Some basic properties of compositions

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This is a companion note to [GriVas22]. The purpose of this note is to prove some elementary properties of integer compositions that are used in [GriVas22]. All of these proofs are elementary and generally quite easy, but they are hard to find written down and often left to the reader to prove.

1. Notations

We let $\mathbb{N} = \{0, 1, 2, \ldots\}$.

A *composition* means a finite list $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers. The set of all compositions will be denoted by Comp.

The *empty composition* is defined to be the composition $()$, which is a 0-tuple. It is denoted by $\varnothing$.

The *length* $\ell(\alpha)$ of a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is defined to be the number k.

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a composition, then the nonnegative integer $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ is called the *size* of α and is denoted by $|\alpha|$. For any $n \in \mathbb{N}$, we define a *composition of* n to be a composition that has size n. We let Comp_n be the set of all compositions of n (for given $n \in \mathbb{N}$). The notation "$\alpha \models n$" is short for "$\alpha \in \operatorname{Comp}_n$". For example, $(1,5,2,1)$ is a composition with size 9 (since $|(1,5,2,1)| = 1+5+2+1 = 9$), so that $(1,5,2,1) \in \operatorname{Comp}_9$, or, in other words, $(1,5,2,1) \models 9$. Note that the empty composition $\varnothing$ is a composition of 0. In other words, $\varnothing \in \operatorname{Comp}_0$.

For any $n \in \mathbb{Z}$, we let $[n]$ denote the set $\{1, 2, \ldots, n\}$. This set is empty whenever $n \leq 0$, and otherwise has size n.

If X is any set, then $\mathcal{P}(X)$ shall denote the *powerset* of X. This is the set of all subsets of X.

2. The maps D and comp

It is well-known that any positive integer n has exactly 2^{n-1} compositions. This has a standard bijective proof ("stars and bars") which relies on the following bijections:

Definition 2.1. Let $n \in \mathbb{N}$.

(a) We define a map $D : \operatorname{Comp}_n \to \mathcal{P}([n-1])$ by setting[1]

$$\begin{aligned} D(\alpha_1, \alpha_2, \ldots, \alpha_k) &= \{\alpha_1 + \alpha_2 + \cdots + \alpha_i \mid i \in [k-1]\} \\ &= \{\alpha_1, \ \alpha_1 + \alpha_2, \ \alpha_1 + \alpha_2 + \alpha_3, \ \ldots, \ \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\} \end{aligned}$$

for each $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in \operatorname{Comp}_n$. (It is easy to see that this map D is well-defined; see [Grinbe15, detailed version, Lemma 10.4] for a detailed proof.)

(b) We define a map $\operatorname{comp} : \mathcal{P}([n-1]) \to \operatorname{Comp}_n$ as follows: For any $I \in \mathcal{P}([n-1])$, we set

$$\operatorname{comp}(I) = (i_1 - i_0, \ i_2 - i_1, \ \ldots, \ i_m - i_{m-1}),$$

where $i_0, i_1, \ldots, i_m$ are the elements of the set $I \cup \{0, n\}$ listed in increasing order (so that $i_0 < i_1 < \cdots < i_m$, therefore $i_0 = 0$ and $i_m = n$ and $\{i_1, i_2, \ldots, i_{m-1}\} = I$). (It is easy to see that this map comp is well-defined; see [Grinbe15, detailed version, Lemma 10.15 (d)] for a detailed proof.)

The maps D and comp are mutually inverse bijections. (See [Grinbe15, detailed version, Proposition 10.17] for a detailed proof of this.)

We note that both of these maps D and comp depend on n. Thus, they should be denoted by D_n or comp_n to avoid ambiguity. Otherwise, for example, the expression "$\operatorname{comp}(\{2,3\})$" would have different meanings depending on whether n is 4 or 5. However, we shall not use the map comp in what follows. As for the map D, we need not be afraid of any ambiguity, since the value of $D(\alpha)$ for a given composition α is uniquely determined (indeed, the expression "$D(\alpha)$" makes sense only for one value of n, namely for $n = |\alpha|$; no other value of n would satisfy $\alpha \in \operatorname{Comp}_n$). Thus, we shall freely use the notation "$D(\alpha)$" without explicitly specifying n.

The notation D we just introduced presumably originates in the word "**d**escent", but the connection between D and actual descents is indirect and rather misleading. I prefer to call D the "partial sum map" (as $D(\alpha)$ consists of the partial sums of the composition α) and its inverse comp the "interstitial map" (as $\operatorname{comp}(I)$ consists of the lengths of the intervals into which the elements of I split the interval $[n]$).

Example 2.2. Let $n = 10$.

(a) The map D defined in Definition 2.1 **(a)** satisfies

$$\begin{aligned} D(1,4,2,3) &= \{1,\ \ 1+4,\ \ 1+4+2\} = \{1,5,7\}; \\ D(3,5,2) &= \{3,\ \ 3+5\} = \{3,8\}; \\ D(1,1,1,1,1,1,1,1,1,1) &= \{1,2,3,4,5,6,7,8,9\} = [9] = [n-1]; \\ D(10) &= \{\} = \varnothing. \end{aligned}$$

(b) The map comp defined in Definition 2.1 **(b)** satisfies

$$\operatorname{comp}(\{2,3,7\}) = (2-0,\ \ 3-2,\ \ 7-3,\ \ 10-7) = (2,1,4,3)$$

(since $0,2,3,7,10$ are the elements of the set $\{2,3,7\} \cup \{0,10\}$ listed in increasing order).

Our first observation about the bijections D and comp concerns the relation between the size of $D(\alpha)$ and the length $\ell(\alpha)$ of α. Namely, we shall show that every composition α of size $|\alpha| > 0$ satisfies $|D(\alpha)| = \ell(\alpha) - 1$:

Proposition 2.3. Let α be a composition such that $|\alpha| > 0$. Then, $|D(\alpha)| = \ell(\alpha) - 1$.

Note that the "$|\alpha| > 0$" assumption in Proposition 2.3 is necessary, since Proposition 2.3 would fail if α was the empty composition $\varnothing = ()$ (because $D(\varnothing) = \varnothing$ and thus $|D(\varnothing)| = 0 \neq \ell(\varnothing) - 1$).

[1]The notation "$D(\alpha_1, \alpha_2, \ldots, \alpha_k)$" means $D((\alpha_1, \alpha_2, \ldots, \alpha_k))$ (that is, the image of the composition $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ under the map D).

Proof of Proposition 2.3. Write the composition α in the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Then, $\ell(\alpha) = k$ (by the definition of $\ell(\alpha)$) and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ (by the definition of $|\alpha|$). If we had $k = 0$, then we would have

$$\begin{aligned} |\alpha| &= \alpha_1 + \alpha_2 + \cdots + \alpha_k = \alpha_1 + \alpha_2 + \cdots + \alpha_0 \qquad (\text{since } k = 0) \\ &= (\text{empty sum}) = 0, \end{aligned}$$

which would contradict $|\alpha| > 0$. Thus, we cannot have $k = 0$. Hence, $k \neq 0$, so that $k \geq 1$ (since $k \in \mathbb{N}$).

From $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, we obtain

$$\begin{aligned} D(\alpha) &= D(\alpha_1, \alpha_2, \ldots, \alpha_k) \\ &= \{\alpha_1,\ \ \alpha_1 + \alpha_2,\ \ \alpha_1 + \alpha_2 + \alpha_3,\ \ \ldots,\ \ \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\} \end{aligned} \tag{1}$$

(by the definition of the map D). However, it is easy to see that the chain of inequalities

$$\alpha_1 < \alpha_1 + \alpha_2 < \alpha_1 + \alpha_2 + \alpha_3 < \cdots < \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}$$

holds[2]. Thus, the $k-1$ numbers $\alpha_1,\ \alpha_1 + \alpha_2,\ \alpha_1 + \alpha_2 + \alpha_3,\ \ldots,\ \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}$ are distinct. Therefore, the set of these $k-1$ numbers has size $k-1$. In other words, we have

$$|\{\alpha_1,\ \ \alpha_1 + \alpha_2,\ \ \alpha_1 + \alpha_2 + \alpha_3,\ \ \ldots,\ \ \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}| = k - 1.$$

In view of (1), we can rewrite this as $|D(\alpha)| = k - 1$. In other words, $|D(\alpha)| = \ell(\alpha) - 1$ (since $\ell(\alpha) = k$). This proves Proposition 2.3. $\square$

The analogue of Proposition 2.3 for $|\alpha| = 0$ is almost trivial:

Proposition 2.4. Let α be a composition such that $|\alpha| = 0$. Then, $\alpha = \varnothing$ and $\ell(\alpha) = 0$ and $D(\alpha) = \varnothing$.

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[2] *Proof.* Let $i \in [k]$. Then, α_i is an entry of α (since $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$).

Recall that α is a composition, i.e., a finite list of positive integers. Hence, α_i is a positive integer (since α_i is an entry of α). Therefore, $\alpha_i > 0$. Hence,

$$\alpha_1 + \alpha_2 + \cdots + \alpha_i = \alpha_1 + \alpha_2 + \cdots + \alpha_{i-1} + \underbrace{\alpha_i}_{>0} > \alpha_1 + \alpha_2 + \cdots + \alpha_{i-1}.$$

In other words, $\alpha_1 + \alpha_2 + \cdots + \alpha_{i-1} < \alpha_1 + \alpha_2 + \cdots + \alpha_i$.

Forget that we fixed i. We thus have proved the inequality $\alpha_1 + \alpha_2 + \cdots + \alpha_{i-1} < \alpha_1 + \alpha_2 + \cdots + \alpha_i$ for each $i \in [k]$. Hence, in particular, this inequality holds for each $i \in \{2, 3, \ldots, k-1\}$. In other words, we have the chain of inequalities

$$\alpha_1 < \alpha_1 + \alpha_2 < \alpha_1 + \alpha_2 + \alpha_3 < \cdots < \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}.$$

Proof of Proposition 2.4. Write the composition α in the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Then, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ (by the definition of $|\alpha|$) and $\ell(\alpha) = k$ (by the definition of $\ell(\alpha)$).

Assume (for the sake of contradiction) that $k \neq 0$. Thus, $k \geq 1$ (since $k \in \mathbb{N}$).

However, α is a composition, i.e., a finite list of positive integers. In other words, $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a finite list of positive integers (since $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$). Thus, $\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers. Therefore, in particular, $\alpha_2, \alpha_3, \ldots, \alpha_k$ are positive integers. Hence, $\alpha_2 + \alpha_3 + \cdots + \alpha_k \geq 0$ (since a sum of positive integers is always ≥ 0). However, from $|\alpha| = 0$, we obtain

$$\begin{aligned} 0 = |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k &= \alpha_1 + \underbrace{(\alpha_2 + \alpha_3 + \cdots + \alpha_k)}_{\geq 0} \qquad (\text{since } k \geq 1) \\ &\geq \alpha_1 > 0 \qquad (\text{since } \alpha_1 \text{ is a positive integer}), \end{aligned}$$

which is absurd. This contradiction shows that our assumption (that $k \neq 0$) was false. Hence, $k = 0$.

Now,

$$\begin{aligned} \alpha &= (\alpha_1, \alpha_2, \ldots, \alpha_k) = (\alpha_1, \alpha_2, \ldots, \alpha_0) \qquad (\text{since } k = 0) \\ &= () = \varnothing \qquad (\text{recall that } \varnothing \text{ denotes the empty composition}). \end{aligned}$$

Moreover, $\ell(\alpha) = k = 0$. Finally, from $\alpha = ()$, we obtain $D(\alpha) = D() = \varnothing$ (by the definition of the map $D : \mathrm{Comp}_0 \to \mathcal{P}([0-1])$). Thus, Proposition 2.4 is proved. $\square$

We can unite Proposition 2.3 with Proposition 2.4 by using the *Iverson bracket notation*:

Convention 2.5. If $\mathcal{A}$ is a logical statement, then $[\mathcal{A}]$ shall denote the truth value of $\mathcal{A}$; this is the integer defined by

$$[\mathcal{A}] = \begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false.} \end{cases}$$

For example, $[2+2=4] = 1$ (since the statement $2+2=4$ is true) and $[2+2=5] = 0$ (since the statement $2+2=5$ is false).

Now, Proposition 2.3 with Proposition 2.4 can be combined into the following:

Corollary 2.6. Let $n \in \mathbb{N}$. Let $\alpha \in \mathrm{Comp}_n$. Then, $\ell(\alpha) = |D(\alpha)| + [n \neq 0]$.

Proof of Corollary 2.6. From $\alpha \in \mathrm{Comp}_n$, we see that α is a composition of n (since Comp_n is the set of all compositions of n). In other words, α is a composition having size n. Therefore, $|\alpha| = n$ (since $|\alpha|$ is the size of α, but we know that α has size n).

We are in one of the following two cases:

Case 1: We have $n = 0$.

Case 2: We have $n \neq 0$.

Let us first consider Case 1. In this case, we have $n = 0$. Hence, we don't have $n \neq 0$. Thus, $[n \neq 0] = 0$.

However, $|\alpha| = n = 0$. Thus, Proposition 2.4 yields $\alpha = \varnothing$ and $\ell(\alpha) = 0$ and $D(\alpha) = \varnothing$. From $D(\alpha) = \varnothing$, we obtain $|D(\alpha)| = |\varnothing| = 0$. Thus, $\underbrace{|D(\alpha)|}_{=0} + \underbrace{[n \neq 0]}_{=0} = 0$. Comparing this with $\ell(\alpha) = 0$, we obtain $\ell(\alpha) = |D(\alpha)| + [n \neq 0]$. Hence, Corollary 2.6 is proved in Case 1.

Let us now consider Case 2. In this case, we have $n \neq 0$. Hence, $[n \neq 0] = 1$. Also, from $n \neq 0$, we obtain $n > 0$ (since $n \in \mathbb{N}$). Thus, $|\alpha| = n > 0$. Hence, Proposition 2.3 yields $|D(\alpha)| = \ell(\alpha) - 1$. Hence, $\ell(\alpha) = |D(\alpha)| + 1$. Comparing this with $|D(\alpha)| + \underbrace{[n \neq 0]}_{=1} = |D(\alpha)| + 1$, we obtain $\ell(\alpha) = |D(\alpha)| + [n \neq 0]$. Thus, Corollary 2.6 is proved in Case 2.

We have now proved Corollary 2.6 in both Cases 1 and 2. Hence, Corollary 2.6 always holds. □

Corollary 2.7. Let $n \in \mathbb{N}$. Let $\alpha \in \operatorname{Comp}_n$ and $\beta \in \operatorname{Comp}_n$. Then, $\ell(\beta) - \ell(\alpha) = |D(\beta)| - |D(\alpha)|$.

Proof of Corollary 2.7. Corollary 2.6 yields $\ell(\alpha) = |D(\alpha)| + [n \neq 0]$. Corollary 2.6 (applied to β instead of α) yields $\ell(\beta) = |D(\beta)| + [n \neq 0]$. Hence,

$$\begin{aligned}
\underbrace{\ell(\beta)}_{=|D(\beta)|+[n\neq 0]} - \underbrace{\ell(\alpha)}_{=|D(\alpha)|+[n\neq 0]} &= (|D(\beta)| + [n \neq 0]) - (|D(\alpha)| + [n \neq 0]) \\
&= |D(\beta)| - |D(\alpha)|.
\end{aligned}$$

This proves Corollary 2.7. □

3. Reversals

We shall now discuss a certain operation on compositions:

Definition 3.1. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a composition, then the *reversal* of α is defined to be the composition $(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$. It is denoted by $\operatorname{rev} \alpha$.

Thus, we have defined a map $\operatorname{rev} : \operatorname{Comp} \to \operatorname{Comp}$ that sends each composition α to the composition $\operatorname{rev} \alpha$.

Example 3.2. We have

$$\begin{aligned}
\operatorname{rev}(2,3,6) &= (6,3,2); \\
\operatorname{rev}(4,1,1,2) &= (2,1,1,4); \\
\operatorname{rev} \varnothing &= \varnothing.
\end{aligned}$$

Proposition 3.3. Let $\alpha \in \operatorname{Comp}$. Then, $\left|\operatorname{rev}\alpha\right| = \left|\alpha\right|$.

Proof of Proposition 3.3. Write the composition α in the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Then, $\operatorname{rev}\alpha = (\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$ (by Definition 3.1) and $\left|\alpha\right| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ (by the definition of $\left|\alpha\right|$). Now,

$$\begin{aligned}
\left|\operatorname{rev}\alpha\right| &= \left|(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)\right| && \left(\text{since } \operatorname{rev}\alpha = (\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)\right) \\
&= \alpha_k + \alpha_{k-1} + \cdots + \alpha_1 && \left(\text{by the definition of } \left|(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)\right|\right) \\
&= \alpha_1 + \alpha_2 + \cdots + \alpha_k \\
&= \left|\alpha\right| \qquad \left(\text{since } \left|\alpha\right| = \alpha_1 + \alpha_2 + \cdots + \alpha_k\right).
\end{aligned}$$

This proves Proposition 3.3. □

Proposition 3.4. Let $\alpha \in \operatorname{Comp}$. Then, $\operatorname{rev}(\operatorname{rev}\alpha) = \alpha$.

Proof of Proposition 3.4. Write the composition α in the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Then, Definition 3.1 yields $\operatorname{rev}\alpha = (\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$. However, Definition 3.1 also yields $\operatorname{rev}(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1) = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Now,

$$\operatorname{rev}\underbrace{(\operatorname{rev}\alpha)}_{=(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)} = \operatorname{rev}(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1) = (\alpha_1, \alpha_2, \ldots, \alpha_k) = \alpha.$$

This proves Proposition 3.4. □

Corollary 3.5. The map

$$\begin{aligned}
\operatorname{Comp} &\to \operatorname{Comp}, \\
\delta &\mapsto \operatorname{rev}\delta
\end{aligned}$$

is a bijection.

Proof of Corollary 3.5. Let us denote this map by rev (since the image of any $\delta \in \operatorname{Comp}$ under this map is already being called $\operatorname{rev}\delta$). Thus, we must prove that this map rev is a bijection.

But this is easy: Every $\alpha \in \operatorname{Comp}$ satisfies

$$\begin{aligned}
(\operatorname{rev} \circ \operatorname{rev})(\alpha) &= \operatorname{rev}(\operatorname{rev}\alpha) = \alpha && (\text{by Proposition 3.4}) \\
&= \operatorname{id}(\alpha).
\end{aligned}$$

Thus, $\operatorname{rev} \circ \operatorname{rev} = \operatorname{id}$. Hence, the map rev is inverse to itself. Thus, the map rev is invertible, i.e., bijective. In other words, it is a bijection. This proves Corollary 3.5. □

We also define a related operation on subsets of $[n-1]$:

Definition 3.6. Let $n \in \mathbb{N}$. For any subset X of $[n-1]$, we let $\operatorname{rev}_n X$ denote the set $\{n-x \mid x \in X\}$.

Example 3.7. If $n = 7$, then

$$\begin{aligned}
\operatorname{rev}_n(\{2,4\}) &= \{7-2,\ 7-4\} = \{5,3\} = \{3,5\};\\
\operatorname{rev}_n(\{1,2,5,6\}) &= \{7-1,\ 7-2,\ 7-5,\ 7-6\} = \{6,5,2,1\} = \{1,2,5,6\};\\
\operatorname{rev}_n(\varnothing) &= \varnothing;\\
\operatorname{rev}_n([6]) &= [6].
\end{aligned}$$

Informally speaking, the set $\operatorname{rev}_n X$ defined in Definition 3.6 is the reflection of the set X across the midpoint of the interval $[n-1]$ (where we regard numbers as points on the number line). From this point of view, all claims of the following theorem are visually obvious:

Theorem 3.8. Let $n \in \mathbb{N}$. Then:

(a) We have $\operatorname{rev}_n X \subseteq [n-1]$ for each subset X of $[n-1]$.

(b) We have $\operatorname{rev}_n(\operatorname{rev}_n X) = X$ for any subset X of $[n-1]$.

(c) If two subsets X and Y of $[n-1]$ satisfy $X \subseteq Y$, then $\operatorname{rev}_n X \subseteq \operatorname{rev}_n Y$.

(d) We have $|\operatorname{rev}_n X| = |X|$ for any subset X of $[n-1]$.

(e) We have $\operatorname{rev}_n X = \{i \in [n-1] \mid n-i \in X\}$ for any subset X of $[n-1]$.

(f) We have $\operatorname{rev}_n(X \setminus Y) = (\operatorname{rev}_n X) \setminus (\operatorname{rev}_n Y)$ for any subsets X and Y of $[n-1]$.

(g) We have $\operatorname{rev}_n(X \cap Y) = (\operatorname{rev}_n X) \cap (\operatorname{rev}_n Y)$ for any subsets X and Y of $[n-1]$.

(h) We have $\operatorname{rev}_n([n-1]) = [n-1]$.

(i) We have $D(\operatorname{rev}\alpha) = \operatorname{rev}_n(D(\alpha))$ for any composition $\alpha \in \operatorname{Comp}_n$.

Proof of Theorem 3.8. **(a)** Let X be a subset of $[n-1]$. Then, $n-x \in [n-1]$ for each $x \in X$ [3]. In other words,

$$\{n-x \mid x \in X\} \subseteq [n-1].$$

[3] *Proof.* Let $x \in X$. Then, $x \in X \subseteq [n-1] = \{1,2,\ldots,n-1\}$, so that $n-x \in \{1,2,\ldots,n-1\} = [n-1]$.

Forget that we fixed x. We thus have shown that $n-x \in [n-1]$ for each $x \in X$.

This rewrites as $\operatorname{rev}_n X \subseteq [n-1]$ (since $\operatorname{rev}_n X$ is defined to be $\{n-x \mid x \in X\}$). This proves Theorem 3.8 **(a)**.

(b) Let X be a subset of $[n-1]$. Let $Y = \operatorname{rev}_n X$.

Let $p \in \operatorname{rev}_n Y$. We shall show that $p \in X$.

We have

$$\begin{aligned} p \in \operatorname{rev}_n Y &= \{n-x \mid x \in Y\} && \text{(by the definition of } \operatorname{rev}_n Y) \\ &= \{n-y \mid y \in Y\} && \text{(here, we have renamed the index } x \text{ as } y). \end{aligned}$$

In other words, $p = n - y$ for some $y \in Y$. Consider this y. Now,

$$y \in Y = \operatorname{rev}_n X = \{n-x \mid x \in X\} \qquad \text{(by the definition of } \operatorname{rev}_n X).$$

In other words, $y = n - x$ for some $x \in X$. Consider this x. Now, $p = n - \underbrace{y}_{=n-x} = n - (n-x) = x \in X$.

Forget that we fixed p. We thus have shown that $p \in X$ for each $p \in \operatorname{rev}_n Y$. In other words, $\operatorname{rev}_n Y \subseteq X$.

On the other hand, let $q \in X$. Then, $n - q$ has the form $n - x$ for some $x \in X$ (namely, for $x = q$). In other words, $n - q \in \{n-x \mid x \in X\}$. Since $Y = \operatorname{rev}_n X = \{n-x \mid x \in X\}$ (by the definition of $\operatorname{rev}_n X$), we can rewrite this as $n - q \in Y$.

Furthermore, $q = n - (n-q)$. Hence, q has the form $n - x$ for some $x \in Y$ (namely, for $x = n - q$). In other words, $q \in \{n-x \mid x \in Y\}$. Since $\operatorname{rev}_n Y = \{n-x \mid x \in Y\}$ (by the definition of $\operatorname{rev}_n Y$), we can rewrite this as $q \in \operatorname{rev}_n Y$.

Forget that we fixed q. We thus have shown that $q \in \operatorname{rev}_n Y$ for each $q \in X$. In other words, $X \subseteq \operatorname{rev}_n Y$.

Combining this with $\operatorname{rev}_n Y \subseteq X$, we obtain $\operatorname{rev}_n Y = X$. In other words, $\operatorname{rev}_n (\operatorname{rev}_n X) = X$ (since $Y = \operatorname{rev}_n X$). This proves Theorem 3.8 **(b)**.

(c) Let X and Y be two subsets of $[n-1]$ that satisfy $X \subseteq Y$. The definition of $\operatorname{rev}_n Y$ yields $\operatorname{rev}_n Y = \{n-x \mid x \in Y\}$.

Let $p \in \operatorname{rev}_n X$. Then, $p \in \operatorname{rev}_n X = \{n-x \mid x \in X\}$ (by the definition of $\operatorname{rev}_n X$). In other words, $p = n - x$ for some $x \in X$. Consider this x, and denote it by z. Thus, $z \in X$ and $p = n - z$.

Now, $z \in X \subseteq Y$ and $p = n - z$. Therefore, $p = n - x$ for some $x \in Y$ (namely, for $x = z$). In other words, $p \in \{n-x \mid x \in Y\}$. This rewrites as $p \in \operatorname{rev}_n Y$ (since $\operatorname{rev}_n Y = \{n-x \mid x \in Y\}$).

Forget that we fixed p. We thus have shown that $p \in \operatorname{rev}_n Y$ for each $p \in \operatorname{rev}_n X$. In other words, $\operatorname{rev}_n X \subseteq \operatorname{rev}_n Y$. This proves Theorem 3.8 **(c)**.

(d) Let X be a subset of $[n-1]$. Let $Y = \operatorname{rev}_n X$.

The definition of $\operatorname{rev}_n X$ yields $\operatorname{rev}_n X = \{n-x \mid x \in X\}$. Thus, the elements of $\operatorname{rev}_n X$ are precisely the numbers $n - x$ for $x \in X$. Clearly, there are at most $|X|$ many such numbers (since there are $|X|$ many elements $x \in X$). Hence, the set $\operatorname{rev}_n X$ has at most $|X|$ many elements. In other words, $|\operatorname{rev}_n X| \leq |X|$.

The same argument (applied to Y instead of X) yields $|\operatorname{rev}_n Y| \leq |Y|$. However, from $Y = \operatorname{rev}_n X$, we obtain $\operatorname{rev}_n Y = \operatorname{rev}_n (\operatorname{rev}_n X) = X$ (by Theorem 3.8 **(b)**). In view of this, we can rewrite $|\operatorname{rev}_n Y| \leq |Y|$ as $|X| \leq |Y|$.

But from $Y = \operatorname{rev}_n X$, we also obtain $|Y| = |\operatorname{rev}_n X| \leq |X|$. Combining this inequality with $|X| \leq |Y|$, we find $|X| = |Y| = |\operatorname{rev}_n X|$. In other words, $|\operatorname{rev}_n X| = |X|$. This proves Theorem 3.8 **(d)**.

(e) Let X be a subset of $[n-1]$. Let $Y = \{i \in [n-1] \mid n-i \in X\}$. We shall show that $\operatorname{rev}_n X = Y$.

Note that $\operatorname{rev}_n X = \{n - x \mid x \in X\}$ (by the definition of $\operatorname{rev}_n X$).

Let $p \in \operatorname{rev}_n X$. Then, $p \in \operatorname{rev}_n X = \{n - x \mid x \in X\}$. In other words, $p = n - x$ for some $x \in X$. Consider this x. Thus, $p = n - x$, so that $n = p + x$. Therefore, $n - p = x \in X$. Also, $p \in \operatorname{rev}_n X \subseteq [n-1]$ (by Theorem 3.8 **(a)**). Hence, p is an element i of $[n-1]$ satisfying $n - i \in X$ (since $n - p \in X$). In other words, $p \in \{i \in [n-1] \mid n-i \in X\}$. In other words, $p \in Y$ (since $Y = \{i \in [n-1] \mid n-i \in X\}$).

Forget that we fixed p. We thus have shown that $p \in Y$ for each $p \in \operatorname{rev}_n X$. In other words, $\operatorname{rev}_n X \subseteq Y$.

Now, let $q \in Y$. Thus, $q \in Y = \{i \in [n-1] \mid n-i \in X\}$. In other words, q is an $i \in [n-1]$ satisfying $n - i \in X$. In other words, $q \in [n-1]$ and $n - q \in X$. Furthermore, $q = n - (n-q)$. Hence, q has the form $n - x$ for some $x \in X$ (namely, for $x = n - q$). In other words, $q \in \{n - x \mid x \in X\}$. This rewrites as $q \in \operatorname{rev}_n X$ (since $\operatorname{rev}_n X = \{n - x \mid x \in X\}$).

Forget that we fixed q. We thus have shown that $q \in \operatorname{rev}_n X$ for each $q \in Y$. In other words, $Y \subseteq \operatorname{rev}_n X$.

Combining this with $\operatorname{rev}_n X \subseteq Y$, we obtain $\operatorname{rev}_n X = Y = \{i \in [n-1] \mid n-i \in X\}$. This proves Theorem 3.8 **(e)**.

(f) Let X and Y be two subsets of $[n-1]$. Then, $X \setminus Y$ is a subset of $[n-1]$ as well (since $X \setminus Y \subseteq X \subseteq [n-1]$). Thus, $\operatorname{rev}_n (X \setminus Y) \subseteq [n-1]$ (by Theorem 3.8 **(a)**, applied to $X \setminus Y$ instead of X). Also, $(\operatorname{rev}_n X) \setminus (\operatorname{rev}_n Y) \subseteq \operatorname{rev}_n X \subseteq [n-1]$ (by Theorem 3.8 **(a)**).

Theorem 3.8 **(e)** yields

$$\operatorname{rev}_n X = \{i \in [n-1] \mid n-i \in X\}.$$

Hence, for any $i \in [n-1]$, we have the logical equivalence

$$(i \in \operatorname{rev}_n X) \iff (n - i \in X). \tag{2}$$

The same argument (applied to Y instead of X) shows that for any $i \in [n-1]$, we have the logical equivalence

$$(i \in \operatorname{rev}_n Y) \iff (n - i \in Y). \tag{3}$$

The same argument (applied to $X \setminus Y$ instead of Y) shows that for any $i \in [n-1]$, we have the logical equivalence

$$(i \in \operatorname{rev}_n (X \setminus Y)) \iff (n - i \in X \setminus Y). \tag{4}$$

Now, for each $i \in [n-1]$, we have the following chain of logical equivalences:

$$\begin{aligned}(i \in \operatorname{rev}_n(X \setminus Y)) &\iff (n - i \in X \setminus Y) \qquad \text{(by (4))} \\ &\iff (n - i \in X \text{ and } n - i \notin Y) \\ &\iff \left(\underbrace{n - i \in X}_{\substack{\iff (i \in \operatorname{rev}_n X) \\ \text{(by (2))}}} \text{ but not } \underbrace{n - i \in Y}_{\substack{\iff (i \in \operatorname{rev}_n Y) \\ \text{(by (3))}}} \right) \\ &\iff (i \in \operatorname{rev}_n X \text{ but not } i \in \operatorname{rev}_n Y) \\ &\iff (i \in \operatorname{rev}_n X \text{ and } i \notin \operatorname{rev}_n Y) \\ &\iff (i \in (\operatorname{rev}_n X) \setminus (\operatorname{rev}_n Y)) . \end{aligned} \tag{5}$$

Now, from $\operatorname{rev}_n(X \setminus Y) \subseteq [n-1]$, we obtain

$$\begin{aligned}\operatorname{rev}_n(X \setminus Y) &= [n-1] \cap (\operatorname{rev}_n(X \setminus Y)) \\ &= \left\{ i \in [n-1] \mid \underbrace{i \in \operatorname{rev}_n(X \setminus Y)}_{\substack{\iff (i \in (\operatorname{rev}_n X) \setminus (\operatorname{rev}_n Y)) \\ \text{(by (5))}}} \right\} \\ &= \{ i \in [n-1] \mid i \in (\operatorname{rev}_n X) \setminus (\operatorname{rev}_n Y) \} . \end{aligned} \tag{6}$$

However, from $(\operatorname{rev}_n X) \setminus (\operatorname{rev}_n Y) \subseteq [n-1]$, we obtain

$$\begin{aligned}(\operatorname{rev}_n X) \setminus (\operatorname{rev}_n Y) &= [n-1] \cap ((\operatorname{rev}_n X) \setminus (\operatorname{rev}_n Y)) \\ &= \{ i \in [n-1] \mid i \in (\operatorname{rev}_n X) \setminus (\operatorname{rev}_n Y) \} . \end{aligned}$$

Comparing this with (6), we find $\operatorname{rev}_n(X \setminus Y) = (\operatorname{rev}_n X) \setminus (\operatorname{rev}_n Y)$. This proves Theorem 3.8 **(f)**.

(g) Recall that

$$A \setminus (A \setminus B) = A \cap B \tag{7}$$

for any two sets A and B.

Let X and Y be two subsets of $[n-1]$. Then, $X \setminus Y$ is a subset of $[n-1]$ as well (since $X \setminus Y \subseteq X \subseteq [n-1]$). Hence, Theorem 3.8 **(f)** (applied to $X \setminus Y$ instead of Y) yields

$$\begin{aligned}\operatorname{rev}_n(X \setminus (X \setminus Y)) &= (\operatorname{rev}_n X) \setminus \underbrace{(\operatorname{rev}_n(X \setminus Y))}_{\substack{= (\operatorname{rev}_n X) \setminus (\operatorname{rev}_n Y) \\ \text{(by Theorem 3.8 \textbf{(f)})}}} \\ &= (\operatorname{rev}_n X) \setminus ((\operatorname{rev}_n X) \setminus (\operatorname{rev}_n Y)) \\ &= (\operatorname{rev}_n X) \cap (\operatorname{rev}_n Y) \end{aligned} \tag{8}$$

(by (7), applied to $A = \operatorname{rev}_n X$ and $B = \operatorname{rev}_n Y$). However, $X \setminus (X \setminus Y) = X \cap Y$ (by (7), applied to $A = X$ and $B = Y$). Thus, we can rewrite (8) as $\operatorname{rev}_n (X \cap Y) = (\operatorname{rev}_n X) \cap (\operatorname{rev}_n Y)$. This proves Theorem 3.8 **(g)**.

(h) The definition of $\operatorname{rev}_n ([n-1])$ yields

$$\begin{aligned}
\operatorname{rev}_n ([n-1]) &= \{n - x \mid x \in [n-1]\} \\
&= \{n - x \mid x \in \{1,2,\ldots,n-1\}\} \qquad (\text{since } [n-1] = \{1,2,\ldots,n-1\}) \\
&= \{n-1,\ n-2,\ \ldots,\ n-(n-1)\} \\
&= \{n-1,\ n-2,\ \ldots,\ 1\} \\
&= \{1,2,\ldots,n-1\} = [n-1].
\end{aligned}$$

This proves Theorem 3.8 **(h)**.

(i) Let $\alpha \in \operatorname{Comp}_n$ be a composition. Write this composition α in the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Then, $\operatorname{rev} \alpha = (\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$ (by the definition of $\operatorname{rev} \alpha$). Also, the definition of $|\alpha|$ yields $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$.

From $\alpha \in \operatorname{Comp}_n$, we see that α is a composition of n (since Comp_n is the set of all compositions of n). In other words, α is a composition having size n. Therefore, $|\alpha| = n$ (since $|\alpha|$ is the size of α, but we know that α has size n).

For each $i \in \{0, 1, \ldots, k\}$, we define two numbers

$$u_i := \alpha_1 + \alpha_2 + \cdots + \alpha_i \qquad \text{and}$$
$$v_i := \alpha_{i+1} + \alpha_{i+2} + \cdots + \alpha_k.$$

Each $i \in \{0, 1, \ldots, k\}$ satisfies

$$\begin{aligned}
&\underbrace{u_i}_{=\alpha_1+\alpha_2+\cdots+\alpha_i} + \underbrace{v_i}_{=\alpha_{i+1}+\alpha_{i+2}+\cdots+\alpha_k} \\
&= (\alpha_1 + \alpha_2 + \cdots + \alpha_i) + (\alpha_{i+1} + \alpha_{i+2} + \cdots + \alpha_k) \\
&= \alpha_1 + \alpha_2 + \cdots + \alpha_k = |\alpha| \qquad (\text{since } |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k) \\
&= n
\end{aligned}$$

and therefore

$$v_i = n - u_i. \tag{9}$$

From $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, we obtain

$$\begin{aligned}
D(\alpha) = D(\alpha_1, \alpha_2, \ldots, \alpha_k) &= \left\{ \underbrace{\alpha_1 + \alpha_2 + \cdots + \alpha_i}_{\substack{=u_i \\ (\text{since } u_i \text{ is defined} \\ \text{to be } \alpha_1+\alpha_2+\cdots+\alpha_i)}} \mid i \in [k-1] \right\} \\
&\qquad (\text{by the definition of } D(\alpha_1, \alpha_2, \ldots, \alpha_k)) \\
&= \{u_i \mid i \in [k-1]\} = \{u_1, u_2, \ldots, u_{k-1}\}.
\end{aligned}$$

The definition of $\operatorname{rev}_n(D(\alpha))$ yields

$$\begin{aligned}
\operatorname{rev}_n(D(\alpha)) &= \{n-x \mid x \in D(\alpha)\} \\
&= \{n-x \mid x \in \{u_1, u_2, \ldots, u_{k-1}\}\} \qquad (\text{since } D(\alpha) = \{u_1, u_2, \ldots, u_{k-1}\}) \\
&= \{n-u_1,\ n-u_2,\ \ldots,\ n-u_{k-1}\} \\
&= \left\{ \underbrace{n-u_i}_{\substack{=v_i \\ (\text{by (9)})}} \mid i \in [k-1] \right\} = \{v_i \mid i \in [k-1]\} \\
&= \{v_1, v_2, \ldots, v_{k-1}\}.
\end{aligned}$$

Comparing this with

$$\begin{aligned}
D(\operatorname{rev}\alpha) &= D(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1) \qquad (\text{since } \operatorname{rev}\alpha = (\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)) \\
&= \left\{ \underbrace{\alpha_k + \alpha_{k-1} + \cdots + \alpha_{k-i+1}}_{\substack{=\alpha_{k-i+1}+\alpha_{k-i+2}+\cdots+\alpha_k \\ =v_{k-i} \\ (\text{since } v_{k-i} \text{ is defined} \\ \text{to be } \alpha_{k-i+1}+\alpha_{k-i+2}+\cdots+\alpha_k)}} \mid i \in [k-1] \right\} \\
&\qquad (\text{by the definition of } D(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)) \\
&= \{v_{k-i} \mid i \in [k-1]\} = \left\{v_{k-1}, v_{k-2}, \ldots, v_{k-(k-1)}\right\} \\
&= \{v_{k-1}, v_{k-2}, \ldots, v_1\} = \{v_1, v_2, \ldots, v_{k-1}\},
\end{aligned}$$

we obtain $D(\operatorname{rev}\alpha) = \operatorname{rev}_n(D(\alpha))$. This proves Theorem 3.8 **(i)**. □

Corollary 3.9. Let $n \in \mathbb{N}$, and let $\alpha \in \operatorname{Comp}_n$. Then, $\operatorname{rev}_n(D(\operatorname{rev}\alpha)) = D(\alpha)$.

Proof of Corollary 3.9. We have $\alpha \in \operatorname{Comp}_n$. In other words, α is a composition of n. That is, α is a composition having size n. In other words, $\alpha \in \operatorname{Comp}$ and $|\alpha| = n$. Hence, Proposition 3.3 yields $|\operatorname{rev}\alpha| = |\alpha| = n$. In other words, the composition $\operatorname{rev}\alpha$ has size n. In other words, $\operatorname{rev}\alpha$ is a composition of n. In other words, $\operatorname{rev}\alpha \in \operatorname{Comp}_n$. Hence, $D(\operatorname{rev}\alpha) \in \mathcal{P}([n-1])$ (since D is a map $\operatorname{Comp}_n \to \mathcal{P}([n-1])$). In other words, $D(\operatorname{rev}\alpha)$ is a subset of $[n-1]$. Hence, $\operatorname{rev}_n(D(\operatorname{rev}\alpha))$ is well-defined.

Furthermore, $D(\alpha) \in \mathcal{P}([n-1])$ (since D is a map $\operatorname{Comp}_n \to \mathcal{P}([n-1])$). In other words, $D(\alpha)$ is a subset of $[n-1]$.

Theorem 3.8 **(i)** yields $D(\operatorname{rev}\alpha) = \operatorname{rev}_n(D(\alpha))$. Thus,

$$\operatorname{rev}_n\left(\underbrace{D(\operatorname{rev}\alpha)}_{=\operatorname{rev}_n(D(\alpha))}\right) = \operatorname{rev}_n(\operatorname{rev}_n(D(\alpha))) = D(\alpha)$$

(by Theorem 3.8 **(b)**, applied to $X = D(\alpha)$). This proves Corollary 3.9. $\square$

Corollary 3.10. Let $n \in \mathbb{N}$. Then, the map

$$\begin{aligned} \mathrm{Comp}_n &\to \mathrm{Comp}_n, \\ \delta &\mapsto \operatorname{rev} \delta \end{aligned}$$

is a bijection.

Proof of Corollary 3.10. Each $\delta \in \mathrm{Comp}_n$ satisfies $\operatorname{rev} \delta \in \mathrm{Comp}_n$ [4]. Hence, the map

$$\begin{aligned} \mathrm{Comp}_n &\to \mathrm{Comp}_n, \\ \delta &\mapsto \operatorname{rev} \delta \end{aligned}$$

is well-defined. It remains to prove that this map is a bijection.

Let us denote this map by rev (since the image of any $\delta \in \mathrm{Comp}$ under this map is already being called $\operatorname{rev} \delta$). Thus, we must prove that this map rev is a bijection.

But this is easy: Every $\alpha \in \mathrm{Comp}_n$ satisfies

$$\begin{aligned} (\operatorname{rev} \circ \operatorname{rev})(\alpha) &= \operatorname{rev}(\operatorname{rev} \alpha) = \alpha \qquad \text{(by Proposition 3.4)} \\ &= \mathrm{id}(\alpha). \end{aligned}$$

Thus, $\operatorname{rev} \circ \operatorname{rev} = \mathrm{id}$. Hence, the map rev is inverse to itself. Thus, the map rev is invertible, i.e., bijective. In other words, it is a bijection. This proves Corollary 3.10. $\square$

Proposition 3.11. Let $n \in \mathbb{N}$. Let $\alpha \in \mathrm{Comp}_n$ and $\beta \in \mathrm{Comp}_n$ be arbitrary. Then, we have the logical equivalence

$$(D(\operatorname{rev} \beta) \subseteq D(\operatorname{rev} \alpha)) \iff (D(\beta) \subseteq D(\alpha)).$$

Proof of Proposition 3.11. We have $\alpha \in \mathrm{Comp}_n$ and thus $D(\alpha) \in \mathcal{P}([n-1])$ (since $D : \mathrm{Comp}_n \to \mathcal{P}([n-1])$ is a map). In other words, $D(\alpha)$ is a subset of $[n-1]$. Similarly, $D(\beta)$ is a subset of $[n-1]$.

Theorem 3.8 **(i)** yields $D(\operatorname{rev} \alpha) = \operatorname{rev}_n(D(\alpha))$. Also, Theorem 3.8 **(i)** (applied to β instead of α) yields $D(\operatorname{rev} \beta) = \operatorname{rev}_n(D(\beta))$.

Now, if $D(\beta) \subseteq D(\alpha)$, then $\operatorname{rev}_n(D(\beta)) \subseteq \operatorname{rev}_n(D(\alpha))$ (by Theorem 3.8 **(c)**, applied to $X = D(\beta)$ and $Y = D(\alpha)$) and therefore

$$D(\operatorname{rev} \beta) = \operatorname{rev}_n(D(\beta)) \subseteq \operatorname{rev}_n(D(\alpha)) = D(\operatorname{rev} \alpha)$$

[4] *Proof.* Let $\delta \in \mathrm{Comp}_n$. Thus, δ is a composition of n. In other words, δ is a composition that has size n. In other words, δ is a composition and satisfies $|\delta| = n$. Now, Proposition 3.3 (applied to $\alpha = \delta$) yields $|\operatorname{rev} \delta| = |\delta| = n$. Hence, $\operatorname{rev} \delta$ is a composition that has size n (since it has size $|\operatorname{rev} \delta| = n$). In other words, $\operatorname{rev} \delta$ is a composition of n. In other words, $\operatorname{rev} \delta \in \mathrm{Comp}_n$. Qed.

(since $D(\operatorname{rev}\alpha) = \operatorname{rev}_n(D(\alpha))$). In other words, the implication

$$(D(\beta) \subseteq D(\alpha)) \implies (D(\operatorname{rev}\beta) \subseteq D(\operatorname{rev}\alpha)) \tag{10}$$

holds.

Proposition 3.4 yields $\operatorname{rev}(\operatorname{rev}\alpha) = \alpha$. Similarly, $\operatorname{rev}(\operatorname{rev}\beta) = \beta$.

However, Corollary 3.10 says that the map

$$\operatorname{Comp}_n \to \operatorname{Comp}_n,$$
$$\delta \mapsto \operatorname{rev}\delta$$

is a bijection. Thus, in particular, this map is well-defined. In other words, for any $\delta \in \operatorname{Comp}_n$, we have $\operatorname{rev}\delta \in \operatorname{Comp}_n$. Applying this to $\delta = \alpha$, we obtain $\operatorname{rev}\alpha \in \operatorname{Comp}_n$ (since $\alpha \in \operatorname{Comp}_n$). Similarly, $\operatorname{rev}\beta \in \operatorname{Comp}_n$. Thus, we can apply the implication (10) to $\operatorname{rev}\alpha$ and $\operatorname{rev}\beta$ instead of α and β. Hence, we obtain the implication

$$(D(\operatorname{rev}\beta) \subseteq D(\operatorname{rev}\alpha)) \implies (D(\operatorname{rev}(\operatorname{rev}\beta)) \subseteq D(\operatorname{rev}(\operatorname{rev}\alpha))).$$

In view of $\operatorname{rev}(\operatorname{rev}\alpha) = \alpha$ and $\operatorname{rev}(\operatorname{rev}\beta) = \beta$, we can rewrite this as

$$(D(\operatorname{rev}\beta) \subseteq D(\operatorname{rev}\alpha)) \implies (D(\beta) \subseteq D(\alpha)).$$

Combining this implication with (10), we obtain the logical equivalence

$$(D(\operatorname{rev}\beta) \subseteq D(\operatorname{rev}\alpha)) \iff (D(\beta) \subseteq D(\alpha)).$$

This proves Proposition 3.11. □

4. The omega operation

Proposition 4.1. Let $n \in \mathbb{N}$. Let $\gamma \in \operatorname{Comp}_n$. Then, there exists a unique composition δ of n satisfying

$$D(\delta) = [n-1] \setminus D(\operatorname{rev}\gamma).$$

Proof of Proposition 4.1. The set $[n-1] \setminus D(\operatorname{rev}\gamma)$ is clearly a subset of $[n-1]$, and thus belongs to $\mathcal{P}([n-1])$. Hence, there exists a unique $\delta \in \operatorname{Comp}_n$ satisfying $D(\delta) = [n-1] \setminus D(\operatorname{rev}\gamma)$ (since the map $D : \operatorname{Comp}_n \to \mathcal{P}([n-1])$ is a bijection). In other words, there exists a unique composition δ of n satisfying $D(\delta) = [n-1] \setminus D(\operatorname{rev}\gamma)$ (because a composition δ of n is the same as an element $\delta \in \operatorname{Comp}_n$). This proves Proposition 4.1. □

We now define another operation on compositions:

Definition 4.2. Let $n \in \mathbb{N}$. For any composition $\gamma \in \operatorname{Comp}_n$, we let $\omega(\gamma)$ denote the unique composition δ of n satisfying

$$D(\delta) = [n-1] \setminus D(\operatorname{rev}\gamma).$$

(This $\omega(\gamma)$ is indeed well-defined, according to Proposition 4.1.)

We observe the following simple properties of these compositions $\omega(\gamma)$:

Proposition 4.3. Let $n \in \mathbb{N}$. Let $\gamma \in \operatorname{Comp}_n$. Then:

(a) We have $\omega(\gamma) \in \operatorname{Comp}_n$.

(b) We have $D(\omega(\gamma)) = [n-1] \setminus D(\operatorname{rev}\gamma)$.

(c) We have $D(\omega(\gamma)) = [n-1] \setminus \operatorname{rev}_n(D(\gamma))$.

(d) We have $\omega(\omega(\gamma)) = \gamma$.

Proof of Proposition 4.3. We have defined $\omega(\gamma)$ to be the unique composition δ of n satisfying $D(\delta) = [n-1] \setminus D(\operatorname{rev}\gamma)$. Thus, $\omega(\gamma)$ is a composition of n and satisfies $D(\omega(\gamma)) = [n-1] \setminus D(\operatorname{rev}\gamma)$. This proves Proposition 4.3 **(b)**. Moreover, we have $\omega(\gamma) \in \operatorname{Comp}_n$ (since $\omega(\gamma)$ is a composition of n); this proves Proposition 4.3 **(a)**.

It remains to prove parts **(c)** and **(d)**.

(c) Theorem 3.8 **(i)** (applied to $\alpha = \gamma$) yields $D(\operatorname{rev}\gamma) = \operatorname{rev}_n(D(\gamma))$. Now,

$$D(\omega(\gamma)) = [n-1] \setminus \underbrace{D(\operatorname{rev}\gamma)}_{=\operatorname{rev}_n(D(\gamma))} = [n-1] \setminus \operatorname{rev}_n(D(\gamma)).$$

This proves Proposition 4.3 **(c)**.

(d) We observe that γ is a composition of n (since $\gamma \in \operatorname{Comp}_n$). In other words, γ is a composition having size n. In other words, $\gamma \in \operatorname{Comp}$ and $|\gamma| = n$. However, Proposition 3.3 (applied to $\alpha = \gamma$) yields $|\operatorname{rev}\gamma| = |\gamma| = n$. Hence, $\operatorname{rev}\gamma$ is a composition having size $|\operatorname{rev}\gamma| = n$. In other words, $\operatorname{rev}\gamma$ is a composition of n. Hence, $\operatorname{rev}\gamma \in \operatorname{Comp}_n$. Thus, $D(\operatorname{rev}\gamma) \in \mathcal{P}([n-1])$ (since D is a map $\operatorname{Comp}_n \to \mathcal{P}([n-1])$). In other words, $D(\operatorname{rev}\gamma)$ is a subset of $[n-1]$. Furthermore, $D(\gamma) \in \mathcal{P}([n-1])$ (since $\gamma \in \operatorname{Comp}_n$ and since D is a map $\operatorname{Comp}_n \to \mathcal{P}([n-1])$). In other words, $D(\gamma)$ is a subset of $[n-1]$. Also, $[n-1]$ is a subset of $[n-1]$ as well.

Theorem 3.8 **(i)** (applied to $\alpha = \omega(\gamma)$) yields

$$\begin{aligned}
& D(\operatorname{rev}(\omega(\gamma))) \\
&= \operatorname{rev}_n\left(\underbrace{D(\omega(\gamma))}_{=[n-1]\setminus D(\operatorname{rev}\gamma)}\right) \\
&= \operatorname{rev}_n([n-1]\setminus D(\operatorname{rev}\gamma)) \\
&= \underbrace{\operatorname{rev}_n([n-1])}_{\substack{=[n-1] \\ \text{(by Theorem 3.8 (h))}}} \setminus \underbrace{\operatorname{rev}_n(D(\operatorname{rev}\gamma))}_{\substack{=D(\gamma) \\ \text{(by Corollary 3.9,} \\ \text{applied to } \alpha=\gamma)}} \\
&\qquad \left(\text{by Theorem 3.8 (f), applied to } X = [n-1] \text{ and } Y = D(\operatorname{rev}\gamma)\right) \\
&= [n-1]\setminus D(\gamma).
\end{aligned}$$

We have $\omega(\omega(\gamma)) \in \operatorname{Comp}_n$ (by Proposition 4.3 **(a)**, applied to $\omega(\gamma)$ instead of γ). Moreover, Proposition 4.3 **(b)** (applied to $\omega(\gamma)$ instead of γ) yields

$$\begin{aligned}
D(\omega(\omega(\gamma))) &= [n-1]\setminus \underbrace{D(\operatorname{rev}(\omega(\gamma)))}_{=[n-1]\setminus D(\gamma)} \\
&= [n-1]\setminus([n-1]\setminus D(\gamma)) \\
&= [n-1]\cap D(\gamma) \qquad \left(\begin{array}{c}\text{since } X\setminus(X\setminus Y) = X\cap Y \text{ for} \\ \text{any two sets } X \text{ and } Y\end{array}\right) \\
&= D(\gamma) \qquad (\text{since } D(\gamma) \text{ is a subset of } [n-1]).
\end{aligned}$$

Recall that the map $D : \operatorname{Comp}_n \to \mathcal{P}([n-1])$ is a bijection. Hence, this map is bijective, therefore injective. Thus, any $\alpha, \beta \in \operatorname{Comp}_n$ satisfying $D(\alpha) = D(\beta)$ must satisfy $\alpha = \beta$. We can apply this to $\alpha = \omega(\omega(\gamma))$ and $\beta = \gamma$ (since $\gamma \in \operatorname{Comp}_n$ and $\omega(\omega(\gamma)) \in \operatorname{Comp}_n$ and $D(\omega(\omega(\gamma))) = D(\gamma)$), and thus we obtain $\omega(\omega(\gamma)) = \gamma$. This proves Proposition 4.3 **(d)**. $\square$

Proposition 4.4. Let n be a positive integer. Let $\alpha \in \operatorname{Comp}_n$ and $\gamma \in \operatorname{Comp}_n$. Then:

(a) We have

$$|D(\omega(\gamma))\cap D(\alpha)| = \ell(\alpha) - 1 - |D(\gamma)\cap D(\operatorname{rev}\alpha)|.$$

(b) We have

$$|D(\omega(\gamma))\setminus D(\alpha)| = n - \ell(\alpha) - |D(\gamma)\setminus D(\operatorname{rev}\alpha)|.$$

Proof of Proposition 4.4. We have $\alpha \in \operatorname{Comp}_n$. In other words, α is a composition of n. That is, α is a composition having size n. In other words, $\alpha \in \operatorname{Comp}$ and $|\alpha| = n$. The same argument (applied to γ instead of α) yields $\gamma \in \operatorname{Comp}$ and $|\gamma| = n$.

We have $n \geq 1$ (since n is a positive integer) and thus $n-1 \in \mathbb{N}$. Hence, $|[n-1]| = n-1$.

Also, we have $|\alpha| = n \geq 1 > 0$. Hence, Proposition 2.3 yields

$$|D(\alpha)| = \ell(\alpha) - 1. \tag{11}$$

Moreover, $D(\alpha) \in \mathcal{P}([n-1])$ (since D is a map $\operatorname{Comp}_n \to \mathcal{P}([n-1])$); in other words, $D(\alpha)$ is a subset of $[n-1]$. The same argument (applied to γ instead of α) shows that $D(\gamma)$ is a subset of $[n-1]$. That is, we have $D(\gamma) \subseteq [n-1]$. Hence, $\operatorname{rev}_n(D(\gamma)) \subseteq [n-1]$ as well (by Theorem 3.8 **(a)**, applied to $X = D(\gamma)$).

Also, $D(\operatorname{rev}\alpha)$ is a subset of $[n-1]$ (this can be easily proved in the same way as in the proof of Corollary 3.9 above).

Proposition 4.3 **(c)** yields $D(\omega(\gamma)) = [n-1] \setminus \operatorname{rev}_n(D(\gamma))$.

However, for any three sets X, Y and Z, we have $(X \setminus Y) \cap Z = (X \cap Z) \setminus Y$. Applying this to $X = [n-1]$ and $Y = \operatorname{rev}_n(D(\gamma))$ and $Z = D(\alpha)$, we obtain

$$\begin{aligned}([n-1] \setminus \operatorname{rev}_n(D(\gamma))) \cap D(\alpha) &= \underbrace{([n-1] \cap D(\alpha))}_{\substack{=D(\alpha) \\ \text{(since } D(\alpha) \text{ is a} \\ \text{subset of } [n-1])}} \setminus \operatorname{rev}_n(D(\gamma)) \\ &= D(\alpha) \setminus \operatorname{rev}_n(D(\gamma)).\end{aligned}$$

Thus,

$$\begin{aligned}\underbrace{D(\omega(\gamma))}_{=[n-1] \setminus \operatorname{rev}_n(D(\gamma))} \cap D(\alpha) &= ([n-1] \setminus \operatorname{rev}_n(D(\gamma))) \cap D(\alpha) \\ &= D(\alpha) \setminus \operatorname{rev}_n(D(\gamma)).\end{aligned}$$

Therefore,

$$\begin{aligned}|D(\omega(\gamma)) \cap D(\alpha)| &= |D(\alpha) \setminus \operatorname{rev}_n(D(\gamma))| \\ &= |D(\alpha)| - |D(\alpha) \cap \operatorname{rev}_n(D(\gamma))| \end{aligned} \tag{12}$$

(since any finite sets X and Y satisfy $|X \setminus Y| = |X| - |X \cap Y|$).

However, Theorem 3.8 **(g)** (applied to $X = D(\gamma)$ and $Y = D(\operatorname{rev}\alpha)$) yields

$$\begin{aligned}\operatorname{rev}_n(D(\gamma) \cap D(\operatorname{rev}\alpha)) &= \operatorname{rev}_n(D(\gamma)) \cap \underbrace{\operatorname{rev}_n(D(\operatorname{rev}\alpha))}_{\substack{=D(\alpha) \\ \text{(by Corollary 3.9)}}} = \operatorname{rev}_n(D(\gamma)) \cap D(\alpha) \\ &= D(\alpha) \cap \operatorname{rev}_n(D(\gamma)).\end{aligned} \tag{13}$$

However, $D(\gamma) \cap D(\operatorname{rev}\alpha)$ is a subset of $[n-1]$ (since $D(\gamma) \cap D(\operatorname{rev}\alpha) \subseteq D(\gamma) \subseteq [n-1]$). Hence, Theorem 3.8 **(d)** (applied to $X = D(\gamma) \cap D(\operatorname{rev}\alpha)$) yields

$$|\operatorname{rev}_n(D(\gamma) \cap D(\operatorname{rev}\alpha))| = |D(\gamma) \cap D(\operatorname{rev}\alpha)|.$$

In view of (13), we can rewrite this as

$$\left|D\left(\alpha\right)\cap\operatorname{rev}_n\left(D\left(\gamma\right)\right)\right| = \left|D\left(\gamma\right)\cap D\left(\operatorname{rev}\alpha\right)\right|.$$

Therefore, (12) becomes

$$\begin{aligned}\left|D\left(\omega\left(\gamma\right)\right)\cap D\left(\alpha\right)\right| &= \underbrace{\left|D\left(\alpha\right)\right|}_{\substack{=\ell(\alpha)-1\\ \text{(by (11))}}} - \underbrace{\left|D\left(\alpha\right)\cap\operatorname{rev}_n\left(D\left(\gamma\right)\right)\right|}_{=\left|D(\gamma)\cap D(\operatorname{rev}\alpha)\right|}\\ &= \ell\left(\alpha\right)-1-\left|D\left(\gamma\right)\cap D\left(\operatorname{rev}\alpha\right)\right|.\end{aligned}$$

This proves Proposition 4.4 **(a)**.

(b) From Proposition 4.3 **(c)**, we obtain $D\left(\omega\left(\gamma\right)\right) = [n-1]\setminus\operatorname{rev}_n\left(D\left(\gamma\right)\right)$.

However, if two finite sets X and Y satisfy $Y\subseteq X$, then $|X\setminus Y| = |X| - |Y|$. Applying this to $X = [n-1]$ and $Y = \operatorname{rev}_n\left(D\left(\gamma\right)\right)$, we obtain

$$\begin{aligned}\left|[n-1]\setminus\operatorname{rev}_n\left(D\left(\gamma\right)\right)\right| &= \underbrace{\left|[n-1]\right|}_{=n-1} - \underbrace{\left|\operatorname{rev}_n\left(D\left(\gamma\right)\right)\right|}_{\substack{=\left|D(\gamma)\right|\\ \text{(by Theorem 3.8 \textbf{(d)},}\\ \text{applied to } X=D(\gamma))}} \qquad \left(\text{since } \operatorname{rev}_n\left(D\left(\gamma\right)\right)\subseteq[n-1]\right)\\ &= n-1-\left|D\left(\gamma\right)\right|.\end{aligned}$$

In view of $D\left(\omega\left(\gamma\right)\right) = [n-1]\setminus\operatorname{rev}_n\left(D\left(\gamma\right)\right)$, we can rewrite this as

$$\left|D\left(\omega\left(\gamma\right)\right)\right| = n-1-\left|D\left(\gamma\right)\right|. \tag{14}$$

Next, recall that $|X\setminus Y| = |X| - |X\cap Y|$ for any two finite sets X and Y. From this equality, we obtain

$$\left|D\left(\omega\left(\gamma\right)\right)\setminus D\left(\alpha\right)\right| = \left|D\left(\omega\left(\gamma\right)\right)\right| - \left|D\left(\omega\left(\gamma\right)\right)\cap D\left(\alpha\right)\right| \tag{15}$$

and

$$\left|D\left(\gamma\right)\setminus D\left(\operatorname{rev}\alpha\right)\right| = \left|D\left(\gamma\right)\right| - \left|D\left(\gamma\right)\cap D\left(\operatorname{rev}\alpha\right)\right|. \tag{16}$$

Adding these two equalities together, we find

$$\begin{aligned}&\left|D\left(\omega\left(\gamma\right)\right)\setminus D\left(\alpha\right)\right| + \left|D\left(\gamma\right)\setminus D\left(\operatorname{rev}\alpha\right)\right|\\ &= \underbrace{\left|D\left(\omega\left(\gamma\right)\right)\right|}_{\substack{=n-1-\left|D(\gamma)\right|\\ \text{(by (14))}}} - \underbrace{\left|D\left(\omega\left(\gamma\right)\right)\cap D\left(\alpha\right)\right|}_{\substack{=\ell(\alpha)-1-\left|D(\gamma)\cap D(\operatorname{rev}\alpha)\right|\\ \text{(by Proposition 4.4 \textbf{(a)})}}} + \left|D\left(\gamma\right)\right| - \left|D\left(\gamma\right)\cap D\left(\operatorname{rev}\alpha\right)\right|\\ &= n-1-\left|D\left(\gamma\right)\right| - \left(\ell\left(\alpha\right)-1-\left|D\left(\gamma\right)\cap D\left(\operatorname{rev}\alpha\right)\right|\right) + \left|D\left(\gamma\right)\right| - \left|D\left(\gamma\right)\cap D\left(\operatorname{rev}\alpha\right)\right|\\ &= n-\ell\left(\alpha\right).\end{aligned}$$

In other words,

$$\left|D\left(\omega\left(\gamma\right)\right)\setminus D\left(\alpha\right)\right| = n-\ell\left(\alpha\right) - \left|D\left(\gamma\right)\setminus D\left(\operatorname{rev}\alpha\right)\right|.$$

This proves Proposition 4.4 **(b)**. $\square$

5. Concatenation

5.1. Definition and basic properties

The simplest binary operation on compositions is concatenation:

Definition 5.1. The *concatenation* of two compositions $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_q)$ is defined to be the composition

$$(\beta_1, \beta_2, \ldots, \beta_p, \gamma_1, \gamma_2, \ldots, \gamma_q).$$

It is denoted by $\beta\gamma$.

It is clear that any composition α satisfies $\alpha\varnothing = \varnothing\alpha = \alpha$ (where $\varnothing$ denotes the empty composition, as before). The next fact is also evident:

Proposition 5.2. Let β and γ be two compositions. Then:

(a) We have $\ell(\beta\gamma) = \ell(\beta) + \ell(\gamma)$.

(b) We have $|\beta\gamma| = |\beta| + |\gamma|$.

Proof of Proposition 5.2. Write the compositions β and γ in the forms $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_q)$. Thus, the definition of $\beta\gamma$ yields

$$\beta\gamma = (\beta_1, \beta_2, \ldots, \beta_p, \gamma_1, \gamma_2, \ldots, \gamma_q).$$

Hence, the definition of $\ell(\beta\gamma)$ yields $\ell(\beta\gamma) = p + q$, whereas the definition of $|\beta\gamma|$ yields

$$|\beta\gamma| = \beta_1 + \beta_2 + \cdots + \beta_p + \gamma_1 + \gamma_2 + \cdots + \gamma_q. \tag{17}$$

However, from $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$, we obtain $\ell(\beta) = p$ and $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_p$. Moreover, from $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_q)$, we obtain $\ell(\gamma) = q$ and $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_q$. Thus,

$$\underbrace{\ell(\beta)}_{=p} + \underbrace{\ell(\gamma)}_{=q} = p + q = \ell(\beta\gamma) \qquad (\text{since } \ell(\beta\gamma) = p + q).$$

This proves Proposition 5.2 **(a)**.

(b) Adding the equalities $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_p$ and $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_q$ together, we obtain

$$|\beta| + |\gamma| = \beta_1 + \beta_2 + \cdots + \beta_p + \gamma_1 + \gamma_2 + \cdots + \gamma_q = |\beta\gamma|$$

(by (17)). This proves Proposition 5.2 **(b)**. □

5.2. Concatenation and reversal

Concatenation and reversal interact in a nice way:

Proposition 5.3. Let β and γ be two compositions. Then, $\operatorname{rev}(\beta\gamma) = (\operatorname{rev}\gamma)(\operatorname{rev}\beta)$.

Proof of Proposition 5.3. Write the compositions β and γ in the forms $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_q)$. Thus, the definition of $\beta\gamma$ yields

$$\beta\gamma = (\beta_1, \beta_2, \ldots, \beta_p, \gamma_1, \gamma_2, \ldots, \gamma_q).$$

Hence, the definition of $\operatorname{rev}(\beta\gamma)$ yields

$$\operatorname{rev}(\beta\gamma) = (\gamma_q, \gamma_{q-1}, \ldots, \gamma_1, \beta_p, \beta_{p-1}, \ldots, \beta_1). \tag{18}$$

However, the definition of $\operatorname{rev}\beta$ yields $\operatorname{rev}\beta = (\beta_p, \beta_{p-1}, \ldots, \beta_1)$ (since $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$). Furthermore, the definition of $\operatorname{rev}\gamma$ yields $\operatorname{rev}\gamma = (\gamma_q, \gamma_{q-1}, \ldots, \gamma_1)$ (since $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_q)$). Thus,

$$\begin{aligned}\underbrace{(\operatorname{rev}\gamma)}_{=(\gamma_q,\gamma_{q-1},\ldots,\gamma_1)}\underbrace{(\operatorname{rev}\beta)}_{=(\beta_p,\beta_{p-1},\ldots,\beta_1)} &= (\gamma_q, \gamma_{q-1}, \ldots, \gamma_1)(\beta_p, \beta_{p-1}, \ldots, \beta_1) \\ &= (\gamma_q, \gamma_{q-1}, \ldots, \gamma_1, \beta_p, \beta_{p-1}, \ldots, \beta_1)\end{aligned}$$

(by the definition of concatenation). Comparing this with (18), we obtain $\operatorname{rev}(\beta\gamma) = (\operatorname{rev}\gamma)(\operatorname{rev}\beta)$. This proves Proposition 5.3. $\square$

5.3. Concatenation and partial sums

We shall next show some less trivial properties of concatenations of compositions. We will need the following notation:

Definition 5.4. If K is a set of integers, and if m is an integer, then we define two sets $K + m$ and $K - m$ by

$$K + m := \{k + m \mid k \in K\},$$
$$K - m := \{k - m \mid k \in K\}.$$

Clearly, both of these sets $K + m$ and $K - m$ are again sets of integers.

For example, $\{2, 3, 5\} + 10 = \{12, 13, 15\}$ and $\{2, 3, 5\} - 1 = \{1, 2, 4\}$. Visually, you can think of $K + m$ as being the set K, moved to the right by m units on the number line. Similarly, $K - m$ is the set K, moved to the left by m units on the number line.

Clearly, if K is any set of integers, and if m is an integer, then $(K+m)-m=K$ and $(K-m)+m=K$.

Now, if we know the sizes and the partial sum sets of two compositions β and γ, then we can compute the partial sum set of their concatenation $\beta\gamma$ as follows:

Proposition 5.5. Let β and γ be two compositions such that $\beta \neq \varnothing$ and $\gamma \neq \varnothing$. Let $m = |\beta|$. Then,

$$D(\beta\gamma) = \{m\} \cup D(\beta) \cup (D(\gamma)+m).$$

Proof of Proposition 5.5. Write the compositions β and γ in the forms $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_q)$. From $\beta \neq \varnothing$, we easily obtain $p \neq 0$ [5]. Similarly, from $\gamma \neq \varnothing$, we obtain $q \neq 0$. Also, $m = |\beta| = \beta_1 + \beta_2 + \cdots + \beta_p$ (by the definition of $|\beta|$, since $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$). Thus, $\beta_1 + \beta_2 + \cdots + \beta_p = m$.

From $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$, we obtain

$$D(\beta) = \{\beta_1,\ \beta_1+\beta_2,\ \beta_1+\beta_2+\beta_3,\ \ldots,\ \beta_1+\beta_2+\cdots+\beta_{p-1}\} \tag{19}$$

(by the definition of $D(\beta)$).

From $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_q)$, we obtain

$$D(\gamma) = \{\gamma_1,\ \gamma_1+\gamma_2,\ \gamma_1+\gamma_2+\gamma_3,\ \ldots,\ \gamma_1+\gamma_2+\cdots+\gamma_{q-1}\}.$$

However, the definition of $D(\gamma)+m$ yields

$$\begin{aligned}
&D(\gamma)+m \\
&= \left\{ \underbrace{k+m}_{=m+k} \mid k \in D(\gamma) \right\} = \{m+k \mid k \in D(\gamma)\} \\
&= \{m+k \mid k \in \{\gamma_1,\ \gamma_1+\gamma_2,\ \gamma_1+\gamma_2+\gamma_3,\ \ldots,\ \gamma_1+\gamma_2+\cdots+\gamma_{q-1}\}\} \\
&\qquad \left(\text{since } D(\gamma) = \{\gamma_1,\ \gamma_1+\gamma_2,\ \gamma_1+\gamma_2+\gamma_3,\ \ldots,\ \gamma_1+\gamma_2+\cdots+\gamma_{q-1}\}\right) \\
&= \{m+\gamma_1,\ m+\gamma_1+\gamma_2,\ m+\gamma_1+\gamma_2+\gamma_3,\ \ldots,\ m+\gamma_1+\gamma_2+\cdots+\gamma_{q-1}\}.
\end{aligned} \tag{20}$$

Now, recall that $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_q)$. Hence, the definition of $\beta\gamma$ yields

$$\beta\gamma = (\beta_1, \beta_2, \ldots, \beta_p, \gamma_1, \gamma_2, \ldots, \gamma_q).$$

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[5] *Proof.* If we had $p = 0$, then we would have

$$\begin{aligned}
\beta &= (\beta_1, \beta_2, \ldots, \beta_p) = (\beta_1, \beta_2, \ldots, \beta_0) \qquad (\text{since } p = 0) \\
&= () = \varnothing,
\end{aligned}$$

which would contradict $\beta \neq \varnothing$. Hence, we cannot have $p = 0$. Thus, we have $p \neq 0$.

Hence, the definition of $D(\beta\gamma)$ yields[6]

$$\begin{aligned}
& D(\beta\gamma) \\
&= \{\beta_1,\ \beta_1+\beta_2,\ \beta_1+\beta_2+\beta_3,\ \ldots,\ \beta_1+\beta_2+\cdots+\beta_{p-1}, \\
&\qquad \beta_1+\beta_2+\cdots+\beta_p, \\
&\qquad \beta_1+\beta_2+\cdots+\beta_p+\gamma_1, \\
&\qquad \beta_1+\beta_2+\cdots+\beta_p+\gamma_1+\gamma_2, \\
&\qquad \beta_1+\beta_2+\cdots+\beta_p+\gamma_1+\gamma_2+\gamma_3, \\
&\qquad \ldots, \\
&\qquad \beta_1+\beta_2+\cdots+\beta_p+\gamma_1+\gamma_2+\cdots+\gamma_{q-1}\} \\
&= \{\beta_1,\ \beta_1+\beta_2,\ \beta_1+\beta_2+\beta_3,\ \ldots,\ \beta_1+\beta_2+\cdots+\beta_{p-1}, \\
&\qquad m,\ m+\gamma_1,\ m+\gamma_1+\gamma_2,\ m+\gamma_1+\gamma_2+\gamma_3,\ \ldots,\ m+\gamma_1+\gamma_2+\cdots+\gamma_{q-1}\} \\
&\qquad\qquad \left(\text{since } \beta_1+\beta_2+\cdots+\beta_p = m\right) \\
&= \underbrace{\left\{\beta_1,\ \beta_1+\beta_2,\ \beta_1+\beta_2+\beta_3,\ \ldots,\ \beta_1+\beta_2+\cdots+\beta_{p-1}\right\}}_{\substack{=D(\beta) \\ \text{(by (19))}}} \\
&\qquad\quad \cup \{m\} \\
&\qquad\quad \cup \underbrace{\left\{m+\gamma_1,\ m+\gamma_1+\gamma_2,\ m+\gamma_1+\gamma_2+\gamma_3,\ \ldots,\ m+\gamma_1+\gamma_2+\cdots+\gamma_{q-1}\right\}}_{\substack{=D(\gamma)+m \\ \text{(by (20))}}} \\
&= D(\beta) \cup \{m\} \cup (D(\gamma)+\{m\}) = \{m\} \cup D(\beta) \cup (D(\gamma)+m).
\end{aligned}$$

This proves Proposition 5.5. □

The following is a variant of Proposition 5.5 that avoids the requirements that $\beta \neq \varnothing$ and $\gamma \neq \varnothing$:

Proposition 5.6. Let β and γ be two compositions. Let $m = |\beta|$ and $n = |\gamma|$. Then,

$$D(\beta\gamma) = (\{m\} \cup D(\beta) \cup (D(\gamma)+m)) \cap [m+n-1].$$

Proof of Proposition 5.6. We know that γ is a composition having size n (since the size of γ is $|\gamma| = n$). In other words, γ is a composition of n. In other words, $\gamma \in \operatorname{Comp}_n$ (since Comp_n is the set of all compositions of n).

We know that β is a composition having size m (since the size of β is $|\beta| = m$). In other words, β is a composition of m. In other words, $\beta \in \operatorname{Comp}_m$ (since Comp_m is the set of all compositions of m).

We have $0 \notin [n-1]$ (since the set $[n-1] = \{1,2,\ldots,n-1\}$ does not contain 0) and $m \notin [m-1]$ (since the set $[m-1] = \{1,2,\ldots,m-1\}$ does not contain m).

[6] Note the tacit use of $p \neq 0$ and $q \neq 0$ in this computation.

We are in one of the following three cases:

Case 1: We have $\beta = \varnothing$.

Case 2: We have $\gamma = \varnothing$.

Case 3: We have neither $\beta = \varnothing$ nor $\gamma = \varnothing$.

Let us first consider Case 1. In this case, we have $\beta = \varnothing$. Thus, $D(\beta) = D(\varnothing) = \varnothing$ (by the definition of the map $D : \mathrm{Comp}_0 \to \mathcal{P}([0-1])$).

Moreover, $m = |\beta|$. In view of $\beta = \varnothing$, this rewrites as $m = |\varnothing| = 0$. Thus, $D(\gamma) + \underbrace{m}_{=0} = D(\gamma) + 0 = D(\gamma)$ (because any set K of integers satisfies $K + 0 = K$).

Recall that D is a map $\mathrm{Comp}_n \to \mathcal{P}([n-1])$. Hence, $D(\gamma) \in \mathcal{P}([n-1])$ (since $\gamma \in \mathrm{Comp}_n$). In other words, $D(\gamma) \subseteq [n-1]$.

Now,

$$\begin{aligned}
&\left(\left\{\underbrace{m}_{=0}\right\} \cup \underbrace{D(\beta)}_{=\varnothing} \cup \underbrace{(D(\gamma)+m)}_{=D(\gamma)}\right) \cap \left[\underbrace{m}_{=0} + n - 1\right] \\
&= \left(\underbrace{\{0\} \cup \varnothing}_{=\{0\}} \cup D(\gamma)\right) \cap \left[\underbrace{0+n-1}_{=n-1}\right] \\
&= (\{0\} \cup D(\gamma)) \cap [n-1] . \qquad (21)
\end{aligned}$$

However, recall that any three sets X_1, X_2, Y satisfy

$$(X_1 \cup X_2) \cap Y = (X_1 \cap Y) \cup (X_2 \cap Y) .$$

Applying this to $X_1 = \{0\}$, $X_2 = D(\gamma)$ and $Y = [n-1]$, we obtain

$$\begin{aligned}
(\{0\} \cup D(\gamma)) \cap [n-1] &= \underbrace{(\{0\} \cap [n-1])}_{\substack{=\varnothing \\ (\text{since } 0 \notin [n-1])}} \cup \underbrace{(D(\gamma) \cap [n-1])}_{\substack{=D(\gamma) \\ (\text{since } D(\gamma) \subseteq [n-1])}} \\
&= \varnothing \cup D(\gamma) = D(\gamma) .
\end{aligned}$$

Thus, (21) rewrites as

$$\begin{aligned}
&(\{m\} \cup D(\beta) \cup (D(\gamma)+m)) \cap [m+n-1] \\
&= D(\gamma) = D(\beta\gamma) \qquad \left(\text{since } \gamma = \beta\gamma \text{ (because } \underbrace{\beta}_{=\varnothing} \gamma = \varnothing\gamma = \gamma)\right) .
\end{aligned}$$

Hence, Proposition 5.6 is proved in Case 1.

Let us now consider Case 2. In this case, we have $\gamma = \varnothing$. Hence, $D(\gamma) = D(\varnothing) = \varnothing$. Hence, $\underbrace{D(\gamma)}_{=\varnothing} + m = \varnothing + m = \varnothing$ (since $\varnothing + k = \varnothing$ for any integer k).

Moreover, $n = |\gamma|$. In view of $\gamma = \varnothing$, this rewrites as $n = |\varnothing| = 0$.

Recall that D is a map $\operatorname{Comp}_m \to \mathcal{P}([m-1])$. Hence, $D(\beta) \in \mathcal{P}([m-1])$ (since $\beta \in \operatorname{Comp}_m$). In other words, $D(\beta) \subseteq [m-1]$.

Now,

$$\begin{aligned}
&\left(\{m\} \cup D(\beta) \cup \underbrace{(D(\gamma)+m)}_{=\varnothing}\right) \cap \left[m + \underbrace{n}_{=0} - 1\right] \\
&= \underbrace{(\{m\} \cup D(\beta) \cup \varnothing)}_{=\{m\}\cup D(\beta)} \cap \left[\underbrace{m+0-1}_{=m-1}\right] \\
&= (\{m\} \cup D(\beta)) \cap [m-1]. \qquad (22)
\end{aligned}$$

However, recall that any three sets X_1, X_2, Y satisfy

$$(X_1 \cup X_2) \cap Y = (X_1 \cap Y) \cup (X_2 \cap Y).$$

Applying this to $X_1 = \{m\}$, $X_2 = D(\beta)$ and $Y = [m-1]$, we obtain

$$\begin{aligned}
(\{m\} \cup D(\beta)) \cap [m-1] &= \underbrace{(\{m\} \cap [m-1])}_{\substack{=\varnothing \\ (\text{since } m \notin [m-1])}} \cup \underbrace{(D(\beta) \cap [m-1])}_{\substack{=D(\beta) \\ (\text{since } D(\beta) \subseteq [m-1])}} \\
&= \varnothing \cup D(\beta) = D(\beta).
\end{aligned}$$

Thus, (22) rewrites as

$$\begin{aligned}
&(\{m\} \cup D(\beta) \cup (D(\gamma)+m)) \cap [m+n-1] \\
&= D(\beta) = D(\beta\gamma) \qquad \left(\text{since } \beta = \beta\gamma \text{ (because } \beta \underbrace{\gamma}_{=\varnothing} = \beta\varnothing = \beta)\right).
\end{aligned}$$

Hence, Proposition 5.6 is proved in Case 2.

Now, let us consider Case 3. In this case, we have neither $\beta = \varnothing$ nor $\gamma = \varnothing$. In other words, we have $\beta \neq \varnothing$ and $\gamma \neq \varnothing$. Thus, Proposition 5.5 yields

$$D(\beta\gamma) = \{m\} \cup D(\beta) \cup (D(\gamma)+m).$$

However, Proposition 5.2 **(b)** yields $|\beta\gamma| = \underbrace{|\beta|}_{=m} + \underbrace{|\gamma|}_{=n} = m+n$. Thus, the composition $\beta\gamma$ has size $|\beta\gamma| = m+n$. In other words, $\beta\gamma$ is a composition of $m+n$. In other words, $\beta\gamma \in \operatorname{Comp}_{m+n}$. Hence, $D(\beta\gamma) \in \mathcal{P}([m+n-1])$ (since D is a map $\operatorname{Comp}_{m+n} \to \mathcal{P}([m+n-1])$). In other words, $D(\beta\gamma) \subseteq [m+n-1]$. Hence, $D(\beta\gamma) \cap [m+n-1] = D(\beta\gamma)$, so that

$$\begin{aligned}
D(\beta\gamma) &= \underbrace{D(\beta\gamma)}_{=\{m\}\cup D(\beta)\cup(D(\gamma)+m)} \cap [m+n-1] \\
&= (\{m\} \cup D(\beta) \cup (D(\gamma)+m)) \cap [m+n-1].
\end{aligned}$$

Therefore, Proposition 5.6 is proved in Case 3.

We have now proved Proposition 5.6 in each of the three Cases 1, 2 and 3. This completes the proof of Proposition 5.6. □

Conversely, given two compositions β and γ, we can reconstruct the partial sum sets $D(\beta)$ and $D(\gamma)$ if we know the size $|\beta|$ and the partial sum set $D(\beta\gamma)$ as follows:

Proposition 5.7. Let β and γ be two compositions. Let $m = |\beta|$. Then:

(a) We have $D(\beta) = D(\beta\gamma) \cap [m-1]$.

(b) We have $D(\gamma) = (D(\beta\gamma) \setminus [m]) - m$.

Proof of Proposition 5.7. Let $n = |\gamma|$. Then, as in the above proof of Proposition 5.6, we can show that $\beta \in \operatorname{Comp}_m$ and $\gamma \in \operatorname{Comp}_n$.

Recall that D is a map $\operatorname{Comp}_n \to \mathcal{P}([n-1])$. Hence, $D(\gamma) \in \mathcal{P}([n-1])$ (since $\gamma \in \operatorname{Comp}_n$). In other words, $D(\gamma) \subseteq [n-1]$. The same argument (applied to β and m instead of γ and n) yields $D(\beta) \subseteq [m-1]$. Also, note that $[m-1] \subseteq [m+n-1]$ (since $m-1 \leq m+n-1$ (because $n \geq 0$)).

(a) Let $x \in D(\beta)$. We shall show that $x \in D(\beta\gamma) \cap [m-1]$.

Indeed, we observe that

$$x \in D(\beta) \subseteq \{m\} \cup D(\beta) \cup (D(\gamma) + m).$$

Combining this with $x \in D(\beta) \subseteq [m-1] \subseteq [m+n-1]$, we obtain

$$\begin{aligned} x &\in (\{m\} \cup D(\beta) \cup (D(\gamma) + m)) \cap [m+n-1] \\ &= D(\beta\gamma) \qquad \text{(by Proposition 5.6)}. \end{aligned}$$

Combining this with $x \in [m-1]$, we obtain $x \in D(\beta\gamma) \cap [m-1]$.

Forget that we fixed x. We thus have shown that $x \in D(\beta\gamma) \cap [m-1]$ for each $x \in D(\beta)$. In other words,

$$D(\beta) \subseteq D(\beta\gamma) \cap [m-1]. \tag{23}$$

On the other hand, let $y \in D(\beta\gamma) \cap [m-1]$. Thus, $y \in D(\beta\gamma)$ and $y \in [m-1]$. From $y \in [m-1] = \{1, 2, \ldots, m-1\}$, we obtain $y \leq m-1 < m$. Thus, we cannot have $y \in \{m\}$ (because $y \in \{m\}$ would entail $y = m$, which would contradict $y < m$). Furthermore, we cannot have $y \in D(\gamma) + m$ (because $y \in D(\gamma) + m$ would entail that $y \geq m$ [7], which would contradict $y < m$).

[7] *Proof.* Assume that $y \in D(\gamma) + m$. We must show that $y \geq m$.

We have $y \in D(\gamma) + m = \{k + m \mid k \in D(\gamma)\}$ (by the definition of $D(\gamma) + m$). In other words, $y = k + m$ for some $k \in D(\gamma)$. Consider this k. From $k \in D(\gamma) \subseteq [n-1] = \{1, 2, \ldots, n-1\}$, we obtain $k \geq 1 > 0$. Hence, $y = \underbrace{k}_{>0} + m > m$, thus $y \geq m$.

However,

$$\begin{aligned} y &\in D(\beta\gamma) \\ &= (\{m\} \cup D(\beta) \cup (D(\gamma)+m)) \cap [m+n-1] \qquad \text{(by Proposition 5.6)} \\ &\subseteq \{m\} \cup D(\beta) \cup (D(\gamma)+m). \end{aligned}$$

In other words, we have $y \in \{m\}$ or $y \in D(\beta)$ or $y \in D(\gamma)+m$. Hence, we must have $y \in D(\beta)$ (since we cannot have $y \in \{m\}$, and we cannot have $y \in D(\gamma)+m$).

Forget that we fixed y. We thus have shown that $y \in D(\beta)$ for each $y \in D(\beta\gamma) \cap [m-1]$. In other words,

$$D(\beta\gamma) \cap [m-1] \subseteq D(\beta).$$

Combining this with (23), we obtain $D(\beta) = D(\beta\gamma) \cap [m-1]$. This proves Proposition 5.7 **(a)**.

(b) The definition of $D(\gamma)+m$ yields

$$D(\gamma)+m = \{k+m \mid k \in D(\gamma)\}. \tag{24}$$

The definition of $(D(\beta\gamma) \setminus [m]) - m$ yields

$$(D(\beta\gamma) \setminus [m]) - m = \{k-m \mid k \in D(\beta\gamma) \setminus [m]\}. \tag{25}$$

Let $x \in D(\gamma)$. We shall show that $x \in (D(\beta\gamma) \setminus [m]) - m$.

Indeed, we have $x \in D(\gamma) \subseteq [n-1] = \{1,2,\ldots,n-1\}$, so that

$$x+m \in \{m+1, m+2, \ldots, m+n-1\} \subseteq \{1,2,\ldots,m+n-1\} = [m+n-1].$$

Also, from $x \in \{1,2,\ldots,n-1\}$, we obtain $x \geq 1 > 0$, and therefore $\underbrace{x}_{>0} + m > m$, so that $x+m \notin [m]$ [8].

Next, we recall that $x \in D(\gamma)$. Thus, the number $x+m$ can be written in the form $k+m$ for some $k \in D(\gamma)$ (namely, for $k = x$). In other words, $x+m \in \{k+m \mid k \in D(\gamma)\}$. In view of (24), we can rewrite this as $x+m \in D(\gamma)+m$. Hence,

$$x+m \in D(\gamma)+m \subseteq \{m\} \cup D(\beta) \cup (D(\gamma)+m).$$

Combining this with $x+m \in [m+n-1]$, we obtain

$$\begin{aligned} x+m &\in (\{m\} \cup D(\beta) \cup (D(\gamma)+m)) \cap [m+n-1] \\ &= D(\beta\gamma) \qquad \text{(by Proposition 5.6)}. \end{aligned}$$

Combining this with $x+m \notin [m]$, we obtain $x+m \in D(\beta\gamma) \setminus [m]$. We also have $x = (x+m)-m$. Therefore, x has the form $k-m$ for some $k \in D(\beta\gamma) \setminus [m]$

[8] *Proof.* If we had $x+m \in [m]$, then we would have $x+m \leq m$ (since $x+m \in [m] = \{1,2,\ldots,m\}$), but this would contradict $x+m > m$. Hence, we cannot have $x+m \in [m]$. In other words, we have $x+m \notin [m]$.

(namely, for $k = x + m$), because $x + m \in D(\beta\gamma) \setminus [m]$. In other words, $x \in \{k - m \mid k \in D(\beta\gamma) \setminus [m]\}$. In view of (25), this rewrites as $x \in (D(\beta\gamma) \setminus [m]) - m$.

Forget that we fixed x. We thus have shown that $x \in (D(\beta\gamma) \setminus [m]) - m$ for each $x \in D(\gamma)$. In other words,

$$D(\gamma) \subseteq (D(\beta\gamma) \setminus [m]) - m. \tag{26}$$

On the other hand, let $y \in (D(\beta\gamma) \setminus [m]) - m$. Thus,

$$y \in (D(\beta\gamma) \setminus [m]) - m = \{k - m \mid k \in D(\beta\gamma) \setminus [m]\}$$

(by (25)). In other words, $y = k - m$ for some $k \in D(\beta\gamma) \setminus [m]$. Consider this k, and denote it by z. Thus, $y = z - m$ and $z \in D(\beta\gamma) \setminus [m]$.

From $z \in D(\beta\gamma) \setminus [m]$, we obtain $z \in D(\beta\gamma)$ and $z \notin [m]$. In particular,

$$\begin{aligned} z &\in D(\beta\gamma) \\ &= (\{m\} \cup D(\beta) \cup (D(\gamma) + m)) \cap [m + n - 1] \qquad \text{(by Proposition 5.6)} \\ &\subseteq [m + n - 1] = \{1, 2, \ldots, m + n - 1\}. \end{aligned}$$

Combining this with $z \notin [m] = \{1, 2, \ldots, m\}$, we obtain

$$z \in \{1, 2, \ldots, m + n - 1\} \setminus \{1, 2, \ldots, m\} = \{m + 1, m + 2, \ldots, m + n - 1\}.$$

Hence, $z \geq m + 1 > m$.

Furthermore,

$$\begin{aligned} z &\in (\{m\} \cup D(\beta) \cup (D(\gamma) + m)) \cap [m + n - 1] \\ &\subseteq \{m\} \cup D(\beta) \cup (D(\gamma) + m). \end{aligned}$$

In other words, we have $z \in \{m\}$ or $z \in D(\beta)$ or $z \in D(\gamma) + m$. However, we cannot have $z \in \{m\}$ [9], and we also cannot have $z \in D(\beta)$ [10]. Hence, we must have $z \in D(\gamma) + m$ (since we have $z \in \{m\}$ or $z \in D(\beta)$ or $z \in D(\gamma) + m$). In view of (24), this rewrites as

$$z \in \{k + m \mid k \in D(\gamma)\}.$$

In other words, $z = k + m$ for some $k \in D(\gamma)$. Consider this k. We have $y = \underbrace{z}_{=k+m} - m = k + m - m = k \in D(\gamma)$.

Forget that we fixed y. We thus have shown that $y \in D(\gamma)$ for each $y \in (D(\beta\gamma) \setminus [m]) - m$. In other words,

$$(D(\beta\gamma) \setminus [m]) - m \subseteq D(\gamma).$$

Combining this with (26), we obtain $(D(\beta\gamma) \setminus [m]) - m = D(\gamma)$. This proves Proposition 5.7 **(b)**. □

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[9] *Proof.* Assume the contrary. Thus, $z \in \{m\}$. Hence, $z = m$, which contradicts $z > m$. This contradiction shows that our assumption was wrong. Thus, we cannot have $z \in \{m\}$.

[10] *Proof.* Assume the contrary. Thus, $z \in D(\beta) \subseteq [m - 1] = \{1, 2, \ldots, m - 1\}$. Hence, $z \leq m - 1 < m$, which contradicts $z > m$. This contradiction shows that our assumption was wrong. Thus, we cannot have $z \in D(\beta)$.

5.4. Further lemmas

The next few propositions and lemmas will be used in a later proof.

Proposition 5.8. Let β, γ, β' and γ' be four compositions such that $|\beta'| = |\beta|$ and $D(\beta') \subseteq D(\beta)$ and $|\gamma'| = |\gamma|$ and $D(\gamma') \subseteq D(\gamma)$. Then, $D(\beta'\gamma') \subseteq D(\beta\gamma)$.

Proof of Proposition 5.8. Let $m = |\beta|$ and $n = |\gamma|$. Thus, $|\beta'| = |\beta| = m$ and $|\gamma'| = |\gamma| = n$.

It is easy to see that if K and L are two sets of integers satisfying $K \subseteq L$, and if k is any integer, then $K + k \subseteq L + k$. Applying this to $K = D(\gamma')$ and $L = D(\gamma)$ and $k = m$, we obtain $D(\gamma') + m \subseteq D(\gamma) + m$ (since $D(\gamma') \subseteq D(\gamma)$).

Now, Proposition 5.6 yields

$$D(\beta\gamma) = (\{m\} \cup D(\beta) \cup (D(\gamma) + m)) \cap [m + n - 1]. \tag{27}$$

Also, we have $m = |\beta'|$ (since $|\beta'| = m$) and $n = |\gamma'|$ (since $|\gamma'| = n$). Hence, Proposition 5.6 (applied to β' and γ' instead of β and γ) yields

$$\begin{aligned} D(\beta'\gamma') &= \left(\{m\} \cup \underbrace{D(\beta')}_{\subseteq D(\beta)} \cup \underbrace{(D(\gamma') + m)}_{\subseteq D(\gamma)+m} \right) \cap [m + n - 1] \\ &\subseteq (\{m\} \cup D(\beta) \cup (D(\gamma) + m)) \cap [m + n - 1] = D(\beta\gamma) \end{aligned}$$

(by (27)). This proves Proposition 5.8. □

Proposition 5.9. Let $\alpha \in \operatorname{Comp}$ be any composition, and let $m \in \mathbb{N}$. Then, there exists at most one pair (β, γ) of compositions such that $|\beta| = m$ and $\beta\gamma = \alpha$.

Proof of Proposition 5.9. Let (β', γ') and (β'', γ'') be two pairs (β, γ) of compositions such that $|\beta| = m$ and $\beta\gamma = \alpha$. Thus, (β', γ') and (β'', γ'') are two pairs of compositions and have the property that $|\beta'| = m$ and $\beta'\gamma' = \alpha$ and $|\beta''| = m$ and $\beta''\gamma'' = \alpha$. Thus, $\alpha = \beta''\gamma''$.

We have $m = |\beta'|$ (since $|\beta'| = m$). Thus, Proposition 5.7 **(a)** (applied to β' and γ' instead of β and γ) yields

$$D(\beta') = D\left(\underbrace{\beta'\gamma'}_{=\alpha}\right) \cap [m - 1] = D(\alpha) \cap [m - 1].$$

The same argument (applied to β'' and γ'' instead of β' and γ') yields

$$D(\beta'') = D(\alpha) \cap [m - 1].$$

Comparing these two equalities, we find $D(\beta') = D(\beta'')$.

Now, β' is a composition having size $|\beta'| = m$. In other words, β' is a composition of m. In other words, $\beta' \in \operatorname{Comp}_m$. Similarly, $\beta'' \in \operatorname{Comp}_m$.

Recall that the map $D : \operatorname{Comp}_n \to \mathcal{P}([n-1])$ is a bijection. Similarly, the map $D : \operatorname{Comp}_m \to \mathcal{P}([m-1])$ is a bijection. Hence, this map D is bijective, thus injective. In other words, if $\varphi, \psi \in \operatorname{Comp}_m$ satisfy $D(\varphi) = D(\psi)$, then $\varphi = \psi$. Applying this to $\varphi = \beta'$ and $\psi = \beta''$, we obtain $\beta' = \beta''$ (since $\beta' \in \operatorname{Comp}_m$ and $\beta'' \in \operatorname{Comp}_m$ and $D(\beta') = D(\beta'')$).

Furthermore, Proposition 5.7 **(b)** (applied to β' and γ' instead of β and γ) yields

$$D(\gamma') = \left(D\left(\underbrace{\beta'\gamma'}_{=\alpha} \right) \setminus [m] \right) - m = (D(\alpha) \setminus [m]) - m.$$

The same argument (applied to β'' and γ'' instead of β' and γ') yields

$$D(\gamma'') = (D(\alpha) \setminus [m]) - m.$$

Comparing these two equalities, we find $D(\gamma') = D(\gamma'')$.

Set $n = |\gamma'|$. Then, from $\beta'\gamma' = \alpha$, we obtain $\alpha = \beta'\gamma'$. Thus,

$$\begin{aligned} |\alpha| = |\beta'\gamma'| &= \underbrace{|\beta'|}_{=m} + \underbrace{|\gamma'|}_{=n} && \left(\begin{array}{c} \text{by Proposition 5.2 } \textbf{(b)}, \\ \text{applied to } \beta = \beta' \text{ and } \gamma = \gamma' \end{array} \right) \\ &= m + n. \end{aligned}$$

Hence,

$$\begin{aligned} m + n = |\alpha| &= |\beta''\gamma''| && (\text{since } \alpha = \beta''\gamma'') \\ &= \underbrace{|\beta''|}_{=m} + |\gamma''| && \left(\begin{array}{c} \text{by Proposition 5.2 } \textbf{(b)}, \\ \text{applied to } \beta = \beta'' \text{ and } \gamma = \gamma'' \end{array} \right) \\ &= m + |\gamma''|. \end{aligned}$$

Subtracting m from this equality, we obtain $n = |\gamma''|$.

Now, γ' is a composition having size $|\gamma'| = n$ (since $n = |\gamma'|$). In other words, γ' is a composition of n. In other words, $\gamma' \in \operatorname{Comp}_n$. Similarly, $\gamma'' \in \operatorname{Comp}_n$ (since $n = |\gamma''|$).

Recall that the map $D : \operatorname{Comp}_n \to \mathcal{P}([n-1])$ is a bijection. Hence, this map D is bijective, thus injective. In other words, if $\varphi, \psi \in \operatorname{Comp}_n$ satisfy $D(\varphi) = D(\psi)$, then $\varphi = \psi$. Applying this to $\varphi = \gamma'$ and $\psi = \gamma''$, we obtain $\gamma' = \gamma''$ (since $\gamma' \in \operatorname{Comp}_n$ and $\gamma'' \in \operatorname{Comp}_n$ and $D(\gamma') = D(\gamma'')$).

Now, $\left(\underbrace{\beta'}_{=\beta''}, \underbrace{\gamma'}_{=\gamma''} \right) = (\beta'', \gamma'')$.

Forget that we fixed (β', γ') and (β'', γ''). We thus have shown that if (β', γ') and (β'', γ'') are two pairs (β, γ) of compositions such that $|\beta| = m$ and $\beta\gamma = \alpha$, then $(\beta', \gamma') = (\beta'', \gamma'')$. In other words, there exists at most one pair (β, γ) of compositions such that $|\beta| = m$ and $\beta\gamma = \alpha$. This proves Proposition 5.9. $\square$

Next, we shall show a nearly trivial lemma:

Lemma 5.10. Let $m \in \mathbb{N}$. Let K be a subset of $\{1, 2, 3, \ldots\}$. Then,

$$(K \cap [m-1]) \cup (K \setminus [m]) = K \setminus \{m\}.$$

Proof of Lemma 5.10. Any element $k \in K$ is an element of $\{1, 2, 3, \ldots\}$ (since K is a subset of $\{1, 2, 3, \ldots\}$) and therefore is a positive integer. Hence, for any element $k \in K$, we have the following chain of logical equivalences:

$$\begin{aligned}(k \in [m-1]) &\iff (k \leq m-1) \qquad \text{(since } k \text{ is a positive integer)} \\ &\iff (k < m) \qquad \text{(since } k \text{ and } m \text{ are integers)}.\end{aligned}$$

Thus,

$$\{k \in K \mid k \in [m-1]\} = \{k \in K \mid k < m\}.$$

Recall again that any element $k \in K$ is a positive integer. Hence, for any element $k \in K$, we have the following chain of logical equivalences:

$$\begin{aligned}(k \notin [m]) &\iff (\text{we don't have } k \in [m]) \\ &\iff (\text{we don't have } k \leq m) \qquad \left(\begin{array}{c}\text{since } k \text{ is a positive integer,} \\ \text{and thus the statement ``} k \in [m] \text{''} \\ \text{is equivalent to ``} k \leq m \text{''}\end{array}\right) \\ &\iff (k > m).\end{aligned}$$

Hence,

$$\{k \in K \mid k \notin [m]\} = \{k \in K \mid k > m\}.$$

Now,

$$\begin{aligned}&\underbrace{(K \cap [m-1])}_{\substack{=\{k \in K \mid k \in [m-1]\} \\ =\{k \in K \mid k < m\}}} \cup \underbrace{(K \setminus [m])}_{\substack{\{k \in K \mid k \notin [m]\} \\ =\{k \in K \mid k > m\}}} \\ &= \{k \in K \mid k < m\} \cup \{k \in K \mid k > m\} \\ &= \{k \in K \mid k < m \text{ or } k > m\} \\ &= \{k \in K \mid k \neq m\} \qquad \left(\begin{array}{c}\text{since the statement ``} k < m \text{ or } k > m \text{''} \\ \text{is equivalent to ``} k \neq m \text{''}\end{array}\right) \\ &= K \setminus \{m\}.\end{aligned}$$

This proves Lemma 5.10. $\square$

Our next proposition characterizes of the concatenation $\varphi\psi$ of two compositions φ and ψ in terms of how its partial sum set $D(\alpha)$ relates to $D(\varphi)$ and $D(\psi)$:

Proposition 5.11. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let $\alpha \in \operatorname{Comp}_{m+n}$ be any composition of $m+n$ such that $m \in D(\alpha) \cup \{0, m+n\}$.

Let $\varphi \in \operatorname{Comp}_m$ be a composition that satisfies $D(\varphi) = D(\alpha) \cap [m-1]$.

Let $\psi \in \operatorname{Comp}_n$ be a composition that satisfies $D(\psi) = (D(\alpha) \setminus [m]) - m$.

Then, $\varphi\psi = \alpha$.

Proof of Proposition 5.11. We have $\varphi \in \operatorname{Comp}_m$. In other words, φ is a composition of m. In other words, φ is a composition having size m. In other words, $\varphi \in \operatorname{Comp}$ and $|\varphi| = m$. Similarly, from $\psi \in \operatorname{Comp}_n$, we obtain $\psi \in \operatorname{Comp}$ and $|\psi| = n$.

Now, Proposition 5.2 **(b)** (applied to $\beta = \varphi$ and $\gamma = \psi$) yields $|\varphi\psi| = \underbrace{|\varphi|}_{=m} + \underbrace{|\psi|}_{=n} = m+n$. Thus, $\varphi\psi$ is a composition having size $|\varphi\psi| = m+n$. In other words, $\varphi\psi$ is a composition of $m+n$. In other words, $\varphi\psi \in \operatorname{Comp}_{m+n}$.

Recall that the map $D : \operatorname{Comp}_{m+n} \to \mathcal{P}([m+n-1])$ is a bijection. Hence, this map D is bijective, thus injective. Furthermore, from $\alpha \in \operatorname{Comp}_{m+n}$, we obtain $D(\alpha) \in \mathcal{P}([m+n-1])$ (since D is a map from $\operatorname{Comp}_{m+n}$ to $\mathcal{P}([m+n-1])$). In other words, $D(\alpha) \subseteq [m+n-1]$. Hence,

$$D(\alpha) \subseteq [m+n-1] \subseteq \{1, 2, 3, \ldots\}.$$

In other words, $D(\alpha)$ is a subset of $\{1, 2, 3, \ldots\}$. Hence, Lemma 5.10 (applied to $K = D(\alpha)$) yields

$$(D(\alpha) \cap [m-1]) \cup (D(\alpha) \setminus [m]) = D(\alpha) \setminus \{m\}. \tag{28}$$

If K is any set of integers, then $(K-m)+m = K$ (indeed, this follows easily from Definition 5.4). Applying this to $K = D(\alpha) \setminus [m]$, we obtain

$$((D(\alpha) \setminus [m]) - m) + m = D(\alpha) \setminus [m].$$

In view of $D(\psi) = (D(\alpha) \setminus [m]) - m$, we can rewrite this as

$$D(\psi) + m = D(\alpha) \setminus [m]. \tag{29}$$

Proposition 5.6 (applied to $\beta = \varphi$ and $\gamma = \psi$) yields

$$\begin{aligned}
D(\varphi\psi) &= \left(\{m\} \cup \underbrace{D(\varphi)}_{=D(\alpha)\cap[m-1]} \cup \underbrace{(D(\psi)+m)}_{\substack{=D(\alpha)\setminus[m] \\ \text{(by (29))}}} \right) \cap [m+n-1] \\
&= \left(\{m\} \cup \underbrace{(D(\alpha)\cap[m-1]) \cup (D(\alpha)\setminus[m])}_{\substack{=D(\alpha)\setminus\{m\} \\ \text{(by (28))}}} \right) \cap [m+n-1] \\
&= \underbrace{(\{m\} \cup (D(\alpha)\setminus\{m\}))}_{\substack{=\{m\}\cup D(\alpha) \\ \text{(since } (X\cup(Y\setminus X))=X\cup Y \\ \text{for any two sets } X \text{ and } Y)}} \cap [m+n-1] \\
&= (\{m\} \cup D(\alpha)) \cap [m+n-1]. \qquad (30)
\end{aligned}$$

Now, we recall that $m \in D(\alpha) \cup \{0, m+n\}$ (by assumption). Hence, $\{m\} \subseteq D(\alpha) \cup \{0, m+n\}$. Thus,

$$\begin{aligned}
\underbrace{\{m\}}_{\subseteq D(\alpha)\cup\{0,m+n\}} \cup D(\alpha) &\subseteq (D(\alpha) \cup \{0, m+n\}) \cup D(\alpha) \\
&= \underbrace{D(\alpha) \cup D(\alpha)}_{=D(\alpha)} \cup \{0, m+n\} \\
&= D(\alpha) \cup \{0, m+n\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\underbrace{(\{m\} \cup D(\alpha))}_{\subseteq D(\alpha)\cup\{0,m+n\}} \cap [m+n-1] \\
&\subseteq (D(\alpha) \cup \{0, m+n\}) \cap [m+n-1] \\
&= (D(\alpha) \cap [m+n-1]) \cup \underbrace{(\{0, m+n\} \cap [m+n-1])}_{\substack{=\varnothing \\ \text{(since neither } 0 \text{ nor } m+n \text{ belongs} \\ \text{to the set } [m+n-1])}} \\
&\qquad \left(\begin{array}{c} \text{since } (X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z) \\ \text{for any three sets } X,\ Y \text{ and } Z \end{array} \right) \\
&= (D(\alpha) \cap [m+n-1]) \cup \varnothing = D(\alpha) \cap [m+n-1] \\
&= D(\alpha) \qquad (\text{since } D(\alpha) \subseteq [m+n-1]).
\end{aligned}$$

Combining this inclusion with

$$\begin{aligned} D(\alpha) &= \underbrace{D(\alpha)}_{\subseteq(\{m\}\cup D(\alpha))} \cap [m+n-1] \qquad (\text{since } D(\alpha) \subseteq [m+n-1]) \\ &\subseteq (\{m\} \cup D(\alpha)) \cap [m+n-1], \end{aligned}$$

we obtain

$$(\{m\} \cup D(\alpha)) \cap [m+n-1] = D(\alpha).$$

Hence, we can rewrite (30) as $D(\varphi\psi) = D(\alpha)$.

Now, recall that the map $D : \operatorname{Comp}_{m+n} \to \mathcal{P}([m+n-1])$ is injective. Hence, if ζ and η are two elements of $\operatorname{Comp}_{m+n}$ satisfying $D(\zeta) = D(\eta)$, then $\zeta = \eta$. Applying this to $\zeta = \varphi\psi$ and $\eta = \alpha$, we obtain $\varphi\psi = \alpha$ (since $\varphi\psi \in \operatorname{Comp}_{m+n}$ and $\alpha \in \operatorname{Comp}_{m+n}$ and $D(\varphi\psi) = D(\alpha)$). This proves Proposition 5.11. □

5.5. Concatenation and coarsenings

We shall next study the interaction between concatenation and coarsenings. First, we define coarsenings:

Definition 5.12. If γ is a composition, then $C(\gamma)$ shall denote the set of all compositions $\beta \in \operatorname{Comp}_{|\gamma|}$ satisfying $D(\beta) \subseteq D(\gamma)$.

The compositions belonging to $C(\gamma)$ are often called the *coarsenings* of γ.

Example 5.13. Let γ be the composition $(4,1,2)$. Then, the set $C(\gamma)$ consists of the compositions $\beta \in \operatorname{Comp}_7$ satisfying $D(\beta) \subseteq D(\gamma) = \{4,5\}$. Thus,

$$C(\gamma) = \{(7),\ (5,2),\ (4,3),\ (4,1,2)\}.$$

So the coarsenings of γ are the four compositions (7), $(5,2)$, $(4,3)$ and $(4,1,2)$.

An equivalent definition of the coarsenings of a composition γ can be informally given as follows: If γ is a composition, then a *coarsening* of γ means a composition obtained by "combining" some groups of consecutive entries of γ (that is, replacing them by their sums). For instance, one of the many coarsenings of a composition $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7)$ is $(\alpha_1 + \alpha_2,\ \alpha_3,\ \alpha_4 + \alpha_5 + \alpha_6,\ \alpha_7)$. We shall not formalize this equivalent definition, as we will not use it.

The following lemma is a trivial consequence of the definition of a coarsening, restated for convenience:

Lemma 5.14. Let γ be a composition.

- **(a)** If $\nu \in C(\gamma)$, then $\nu \in \operatorname{Comp}$ and $|\nu| = |\gamma|$ and $D(\nu) \subseteq D(\gamma)$.
- **(b)** If $\nu \in \operatorname{Comp}$ is a composition that satisfies $|\nu| = |\gamma|$ and $D(\nu) \subseteq D(\gamma)$, then $\nu \in C(\gamma)$.

Proof. **(a)** Assume that $\nu \in C(\gamma)$. According to the definition of $C(\gamma)$, this means that ν is a composition $\beta \in \operatorname{Comp}_{|\gamma|}$ satisfying $D(\beta) \subseteq D(\gamma)$. In other words, $\nu \in \operatorname{Comp}_{|\gamma|}$ and $D(\nu) \subseteq D(\gamma)$. From $\nu \in \operatorname{Comp}_{|\gamma|}$, we obtain $\nu \in \operatorname{Comp}$ and $|\nu| = |\gamma|$. Thus, we have $\nu \in \operatorname{Comp}$ and $|\nu| = |\gamma|$ and $D(\nu) \subseteq D(\gamma)$. This proves Lemma 5.14 **(a)**.

(b) Assume that $\nu \in \operatorname{Comp}$ is a composition that satisfies $|\nu| = |\gamma|$ and $D(\nu) \subseteq D(\gamma)$. From $\nu \in \operatorname{Comp}$ and $|\nu| = |\gamma|$, we obtain $\nu \in \operatorname{Comp}_{|\gamma|}$. Thus, ν is a composition $\beta \in \operatorname{Comp}_{|\gamma|}$ satisfying $D(\beta) \subseteq D(\gamma)$ (since $D(\nu) \subseteq D(\gamma)$). In other words, $\nu \in C(\gamma)$ (by the definition of $C(\gamma)$). This proves Lemma 5.14 **(b)**. $\square$

We can now restate Proposition 5.8 in terms of coarsenings:

Proposition 5.15. Let β and γ be two compositions. Let $\mu \in C(\beta)$ and $\nu \in C(\gamma)$. Then, $\mu\nu \in C(\beta\gamma)$.

Proof of Proposition 5.15. We have $\nu \in C(\gamma)$. Thus, Lemma 5.14 **(a)** yields that $\nu \in \operatorname{Comp}$ and $|\nu| = |\gamma|$ and $D(\nu) \subseteq D(\gamma)$. The same argument (applied to μ and β instead of ν and γ) shows that $\mu \in \operatorname{Comp}$ and $|\mu| = |\beta|$ and $D(\mu) \subseteq D(\beta)$. Hence, Proposition 5.8 (applied to $\beta' = \mu$ and $\gamma' = \nu$) yields $D(\mu\nu) \subseteq D(\beta\gamma)$.

However, Proposition 5.2 **(b)** yields $|\beta\gamma| = |\beta| + |\gamma|$.

Furthermore, Proposition 5.2 **(b)** (applied to μ and ν instead of β and γ) yields

$$|\mu\nu| = \underbrace{|\mu|}_{=|\beta|} + \underbrace{|\nu|}_{=|\gamma|} = |\beta| + |\gamma| = |\beta\gamma| \qquad (\text{since } |\beta\gamma| = |\beta| + |\gamma|).$$

Thus, we now know that $\mu\nu \in \operatorname{Comp}$ and $|\mu\nu| = |\beta\gamma|$ and $D(\mu\nu) \subseteq D(\beta\gamma)$. Hence, Lemma 5.14 **(b)** (applied to $\beta\gamma$ and $\mu\nu$ instead of γ and ν) yields that $\mu\nu \in C(\beta\gamma)$. This proves Proposition 5.15. $\square$

The following proposition is a sort of converse to Proposition 5.15:

Proposition 5.16. Let α be a composition. Let μ and ν be two compositions satisfying $\mu\nu \in C(\alpha)$. Then, there exists a unique pair $(\beta, \gamma) \in \operatorname{Comp} \times \operatorname{Comp}$ of compositions satisfying $\beta\gamma = \alpha$ and $\mu \in C(\beta)$ and $\nu \in C(\gamma)$.

Proof of Proposition 5.16. Let $m = |\mu|$ and $n = |\nu|$. Then, $\mu \in \operatorname{Comp}_m$ (since μ is a composition that satisfies $|\mu| = m$) and $\nu \in \operatorname{Comp}_n$ (since ν is a composition that satisfies $|\nu| = n$). Also, from $\mu\nu \in C(\alpha)$, we conclude (by the definition of $C(\alpha)$) that $\mu\nu \in \operatorname{Comp}_{|\alpha|}$ and $D(\mu\nu) \subseteq D(\alpha)$. Now, from $\mu\nu \in \operatorname{Comp}_{|\alpha|}$, we obtain $|\mu\nu| = |\alpha|$. Thus, $|\alpha| = |\mu\nu|$.

On the other hand, Proposition 5.2 **(b)** (applied to $\beta = \mu$ and $\gamma = \nu$) yields

$$|\mu\nu| = \underbrace{|\mu|}_{=m} + \underbrace{|\nu|}_{=n} = m + n.$$

Hence, $|\alpha| = |\mu\nu| = m + n$, so that $\alpha \in \operatorname{Comp}_{m+n}$. Thus, $D(\alpha) \in \mathcal{P}([m+n-1])$ (since $D : \operatorname{Comp}_{m+n} \to \mathcal{P}([m+n-1])$ is a bijection). In other words, $D(\alpha) \subseteq [m+n-1]$.

It is furthermore easy to see that

$$m \in D(\alpha) \cup \{0, m+n\}$$

[11].

We have $D(\alpha) \cap [m-1] \subseteq [m-1]$, so that $D(\alpha) \cap [m-1] \in \mathcal{P}([m-1])$.

Furthermore, it is easy to see that $(D(\alpha) \setminus [m]) - m \in \mathcal{P}([n-1])$ [12].

Recall that the map $D : \operatorname{Comp}_m \to \mathcal{P}([m-1])$ is a bijection. Hence, it is bijective, thus surjective. Therefore, there exists some composition $\varphi \in \operatorname{Comp}_m$ that satisfies

$$D(\varphi) = D(\alpha) \cap [m-1] \tag{32}$$

[11]*Proof.* We are in one of the following three cases:

Case 1: We have $m = 0$.

Case 2: We have $n = 0$.

Case 3: We have neither $m = 0$ nor $n = 0$.

Let us first consider Case 1. In this case, we have $m = 0$. Hence, $m = 0 \in \{0, m+n\} \subseteq D(\alpha) \cup \{0, m+n\}$. Thus, $m \in D(\alpha) \cup \{0, m+n\}$ is proved in Case 1.

Let us next consider Case 2. In this case, we have $n = 0$. Hence, $m + \underbrace{n}_{=0} = m$, so that $m = m + n \in \{0, m+n\} \subseteq D(\alpha) \cup \{0, m+n\}$. Thus, $m \in D(\alpha) \cup \{0, m+n\}$ is proved in Case 2.

Now, let us consider Case 3. In this case, we have neither $m = 0$ nor $n = 0$. Hence, $m \neq 0$ and $n \neq 0$. Therefore, $\mu \neq \varnothing$ (since $|\mu| = m \neq 0 = |\varnothing|$) and $\nu \neq \varnothing$ (since $|\nu| = n \neq 0 = |\varnothing|$). Hence, Proposition 5.5 (applied to μ and ν instead of β and γ) yields

$$D(\mu\nu) = \{m\} \cup D(\mu) \cup (D(\nu) + m). \tag{31}$$

Now,

$$\begin{aligned} m &\in \{m\} \subseteq \{m\} \cup D(\mu) \cup (D(\nu) + m) \\ &= D(\mu\nu) \qquad \text{(by (31))} \\ &\subseteq D(\alpha) \subseteq D(\alpha) \cup \{0, m+n\}. \end{aligned}$$

Thus, $m \in D(\alpha) \cup \{0, m+n\}$ is proved in Case 3.

Hence, we have proved $m \in D(\alpha) \cup \{0, m+n\}$ in all three Cases 1, 2 and 3. Thus, $m \in D(\alpha) \cup \{0, m+n\}$ always holds.

[12]*Proof.* Let $g \in (D(\alpha) \setminus [m]) - m$. We shall show that $g \in [n-1]$.

Indeed,

$$g \in (D(\alpha) \setminus [m]) - m = \{k - m \mid k \in D(\alpha) \setminus [m]\}$$

(by the definition of $(D(\alpha) \setminus [m]) - m$). In other words, $g = k - m$ for some $k \in D(\alpha) \setminus [m]$. Consider this k.

We have $k \in D(\alpha) \setminus [m]$, so that $k \in D(\alpha)$ and $k \notin [m]$. From $k \in D(\alpha) \subseteq [m+n-1]$, we obtain $1 \leq k \leq m+n-1$. If we had $k \leq m$, then we would have $k \in [m]$ (since $1 \leq k \leq m$), which would contradict $k \notin [m]$. Thus, we cannot have $k \leq m$. Hence, we must have $k > m$. Thus, $k \geq m+1$ (since k and m are integers), so that $k - m \geq 1$. Furthermore, from $k \leq m+n-1$, we obtain $k - m \leq n-1$. Combining $k - m \geq 1$ with $k - m \leq n-1$, we find $k - m \in \{1, 2, \ldots, n-1\} = [n-1]$. Thus, $g = k - m \in [n-1]$.

Forget now that we fixed g. We thus have shown that $g \in [n-1]$ for each $g \in (D(\alpha) \setminus [m]) - m$. In other words, $(D(\alpha) \setminus [m]) - m \subseteq [n-1]$. In other words, $(D(\alpha) \setminus [m]) - m \in \mathcal{P}([n-1])$.

(since $D(\alpha) \cap [m-1] \in \mathcal{P}([m-1])$). Consider this φ.

Recall that the map $D : \operatorname{Comp}_n \to \mathcal{P}([n-1])$ is a bijection. Hence, it is bijective, thus surjective. Therefore, there exists some composition $\psi \in \operatorname{Comp}_n$ that satisfies

$$D(\psi) = (D(\alpha) \setminus [m]) - m \tag{33}$$

(since $(D(\alpha) \setminus [m]) - m \in \mathcal{P}([n-1])$). Consider this ψ.

Proposition 5.11 yields that

$$\varphi\psi = \alpha.$$

Also, $\varphi \in \operatorname{Comp}_m \subseteq \operatorname{Comp}$ and $|\varphi| = m$ (since $\varphi \in \operatorname{Comp}_m$). Furthermore, $\psi \in \operatorname{Comp}_n \subseteq \operatorname{Comp}$ and $|\psi| = n$ (since $\psi \in \operatorname{Comp}_n$). From $\varphi \in \operatorname{Comp}$ and $\psi \in \operatorname{Comp}$, we obtain $(\varphi, \psi) \in \operatorname{Comp} \times \operatorname{Comp}$.

Proposition 5.7 **(a)** (applied to $\beta = \mu$ and $\gamma = \nu$) yields

$$D(\mu) = \underbrace{D(\mu\nu)}_{\subseteq D(\alpha)} \cap [m-1] \subseteq D(\alpha) \cap [m-1] = D(\varphi)$$

(by (32)). Also, we have $|\mu| = m = |\varphi|$ (since $|\varphi| = m$), so that $\mu \in \operatorname{Comp}_{|\varphi|}$. Thus, μ is a composition $\beta \in \operatorname{Comp}_{|\varphi|}$ satisfying $D(\beta) \subseteq D(\varphi)$ (since we have shown that $D(\mu) \subseteq D(\varphi)$). In other words, $\mu \in C(\varphi)$ (by the definition of $C(\varphi)$).

Proposition 5.7 **(b)** (applied to $\beta = \mu$ and $\gamma = \nu$) yields

$$D(\nu) = (D(\mu\nu) \setminus [m]) - m. \tag{34}$$

However, $\underbrace{D(\mu\nu)}_{\subseteq D(\alpha)} \setminus [m] \subseteq D(\alpha) \setminus [m]$. But it is easy to see that if k is any integer, and if K and K' are two sets of integers satisfying $K \subseteq K'$, then $K - k \subseteq K' - k$. Applying this to $k = m$ and $K = D(\mu\nu) \setminus [m]$ and $K' = D(\alpha) \setminus [m]$, we conclude that $(D(\mu\nu) \setminus [m]) - m \subseteq (D(\alpha) \setminus [m]) - m$ (since $D(\mu\nu) \setminus [m] \subseteq D(\alpha) \setminus [m]$). In view of (34), we can rewrite this as

$$D(\nu) = (D(\alpha) \setminus [m]) - m = D(\psi)$$

(by (33)). Also, we have $|\nu| = n = |\psi|$ (since $|\psi| = n$), so that $\nu \in \operatorname{Comp}_{|\psi|}$. Thus, ν is a composition $\beta \in \operatorname{Comp}_{|\psi|}$ satisfying $D(\beta) \subseteq D(\psi)$ (since we have shown that $D(\nu) \subseteq D(\psi)$). In other words, $\nu \in C(\psi)$ (by the definition of $C(\psi)$).

We have now shown that $(\varphi, \psi) \in \operatorname{Comp} \times \operatorname{Comp}$ is a pair of compositions satisfying $\varphi\psi = \alpha$ and $\mu \in C(\varphi)$ and $\nu \in C(\psi)$. Hence, there exists **at least** one pair $(\beta, \gamma) \in \operatorname{Comp} \times \operatorname{Comp}$ of compositions satisfying $\beta\gamma = \alpha$ and $\mu \in C(\beta)$ and $\nu \in C(\gamma)$ (namely, the pair (φ, ψ)).

It remains to show that there exists **only** one such pair. So let us show this now.

Indeed, let (β', γ') and (β'', γ'') be two pairs $(\beta, \gamma) \in \operatorname{Comp} \times \operatorname{Comp}$ of compositions satisfying $\beta\gamma = \alpha$ and $\mu \in C(\beta)$ and $\nu \in C(\gamma)$. We must prove that $(\beta', \gamma') = (\beta'', \gamma'')$.

We know that (β', γ') is a pair $(\beta, \gamma) \in \text{Comp} \times \text{Comp}$ of compositions satisfying $\beta\gamma = \alpha$ and $\mu \in C(\beta)$ and $\nu \in C(\gamma)$. In other words, $(\beta', \gamma') \in \text{Comp} \times \text{Comp}$ is a pair of compositions satisfying $\beta'\gamma' = \alpha$ and $\mu \in C(\beta')$ and $\nu \in C(\gamma')$. From $\mu \in C(\beta')$, we easily obtain $|\beta'| = m$ [13].

We have now shown that $|\beta'| = m$ and $\beta'\gamma' = \alpha$. In other words, (β', γ') is a pair (β, γ) of compositions such that $|\beta| = m$ and $\beta\gamma = \alpha$. The same argument (applied to (β'', γ'') instead of (β', γ')) shows that (β'', γ'') is such a pair as well.

However, Proposition 5.9 shows that there exists at most one pair (β, γ) of compositions such that $|\beta| = m$ and $\beta\gamma = \alpha$. Hence, any two such pairs (β, γ) must be equal. Since (β', γ') and (β'', γ'') are two such pairs (as we have shown in the previous paragraph), we thus can conclude that (β', γ') and (β'', γ'') must be equal. In other words, $(\beta', \gamma') = (\beta'', \gamma'')$.

Now, forget that we fixed (β', γ') and (β'', γ''). We thus have shown that if (β', γ') and (β'', γ'') are two pairs $(\beta, \gamma) \in \text{Comp} \times \text{Comp}$ of compositions satisfying $\beta\gamma = \alpha$ and $\mu \in C(\beta)$ and $\nu \in C(\gamma)$, then $(\beta', \gamma') = (\beta'', \gamma'')$. In other words, any two pairs $(\beta, \gamma) \in \text{Comp} \times \text{Comp}$ of compositions satisfying $\beta\gamma = \alpha$ and $\mu \in C(\beta)$ and $\nu \in C(\gamma)$ must be equal. In other words, there exists **at most one** such pair (β, γ). Since we also know that there exists **at least one** such pair (β, γ) (because we have proved this further above), we thus conclude that there exists a **unique** such pair (β, γ). This proves Proposition 5.16. □

We can combine Propositions 5.15 and 5.16 into a convenient package:

Proposition 5.17. Let $(A, +, 0)$ be an abelian group. Let $u_{\mu,\nu}$ be an element of A for each pair $(\mu, \nu) \in \text{Comp} \times \text{Comp}$ of two compositions. Let $\alpha \in \text{Comp}$. Then,

$$\sum_{\substack{(\mu,\nu) \in \text{Comp} \times \text{Comp}; \\ \mu\nu \in C(\alpha)}} u_{\mu,\nu} = \sum_{\substack{(\beta,\gamma) \in \text{Comp} \times \text{Comp}; \\ \beta\gamma = \alpha}} \sum_{\mu \in C(\beta)} \sum_{\nu \in C(\gamma)} u_{\mu,\nu}.$$

[13] *Proof.* We have $\mu \in C(\beta')$. By the definition of $C(\beta')$, this means that μ is a composition $\beta \in \text{Comp}_{|\beta'|}$ satisfying $D(\beta) \subseteq D(\beta')$. In other words, $\mu \in \text{Comp}_{|\beta'|}$ and $D(\mu) \subseteq D(\beta')$. Hence, $\mu \in \text{Comp}_{|\beta'|}$, so that $|\mu| = |\beta'|$. Thus, $|\beta'| = |\mu| = m$.

Proof of Proposition 5.17. We have

$$\begin{aligned}
&\sum_{\substack{(\beta,\gamma)\in\operatorname{Comp}\times\operatorname{Comp};\\ \beta\gamma=\alpha}} \underbrace{\sum_{\mu\in C(\beta)}}_{\substack{=\sum_{\substack{\mu\in\operatorname{Comp};\\ \mu\in C(\beta)}}\\ \text{(since } C(\beta)\subseteq\operatorname{Comp}\text{)}}} \underbrace{\sum_{\nu\in C(\gamma)}}_{\substack{=\sum_{\substack{\nu\in\operatorname{Comp};\\ \nu\in C(\gamma)}}\\ \text{(since } C(\gamma)\subseteq\operatorname{Comp}\text{)}}} u_{\mu,\nu}\\
&= \underbrace{\sum_{\substack{(\beta,\gamma)\in\operatorname{Comp}\times\operatorname{Comp};\\ \beta\gamma=\alpha}} \sum_{\substack{\mu\in\operatorname{Comp};\\ \mu\in C(\beta)}} \sum_{\substack{\nu\in\operatorname{Comp};\\ \nu\in C(\gamma)}}}_{=\sum_{\mu\in\operatorname{Comp}}\sum_{\nu\in\operatorname{Comp}}\sum_{\substack{(\beta,\gamma)\in\operatorname{Comp}\times\operatorname{Comp};\\ \beta\gamma=\alpha;\\ \mu\in C(\beta);\\ \nu\in C(\gamma)}}} u_{\mu,\nu}\\
&= \underbrace{\sum_{\mu\in\operatorname{Comp}} \sum_{\nu\in\operatorname{Comp}}}_{=\sum_{(\mu,\nu)\in\operatorname{Comp}\times\operatorname{Comp}}} \sum_{\substack{(\beta,\gamma)\in\operatorname{Comp}\times\operatorname{Comp};\\ \beta\gamma=\alpha;\\ \mu\in C(\beta);\\ \nu\in C(\gamma)}} u_{\mu,\nu}\\
&= \sum_{(\mu,\nu)\in\operatorname{Comp}\times\operatorname{Comp}} \sum_{\substack{(\beta,\gamma)\in\operatorname{Comp}\times\operatorname{Comp};\\ \beta\gamma=\alpha;\\ \mu\in C(\beta);\\ \nu\in C(\gamma)}} u_{\mu,\nu}. \qquad (35)
\end{aligned}$$

Now, we claim the following:

Claim 1: Let $(\mu,\nu)\in\operatorname{Comp}\times\operatorname{Comp}$ be such that $\mu\nu\in C(\alpha)$. Then,

$$\sum_{\substack{(\beta,\gamma)\in\operatorname{Comp}\times\operatorname{Comp};\\ \beta\gamma=\alpha;\\ \mu\in C(\beta);\\ \nu\in C(\gamma)}} u_{\mu,\nu} = u_{\mu,\nu}.$$

[*Proof of Claim 1:* Proposition 5.16 shows that there exists a unique pair $(\beta,\gamma)\in\operatorname{Comp}\times\operatorname{Comp}$ of compositions satisfying $\beta\gamma=\alpha$ and $\mu\in C(\beta)$ and $\nu\in C(\gamma)$. In other words, the sum $\sum_{\substack{(\beta,\gamma)\in\operatorname{Comp}\times\operatorname{Comp};\\ \beta\gamma=\alpha;\\ \mu\in C(\beta);\\ \nu\in C(\gamma)}} u_{\mu,\nu}$ has exactly one addend. Hence, this sum equals $u_{\mu,\nu}$. This proves Claim 1.]

Claim 2: Let $(\mu,\nu)\in\operatorname{Comp}\times\operatorname{Comp}$ be such that $\mu\nu\notin C(\alpha)$. Then,

$$\sum_{\substack{(\beta,\gamma)\in\operatorname{Comp}\times\operatorname{Comp};\\ \beta\gamma=\alpha;\\ \mu\in C(\beta);\\ \nu\in C(\gamma)}} u_{\mu,\nu} = 0.$$

[*Proof of Claim 2:* If $(\beta,\gamma) \in \operatorname{Comp} \times \operatorname{Comp}$ is a pair of compositions satisfying $\beta\gamma = \alpha$ and $\mu \in C(\beta)$ and $\nu \in C(\gamma)$, then Proposition 5.15 shows that $\mu\nu \in C\left(\underbrace{\beta\gamma}_{=\alpha}\right) = C(\alpha)$, which contradicts $\mu\nu \notin C(\alpha)$. Hence, there exists no pair $(\beta,\gamma) \in \operatorname{Comp} \times \operatorname{Comp}$ of compositions satisfying $\beta\gamma = \alpha$ and $\mu \in C(\beta)$ and $\nu \in C(\gamma)$. In other words, the sum $\sum\limits_{\substack{(\beta,\gamma)\in\operatorname{Comp}\times\operatorname{Comp};\\ \beta\gamma=\alpha;\\ \mu\in C(\beta);\\ \nu\in C(\gamma)}} u_{\mu,\nu}$ is empty. Therefore, this sum equals 0. This proves Claim 2.]

Now, each pair $(\mu,\nu) \in \operatorname{Comp} \times \operatorname{Comp}$ satisfies either $\mu\nu \in C(\alpha)$ or $\mu\nu \notin C(\alpha)$ (but not both). Hence, we can split the outer sum on the right hand side of (35) as follows:

$$\begin{aligned}
&\sum_{(\mu,\nu)\in\operatorname{Comp}\times\operatorname{Comp}} \quad \sum_{\substack{(\beta,\gamma)\in\operatorname{Comp}\times\operatorname{Comp};\\ \beta\gamma=\alpha;\\ \mu\in C(\beta);\\ \nu\in C(\gamma)}} u_{\mu,\nu}\\
&= \sum_{\substack{(\mu,\nu)\in\operatorname{Comp}\times\operatorname{Comp};\\ \mu\nu\in C(\alpha)}} \underbrace{\sum_{\substack{(\beta,\gamma)\in\operatorname{Comp}\times\operatorname{Comp};\\ \beta\gamma=\alpha;\\ \mu\in C(\beta);\\ \nu\in C(\gamma)}} u_{\mu,\nu}}_{\substack{=u_{\mu,\nu}\\ \text{(by Claim 1)}}} + \sum_{\substack{(\mu,\nu)\in\operatorname{Comp}\times\operatorname{Comp};\\ \mu\nu\notin C(\alpha)}} \underbrace{\sum_{\substack{(\beta,\gamma)\in\operatorname{Comp}\times\operatorname{Comp};\\ \beta\gamma=\alpha;\\ \mu\in C(\beta);\\ \nu\in C(\gamma)}} u_{\mu,\nu}}_{\substack{=0\\ \text{(by Claim 2)}}}\\
&= \sum_{\substack{(\mu,\nu)\in\operatorname{Comp}\times\operatorname{Comp};\\ \mu\nu\in C(\alpha)}} u_{\mu,\nu} + \underbrace{\sum_{\substack{(\mu,\nu)\in\operatorname{Comp}\times\operatorname{Comp};\\ \mu\nu\notin C(\alpha)}} 0}_{=0} = \sum_{\substack{(\mu,\nu)\in\operatorname{Comp}\times\operatorname{Comp};\\ \mu\nu\in C(\alpha)}} u_{\mu,\nu}.
\end{aligned}$$

Hence, we can rewrite (35) as

$$\sum_{\substack{(\beta,\gamma)\in\operatorname{Comp}\times\operatorname{Comp};\\ \beta\gamma=\alpha}} \quad \sum_{\mu\in C(\beta)} \quad \sum_{\nu\in C(\gamma)} u_{\mu,\nu} = \sum_{\substack{(\mu,\nu)\in\operatorname{Comp}\times\operatorname{Comp};\\ \mu\nu\in C(\alpha)}} u_{\mu,\nu}.$$

This proves Proposition 5.17. □

References

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