&=(t\otimes \mathrm{id})\circ(\mathrm{id}\otimes t)-\underbrace{(\mathrm{id}\otimes t)}_{=\mathrm{id}_{H\otimes H}\circ(\mathrm{id}\otimes t)}+(\mathrm{id}\otimes t)-\mathrm{id}_{H\otimes H}\\
&=(t\otimes \mathrm{id})\circ(\mathrm{id}\otimes t)-\underbrace{\mathrm{id}_{H\otimes H}}_{=\mathrm{id}\otimes\mathrm{id}}\circ(\mathrm{id}\otimes t)+(\mathrm{id}\otimes t)-\underbrace{\mathrm{id}_{H\otimes H}}_{=\mathrm{id}\otimes\mathrm{id}}\\
&=\underbrace{(t\otimes \mathrm{id})\circ(\mathrm{id}\otimes t)-(\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes t)}_{\substack{=(t\otimes\mathrm{id}-\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes t)\\ \text{(since composition of }k\text{-linear maps}\\ \text{is }k\text{-bilinear)}}}+\underbrace{(\mathrm{id}\otimes t)-(\mathrm{id}\otimes\mathrm{id})}_{\substack{=\mathrm{id}\otimes(t-\mathrm{id})\\ \text{(since tensoring of }k\text{-linear maps}\\ \text{is }k\text{-bilinear)}}}\\
&=\underbrace{(t\otimes\mathrm{id}-\mathrm{id}\otimes\mathrm{id})}_{\substack{=(t-\mathrm{id})\otimes\mathrm{id}\\ \text{(since tensoring of }k\text{-linear maps}\\ \text{is }k\text{-bilinear)}}}\circ(\mathrm{id}\otimes t)-\mathrm{id}\otimes(t-\mathrm{id})\\
&=((t-\mathrm{id})\otimes\mathrm{id})\circ(\mathrm{id}\otimes t)-\mathrm{id}\otimes(t-\mathrm{id}),
\end{aligned}$$

this becomes

$$\begin{aligned}
\Delta\circ(t-\mathrm{id})&=\left(\underbrace{(t\otimes \mathrm{id})\circ(\mathrm{id}\otimes t)-\mathrm{id}_{H\otimes H}}_{=((t-\mathrm{id})\otimes\mathrm{id})\circ(\mathrm{id}\otimes t)-\mathrm{id}\otimes(t-\mathrm{id})}\right)\circ\Delta\\
&=(((t-\mathrm{id})\otimes\mathrm{id})\circ(\mathrm{id}\otimes t)-\mathrm{id}\otimes(t-\mathrm{id}))\circ\Delta.
\end{aligned}$$

Thus,

$$\begin{aligned}
(\Delta\circ(t-\mathrm{id}))(H)&=((((t-\mathrm{id})\otimes\mathrm{id})\circ(\mathrm{id}\otimes t)-\mathrm{id}\otimes(t-\mathrm{id}))\circ\Delta)(H)\\
&=(((t-\mathrm{id})\otimes\mathrm{id})\circ(\mathrm{id}\otimes t)-\mathrm{id}\otimes(t-\mathrm{id}))\underbrace{(\Delta(H))}_{\subseteq H\otimes H}\\
&\subseteq(((t-\mathrm{id})\otimes\mathrm{id})\circ(\mathrm{id}\otimes t)-\mathrm{id}\otimes(t-\mathrm{id}))(H\otimes H)\\
&\subseteq\underbrace{(((t-\mathrm{id})\otimes\mathrm{id})\circ(\mathrm{id}\otimes t))(H\otimes H)}_{=((t-\mathrm{id})\otimes\mathrm{id})((\mathrm{id}\otimes t)(H\otimes H))}-\underbrace{(\mathrm{id}\otimes(t-\mathrm{id}))(H\otimes H)}_{=\mathrm{id}(H)\otimes(t-\mathrm{id})(H)}\\
&=((t-\mathrm{id})\otimes\mathrm{id})\underbrace{((\mathrm{id}\otimes t)(H\otimes H))}_{\subseteq H\otimes H}-\underbrace{\mathrm{id}(H)}_{=H}\otimes(t-\mathrm{id})(H)\\
&\subseteq\underbrace{((t-\mathrm{id})\otimes\mathrm{id})(H\otimes H)}_{=(t-\mathrm{id})(H)\otimes\mathrm{id}(H)}-H\otimes(t-\mathrm{id})(H)\\
&=(t-\mathrm{id})(H)\otimes\underbrace{\mathrm{id}(H)}_{=H}-H\otimes(t-\mathrm{id})(H)\\
&=\underbrace{(t-\mathrm{id})(H)}_{\subseteq\mathrm{Inv}_t H}\otimes H-H\otimes\underbrace{(t-\mathrm{id})(H)}_{\substack{\subseteq\mathrm{Inv}_t H=-\mathrm{Inv}_t H\\ \text{(since }\mathrm{Inv}_t H\text{ is a }k\text{-module)}}}\\
&\subseteq(\mathrm{Inv}_t H)\otimes H-H\otimes(-\mathrm{Inv}_t H)=(\mathrm{Inv}_t H)\otimes H+H\otimes(\mathrm{Inv}_t H).
\end{aligned}$$

Now, it is a known (and very easy) fact that if $\mathfrak{A}$ and $\mathfrak{B}$ are two k-algebras, and $\mathfrak{I}$ is an ideal of $\mathfrak{A}$, then $\mathfrak{I}\otimes\mathfrak{B}$ is an ideal of the k-algebra $\mathfrak{A}\otimes\mathfrak{B}$. Applied to $\mathfrak{A}=H$,

$\mathfrak{B} = H$ and $\mathfrak{I} = \operatorname{Inv}_t H$, this yields that $(\operatorname{Inv}_t H) \otimes H$ is an ideal of $H \otimes H$. Similarly, $H \otimes (\operatorname{Inv}_t H)$ is an ideal of $H \otimes H$.

So we know that $(\operatorname{Inv}_t H) \otimes H$ and $H \otimes (\operatorname{Inv}_t H)$ are ideals of $H \otimes H$. Thus, $(\operatorname{Inv}_t H) \otimes H + H \otimes (\operatorname{Inv}_t H)$ is an ideal of $H \otimes H$ (because the sum of two ideals is always an ideal).

Now, $\operatorname{Inv}_t H = H \cdot ((t - \operatorname{id})(H)) \cdot H$, so that

$$\begin{aligned}
\Delta(\operatorname{Inv}_t H) &= \Delta(H \cdot ((t-\operatorname{id})(H)) \cdot H) = \underbrace{(\Delta(H))}_{\subseteq H\otimes H} \cdot \underbrace{(\Delta((t-\operatorname{id})(H)))}_{\substack{=(\Delta\circ(t-\operatorname{id}))(H) \\ \subseteq(\operatorname{Inv}_t H)\otimes H + H\otimes(\operatorname{Inv}_t H)}} \cdot \underbrace{(\Delta(H))}_{\subseteq H\otimes H} \\
&\qquad \left(\begin{array}{c}\text{since } H \text{ is a bialgebra, so (by the axioms of a bialgebra)} \\ \Delta \text{ is a } k\text{-algebra homomorphism}\end{array}\right) \\
&\subseteq (H\otimes H) \cdot \underbrace{((\operatorname{Inv}_t H) \otimes H + H \otimes (\operatorname{Inv}_t H)) \cdot (H \otimes H)}_{\substack{\subseteq(\operatorname{Inv}_t H)\otimes H + H\otimes(\operatorname{Inv}_t H) \\ \text{(since } (\operatorname{Inv}_t H)\otimes H + H\otimes(\operatorname{Inv}_t H) \text{ is an ideal of } H\otimes H)}} \\
&\subseteq (H\otimes H) \cdot ((\operatorname{Inv}_t H) \otimes H + H \otimes (\operatorname{Inv}_t H)) \subseteq (\operatorname{Inv}_t H) \otimes H + H \otimes (\operatorname{Inv}_t H)
\end{aligned}$$

(since $(\operatorname{Inv}_t H) \otimes H + H \otimes (\operatorname{Inv}_t H)$ is an ideal of $H \otimes H$).

Moreover,

$$\begin{aligned}
\varepsilon \circ (t - \operatorname{id}) &= \underbrace{\varepsilon \circ t}_{\substack{=\varepsilon \\ \text{(since } t \text{ is a} \\ \text{coalgebra homomorphism)}}} - \underbrace{\varepsilon \circ \operatorname{id}}_{=\varepsilon} \qquad (\text{since composition of } k\text{-linear maps is } k\text{-bilinear}) \\
&= \varepsilon - \varepsilon = 0.
\end{aligned}$$

Now, $\operatorname{Inv}_t H = H \cdot ((t - \operatorname{id})(H)) \cdot H$, so that

$$\begin{aligned}
\varepsilon(\operatorname{Inv}_t H) &= \varepsilon(H \cdot ((t-\operatorname{id})(H)) \cdot H) = (\varepsilon(H)) \cdot \underbrace{(\varepsilon((t-\operatorname{id})(H)))}_{=(\varepsilon\circ(t-\operatorname{id}))(H)=0} \cdot (\varepsilon(H)) \\
&\qquad \left(\begin{array}{c}\text{since } H \text{ is a bialgebra, so (by the axioms of a bialgebra)} \\ \varepsilon \text{ is a } k\text{-algebra homomorphism}\end{array}\right) \\
&= (\varepsilon(H)) \cdot 0 \cdot (\varepsilon(H)) = 0.
\end{aligned}$$

So we have shown that $\Delta(\operatorname{Inv}_t H) \subseteq (\operatorname{Inv}_t H) \otimes H + H \otimes (\operatorname{Inv}_t H)$ and $\varepsilon(\operatorname{Inv}_t H) = 0$. In other words, $\operatorname{Inv}_t H$ is a coideal of H.

Since $\operatorname{Inv}_t H$ is both an ideal and a coideal of H, it follows that $\operatorname{Inv}_t H$ is a biideal of H. This proves Theorem 3.

Proof of Theorem 2. **(a)** It is well-known that the antipode S is an anti-coalgebra homomorphism. Hence, $S \circ S$ is a coalgebra homomorphism (because the composition of any two anti-coalgebra homomorphisms is a coalgebra homomorphism). In other words, S^2 is a coalgebra homomorphism (since $S^2 = S \circ S$). Hence, we can apply Theorem 3 to $t = S^2$. This yields that the k-submodule $\operatorname{Inv}_{S^2} H$ is a biideal of H (where $\operatorname{Inv}_{S^2} H$ is defined as in Theorem 3). Since

$$\begin{aligned}
\operatorname{Inv}_{S^2} H &= H \cdot \left(\left(S^2 - \operatorname{id}\right)(H)\right) \cdot H \qquad (\text{by the definition of } \operatorname{Inv}_{S^2} H) \\
&= \operatorname{Inv} H,
\end{aligned}$$

this yields that $\operatorname{Inv} H$ is a biideal of H.

Now,

$$\begin{aligned}
S\circ\left(S^2-\mathrm{id}\right) &= \underbrace{S\circ S^2}_{=S^3=S^2\circ S} - \underbrace{S\circ\mathrm{id}}_{=S=\mathrm{id}\circ S} \qquad \text{(since composition of k-linear maps is k-bilinear)}\\
&= S^2\circ S-\mathrm{id}\circ S=\left(S^2-\mathrm{id}\right)\circ S \qquad \text{(since composition of k-linear maps is k-bilinear)},
\end{aligned}$$

so that

$$\left(S\circ\left(S^2-\mathrm{id}\right)\right)(H)=\left(\left(S^2-\mathrm{id}\right)\circ S\right)(H)=\left(S^2-\mathrm{id}\right)\left(\underbrace{S(H)}_{\subseteq H}\right)\subseteq\left(S^2-\mathrm{id}\right)(H).$$

But $\operatorname{Inv} H=H\cdot\left(\left(S^2-\mathrm{id}\right)(H)\right)\cdot H$ yields

$$\begin{aligned}
S(\operatorname{Inv} H) &= S\left(H\cdot\left(\left(S^2-\mathrm{id}\right)(H)\right)\cdot H\right)=\underbrace{(S(H))}_{\subseteq H}\cdot\underbrace{\left(S\left(\left(S^2-\mathrm{id}\right)(H)\right)\right)}_{\substack{=\left(S\circ\left(S^2-\mathrm{id}\right)\right)(H)\\ \subseteq\left(S^2-\mathrm{id}\right)(H)}}\cdot\underbrace{(S(H))}_{\subseteq H}\\
&\qquad \text{(since the antipode S is an anti-algebra homomorphism)}\\
&\subseteq H\cdot\left(\left(S^2-\mathrm{id}\right)(H)\right)\cdot H=\operatorname{Inv} H.
\end{aligned}$$

Thus, $\operatorname{Inv} H$ is a biideal of H satisfying $S(\operatorname{Inv} H)\subseteq\operatorname{Inv} H$. In other words, $\operatorname{Inv} H$ is a Hopf ideal of H. This proves Theorem 2 **(a)**.

(b) For every $y\in H$, let $\overline{y}$ denote the residue class of y modulo $\operatorname{Inv} H$.

Note that

$$\operatorname{Inv} H=\underbrace{H}_{\ni 1}\cdot\left(\left(S^2-\mathrm{id}\right)(H)\right)\cdot\underbrace{H}_{\ni 1}\supseteq 1\cdot\left(\left(S^2-\mathrm{id}\right)(H)\right)\cdot 1=\left(S^2-\mathrm{id}\right)(H).$$

Let $\overline{S}$ denote the antipode of the quotient Hopf algebra $H\diagup(\operatorname{Inv} H)$. By the definition of the antipode of a quotient Hopf algebra, we have $\overline{S}(\overline{x})=\overline{S(x)}$ for every $x\in H$. Thus, every $y\in H$ satisfies

$$\begin{aligned}
\overline{S}^2(\overline{y}) &= \overline{S}\left(\overline{S}(\overline{y})\right)=\overline{S}\left(\overline{S(y)}\right)\\
&\qquad \left(\text{since the formula } \overline{S}(\overline{x})=\overline{S(x)} \text{ (applied to } x=y\text{) yields } \overline{S}(\overline{y})=\overline{S(y)}\right)\\
&= \overline{S(S(y))} \qquad \left(\text{by the formula } \overline{S}(\overline{x})=\overline{S(x)} \text{ (applied to } x=S(y)\text{)}\right)\\
&= \overline{y+\left(S^2-\mathrm{id}\right)(y)}\\
&\qquad \left(\begin{array}{c}\text{since}\\ S(S(y))=S^2(y)=y+S^2(y)-\underbrace{y}_{=\mathrm{id}(y)}=y+\underbrace{S^2(y)-\mathrm{id}(y)}_{=\left(S^2-\mathrm{id}\right)(y)}=y+\left(S^2-\mathrm{id}\right)(y)\end{array}\right)\\
&= \overline{y}\\
&\qquad \left(\text{since } \left(S^2-\mathrm{id}\right)\underbrace{(y)}_{\in H}\in\left(S^2-\mathrm{id}\right)(H)\subseteq\operatorname{Inv} H, \text{ so that } y+\left(S^2-\mathrm{id}\right)(y)\equiv y \bmod \operatorname{Inv} H\right).
\end{aligned}$$

Thus, every $z \in H\diagup(\operatorname{Inv} H)$ satisfies $\overline{S}^2(z) = \operatorname{id}_{H\diagup(\operatorname{Inv} H)}(z)$ [2]. In other words, $\overline{S}^2 = \operatorname{id}_{H\diagup(\operatorname{Inv} H)}$. This shows that the quotient Hopf algebra $H\diagup(\operatorname{Inv} H)$ is involutive. Theorem 2 **(b)** is proven.

(c) Let G be an involutive Hopf algebra and $f : H \to G$ be a Hopf algebra homomorphism. Then, since f is a Hopf algebra homomorphism, we have $f \circ S = S_G \circ f$, where S_G denotes the antipode of G. Thus,

$$f \circ \underbrace{S^2}_{=S\circ S} = \underbrace{f \circ S}_{=S_G\circ f} \circ S = S_G \circ \underbrace{f \circ S}_{=S_G\circ f} = \underbrace{S_G \circ S_G}_{\substack{=S_G^2=\operatorname{id}_G \\ \text{(since } G \text{ is involutive)}}} \circ f = f,$$

so that

$$\begin{aligned} f \circ \left(S^2 - \operatorname{id}\right) &= \underbrace{f \circ S^2}_{=f} - \underbrace{f \circ \operatorname{id}}_{=f} \qquad \text{(since composition of } k\text{-linear maps is } k\text{-bilinear)} \\ &= f - f = 0. \end{aligned}$$

Hence, $(f \circ (S^2 - \operatorname{id}))(H) = 0$. Now, $\operatorname{Inv} H = H \cdot ((S^2 - \operatorname{id})(H)) \cdot H$ yields

$$\begin{aligned} f(\operatorname{Inv} H) &= f\left(H \cdot \left(\left(S^2 - \operatorname{id}\right)(H)\right) \cdot H\right) = (f(H)) \cdot \underbrace{\left(f\left(\left(S^2 - \operatorname{id}\right)(H)\right)\right)}_{=(f\circ(S^2-\operatorname{id}))(H)=0} \cdot (f(H)) \\ &\qquad \text{(since } f \text{ is a } k\text{-algebra homomorphism)} \\ &= (f(H)) \cdot 0 \cdot (f(H)) = 0. \end{aligned}$$

Hence, the homomorphism f factors through $H\diagup(\operatorname{Inv} H)$. Theorem 2 **(c)** is thus proven.

Parts **(d)** and **(e)** of Theorem 2 are completely straightforward and thus left to the reader. The proof of Theorem 2 is thus complete.

We notice that we could generalize Theorems 2 and 3 by replacing "field" by "commutative ring". These generalizations are proven in exactly the same way as we verified Theorems 2 and 3, with the only difference that we would have to use more complicated notations, because we couldn't anymore identify (for instance) $(\operatorname{Inv}_t H) \otimes H$ with a k-submodule of $H \otimes H$ in the proof of Theorem 3 (because if A, B and C are three k-modules over a commutative ring k such that A is a submodule of B, then $A \otimes C$ is *in general* not a submodule of $B \otimes C$), so we would have to explicitly work with inclusion maps instead.

§2. The Dynkin operator

We can define a Dynkin operator for an arbitrary graded Hopf algebra, even though it enjoys most of its interesting properties in less general cases (for instance, when H is commutative or cocommutative).

Definition 4. Let H be a graded Hopf algebra over a field k. For every $n \in \mathbb{N}$, the denote by H_n the n-th graded component of H.

[2] *Proof.* Let $z \in H\diagup(\operatorname{Inv} H)$. Then, there exists some $y \in H$ such that $z = \overline{y}$. Consider this y. Then, $z = \overline{y}$ yields $\overline{S}^2(z) = \overline{S}^2(\overline{y}) = \overline{y} = z = \operatorname{id}_{H\diagup(\operatorname{Inv} H)}(z)$, qed.

(a) Let $E : H \to H$ be the k-linear map which sends every $x \in H_n$ to nx for every $n \in \mathbb{N}$. Note that this map E is easily seen to be a derivation (i. e., it satisfies $E(xy) = E(x)y + xE(y)$ for all $x \in H$ and $y \in H$, or, equivalently, it satisfies the identity $E \circ \mu = \mu \circ (E \otimes \mathrm{id} + \mathrm{id} \otimes E)$ with μ denoting the multiplication map of H) and a coderivation (i. e., it satisfies $\Delta \circ E = (E \otimes \mathrm{id} + \mathrm{id} \otimes E) \circ \Delta$ with Δ denoting the comultiplication map of H).

(b) The *Dynkin operator* of H is defined to mean the convolution $E * S$ of the maps $E : H \to H$ and $S : H \to H$ (where S denotes the antipode of H, as usual).

Note that $E * S$ is not the only "Dynkin operator" around. One can just as well call $S * E$ a "Dynkin operator" (it is a kind of mirror version of $E * S$), and for a field k of characteristic 0 one can even "interpolate" between these two "Dynkin operators" $E * S$ and $S * E$ by introducing a "Dynkin operator" $S^{\alpha} * E * S^{\beta}$ for any two elements α and β of k satisfying $\alpha + \beta = 1$. See the Remark in §3 of [2] (where our E is denoted by D) for this definition (which is due to Claudio Procesi).

A well-known theorem states that:

Theorem 5. Let H be a graded Hopf algebra over k.

(a) If H is cocommutative, then the Dynkin operator $E * S$ (defined in Definition 4) satisfies $(E * S) \circ (E * S) = E \circ (E * S)$, $(E * S) \mid_{\mathrm{Prim}\, H} = E \mid_{\mathrm{Prim}\, H}$ and $(E * S)(H) \subseteq \mathrm{Prim}\, H$. (Here, $\mathrm{Prim}\, H$ denotes the subspace of H formed by all primitive elements of H.)

(b) If H is commutative, then the Dynkin operator $E * S$ (defined in Definition 4) satisfies $(E * S) \circ (E * S) = E \circ (E * S)$, $\mathrm{Ker}\,(E * S) \subseteq \mathrm{Ker}\, E + (H^+)^2$ and $(E * S)\left((H^+)^2\right) = 0$. (Here, H^+ denotes the ideal $\mathrm{Ker}\, \varepsilon$ of H.)

Our goal in the next section (§3) is to prove that $(E * S) \circ (E * S) = E \circ (E * S)$ is not necessarily true for an *involutive (but not necessarily commutative or cocommutative) connected graded Hopf algebra H*. (Note that tensor products of commutative with cocommutative Hopf algebras are easily seen to satisfy $(E * S) \circ (E * S) = E \circ (E * S)$, so we need a more subtle counterexample).

§3. The counterexample

To construct a counterexample, we will need the Hopf algebra $\mathbf{H}_o$ defined by Foissy in [1], Definition 2. This is the Hopf algebra of ordered (rooted) forests, where "ordered" means that the set of vertices is totally ordered (but not necessarily heap-ordered, i. e., a child needs not be greater than its father). We recall its definition (mostly quoted from [1]):

Definition 6. **(a)** An *ordered forest* means a rooted forest endowed with a total order on the set of its vertices. We are going to represent such an ordered forest pictorially by drawing the forest and decorating every vertex with its position number[3]. The empty forest is denoted by 1 (as it will later become the unity of an algebra).

(b) For every $n \in \mathbb{N}$, let $\mathcal{F}_o(n)$ denote the set of all ordered forests with n vertices. The set of all ordered forests will be called $\mathcal{F}_o$; it is the union of the mutually disjoint sets $\mathcal{F}_o(n)$ over all $n \in \mathbb{N}$.

(c) Let k be a field. We denote by $\mathbf{H}_o$ the free k-vector space with basis $\mathcal{F}_o$; it is canonically graded, with the n-th graded component $\mathbf{H}_{o,n}$ being the free k-vector space with basis $\mathcal{F}_o(n)$.

(d) We make $\mathbf{H}_o$ into a graded k-algebra as follows: For any two ordered forests a and b, let ab be the ordered forest whose underlying forest is the disjoint union of the forests a and b, and the order on which is defined by letting each vertex of a be smaller than each vertex of b (but the order of the vertices of a is kept from a, and the order of the vertices of b is kept from b). The unity of this k-algebra is 1 (the empty forest).

(e) For every forest F, let $V(F)$ denote the set of the vertices of F. If F is a forest and $\mathbf{v}$ is a subset of $V(F)$, then we write $\mathbf{v} \models V(F)$ (and we say that $\mathbf{v}$ is an *admissible cut* of F) if and only if no element of $\mathbf{v}$ is a descendant[4] of another element of $\mathbf{v}$ except of itself. If F is a forest and $\mathbf{v}$ is a subset of $V(F)$ satisfying $\mathbf{v} \models V(F)$, then we denote by $Lea_{\mathbf{v}}F$ the rooted subforest of F obtained by keeping only the vertices above $\mathbf{v}$ (where a vertex of F is said to be "above $\mathbf{v}$" if it is a descendant of an element of $\mathbf{v}$), and we denote by $Roo_{\mathbf{v}}F$ the rooted subforest of F obtained by keeping only the other vertices. If F is an ordered forest, then $Lea_{\mathbf{v}}F$ and $Roo_{\mathbf{v}}F$ canonically become ordered forests.

(f) We now make $\mathbf{H}_o$ into a k-bialgebra by defining

$$\Delta(F) = \sum_{\mathbf{v} \models V(F)} Lea_{\mathbf{v}}F \otimes Roo_{\mathbf{v}}F \qquad \text{for every forest } F$$

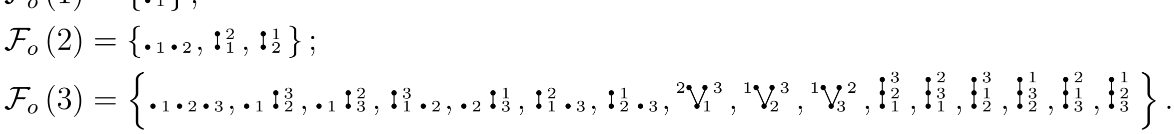
and

$$\varepsilon(F) = \begin{cases} 1, \text{ if } F = 1; \\ 0, \text{ if } F \neq 1 \end{cases} \qquad \text{for every forest } F.$$

Theorem 7. Definition 6 makes $\mathbf{H}_o$ into a connected graded k-Hopf algebra.

For the proof of Theorem 7, see the references in §1.2 of [1]. Here are the $\mathcal{F}_o(n)$ for $0 \leq n \leq 3$ (this table is copied from [1]):

$\mathcal{F}_o(0) = \{1\}$;
$\mathcal{F}_o(1) = \{\bullet_1\}$;



[3] By a *position number* of an element a in a totally ordered finite set T, I mean the number n such that a is the n-th smallest element of T.

[4] Descendants mean direct or indirect descendants. (In particular, every vertex is a descendant of itself.)

Here are some examples of how multiplication, comultiplication and the antipode look on $\mathbf{H}_o$:

Multiplication:

$$\mathrm{I}^2_1 \cdot {}^1\mathrm{V}^3_2 = \mathrm{I}^2_1\, {}^3\mathrm{V}^5_4 ;$$
$${}^1\mathrm{V}^3_2 \cdot \mathrm{I}^2_1 = {}^1\mathrm{V}^3_2\, \mathrm{I}^5_4 .$$

Comultiplication:

$$\Delta(\bullet_1) = \bullet_1 \otimes 1 + 1 \otimes \bullet_1 ;$$
$$\Delta(\bullet_1\bullet_2) = \bullet_1\bullet_2 \otimes 1 + 2\bullet_1 \otimes \bullet_1 + 1 \otimes \bullet_1\bullet_2 ;$$
$$\Delta(\mathrm{I}^1_2) = \mathrm{I}^1_2 \otimes 1 + \bullet_1 \otimes \bullet_1 + 1 \otimes \mathrm{I}^1_2 ;$$
$$\Delta(\mathrm{I}^2_1) = \mathrm{I}^2_1 \otimes 1 + \bullet_1 \otimes \bullet_1 + 1 \otimes \mathrm{I}^2_1 ;$$
$$\Delta\left(\mathrm{I}\substack{1\\2\\3}\right) = \mathrm{I}\substack{1\\2\\3} \otimes 1 + \mathrm{I}^1_2 \otimes \bullet_1 + \bullet_1 \otimes \mathrm{I}^1_2 + 1 \otimes \mathrm{I}\substack{1\\2\\3} ;$$
$$\Delta\left(\mathrm{I}\substack{1\\3\\2}\right) = \mathrm{I}\substack{1\\3\\2} \otimes 1 + \mathrm{I}^1_2 \otimes \bullet_1 + \bullet_1 \otimes \mathrm{I}^2_1 + 1 \otimes \mathrm{I}\substack{1\\3\\2} ;$$
$$\Delta(\mathrm{I}^3_2\bullet_1) = \mathrm{I}^3_2\bullet_1 \otimes 1 + \bullet_1\bullet_2 \otimes \bullet_1 + \mathrm{I}^2_1 \otimes \bullet_1 + \bullet_1 \otimes \bullet_1\bullet_2 + \bullet_1 \otimes \mathrm{I}^2_1 + 1 \otimes \mathrm{I}^3_2\bullet_1 ;$$
$$\Delta\left({}^1\mathrm{V}^3_2\right) = {}^1\mathrm{V}^3_2 \otimes 1 + \bullet_1\bullet_2 \otimes \bullet_1 + \bullet_1 \otimes \mathrm{I}^1_2 + \bullet_1 \otimes \mathrm{I}^2_1 + 1 \otimes {}^1\mathrm{V}^3_2 ;$$
$$\Delta\left({}^{1}_{4}\mathrm{V}^3_2\right) = {}^{1}_{4}\mathrm{V}^3_2 \otimes 1 + 1 \otimes {}^{1}_{4}\mathrm{V}^3_2 + \bullet_1 \otimes {}^2\mathrm{V}^3_1 + \mathrm{I}^1_2 \otimes \mathrm{I}^2_1 + \bullet_1 \otimes \mathrm{I}\substack{1\\3\\2} + \bullet_1\bullet_2 \otimes \mathrm{I}^2_1 + \mathrm{I}^1_3\bullet_2 \otimes \bullet_1 ;$$
$$\Delta\left(\mathrm{I}\substack{1\\2\\4\\3}\right) = \mathrm{I}\substack{1\\2\\4\\3} \otimes 1 + \mathrm{I}\substack{1\\2\\3} \otimes \bullet_1 + \mathrm{I}^1_2 \otimes \mathrm{I}^2_1 + \bullet_1 \otimes \mathrm{I}\substack{1\\3\\2} + 1 \otimes \mathrm{I}\substack{1\\2\\4\\3} .$$

Antipode:[5]

$$S(\bullet_1) = -\bullet_1 \qquad \text{(no wonder, since } \bullet_1 \text{ is primitive)} ;$$
$$S(\mathrm{I}^1_2) = \bullet_1\bullet_2 - \mathrm{I}^1_2 ;$$
$$S(\mathrm{I}^2_1) = \bullet_1\bullet_2 - \mathrm{I}^2_1 ;$$
$$S\left(\mathrm{I}\substack{1\\2\\3}\right) = -\bullet_1\bullet_2\bullet_3 + \mathrm{I}^1_2\bullet_3 + \bullet_1\mathrm{I}^2_3 - \mathrm{I}\substack{1\\2\\3} ;$$
$$S\left(\mathrm{I}\substack{1\\3\\2}\right) = -\bullet_1\bullet_2\bullet_3 + \mathrm{I}^1_2\bullet_3 + \bullet_1\mathrm{I}^3_2 - \mathrm{I}\substack{1\\3\\2} .$$

Note that, at this place, our notations are slightly ambiguous: For example, what does $\mathrm{I}^2_1\,{}^4\mathrm{V}^3_5\,\mathrm{I}^1_2$ mean? Does it mean $\left(\mathrm{I}^2_1\,{}^4\mathrm{V}^3_5\right)\mathrm{I}^1_2 = \mathrm{I}^2_1\,{}^4\mathrm{V}^3_5\,\mathrm{I}^6_7$ or does it mean $\mathrm{I}^2_1\left({}^4\mathrm{V}^3_5\,\mathrm{I}^1_2\right) = \mathrm{I}^2_1\,{}^6\mathrm{V}^5_7\,\mathrm{I}^3_4$? The ambiguity is due to the fact that when we write two trees one after the other, it is not clear whether we mean to multiply these two trees as elements of $\mathbf{H}_o$ or join these two trees into a forest. One way to get rid of this ambiguity is never to suppress the product sign (so that when two trees are written one after the other, it cannot mean multiplication, but can only mean the forest). This is what we are

[5]Computing the antipode can be done by recurrently solving the equation $(\mathrm{id} * S)(x) = \varepsilon(x) \cdot 1$ (or the equation $(S * \mathrm{id})(x) = \varepsilon(x) \cdot 1$), but in some cases one can simplify one's work by using the fact that the antipode is an anti-algebra homomorphism.

going to do in the following. But let us also note that we will use $\times$ instead of $\cdot$ as our product sign, because $\cdot$ is too similar to a one-vertex tree.

Let now R be the k-subalgebra of $\mathbf{H}_o$ generated by the ordered trees $\bullet_1$, $\begin{smallmatrix}\bullet 1\\ \bullet 2\end{smallmatrix}$, $\begin{smallmatrix}\bullet 2\\ \bullet 1\end{smallmatrix}$, $\begin{smallmatrix}\bullet 1\\ \bullet 2\\ \bullet 3\end{smallmatrix}$, $\begin{smallmatrix}\bullet 1\\ \bullet 3\\ \bullet 2\end{smallmatrix}$ and $\begin{smallmatrix}\bullet 1\\ \bullet 2\\ \bullet 4\\ \bullet 3\end{smallmatrix}$. It is very easy to see that R is a connected graded Hopf subalgebra of $\mathbf{H}_o$ [6]. Let R' be the involutive Hopf algebra $R \diagup (\operatorname{Inv} R)$. Assume that the field k has characteristic 0. Then we are going to prove that $(E * S) \circ (E * S) \neq E \circ (E * S)$ in R'. This will answer the Concrete Question in [3] negatively.

We denote by Δ, ε, μ, η and S the usual operations of the Hopf algebra R (in this order: comultiplication, counit, multiplication map, unit map and antipode). Also, let $[\cdot, \cdot]$ denote the commutator.

First, a very simple lemma:

Lemma 8. Let H be a graded Hopf algebra over k. Let E and S be defined as in Theorem 5. Then:

(a) Every graded map $f : H \to H$ satisfies $E \circ f = f \circ E$.

(b) Let S be the antipode of H, let ε be the counit map of H, and let η be the unit map of H. Then,

$$(E * S) \circ (E * S) - E \circ (E * S) = (E * S) \circ \left(E * \overline{S}\right),$$

where $\overline{S}$ denotes the map $S - \eta \circ \varepsilon : H \to H$.

Proof of Lemma 8 (sketched). Lemma 8 **(a)** is left to the reader.

(b) By Lemma 8 **(a)** (applied to $f = E * S$, which is easily seen to be graded), we have $E \circ (E * S) = (E * S) \circ E$. Hence,

$$\begin{aligned}(E * S) \circ (E * S) - E \circ (E * S) &= (E * S) \circ (E * S) - (E * S) \circ E = (E * S) \circ \left(E * S - \underbrace{E}_{=E*(\eta\circ\varepsilon)}\right) \\ &\qquad \text{(since composition of } k\text{-linear maps is } k\text{-bilinear)} \\ &= (E * S) \circ \underbrace{(E * S - E * (\eta \circ \varepsilon))}_{\substack{=E*(S-\eta\circ\varepsilon) \\ \text{(since convolution of } k\text{-linear maps is } k\text{-bilinear)}}} \\ &= (E * S) \circ \left(E * \underbrace{(S - \eta \circ \varepsilon)}_{=\overline{S}}\right) = (E * S) \circ \left(E * \overline{S}\right),\end{aligned}$$

proving Lemma 8 **(b)**.

Now, let us come back to our R. In order to prove that $(E * S) \circ (E * S) \neq E \circ (E * S)$ in R', it is enough to show that $(E * S) \circ \left(E * \overline{S}\right) \neq 0$ in R' (by Lemma 8 **(b)**, applied to $H = R'$). We will achieve this by showing that $\left((E * S) \circ \left(E * \overline{S}\right)\right) \left(\begin{smallmatrix}\bullet 1\\ \bullet 2\\ \bullet 4\\ \bullet 3\end{smallmatrix}\right) \neq$

[6] More generally: If S is a set of ordered trees which, for every ordered tree $s \in S$, contains each ordered subtree of S, then the k-subalgebra of $\mathbf{H}_o$ generated by S is a connected graded Hopf subalgebra of $\mathbf{H}_o$.

0 in R'. This means showing that

$$\left(\left(E * S\right) \circ \left(E * \overline{S}\right)\right) \left(\mathrm{I}\begin{smallmatrix}1\\2\\4\\3\end{smallmatrix}\right) \notin \operatorname{Inv} R. \tag{1}$$

We will prove (1) by showing something stronger: We will show that

$$\left(\left(E * S\right) \circ \left(E * \overline{S}\right)\right) \left(\mathrm{I}\begin{smallmatrix}1\\2\\4\\3\end{smallmatrix}\right) \notin \operatorname{Inv} R + \left(R^{+}\right)^{3}. \tag{2}$$

Here, for every coalgebra C, we denote by C^{+} the subspace $\operatorname{Ker} \varepsilon$ of C. We notice that R^{+} is an ideal of R, so that $\left(R^{+}\right)^{3}$ is an ideal as well. Since $R \otimes R$ also is a bialgebra, we have a well-defined ideal $\left(R \otimes R\right)^{+}$ of $R \otimes R$ as well, and it is important to notice that $\left(R \otimes R\right)^{+} = R^{+} \otimes R + R \otimes R^{+}$, so that every $n \in \mathbb{N}$ satisfies $\left(\left(R \otimes R\right)^{+}\right)^{n} = \sum\limits_{i=0}^{n} \left(R^{+}\right)^{i} \otimes \left(R^{+}\right)^{n-i}$. As a consequence, every $n \in \mathbb{N}$ satisfies $\mu\left(\left(\left(R \otimes R\right)^{+}\right)^{n}\right) \subseteq \left(R^{+}\right)^{n}$. Also, every $n \in \mathbb{N}$ satisfies $\Delta\left(\left(R^{+}\right)^{n}\right) \subseteq \left(\left(R \otimes R\right)^{+}\right)^{n}$ (since Δ is an algebra homomorphism and commutes with the counity maps of R and $R \otimes R$), $S\left(\left(R^{+}\right)^{n}\right) \subseteq \left(R^{+}\right)^{n}$ (since S is an anti-algebra homomorphism and $S\left(R^{+}\right) \subseteq R^{+}$) and $E\left(\left(R^{+}\right)^{n}\right) \subseteq \left(R^{+}\right)^{n}$ (very easy to check). As a consequence, $\left(E * S\right)\left(\left(R^{+}\right)^{3}\right) \subseteq \left(R^{+}\right)^{3}$ (since $E * S = \mu \circ \left(E \otimes S\right) \circ \Delta$ by the definition of convolution).

This allows us to work modulo $\left(R^{+}\right)^{3}$ (when we are working in R) and modulo $\left(\left(R \otimes R\right)^{+}\right)^{3}$ (when we are working in $R \otimes R$), i. e., to forget all terms which include more than 2 (disjoint) trees (because $\left(R^{+}\right)^{3}$ is the k-vector subspace of R which is formed by products of more than 2 disjoint trees). Thus, proving (2) will require less computations than proving (1), even though (2) is a slightly stronger result.

First of all, we are always going to compute in the first 5 degrees of R. Let us give bases for these degrees:

$$\begin{aligned}
R_0 &= \left\langle 1 \right\rangle; \\
R_1 &= \left\langle \bullet_1 \right\rangle; \\
R_2 &= \left\langle \bullet_1 \times \bullet_1,\quad \mathrm{I}\begin{smallmatrix}1\\2\end{smallmatrix},\quad \mathrm{I}\begin{smallmatrix}2\\1\end{smallmatrix} \right\rangle \\
R_3 &= \left\langle \bullet_1 \times \bullet_1 \times \bullet_1,\quad \bullet_1 \times \mathrm{I}\begin{smallmatrix}1\\2\end{smallmatrix},\quad \mathrm{I}\begin{smallmatrix}1\\2\end{smallmatrix} \times \bullet_1,\quad \bullet_1 \times \mathrm{I}\begin{smallmatrix}2\\1\end{smallmatrix},\quad \mathrm{I}\begin{smallmatrix}2\\1\end{smallmatrix} \times \bullet_1,\quad \mathrm{I}\begin{smallmatrix}1\\2\\3\end{smallmatrix},\quad \mathrm{I}\begin{smallmatrix}1\\3\\2\end{smallmatrix} \right\rangle \\
R_4 &= \left\langle \begin{array}{c}
\bullet_1 \times \bullet_1 \times \bullet_1 \times \bullet_1,\quad \bullet_1 \times \bullet_1 \times \mathrm{I}\begin{smallmatrix}1\\2\end{smallmatrix},\quad \bullet_1 \times \mathrm{I}\begin{smallmatrix}1\\2\end{smallmatrix} \times \bullet_1,\quad \mathrm{I}\begin{smallmatrix}1\\2\end{smallmatrix} \times \bullet_1 \times \bullet_1, \\
\bullet_1 \times \bullet_1 \times \mathrm{I}\begin{smallmatrix}2\\1\end{smallmatrix},\quad \bullet_1 \times \mathrm{I}\begin{smallmatrix}2\\1\end{smallmatrix} \times \bullet_1,\quad \mathrm{I}\begin{smallmatrix}2\\1\end{smallmatrix} \times \bullet_1 \times \bullet_1,\quad \mathrm{I}\begin{smallmatrix}1\\2\end{smallmatrix} \times \mathrm{I}\begin{smallmatrix}1\\2\end{smallmatrix},\quad \mathrm{I}\begin{smallmatrix}1\\2\end{smallmatrix} \times \mathrm{I}\begin{smallmatrix}2\\1\end{smallmatrix}, \\
\mathrm{I}\begin{smallmatrix}2\\1\end{smallmatrix} \times \mathrm{I}\begin{smallmatrix}1\\2\end{smallmatrix},\quad \mathrm{I}\begin{smallmatrix}2\\1\end{smallmatrix} \times \mathrm{I}\begin{smallmatrix}2\\1\end{smallmatrix},\quad \bullet_1 \times \mathrm{I}\begin{smallmatrix}1\\2\\3\end{smallmatrix},\quad \mathrm{I}\begin{smallmatrix}1\\2\\3\end{smallmatrix} \times \bullet_1,\quad \bullet_1 \times \mathrm{I}\begin{smallmatrix}1\\3\\2\end{smallmatrix}, \\
\mathrm{I}\begin{smallmatrix}1\\3\\2\end{smallmatrix} \times \bullet_1,\quad \mathrm{I}\begin{smallmatrix}1\\2\\4\\3\end{smallmatrix}
\end{array} \right\rangle.
\end{aligned}$$

To be able to prove (2), we need to compute a basis for $\left(\operatorname{Inv} R + \left(R^{+}\right)^{3}\right) \cap R_4$. Since everything is graded, we have $\left(\operatorname{Inv} R + \left(R^{+}\right)^{3}\right) \cap R_4 = \left(\operatorname{Inv} R\right) \cap R_4 + \left(R^{+}\right)^{3} \cap R_4$ (see a more detailed proof of this further below). It is clear that $\left(R^{+}\right)^{3} \cap R_4$ is the k-vector subspace of R which is formed by products of more than 2 disjoint trees, so a basis is obvious:

$$\left(R^{+}\right)^{3} \cap R_4 = \left\langle \begin{array}{c}
\bullet_1 \times \bullet_1 \times \bullet_1 \times \bullet_1,\quad \bullet_1 \times \bullet_1 \times \mathrm{I}\begin{smallmatrix}1\\2\end{smallmatrix},\quad \bullet_1 \times \mathrm{I}\begin{smallmatrix}1\\2\end{smallmatrix} \times \bullet_1,\quad \mathrm{I}\begin{smallmatrix}1\\2\end{smallmatrix} \times \bullet_1 \times \bullet_1, \\
\bullet_1 \times \bullet_1 \times \mathrm{I}\begin{smallmatrix}2\\1\end{smallmatrix},\quad \bullet_1 \times \mathrm{I}\begin{smallmatrix}2\\1\end{smallmatrix} \times \bullet_1,\quad \mathrm{I}\begin{smallmatrix}2\\1\end{smallmatrix} \times \bullet_1 \times \bullet_1
\end{array} \right\rangle.$$

It remains to compute $(\operatorname{Inv} R) \cap R_4$. For this we first notice that:

Lemma 9. Let H be a Hopf algebra over a field k. Then, $\operatorname{Ker}(S^2 - \operatorname{id})$ is a k-subalgebra of H.

Proof of Lemma 9. Notice that the antipode S is an anti-algebra homomorphism. Hence, $S(1) = 1$, so that $S(S(1)) = S(1) = 1$. Hence, $(S^2 - \operatorname{id})(1) = \underbrace{S^2(1)}_{=S(S(1))=1} - \underbrace{\operatorname{id}(1)}_{=1} = 1 - 1 = 0$, so that $1 \in \operatorname{Ker}(S^2 - \operatorname{id})$.

Let $a \in \operatorname{Ker}(S^2 - \operatorname{id})$ and $b \in \operatorname{Ker}(S^2 - \operatorname{id})$. Then, $S^2(a) - \underbrace{a}_{=\operatorname{id}(a)} = S^2(a) - \operatorname{id}(a) = (S^2 - \operatorname{id})(a) = 0$ (since $a \in \operatorname{Ker}(S^2 - \operatorname{id})$), so that $S^2(a) = a$. Similarly, $S^2(b) = b$.

We know that the antipode S is an anti-algebra homomorphism. Hence, $S(ab) = S(b)S(a)$ and $S(S(b)S(a)) = S(S(a))S(S(b))$. Thus,

$$S^2(ab) = S\left(\underbrace{S(ab)}_{=S(b)S(a)}\right) = S(S(b)S(a)) = \underbrace{S(S(a))}_{=S^2(a)=a}\underbrace{S(S(b))}_{=S^2(b)=b} = ab,$$

so that $(S^2 - \operatorname{id})(ab) = \underbrace{S^2(ab)}_{=ab} - \underbrace{\operatorname{id}(ab)}_{=ab} = ab - ab = 0$. In other words, $ab \in \operatorname{Ker}(S^2 - \operatorname{id})$.

So we have shown that any $a \in \operatorname{Ker}(S^2 - \operatorname{id})$ and $b \in \operatorname{Ker}(S^2 - \operatorname{id})$ satisfy $ab \in \operatorname{Ker}(S^2 - \operatorname{id})$. Combined with the fact that $1 \in \operatorname{Ker}(S^2 - \operatorname{id})$, this yields that $\operatorname{Ker}(S^2 - \operatorname{id})$ is a k-subalgebra of H (since $\operatorname{Ker}(S^2 - \operatorname{id})$ is clearly a k-submodule of H). Lemma 9 is proven.

Next we notice that $R_0 + R_1 + R_2 \subseteq \operatorname{Ker}(S^2 - \operatorname{id})$. This is very easy to check manually, but actually is a particular case of something general:

Lemma 10. Let H be a connected graded Hopf algebra over a field k such that any $a \in H_1$ and $b \in H_1$ satisfy $ab = ba$. Then, $H_0 + H_1 + H_2 \subseteq \operatorname{Ker}(S^2 - \operatorname{id})$.

The proof of Lemma 10 is left to the reader; it can be applied to $H = R$ since R_1 is one-dimensional.

Now we notice that $\operatorname{Inv} R = R \cdot ((S^2 - \operatorname{id})(R)) \cdot R$ (by the definition of $\operatorname{Inv} R$), so that

$$\begin{aligned}
&(\operatorname{Inv} R) \cap R_4 \\
&= \left(R \cdot \left(\left(S^2 - \operatorname{id}\right)(R)\right) \cdot R\right) \cap R_4 = \sum_{\substack{(i,j,\ell) \in \mathbb{N}^3; \\ i+j+\ell=4}} R_i \cdot \left(\left(S^2 - \operatorname{id}\right)(R_j)\right) \cdot R_\ell \qquad \text{(since } R \text{ is graded)} \\
&= \sum_{\substack{(i,j,\ell) \in \mathbb{N}^3; \\ i+j+\ell=4; \\ j>2}} R_i \cdot \left(\left(S^2 - \operatorname{id}\right)(R_j)\right) \cdot R_\ell \qquad \left(\begin{array}{c}\text{here we removed all terms with } j \leq 2 \text{ from the sum,} \\ \text{since } R_0 + R_1 + R_2 \subseteq \operatorname{Ker}(S^2 - \operatorname{id})\end{array}\right) \\
&= \left(S^2 - \operatorname{id}\right)(R_4) + \left(S^2 - \operatorname{id}\right)(R_3) \cdot R_1 + R_1 \cdot \left(S^2 - \operatorname{id}\right)(R_3). \qquad (3)
\end{aligned}$$

We need now to compute $\left(S^2-\mathrm{id}\right)(R_3)$ and $\left(S^2-\mathrm{id}\right)(R_4)$.

Computation of $\left(S^2-\mathrm{id}\right)(R_3)$*:* We know that

$$R_3=\left\langle \bullet_1\times\bullet_1\times\bullet_1,\ \ \bullet_1\times\mathrm{I}\substack{1\\2},\ \ \mathrm{I}\substack{1\\2}\times\bullet_1,\ \ \bullet_1\times\mathrm{I}\substack{2\\1},\ \ \mathrm{I}\substack{2\\1}\times\bullet_1,\ \ \mathrm{I}\substack{1\\2\\3},\ \ \mathrm{I}\substack{1\\3\\2}\right\rangle .$$

Hence, to compute $\left(S^2-\mathrm{id}\right)(R_3)$, we need to apply $S^2-\mathrm{id}$ to each of these generators. But since

$$\text{any product of elements of } R_0+R_1+R_2 \text{ lies in } \operatorname{Ker}\left(S^2-\mathrm{id}\right) \tag{4}$$

(by $R_0+R_1+R_2\subseteq\operatorname{Ker}\left(S^2-\mathrm{id}\right)$ and Lemma 9), it is clear that $\bullet_1\times\bullet_1\times\bullet_1$, $\bullet_1\times\mathrm{I}\substack{1\\2}$, $\mathrm{I}\substack{1\\2}\times\bullet_1$, $\bullet_1\times\mathrm{I}\substack{2\\1}$ and $\mathrm{I}\substack{2\\1}\times\bullet_1$ all lie in $\operatorname{Ker}\left(S^2-\mathrm{id}\right)$, so we only need to apply $S^2-\mathrm{id}$ to $\mathrm{I}\substack{1\\2\\3}$ and $\mathrm{I}\substack{1\\3\\2}$ only.

It is easy to check from the recurrent definition of S that

$$S\left(\mathrm{I}\substack{1\\2\\3}\right)=-\bullet_1\times\bullet_1\times\bullet_1+\mathrm{I}\substack{1\\2}\times\bullet_1+\bullet_1\times\mathrm{I}\substack{1\\2}-\mathrm{I}\substack{1\\2\\3}; \tag{5}$$

$$S\left(\mathrm{I}\substack{1\\3\\2}\right)=-\bullet_1\times\bullet_1\times\bullet_1+\mathrm{I}\substack{1\\2}\times\bullet_1+\bullet_1\times\mathrm{I}\substack{2\\1}-\mathrm{I}\substack{1\\3\\2}. \tag{6}$$

Now,

$$\begin{aligned}
S^2\left(\mathrm{I}\substack{1\\2\\3}\right)&=S\left(S\left(\mathrm{I}\substack{1\\2\\3}\right)\right)=S\left(-\bullet_1\times\bullet_1\times\bullet_1+\mathrm{I}\substack{1\\2}\times\bullet_1+\bullet_1\times\mathrm{I}\substack{1\\2}-\mathrm{I}\substack{1\\2\\3}\right)\\
&\qquad\text{(by (5))}\\
&=-S(\bullet_1)\times S(\bullet_1)\times S(\bullet_1)+S(\bullet_1)\times S\left(\mathrm{I}\substack{1\\2}\right)+S\left(\mathrm{I}\substack{1\\2}\right)\times S(\bullet_1)-S\left(\mathrm{I}\substack{1\\2\\3}\right)\\
&\qquad\text{(since } S \text{ is an anti-algebra homomorphism)}\\
&=\bullet_1\times\bullet_1\times\bullet_1-\bullet_1\times\left(\bullet_1\times\bullet_1-\mathrm{I}\substack{1\\2}\right)-\left(\bullet_1\times\bullet_1-\mathrm{I}\substack{1\\2}\right)\times\bullet_1\\
&\qquad-\left(-\bullet_1\times\bullet_1\times\bullet_1+\mathrm{I}\substack{1\\2}\times\bullet_1+\bullet_1\times\mathrm{I}\substack{1\\2}-\mathrm{I}\substack{1\\2\\3}\right)\\
&\qquad\text{(using already-known values of } S\text{)}\\
&=\mathrm{I}\substack{1\\2\\3}.
\end{aligned}$$

Hence, $(S^2 - \mathrm{id})\left(\begin{smallmatrix}\bullet 1\\ \bullet 2\\ \bullet 3\end{smallmatrix}\right) = 0.$ [7] Besides,

$$\begin{aligned}
S^2\left(\begin{smallmatrix}\bullet 1\\ \bullet 3\\ \bullet 2\end{smallmatrix}\right) &= S\left(S\left(\begin{smallmatrix}\bullet 1\\ \bullet 3\\ \bullet 2\end{smallmatrix}\right)\right) = S\left(-\bullet_1 \times \bullet_1 \times \bullet_1 + \begin{smallmatrix}\bullet 1\\ \bullet 2\end{smallmatrix} \times \bullet_1 + \bullet_1 \times \begin{smallmatrix}\bullet 2\\ \bullet 1\end{smallmatrix} - \begin{smallmatrix}\bullet 1\\ \bullet 3\\ \bullet 2\end{smallmatrix}\right) \\
&\qquad \text{(by (6))} \\
&= -S(\bullet_1) \times S(\bullet_1) \times S(\bullet_1) + S(\bullet_1) \times S\left(\begin{smallmatrix}\bullet 1\\ \bullet 2\end{smallmatrix}\right) + S\left(\begin{smallmatrix}\bullet 2\\ \bullet 1\end{smallmatrix}\right) \times S(\bullet_1) - S\left(\begin{smallmatrix}\bullet 1\\ \bullet 3\\ \bullet 2\end{smallmatrix}\right) \\
&\qquad \text{(since } S \text{ is an anti-algebra homomorphism)} \\
&= \bullet_1 \times \bullet_1 \times \bullet_1 - \bullet_1 \times \left(\bullet_1 \times \bullet_1 - \begin{smallmatrix}\bullet 1\\ \bullet 2\end{smallmatrix}\right) - \left(\bullet_1 \times \bullet_1 - \begin{smallmatrix}\bullet 2\\ \bullet 1\end{smallmatrix}\right) \times \bullet_1 \\
&\qquad - \left(-\bullet_1 \times \bullet_1 \times \bullet_1 + \begin{smallmatrix}\bullet 1\\ \bullet 2\end{smallmatrix} \times \bullet_1 + \bullet_1 \times \begin{smallmatrix}\bullet 2\\ \bullet 1\end{smallmatrix} - \begin{smallmatrix}\bullet 1\\ \bullet 3\\ \bullet 2\end{smallmatrix}\right) \\
&\qquad \text{(using already-known values of } S\text{)} \\
&= \left[\bullet_1,\ \begin{smallmatrix}\bullet 1\\ \bullet 2\end{smallmatrix} - \begin{smallmatrix}\bullet 2\\ \bullet 1\end{smallmatrix}\right] + \begin{smallmatrix}\bullet 1\\ \bullet 3\\ \bullet 2\end{smallmatrix},
\end{aligned}$$

so that $(S^2 - \mathrm{id})\left(\begin{smallmatrix}\bullet 1\\ \bullet 3\\ \bullet 2\end{smallmatrix}\right) = \left[\bullet_1,\ \begin{smallmatrix}\bullet 1\\ \bullet 2\end{smallmatrix} - \begin{smallmatrix}\bullet 2\\ \bullet 1\end{smallmatrix}\right]$.

As a result of this all,

$$\left(S^2 - \mathrm{id}\right)(R_3) = \left\langle \left[\bullet_1,\ \begin{smallmatrix}\bullet 1\\ \bullet 2\end{smallmatrix} - \begin{smallmatrix}\bullet 2\\ \bullet 1\end{smallmatrix}\right] \right\rangle. \tag{7}$$

Computation of $(S^2 - \mathrm{id})(R_4)$*:* We have

$$R_4 = \left\langle \begin{matrix} \bullet_1 \times \bullet_1 \times \bullet_1 \times \bullet_1,\quad \bullet_1 \times \bullet_1 \times \begin{smallmatrix}\bullet 1\\ \bullet 2\end{smallmatrix},\quad \bullet_1 \times \begin{smallmatrix}\bullet 1\\ \bullet 2\end{smallmatrix} \times \bullet_1,\quad \begin{smallmatrix}\bullet 1\\ \bullet 2\end{smallmatrix} \times \bullet_1 \times \bullet_1, \\ \bullet_1 \times \bullet_1 \times \begin{smallmatrix}\bullet 2\\ \bullet 1\end{smallmatrix},\quad \bullet_1 \times \begin{smallmatrix}\bullet 2\\ \bullet 1\end{smallmatrix} \times \bullet_1,\quad \begin{smallmatrix}\bullet 2\\ \bullet 1\end{smallmatrix} \times \bullet_1 \times \bullet_1,\quad \begin{smallmatrix}\bullet 1\\ \bullet 2\end{smallmatrix} \times \begin{smallmatrix}\bullet 1\\ \bullet 2\end{smallmatrix},\quad \begin{smallmatrix}\bullet 1\\ \bullet 2\end{smallmatrix} \times \begin{smallmatrix}\bullet 2\\ \bullet 1\end{smallmatrix}, \\ \begin{smallmatrix}\bullet 2\\ \bullet 1\end{smallmatrix} \times \begin{smallmatrix}\bullet 1\\ \bullet 2\end{smallmatrix},\quad \begin{smallmatrix}\bullet 2\\ \bullet 1\end{smallmatrix} \times \begin{smallmatrix}\bullet 2\\ \bullet 1\end{smallmatrix},\quad \bullet_1 \times \begin{smallmatrix}\bullet 1\\ \bullet 2\\ \bullet 3\end{smallmatrix},\quad \begin{smallmatrix}\bullet 1\\ \bullet 2\\ \bullet 3\end{smallmatrix} \times \bullet_1,\quad \bullet_1 \times \begin{smallmatrix}\bullet 1\\ \bullet 3\\ \bullet 2\end{smallmatrix}, \\ \begin{smallmatrix}\bullet 1\\ \bullet 3\\ \bullet 2\end{smallmatrix} \times \bullet_1,\quad \begin{smallmatrix}\bullet 1\\ \bullet 2\\ \bullet 4\\ \bullet 3\end{smallmatrix} \end{matrix} \right\rangle.$$

As in the case of R_3, we don't have to apply $S^2 - \mathrm{id}$ to each of these generators because of (4). We only need to apply $S^2 - \mathrm{id}$ to the five elements $\bullet_1 \times \begin{smallmatrix}\bullet 1\\ \bullet 2\\ \bullet 3\end{smallmatrix}$, $\begin{smallmatrix}\bullet 1\\ \bullet 2\\ \bullet 3\end{smallmatrix} \times \bullet_1$, $\bullet_1 \times \begin{smallmatrix}\bullet 1\\ \bullet 3\\ \bullet 2\end{smallmatrix}$, $\begin{smallmatrix}\bullet 1\\ \bullet 3\\ \bullet 2\end{smallmatrix} \times \bullet_1$, $\begin{smallmatrix}\bullet 1\\ \bullet 2\\ \bullet 4\\ \bullet 3\end{smallmatrix}$. We notice that whenever a and b are two elements of a Hopf algebra H with a being primitive, then

$$\begin{aligned}
S^2(ab) &= S\Bigg(\underbrace{S(ab)}_{=S(b)S(a)}\Bigg) = S(S(b)\,S(a)) = \underbrace{S(S(a))}_{\substack{=a \\ \text{(since } a \text{ is primitive, thus } S(a)=-a, \text{ and } -a \\ \text{is primitive again, so that } S(-a)=a)}} \underbrace{S(S(b))}_{=S^2(b)} \\
&= aS^2(b),
\end{aligned}$$

so that $(S^2 - \mathrm{id})(ab) = a\,(S^2 - \mathrm{id})(b)$. Applying this to $a = \bullet_1$ and $b = \begin{smallmatrix}\bullet 1\\ \bullet 2\\ \bullet 3\end{smallmatrix}$, we obtain

$$\left(S^2 - \mathrm{id}\right)\left(\bullet_1 \times \begin{smallmatrix}\bullet 1\\ \bullet 2\\ \bullet 3\end{smallmatrix}\right) = \bullet_1 \times \underbrace{\left(S^2 - \mathrm{id}\right)\left(\begin{smallmatrix}\bullet 1\\ \bullet 2\\ \bullet 3\end{smallmatrix}\right)}_{=0} = 0.$$

[7] This result should not surprise us in the least, and we could actually have found it without any computation: It is easy to see that the subalgebra of R generated by ladders ordered from top to bottom (i. e., trees of the form $\bullet_1$, $\begin{smallmatrix}\bullet 1\\ \bullet 2\end{smallmatrix}$, $\begin{smallmatrix}\bullet 1\\ \bullet 2\\ \bullet 3\end{smallmatrix}$, $\begin{smallmatrix}\bullet 1\\ \bullet 2\\ \bullet 3\\ \bullet 4\end{smallmatrix}$, ...) is a Hopf subalgebra of R, and this Hopf subalgebra is cocommutative, so it satisfies $S^2 = \mathrm{id}$. Thus, $\left(S^2 - \mathrm{id}\right)\left(\begin{smallmatrix}\bullet 1\\ \bullet 2\\ \bullet 3\end{smallmatrix}\right) = 0$.

Similarly,

$$\begin{aligned}
\left(S^2-\mathrm{id}\right)\left(\ell_{\substack{1\\2\\3}} \times \bullet_1\right) &= 0;\\
\left(S^2-\mathrm{id}\right)\left(\bullet_1 \times \ell_{\substack{1\\3\\2}}\right) &= \bullet_1 \times \left[\bullet_1,\ \ell_{\substack{1\\2}} - \ell_{\substack{2\\1}}\right];\\
\left(S^2-\mathrm{id}\right)\left(\ell_{\substack{1\\3\\2}} \times \bullet_1\right) &= \left[\bullet_1,\ \ell_{\substack{1\\2}} - \ell_{\substack{2\\1}}\right] \times \bullet_1 .
\end{aligned}$$

It is the fifth element, $\ell_{\substack{1\\2\\4\\3}}$, that makes us the most trouble. From

$$\Delta\left(\ell_{\substack{1\\2\\4\\3}}\right) = \ell_{\substack{1\\2\\4\\3}} \otimes 1 + \ell_{\substack{1\\2\\3}} \otimes \bullet_1 + \ell_{\substack{1\\2}} \otimes \ell_{\substack{2\\1}} + \bullet_1 \otimes \ell_{\substack{1\\3\\2}} + 1 \otimes \ell_{\substack{1\\2\\4\\3}}$$

and the recursive interpretation of $S * \mathrm{id} = \eta \circ \varepsilon$, we obtain

$$\begin{aligned}
0 &= S\left(\ell_{\substack{1\\2\\4\\3}}\right) \times 1 + \underbrace{S\left(\ell_{\substack{1\\2\\3}}\right)}_{=-\bullet_1\times\bullet_1\times\bullet_1+\ell_{\substack{1\\2}}\times\bullet_1+\bullet_1\times\ell_{\substack{1\\2}}-\ell_{\substack{1\\2\\3}}} \times \bullet_1 + \underbrace{S\left(\ell_{\substack{1\\2}}\right)}_{=\bullet_1\times\bullet_1-\ell_{\substack{1\\2}}} \times \ell_{\substack{2\\1}}\\
&\qquad + \underbrace{S\left(\bullet_1\right)}_{=-\bullet_1} \times \ell_{\substack{1\\3\\2}} + \underbrace{S\left(1\right)}_{=1} \times \ell_{\substack{1\\2\\4\\3}}\\
&= S\left(\ell_{\substack{1\\2\\4\\3}}\right) \times 1 - \bullet_1 \times \bullet_1 \times \bullet_1 \times \bullet_1 + \ell_{\substack{1\\2}} \times \bullet_1 \times \bullet_1 + \bullet_1 \times \ell_{\substack{1\\2}} \times \bullet_1 - \ell_{\substack{1\\2\\3}} \times \bullet_1\\
&\qquad + \bullet_1 \times \bullet_1 \times \ell_{\substack{2\\1}} - \ell_{\substack{1\\2}} \times \ell_{\substack{2\\1}} - \bullet_1 \times \ell_{\substack{1\\3\\2}} + \ell_{\substack{1\\2\\4\\3}},
\end{aligned}$$

so that

$$\begin{aligned}
S\left(\ell_{\substack{1\\2\\4\\3}}\right) &= \bullet_1 \times \bullet_1 \times \bullet_1 \times \bullet_1 - \ell_{\substack{1\\2}} \times \bullet_1 \times \bullet_1 - \bullet_1 \times \ell_{\substack{1\\2}} \times \bullet_1 + \ell_{\substack{1\\2\\3}} \times \bullet_1\\
&\qquad - \bullet_1 \times \bullet_1 \times \ell_{\substack{2\\1}} + \ell_{\substack{1\\2}} \times \ell_{\substack{2\\1}} + \bullet_1 \times \ell_{\substack{1\\3\\2}} - \ell_{\substack{1\\2\\4\\3}} . \qquad (8)
\end{aligned}$$

Applying S to this again (and using that S is an anti-algebra homomorphism), we get

$$\begin{aligned}
&S^2\left(\ell_{\substack{1\\2\\4\\3}}\right)\\
&= S\left(\bullet_1\right) \times S\left(\bullet_1\right) \times S\left(\bullet_1\right) \times S\left(\bullet_1\right) - S\left(\bullet_1\right) \times S\left(\bullet_1\right) \times S\left(\ell_{\substack{1\\2}}\right) - S\left(\bullet_1\right) \times S\left(\ell_{\substack{1\\2}}\right) \times S\left(\bullet_1\right)\\
&\qquad + S\left(\bullet_1\right) \times S\left(\ell_{\substack{1\\2\\3}}\right) - S\left(\ell_{\substack{2\\1}}\right) \times S\left(\bullet_1\right) \times S\left(\bullet_1\right) + S\left(\ell_{\substack{2\\1}}\right) \times S\left(\ell_{\substack{1\\2}}\right) + S\left(\ell_{\substack{1\\3\\2}}\right) \times S\left(\bullet_1\right) - S\left(\ell_{\substack{1\\2\\4\\3}}\right)\\
&= \left[\bullet_1, \ell_{\substack{1\\2\\3}}\right] + \bullet_1 \times \left[\bullet_1, \ell_{\substack{2\\1}} - \ell_{\substack{1\\2}}\right] - \left[\bullet_1, \ell_{\substack{1\\3\\2}}\right] - \left[\ell_{\substack{1\\2}}, \ell_{\substack{2\\1}}\right] + \ell_{\substack{1\\2\\4\\3}}\\
&\qquad \left(\begin{array}{c} \text{using the known values } S\left(\bullet_1\right) = -\bullet_1,\ S\left(\ell_{\substack{1\\2}}\right) = \bullet_1 \times \bullet_1 - \ell_{\substack{1\\2}},\ S\left(\ell_{\substack{2\\1}}\right) = \bullet_1 \times \bullet_1 - \ell_{\substack{2\\1}},\\ \text{and the values of } S\left(\ell_{\substack{1\\2\\3}}\right),\ S\left(\ell_{\substack{1\\3\\2}}\right) \text{ and } S\left(\ell_{\substack{1\\2\\4\\3}}\right) \text{ given by (1), (2) and (8)} \end{array}\right),
\end{aligned}$$

so that

$$\left(S^2-\mathrm{id}\right)\left(\bullet\substack{1\\2\\4\\3}\right) = \left[\bullet_1, \bullet\substack{1\\2\\3}\right] + \bullet_1 \times \left[\bullet_1, \bullet\substack{2\\1} - \bullet\substack{1\\2}\right] - \left[\bullet_1, \bullet\substack{1\\3\\2}\right] - \left[\bullet\substack{1\\2}, \bullet\substack{2\\1}\right].$$

Hence, altogether,

$$\begin{aligned}
&\left(S^2-\mathrm{id}\right)(R_4)\\
&= \left\langle \begin{array}{l} 0,\quad 0,\quad \bullet_1 \times \left[\bullet_1,\ \bullet\substack{1\\2} - \bullet\substack{2\\1}\right],\quad \left[\bullet_1,\ \bullet\substack{1\\2} - \bullet\substack{2\\1}\right] \times \bullet_1, \\ \left[\bullet_1, \bullet\substack{1\\2\\3}\right] + \bullet_1 \times \left[\bullet_1, \bullet\substack{2\\1} - \bullet\substack{1\\2}\right] - \left[\bullet_1, \bullet\substack{1\\3\\2}\right] - \left[\bullet\substack{1\\2}, \bullet\substack{2\\1}\right] \end{array} \right\rangle\\
&= \left\langle \bullet_1 \times \left[\bullet_1,\ \bullet\substack{1\\2} - \bullet\substack{2\\1}\right],\ \left[\bullet_1,\ \bullet\substack{1\\2} - \bullet\substack{2\\1}\right] \times \bullet_1,\ \left[\bullet_1, \bullet\substack{1\\2\\3}\right] - \left[\bullet_1, \bullet\substack{1\\3\\2}\right] - \left[\bullet\substack{1\\2}, \bullet\substack{2\\1}\right] \right\rangle . \qquad (9)
\end{aligned}$$

Substituting (7) and (9) (and $R_1 = \langle \bullet_1 \rangle$) in (3), we obtain

$$\begin{aligned}
&(\operatorname{Inv} R) \cap R_4\\
&= \left\langle \bullet_1 \times \left[\bullet_1,\ \bullet\substack{1\\2} - \bullet\substack{2\\1}\right],\ \left[\bullet_1,\ \bullet\substack{1\\2} - \bullet\substack{2\\1}\right] \times \bullet_1,\ \left[\bullet_1, \bullet\substack{1\\2\\3}\right] - \left[\bullet_1, \bullet\substack{1\\3\\2}\right] - \left[\bullet\substack{1\\2}, \bullet\substack{2\\1}\right] \right\rangle\\
&\qquad + \left\langle \left[\bullet_1,\ \bullet\substack{1\\2} - \bullet\substack{2\\1}\right] \right\rangle \times \langle \bullet_1 \rangle + \langle \bullet_1 \rangle \times \left\langle \left[\bullet_1,\ \bullet\substack{1\\2} - \bullet\substack{2\\1}\right] \right\rangle\\
&= \left\langle \bullet_1 \times \left[\bullet_1,\ \bullet\substack{1\\2} - \bullet\substack{2\\1}\right],\ \left[\bullet_1,\ \bullet\substack{1\\2} - \bullet\substack{2\\1}\right] \times \bullet_1,\ \left[\bullet_1, \bullet\substack{1\\2\\3}\right] - \left[\bullet_1, \bullet\substack{1\\3\\2}\right] - \left[\bullet\substack{1\\2}, \bullet\substack{2\\1}\right] \right\rangle \qquad (10)\\
&= \underbrace{\left\langle \bullet_1 \times \left[\bullet_1,\ \bullet\substack{1\\2} - \bullet\substack{2\\1}\right],\ \left[\bullet_1,\ \bullet\substack{1\\2} - \bullet\substack{2\\1}\right] \times \bullet_1 \right\rangle}_{\subseteq (R^+)^3 \cap R_4} + \left\langle \left[\bullet_1, \bullet\substack{1\\2\\3}\right] - \left[\bullet_1, \bullet\substack{1\\3\\2}\right] - \left[\bullet\substack{1\\2}, \bullet\substack{2\\1}\right] \right\rangle\\
&\subseteq \left(R^+\right)^3 \cap R_4 + \left\langle \left[\bullet_1, \bullet\substack{1\\2\\3}\right] - \left[\bullet_1, \bullet\substack{1\\3\\2}\right] - \left[\bullet\substack{1\\2}, \bullet\substack{2\\1}\right] \right\rangle ,
\end{aligned}$$

so that

$$\begin{aligned}
&(\operatorname{Inv} R) \cap R_4 + \left(R^+\right)^3 \cap R_4\\
&= \left\langle \left[\bullet_1, \bullet\substack{1\\2\\3}\right] - \left[\bullet_1, \bullet\substack{1\\3\\2}\right] - \left[\bullet\substack{1\\2}, \bullet\substack{2\\1}\right] \right\rangle + \left(R^+\right)^3 \cap R_4. \qquad (11)
\end{aligned}$$

But since the ideals $\operatorname{Inv} R$ and $\left(R^+\right)^3$ are homogeneous, we have $(\operatorname{Inv} R) \cap R_4 + \left(R^+\right)^3 \cap R_4 = \left((\operatorname{Inv} R) + \left(R^+\right)^3\right) \cap R_4$ (because the sum of the intersections of two homogeneous ideals with R_4 equals the intersection of their sum with R_4 [8]). Hence, (11) becomes

$$\begin{aligned}
&\left((\operatorname{Inv} R) + \left(R^+\right)^3\right) \cap R_4\\
&= \left\langle \left[\bullet_1, \bullet\substack{1\\2\\3}\right] - \left[\bullet_1, \bullet\substack{1\\3\\2}\right] - \left[\bullet\substack{1\\2}, \bullet\substack{2\\1}\right] \right\rangle + \left(R^+\right)^3 \cap R_4. \qquad (12)
\end{aligned}$$

We now take aim at (2) and compute $\left(E * \overline{S}\right)\left(\bullet\substack{1\\2\\4\\3}\right)$ modulo $\left(R^+\right)^3$: Since

$$\Delta\left(\bullet\substack{1\\2\\4\\3}\right) = \bullet\substack{1\\2\\4\\3} \otimes 1 + \bullet\substack{1\\2\\3} \otimes \bullet_1 + \bullet\substack{1\\2} \otimes \bullet\substack{2\\1} + \bullet_1 \otimes \bullet\substack{1\\3\\2} + 1 \otimes \bullet\substack{1\\2\\4\\3},$$

[8] *Proof (sketched).* The intersection of a homogeneous ideal with R_4 is the same as the projection of this ideal on R_4. But the sum of the projections of two ideals on R_4 clearly equals the projection of their sum on R_4, qed.

we have (using the fact that $\overline{S}(x) = S(x)$ for every $x \in R^+$) the following computation:

$$
\begin{aligned}
&\left(E * \overline{S}\right)\left(\left\vert\begin{smallmatrix}1\\2\\4\\3\end{smallmatrix}\right.\right) \\
&= E\left(\left\vert\begin{smallmatrix}1\\2\\4\\3\end{smallmatrix}\right.\right) \times \underbrace{\overline{S}(1)}_{=0} + \underbrace{E\left(\left\vert\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right.\right)}_{=3\left\vert\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right.} \times \underbrace{\overline{S}(\bullet_1)}_{=-\bullet_1} + \underbrace{E\left(\left\vert\begin{smallmatrix}1\\2\end{smallmatrix}\right.\right)}_{=2\left\vert\begin{smallmatrix}1\\2\end{smallmatrix}\right.} \times \underbrace{\overline{S}\left(\left\vert\begin{smallmatrix}2\\1\end{smallmatrix}\right.\right)}_{=\bullet_1 \times \bullet_1 - \left\vert\begin{smallmatrix}2\\1\end{smallmatrix}\right.} \\
&\qquad + \underbrace{E(\bullet_1)}_{=1\bullet_1} \times \underbrace{\overline{S}\left(\left\vert\begin{smallmatrix}1\\3\\2\end{smallmatrix}\right.\right)}_{=-\bullet_1 \times \bullet_1 \times \bullet_1 + \left\vert\begin{smallmatrix}1\\2\end{smallmatrix}\right. \times \bullet_1 + \bullet_1 \times \left\vert\begin{smallmatrix}2\\1\end{smallmatrix}\right. - \left\vert\begin{smallmatrix}1\\3\\2\end{smallmatrix}\right.} + \underbrace{E(1)}_{=0\cdot 1} \times \overline{S}\left(\left\vert\begin{smallmatrix}1\\2\\4\\3\end{smallmatrix}\right.\right) \\
&= -3\left\vert\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right. \times \bullet_1 + 2\left\vert\begin{smallmatrix}1\\2\end{smallmatrix}\right. \times \left(\bullet_1 \times \bullet_1 - \left\vert\begin{smallmatrix}2\\1\end{smallmatrix}\right.\right) + \bullet_1 \times \left(-\bullet_1 \times \bullet_1 \times \bullet_1 + \left\vert\begin{smallmatrix}1\\2\end{smallmatrix}\right. \times \bullet_1 + \bullet_1 \times \left\vert\begin{smallmatrix}2\\1\end{smallmatrix}\right. - \left\vert\begin{smallmatrix}1\\3\\2\end{smallmatrix}\right.\right) \\
&\equiv -3\left\vert\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right. \times \bullet_1 - 2\left\vert\begin{smallmatrix}1\\2\end{smallmatrix}\right. \times \left\vert\begin{smallmatrix}2\\1\end{smallmatrix}\right. - \bullet_1 \times \left\vert\begin{smallmatrix}1\\3\\2\end{smallmatrix}\right. \mod \left(R^+\right)^3 .
\end{aligned}
\tag{13}
$$

We want to apply $E * S$ to (13), but for this we need to compute $(E * S)\left(\left\vert\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right. \times \bullet_1\right)$, $(E * S)\left(\left\vert\begin{smallmatrix}1\\2\end{smallmatrix}\right. \times \left\vert\begin{smallmatrix}2\\1\end{smallmatrix}\right.\right)$ and $(E * S)\left(\bullet_1 \times \left\vert\begin{smallmatrix}1\\3\\2\end{smallmatrix}\right.\right)$. This is a routine computation, but we can simplify it using the following lemma:

> **Lemma 11.** Let H be a connected graded Hopf algebra over a field k. We will use the notations of Definition 4. Let $x \in H_1$ and $y \in H^+$.
>
> **(a)** Then, $(E * S)(xy) = [x, (E * S)(y)]$.
>
> **(b)** Besides, $(E * S)(yx) = y_{(1)} x S\left(y_{(2)}\right)$ using the sumfree Sweedler notation.

Note that Lemma 11 **(a)** is a particular case of Theorem 6 in [2] (using Theorem 4 in [2]).

Proof of Lemma 11 (sketched). We will use the sumfree Sweedler notation.

It is known enough that every element of H_1 is primitive (due to the connectedness of H). Thus, in particular, x is primitive, so that $x_{(1)} \otimes x_{(2)} = x \otimes 1 + 1 \otimes x$. Also, $x \in H_1$ yields $E(x) = 1x = x$.

(a) We have

$$(E*S)(xy) = E\left((xy)_{(1)}\right) S\left((xy)_{(2)}\right) = \underbrace{E\left(x_{(1)}y_{(1)}\right)}_{\substack{=E\left(x_{(1)}\right)y_{(1)}+x_{(1)}E\left(y_{(1)}\right) \\ \text{(since } E \text{ is a derivation)}}} \underbrace{S\left(x_{(2)}y_{(2)}\right)}_{\substack{=S\left(y_{(2)}\right)S\left(x_{(2)}\right) \\ \text{(since } S \text{ is an anti-algebra homomorphism)}}}$$

$$= \left(E\left(x_{(1)}\right) y_{(1)} + x_{(1)} E\left(y_{(1)}\right)\right) S\left(y_{(2)}\right) S\left(x_{(2)}\right)$$

$$= E\left(x_{(1)}\right) \underbrace{y_{(1)} S\left(y_{(2)}\right)}_{=(\mathrm{id}*S)(y)=\varepsilon(y)1} S\left(x_{(2)}\right) + x_{(1)} E\left(y_{(1)}\right) S\left(y_{(2)}\right) S\left(x_{(2)}\right)$$

$$= E\left(x_{(1)}\right) S\left(x_{(2)}\right) \varepsilon(y) 1 + x_{(1)} E\left(y_{(1)}\right) S\left(y_{(2)}\right) S\left(x_{(2)}\right)$$

$$= \underbrace{E(x)}_{=x} \underbrace{S(1)}_{=1} \underbrace{\varepsilon(y)}_{\substack{=0 \\ \text{(since } y \in H^{+})}} 1 + \underbrace{E(1)}_{=0} S(x) \varepsilon(y) 1$$

$$+ xE\left(y_{(1)}\right) S\left(y_{(2)}\right) \underbrace{S(1)}_{=1} + 1E\left(y_{(1)}\right) S\left(y_{(2)}\right) \underbrace{S(x)}_{\substack{=-x \\ \text{(since } x \text{ is primitive)}}}$$

$$\left(\text{since } x_{(1)} \otimes x_{(2)} = x \otimes 1 + 1 \otimes x\right)$$

$$= xE\left(y_{(1)}\right) S\left(y_{(2)}\right) - E\left(y_{(1)}\right) S\left(y_{(2)}\right) x = \left[x, \underbrace{E\left(y_{(1)}\right) S\left(y_{(2)}\right)}_{=(E*S)(y)}\right] = [x, (E*S)(y)].$$

Lemma 11 **(a)** is thus proven.

(b) We have

$$(E*S)(yx) = E\left((yx)_{(1)}\right) S\left((yx)_{(2)}\right) = \underbrace{E\left(y_{(1)}x_{(1)}\right)}_{\substack{=E\left(y_{(1)}\right)x_{(1)}+y_{(1)}E\left(x_{(1)}\right) \\ \text{(since } E \text{ is a derivation)}}} \underbrace{S\left(y_{(2)}x_{(2)}\right)}_{\substack{=S\left(x_{(2)}\right)S\left(y_{(2)}\right) \\ \text{(since } S \text{ is an anti-algebra homomorphism)}}}$$

$$= \left(E\left(y_{(1)}\right) x_{(1)} + y_{(1)} E\left(x_{(1)}\right)\right) S\left(x_{(2)}\right) S\left(y_{(2)}\right)$$

$$= E\left(y_{(1)}\right) \underbrace{x_{(1)} S\left(x_{(2)}\right)}_{\substack{=(\mathrm{id}*S)(x)=\varepsilon(x)1=0 \\ \text{(since } x \in H_1 \subseteq H^{+})}} S\left(y_{(2)}\right) + y_{(1)} E\left(x_{(1)}\right) S\left(x_{(2)}\right) S\left(y_{(2)}\right)$$

$$= y_{(1)} E\left(x_{(1)}\right) S\left(x_{(2)}\right) S\left(y_{(2)}\right) = y_{(1)} \underbrace{E(x)}_{=x} \underbrace{S(1)}_{=1} S\left(y_{(2)}\right) + y_{(1)} \underbrace{E(1)}_{=0} S(x) S\left(y_{(2)}\right)$$

$$\left(\text{since } x_{(1)} \otimes x_{(2)} = x \otimes 1 + 1 \otimes x\right)$$

$$= y_{(1)} x S\left(y_{(2)}\right).$$

Lemma 11 **(b)** is thus proven.

Back to work: First, let us compute $(E*S)\left(\mathsf{I}^{1}_{2} \times \mathsf{I}^{2}_{1}\right)$. This is where Lemma 11 doesn't help. Since Δ is an algebra homomorphism,

$$\Delta\left(\mathsf{I}^{1}_{2} \times \mathsf{I}^{2}_{1}\right) = \underbrace{\Delta\left(\mathsf{I}^{1}_{2}\right)}_{=\mathsf{I}^{1}_{2} \otimes 1 + \bullet_1 \otimes \bullet_1 + 1 \otimes \mathsf{I}^{1}_{2}} \times \underbrace{\Delta\left(\mathsf{I}^{2}_{1}\right)}_{=\mathsf{I}^{2}_{1} \otimes 1 + \bullet_1 \otimes \bullet_1 + 1 \otimes \mathsf{I}^{2}_{1}}$$

$$= \left(\mathsf{I}^{1}_{2} \otimes 1 + \bullet_1 \otimes \bullet_1 + 1 \otimes \mathsf{I}^{1}_{2}\right) \times \left(\mathsf{I}^{2}_{1} \otimes 1 + \bullet_1 \otimes \bullet_1 + 1 \otimes \mathsf{I}^{2}_{1}\right)$$

$$= \mathsf{I}^{1}_{2} \times \mathsf{I}^{2}_{1} \otimes 1 + \mathsf{I}^{1}_{2} \times \bullet_1 \otimes \bullet_1 + \mathsf{I}^{1}_{2} \otimes \mathsf{I}^{2}_{1}$$

$$+ \bullet_1 \times \mathsf{I}^{2}_{1} \otimes \bullet_1 + \bullet_1 \times \bullet_1 \otimes \bullet_1 \times \bullet_1 + \bullet_1 \otimes \bullet_1 \times \mathsf{I}^{2}_{1}$$

$$+ \mathsf{I}^{2}_{1} \otimes \mathsf{I}^{1}_{2} + \bullet_1 \otimes \mathsf{I}^{1}_{2} \times \bullet_1 + 1 \otimes \mathsf{I}^{1}_{2} \times \mathsf{I}^{2}_{1}.$$

Hence,

$$
\begin{aligned}
&(E * S)\left(\ell_{\substack{1\\2}} \times \ell_{\substack{2\\1}}\right) \\
&= \underbrace{E\left(\ell_{\substack{1\\2}} \times \ell_{\substack{2\\1}}\right)}_{=4\ell_{\substack{1\\2}} \times \ell_{\substack{2\\1}}} \times \underbrace{S(1)}_{=1} + \underbrace{E\left(\ell_{\substack{1\\2}} \times \bullet_1\right)}_{=3\ell_{\substack{1\\2}} \times \bullet_1} \times \underbrace{S(\bullet_1)}_{=-\bullet_1} + \underbrace{E\left(\ell_{\substack{1\\2}}\right)}_{=2\ell_{\substack{1\\2}}} \times \underbrace{S\left(\ell_{\substack{2\\1}}\right)}_{=\bullet_1 \times \bullet_1 - \ell_{\substack{2\\1}}} \\
&\qquad + \underbrace{E\left(\bullet_1 \times \ell_{\substack{2\\1}}\right)}_{=3\bullet_1 \times \ell_{\substack{2\\1}}} \times \underbrace{S(\bullet_1)}_{=-\bullet_1} + \underbrace{E(\bullet_1 \times \bullet_1)}_{=2\bullet_1 \times \bullet_1} \times \underbrace{S(\bullet_1 \times \bullet_1)}_{=\bullet_1 \times \bullet_1} + \underbrace{E(\bullet_1)}_{=1\bullet_1} \times \underbrace{S\left(\bullet_1 \times \ell_{\substack{2\\1}}\right)}_{\substack{=S\left(\ell_{\substack{2\\1}}\right) \times S(\bullet_1) \\ =\left(\bullet_1 \times \bullet_1 - \ell_{\substack{2\\1}}\right) \times (-\bullet_1)}} \\
&\qquad + \underbrace{E\left(\ell_{\substack{2\\1}}\right)}_{=2\ell_{\substack{2\\1}}} \times \underbrace{S\left(\ell_{\substack{1\\2}}\right)}_{=\bullet_1 \times \bullet_1 - \ell_{\substack{1\\2}}} + \underbrace{E(\bullet_1)}_{=1\bullet_1} \times \underbrace{S\left(\ell_{\substack{1\\2}} \times \bullet_1\right)}_{\substack{=S(\bullet_1) \times S\left(\ell_{\substack{1\\2}}\right) \\ =(-\bullet_1) \times \left(\bullet_1 \times \bullet_1 - \ell_{\substack{1\\2}}\right)}} + \underbrace{E(1)}_{=0} \times S\left(\ell_{\substack{1\\2}} \times \ell_{\substack{2\\1}}\right) \\
&= 2\left[\ell_{\substack{1\\2}}, \ell_{\substack{2\\1}}\right] - \ell_{\substack{1\\2}} \times \bullet_1 \times \bullet_1 - 2\bullet_1 \times \ell_{\substack{2\\1}} \times \bullet_1 + 2\ell_{\substack{2\\1}} \times \bullet_1 \times \bullet_1 + \bullet_1 \times \bullet_1 \times \ell_{\substack{1\\2}} \\
&\equiv 2\left[\ell_{\substack{1\\2}}, \ell_{\substack{2\\1}}\right] \mathrm{mod} \left(R^+\right)^3 .
\end{aligned}
\tag{14}
$$

Now, the easy parts: Using the sumfree Sweedler notation,

$$
\begin{aligned}
&(E * S)\left(\ell_{\substack{1\\2\\3}} \times \bullet_1\right) \\
&= \ell_{\substack{1\\2\\3}\,(1)} \times \bullet_1 \times S\left(\ell_{\substack{1\\2\\3}\,(2)}\right) \qquad \left(\text{by Lemma 11 \textbf{(b)}, applied to } H = R,\ x = \bullet_1 \text{ and } y = \ell_{\substack{1\\2\\3}}\right) \\
&= \ell_{\substack{1\\2\\3}} \times \bullet_1 \times \underbrace{S(1)}_{=1} + \ell_{\substack{1\\2}} \times \bullet_1 \times \underbrace{S(\bullet_1)}_{=-\bullet_1} + \bullet_1 \times \bullet_1 \times \underbrace{S\left(\ell_{\substack{1\\2}}\right)}_{=\bullet_1 \times \bullet_1 - \ell_{\substack{1\\2}}} + 1 \times \bullet_1 \times \underbrace{S\left(\ell_{\substack{1\\2\\3}}\right)}_{=-\bullet_1 \times \bullet_1 \times \bullet_1 + \ell_{\substack{1\\2}} \times \bullet_1 + \bullet_1 \times \ell_{\substack{1\\2}} - \ell_{\substack{1\\2\\3}}} \\
&\qquad \left(\text{since } \Delta\left(\ell_{\substack{1\\2\\3}}\right) = \ell_{\substack{1\\2\\3}} \otimes 1 + \ell_{\substack{1\\2}} \otimes \bullet_1 + \bullet_1 \otimes \ell_{\substack{1\\2}} + 1 \otimes \ell_{\substack{1\\2\\3}}\right) \\
&= -\left[\bullet_1, \ell_{\substack{1\\2\\3}}\right] + \left[\bullet_1, \ell_{\substack{1\\2}}\right] \times \bullet_1 \equiv -\left[\bullet_1, \ell_{\substack{1\\2\\3}}\right] \mathrm{mod} \left(R^+\right)^3 .
\end{aligned}
\tag{15}
$$

Also,

$$
\begin{aligned}
(E * S)\left(\ell_{\substack{1\\3\\2}}\right) &= \underbrace{E\left(\ell_{\substack{1\\3\\2}}\right)}_{=3\ell_{\substack{1\\3\\2}}} \times \underbrace{S(1)}_{=1} + \underbrace{E\left(\ell_{\substack{1\\2}}\right)}_{=2\ell_{\substack{1\\2}}} \times \underbrace{S(\bullet_1)}_{=-\bullet_1} + \underbrace{E(\bullet_1)}_{=1\bullet_1} \times \underbrace{S\left(\ell_{\substack{2\\1}}\right)}_{=\bullet_1 \times \bullet_1 - \ell_{\substack{2\\1}}} + \underbrace{E(1)}_{=0} \times S\left(\ell_{\substack{1\\3\\2}}\right) \\
&\qquad \left(\text{since } \Delta\left(\ell_{\substack{1\\3\\2}}\right) = \ell_{\substack{1\\3\\2}} \otimes 1 + \ell_{\substack{1\\2}} \otimes \bullet_1 + \bullet_1 \otimes \ell_{\substack{2\\1}} + 1 \otimes \ell_{\substack{1\\3\\2}}\right) \\
&= 3\ell_{\substack{1\\3\\2}} - 2\ell_{\substack{1\\2}} \times \bullet_1 + \bullet_1 \times \left(\bullet_1 \times \bullet_1 - \ell_{\substack{2\\1}}\right) \\
&= 3\ell_{\substack{1\\3\\2}} - 2\ell_{\substack{1\\2}} \times \bullet_1 + \bullet_1 \times \bullet_1 \times \bullet_1 - \bullet_1 \times \ell_{\substack{2\\1}} ,
\end{aligned}
$$

and now

$$(E*S)\left(\bullet_1 \times \begin{smallmatrix}\bullet_1\\\bullet_3\\\bullet_2\end{smallmatrix}\right) = \Bigg[\bullet_1,\ \underbrace{(E*S)\left(\begin{smallmatrix}\bullet_1\\\bullet_3\\\bullet_2\end{smallmatrix}\right)}_{=3\begin{smallmatrix}\bullet_1\\\bullet_3\\\bullet_2\end{smallmatrix}-2\begin{smallmatrix}\bullet_1\\\bullet_2\end{smallmatrix}\times\bullet_1+\bullet_1\times\bullet_1\times\bullet_1-\bullet_1\times\begin{smallmatrix}\bullet_2\\\bullet_1\end{smallmatrix}}\Bigg]$$

$$\left(\text{by Lemma 11 } \textbf{(a)}\text{, applied to } H=R,\ x=\bullet_1 \text{ and } y=\begin{smallmatrix}\bullet_1\\\bullet_2\\\bullet_3\end{smallmatrix}\right)$$

$$\begin{aligned}
&= \left[\bullet_1,\ 3\begin{smallmatrix}\bullet_1\\\bullet_3\\\bullet_2\end{smallmatrix} - 2\begin{smallmatrix}\bullet_1\\\bullet_2\end{smallmatrix}\times\bullet_1 + \bullet_1\times\bullet_1\times\bullet_1 - \bullet_1\times\begin{smallmatrix}\bullet_2\\\bullet_1\end{smallmatrix}\right]\\
&= 3\left[\bullet_1,\ \begin{smallmatrix}\bullet_1\\\bullet_3\\\bullet_2\end{smallmatrix}\right] - 2\left[\bullet_1,\ \begin{smallmatrix}\bullet_1\\\bullet_2\end{smallmatrix}\times\bullet_1\right] + \underbrace{\left[\bullet_1,\ \bullet_1\times\bullet_1\times\bullet_1\right]}_{=0} - \left[\bullet_1,\ \bullet_1\times\begin{smallmatrix}\bullet_2\\\bullet_1\end{smallmatrix}\right]\\
&= 3\left[\bullet_1,\ \begin{smallmatrix}\bullet_1\\\bullet_3\\\bullet_2\end{smallmatrix}\right] - 2\left[\bullet_1,\ \begin{smallmatrix}\bullet_1\\\bullet_2\end{smallmatrix}\times\bullet_1\right] - \left[\bullet_1,\ \bullet_1\times\begin{smallmatrix}\bullet_2\\\bullet_1\end{smallmatrix}\right]\\
&\equiv 3\left[\bullet_1,\ \begin{smallmatrix}\bullet_1\\\bullet_3\\\bullet_2\end{smallmatrix}\right] \operatorname{mod} \left(R^+\right)^3. \qquad (16)
\end{aligned}$$

Now, applying $E*S$ to (13), we get

$$\begin{aligned}
&\left((E*S)\circ\left(E*\overline{S}\right)\right)\left(\begin{smallmatrix}\bullet_1\\\bullet_2\\\bullet_4\\\bullet_3\end{smallmatrix}\right)\\
&\equiv -3\underbrace{(E*S)\left(\begin{smallmatrix}\bullet_1\\\bullet_2\\\bullet_3\end{smallmatrix}\times\bullet_1\right)}_{\substack{\equiv-\left[\bullet_1,\ \begin{smallmatrix}\bullet_1\\\bullet_2\\\bullet_3\end{smallmatrix}\right]\operatorname{mod}\left(R^+\right)^3\\ \text{(by (15))}}} - 2\underbrace{(E*S)\left(\begin{smallmatrix}\bullet_1\\\bullet_2\end{smallmatrix}\times\begin{smallmatrix}\bullet_2\\\bullet_1\end{smallmatrix}\right)}_{\substack{\equiv 2\left[\begin{smallmatrix}\bullet_1\\\bullet_2\end{smallmatrix},\ \begin{smallmatrix}\bullet_2\\\bullet_1\end{smallmatrix}\right]\operatorname{mod}\left(R^+\right)^3.\\ \text{(by (14))}}} - \underbrace{(E*S)\left(\bullet_1\times\begin{smallmatrix}\bullet_1\\\bullet_3\\\bullet_2\end{smallmatrix}\right)}_{\substack{\equiv 3\left[\bullet_1,\ \begin{smallmatrix}\bullet_1\\\bullet_3\\\bullet_2\end{smallmatrix}\right]\operatorname{mod}\left(R^+\right)^3\\ \text{(by (16))}}}\\
&\equiv -3\left(-\left[\bullet_1,\ \begin{smallmatrix}\bullet_1\\\bullet_2\\\bullet_3\end{smallmatrix}\right]\right) - 2\left(2\left[\begin{smallmatrix}\bullet_1\\\bullet_2\end{smallmatrix},\ \begin{smallmatrix}\bullet_2\\\bullet_1\end{smallmatrix}\right]\right) - 3\left[\bullet_1,\ \begin{smallmatrix}\bullet_1\\\bullet_3\\\bullet_2\end{smallmatrix}\right]\\
&= 3\left[\bullet_1,\ \begin{smallmatrix}\bullet_1\\\bullet_2\\\bullet_3\end{smallmatrix}\right] - 4\left[\begin{smallmatrix}\bullet_1\\\bullet_2\end{smallmatrix},\ \begin{smallmatrix}\bullet_2\\\bullet_1\end{smallmatrix}\right] - 3\left[\bullet_1,\ \begin{smallmatrix}\bullet_1\\\bullet_3\\\bullet_2\end{smallmatrix}\right] \operatorname{mod}\left(R^+\right)^3. \qquad (17)
\end{aligned}$$

This yields

$$\left((E*S)\circ\left(E*\overline{S}\right)\right)\left(\begin{smallmatrix}\bullet_1\\\bullet_2\\\bullet_4\\\bullet_3\end{smallmatrix}\right) \equiv 3\left[\bullet_1,\ \begin{smallmatrix}\bullet_1\\\bullet_2\\\bullet_3\end{smallmatrix}\right] - 4\left[\begin{smallmatrix}\bullet_1\\\bullet_2\end{smallmatrix},\ \begin{smallmatrix}\bullet_2\\\bullet_1\end{smallmatrix}\right] - 3\left[\bullet_1,\ \begin{smallmatrix}\bullet_1\\\bullet_3\\\bullet_2\end{smallmatrix}\right] \operatorname{mod}\left(\left(R^+\right)^3\cap R_4\right)$$

(because both sides $\left((E*S)\circ\left(E*\overline{S}\right)\right)\left(\begin{smallmatrix}\bullet_1\\\bullet_2\\\bullet_4\\\bullet_3\end{smallmatrix}\right)$ and $3\left[\bullet_1,\ \begin{smallmatrix}\bullet_1\\\bullet_2\\\bullet_3\end{smallmatrix}\right] - 4\left[\begin{smallmatrix}\bullet_1\\\bullet_2\end{smallmatrix},\ \begin{smallmatrix}\bullet_2\\\bullet_1\end{smallmatrix}\right] - 3\left[\bullet_1,\ \begin{smallmatrix}\bullet_1\\\bullet_3\\\bullet_2\end{smallmatrix}\right]$ lie in R_4, and thus so does their difference).

Thus, $\left((E*S)\circ\left(E*\overline{S}\right)\right)\left(\begin{smallmatrix}\bullet_1\\\bullet_2\\\bullet_4\\\bullet_3\end{smallmatrix}\right)$ is *not* in $\left((\operatorname{Inv} R)+\left(R^+\right)^3\right)\cap R_4$ (according to (12)). Hence, $\left((E*S)\circ\left(E*\overline{S}\right)\right)\left(\begin{smallmatrix}\bullet_1\\\bullet_2\\\bullet_4\\\bullet_3\end{smallmatrix}\right)$ lies in R_4 (because the maps $E*S$ and $E*\overline{S}$ are graded) but not in $\left((\operatorname{Inv} R)+\left(R^+\right)^3\right)\cap R_4$. Consequently, $\left((E*S)\circ\left(E*\overline{S}\right)\right)\left(\begin{smallmatrix}\bullet_1\\\bullet_2\\\bullet_4\\\bullet_3\end{smallmatrix}\right) \notin \operatorname{Inv} R + \left(R^+\right)^3$. This proves (2). Thus, we have shown that $(E*S)\circ(E*S) \neq E\circ(E*S)$ in R'.

§4. A shortcut

The proof that $(E * S) \circ (E * S) \neq E \circ (E * S)$ in R' is complete, but let us give an alternative shortcut from (13) to (17), without requiring that much computation:

We will prove that

$$(E * S)(a \times b) \equiv \deg b \cdot [a, b] \operatorname{mod} \left(R^+\right)^3 \qquad \text{for any two ordered trees } a \text{ and } b. \quad (18)$$

(Here, $\deg b$ means the number of vertices of the tree b, or, equivalently, the degree of b in the graded Hopf algebra R.)

Proof of (18). Let t be an ordered tree. We first notice that, from the definition of Δ, it is pretty much clear that

$$\Delta(t) \equiv t \otimes 1 + 1 \otimes t \operatorname{mod} R^+ \otimes R^+.$$

Applying $\operatorname{id} \otimes S$ to this congruence, we get

$$(\operatorname{id} \otimes S)(\Delta(t)) \equiv t \otimes \underbrace{S(1)}_{=1} + 1 \otimes S(t) = t \otimes 1 + 1 \otimes S(t) \operatorname{mod} (\operatorname{id} \otimes S)\left(R^+ \otimes R^+\right).$$

Since $(\operatorname{id} \otimes S)(R^+ \otimes R^+) = R^+ \otimes \underbrace{S\left(R^+\right)}_{\subseteq R^+} \subseteq R^+ \otimes R^+$, this becomes

$$(\operatorname{id} \otimes S)(\Delta(t)) \equiv t \otimes 1 + 1 \otimes S(t) \operatorname{mod} R^+ \otimes R^+.$$

Since $\mu(R^+ \otimes R^+) = (R^+)^2$, we can apply μ to this congruence, and obtain

$$\mu((\operatorname{id} \otimes S)(\Delta(t))) \equiv t + S(t) \operatorname{mod} \left(R^+\right)^2.$$

Since (by the axioms of a Hopf algebra) we have $\mu((\operatorname{id} \otimes S)(\Delta(t))) = \varepsilon(t) = 0$, this becomes $0 \equiv t + S(t) \operatorname{mod} (R^+)^2$, so that $S(t) \equiv -t \operatorname{mod} (R^+)^2$. We have thus proven that

$$S(t) \equiv -t \operatorname{mod} \left(R^+\right)^2 \qquad \text{for every ordered tree } t. \quad (19)$$

Now, let a and b be two ordered trees. Then,

$$\Delta(a \times b) \equiv (a \times b) \otimes 1 + 1 \otimes (a \times b) + a \otimes b + b \otimes a \operatorname{mod} \left(\left(R^+\right)^2 \otimes R^+ + R^+ \otimes \left(R^+\right)^2\right)$$

(this is, again, easy to see from the definition of Δ: for every $\mathbf{v} \models V(a \times b)$, we have either $\mathbf{v} \in \{V(a \times b), V(a), V(b), \varnothing\}$ or $Lea_{\mathbf{v}} F \otimes Roo_{\mathbf{v}} F \in (R^+)^2 \otimes R^+ + R^+ \otimes (R^+)^2$). Thus,

$$\begin{aligned}
(E \otimes S)(\Delta(a \times b)) \equiv\ & \underbrace{E(a \times b)}_{\substack{=\deg(a \times b) \cdot a \times b \\ =(\deg a + \deg b) \cdot a \times b}} \otimes \underbrace{S(1)}_{=1} + \underbrace{E(1)}_{=0} \otimes S(a \times b) \\
& + \underbrace{E(a)}_{=\deg a \cdot a} \otimes S(b) + \underbrace{E(b)}_{=\deg b \cdot b} \otimes S(a) \\
\equiv\ & (\deg a + \deg b) \cdot (a \times b) \otimes 1 + \deg a \cdot a \otimes S(b) + \deg b \cdot b \otimes S(a) \\
& \operatorname{mod} (E \otimes S)\left(\left(R^+\right)^2 \otimes R^+ + R^+ \otimes \left(R^+\right)^2\right).
\end{aligned}$$

Since $(E \otimes S)\left((R^+)^2 \otimes R^+ + R^+ \otimes (R^+)^2\right) \subseteq (R^+)^2 \otimes R^+ + R^+ \otimes (R^+)^2$ (very easily seen), this becomes

$$\begin{aligned}
&(E \otimes S)(\Delta(a \times b)) \\
&\equiv (\deg a + \deg b) \cdot (a \times b) \otimes 1 + \deg a \cdot \underbrace{a \otimes S(b)}_{\substack{\equiv a \otimes (-b) \bmod R^+ \otimes \left(R^+\right)^2 \\ \left(\text{since } a \in R^+ \text{ and } S(b) \equiv -b \bmod \left(R^+\right)^2 \text{ (by (19))}\right)}} \\
&\qquad + \deg b \cdot \underbrace{b \otimes S(a)}_{\substack{\equiv b \otimes (-a) \bmod R^+ \otimes \left(R^+\right)^2 \\ \left(\text{since } b \in R^+ \text{ and } S(a) \equiv -a \bmod \left(R^+\right)^2 \text{ (by (19))}\right)}} \\
&\equiv (\deg a + \deg b) \cdot (a \times b) \otimes 1 + \deg a \cdot a \otimes (-b) + \deg b \cdot b \otimes (-a) \\
&\qquad \bmod \left(\left(R^+\right)^2 \otimes R^+ + R^+ \otimes \left(R^+\right)^2\right).
\end{aligned}$$

Applying μ to this equation, and using $\mu\left((R^+)^2 \otimes R^+ + R^+ \otimes (R^+)^2\right) = (R^+)^3$, we obtain

$$\begin{aligned}
\mu((E \otimes S)(\Delta(a \times b))) &\equiv (\deg a + \deg b) \cdot a \times b + \deg a \cdot a \times (-b) + \deg b \cdot b \times (-a) \\
&= \deg b \cdot [a, b] \bmod \left(R^+\right)^3.
\end{aligned}$$

Since $\mu((E \otimes S)(\Delta(a \times b))) = (E * S)(a \times b)$, this proves (18).

Now, applying $E * S$ to (13) (and using that $(E * S)\left((R^+)^3\right) \subseteq (R^+)^3$), we obtain

$$\begin{aligned}
&\left((E * S) \circ \left(E * \overline{S}\right)\right) \left(\left|\substack{1\\2\\4\\3}\right.\right) \\
&\equiv (E * S)\left(-3 \left|\substack{1\\2\\3}\right. \times \bullet_1 - 2 \left|\substack{1\\2}\right. \times \left|\substack{2\\1}\right. - \bullet_1 \times \left|\substack{1\\3\\2}\right.\right) \\
&= -3 \underbrace{(E * S)\left(\left|\substack{1\\2\\3}\right. \times \bullet_1\right)}_{\substack{\equiv 1 \left[\left|\substack{1\\2\\3}\right., \bullet_1\right] \bmod \left(R^+\right)^3 \\ \text{(after (18))}}} - 2 \underbrace{(E * S)\left(\left|\substack{1\\2}\right. \times \left|\substack{2\\1}\right.\right)}_{\substack{\equiv 2 \left[\left|\substack{1\\2}\right., \left|\substack{2\\1}\right.\right] \bmod \left(R^+\right)^3 \\ \text{(after (18))}}} - \underbrace{(E * S)\left(\bullet_1 \times \left|\substack{1\\3\\2}\right.\right)}_{\substack{\equiv 3 \left[\bullet_1, \left|\substack{1\\3\\2}\right.\right] \bmod \left(R^+\right)^3 \\ \text{(after (18))}}} \\
&\equiv -3 \underbrace{\left[\left|\substack{1\\2\\3}\right., \bullet_1\right]}_{= -\left[\bullet_1, \left|\substack{1\\2\\3}\right.\right]} - 4 \left[\left|\substack{1\\2}\right., \left|\substack{2\\1}\right.\right] - 3 \left[\bullet_1, \left|\substack{1\\3\\2}\right.\right] \\
&= 3 \left[\bullet_1, \left|\substack{1\\2\\3}\right.\right] - 4 \left[\left|\substack{1\\2}\right., \left|\substack{2\\1}\right.\right] - 3 \left[\bullet_1, \left|\substack{1\\3\\2}\right.\right] \bmod \left(R^+\right)^3.
\end{aligned}$$

Hence, (17) is proven once again.

References

[1] Loic Foissy, *Ordered forests and parking functions*, arXiv:1007.1547v3.
`http://arxiv.org/abs/1007.1547v3`

[2] Frédéric Patras, Christophe Reutenauer, *On Dynkin and Klyachko idempotents in graded bialgebras*.
http://www-irma.u-strasbg.fr/annexes/publications/pdf/01029.pdf

[3] Darij Grinberg, *MathOverflow question #84345*.
http://mathoverflow.net/questions/84345