On Injectivity of Coalgebra Morphisms

GPT-5.6 Sol, with prompting by Darij Grinberg

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Abstract

Abstract. We construct a surjective filtered morphism of connected graded coalgebras over $\mathbb{Z}$ that is injective on primitive elements but is not injective. As a consequence, we construct a nonzero coideal containing no nonzero primitive element. All tensor factors occurring in the coproduct of the source are direct summands, so the construction does not rely on any ambiguity in the notion of a subcoalgebra in the absence of flatness.

Manifest. This is a negative answer to https://mathoverflow.net/questions/140357, generated by GPT-5.6 Sol. – DG[1]

1 Statement of the counterexample

Let R be a commutative ring, and let C be a connected filtered R-coalgebra. We recall that "connected" means that the 0-th part of the filtration on C is isomorphic to R. The unique preimage of $1_R \in R$ under this isomorphism shall be denoted by 1_C or, for short, by 1. We write

$$\operatorname{Prim}(C) = \{x \in C \mid \Delta(x) = 1 \otimes x + x \otimes 1\}$$

for the R-module of primitive elements of C.

Recall that a *coideal* of C is an R-submodule $I \subseteq C$ satisfying

$$\varepsilon(I) = 0$$

and

$$\Delta(I) \subseteq \operatorname{im}(I \otimes_R C \longrightarrow C \otimes_R C) + \operatorname{im}(C \otimes_R I \longrightarrow C \otimes_R C).$$

(The latter relation is often written as $\Delta(I) \subseteq I \otimes_R C + C \otimes_R I$, understanding that the tensor products on the right hand side actually refer to their images in $C \otimes_R C$. We shall avoid this abuse of notation here.)

We first recall a well-known result for connected **graded** coalgebras (see, e.g., [GR26, Exercise 1.4.35]):

Proposition 1.1 (The graded case).

[1] This work is in the public domain.

(a) Let C be a connected graded R-coalgebra, and let I be a graded coideal of C. If

$$I \cap \operatorname{Prim}(C) = 0,$$

then $I = 0$.

(b) Let C and D be connected graded R-coalgebras, and let $f : C \twoheadrightarrow D$ be a surjective graded coalgebra morphism. If the restriction

$$f|_{\operatorname{Prim}(C)} : \operatorname{Prim}(C) \longrightarrow \operatorname{Prim}(D)$$

is injective, then f is injective.

Proof. (a) Assume, for contradiction, that $I \neq 0$. Since I is graded, there exists a smallest $n \geq 0$ such that the degree-n component I_n is nonzero. Consider this n, and choose a nonzero element $x \in I_n$. Since $\varepsilon(I) = 0$ and $C_0 = R1$, we have $I_0 = 0$, so that $n > 0$.

Let us spell out the grading on the tensor product $C \otimes_R C$. Since

$$C = \bigoplus_{k \geq 0} C_k,$$

the tensor product has the direct-sum decomposition

$$C \otimes_R C = \bigoplus_{i,j \geq 0} C_i \otimes_R C_j. \tag{1}$$

An element of $C_i \otimes_R C_j$ is said to have *bidegree* (i, j); its *total degree* is $i + j$. Thus, the degree-n component of $C \otimes_R C$ for the total grading is

$$(C \otimes_R C)_n = \bigoplus_{i+j=n} C_i \otimes_R C_j.$$

By definition, the coproduct of a graded coalgebra is homogeneous of degree 0 with respect to this total grading. In other words,

$$\Delta(C_n) \subseteq (C \otimes_R C)_n = \bigoplus_{i+j=n} C_i \otimes_R C_j.$$

Since $x \in C_n$, this gives

$$\Delta(x) \in \bigoplus_{i+j=n} C_i \otimes_R C_j. \tag{2}$$

Thus, all (i, j)-bidegree components of $\Delta(x)$ are zero except for those with $i + j = n$.

On the other hand, x belongs to the coideal I, so that $\Delta(x)$ belongs to

$$\operatorname{im}(I \otimes_R C \longrightarrow C \otimes_R C) + \operatorname{im}(C \otimes_R I \longrightarrow C \otimes_R C). \tag{3}$$

Since I is graded, we have

$$I = \bigoplus_{k \geq 0} I_k, \qquad \text{where } I_k = I \cap C_k.$$

Consequently, the two maps occurring in (3) respect the bigrading (1): the first sends $I_i \otimes_R C_j$ into $C_i \otimes_R C_j$, whereas the second sends $C_i \otimes_R I_j$ into $C_i \otimes_R C_j$. Hence, for any $i, j \geq 0$, the bidegree-(i, j) component of the module in (3) is contained in

$$\operatorname{im}(I_i \otimes_R C_j \longrightarrow C_i \otimes_R C_j) + \operatorname{im}(C_i \otimes_R I_j \longrightarrow C_i \otimes_R C_j).$$

If $i < n$ and $j < n$, then this component must therefore be zero (since the minimality of n yields $I_i = I_j = 0$), and therefore the bidegree-(i, j) component of $\Delta(x)$ must be 0 (since $\Delta(x)$ lies in this module). But we have previously also shown (using (2)) that all (i, j)-bidegree components of $\Delta(x)$ are zero except for those with $i + j = n$. Combining these two observations, we conclude that the only (i, j)-bidegree components of $\Delta(x)$ that can be nonzero are those that satisfy $i + j = n$ but not $i < n$ and $j < n$. In other words, they are the components with bidegrees $(n, 0)$ and $(0, n)$. Therefore,

$$\Delta(x) \in C_n \otimes_R C_0 + C_0 \otimes_R C_n.$$

Since $C_0 = R1$, this means that there exist $y, z \in C_n$ such that

$$\Delta(x) = y \otimes 1 + 1 \otimes z. \tag{4}$$

Because $n > 0$ and $y, z \in C_n$, we have $\varepsilon(y) = \varepsilon(z) = 0$. Applying $\operatorname{id} \otimes \varepsilon$ to the equality (4), we obtain $x = y\varepsilon(1) + 1\varepsilon(z) = y$, so that $y = x$; similarly, applying $\varepsilon \otimes \operatorname{id}$ to (4), we obtain $z = x$. Hence (4) rewrites as

$$\Delta(x) = x \otimes 1 + 1 \otimes x,$$

so $x \in \operatorname{Prim}(C)$. This contradicts $I \cap \operatorname{Prim}(C) = 0$. Thus $I = 0$.

(b) Set $I = \ker f$. Since f is graded, this submodule I is graded. Since f is surjective, right exactness of tensor products shows that

$$\ker(f \otimes f) = \operatorname{im}(I \otimes C \longrightarrow C \otimes C) + \operatorname{im}(C \otimes I \longrightarrow C \otimes C).$$

But f is a coalgebra morphism, so that $(f \otimes f) \circ \Delta = \Delta \circ f$, and thus $(f \otimes f)(\Delta(I)) = \Delta(f(I)) = \Delta(0) = 0$, so that

$$\Delta(I) \subseteq \ker(f \otimes f) = \operatorname{im}(I \otimes C \longrightarrow C \otimes C) + \operatorname{im}(C \otimes I \longrightarrow C \otimes C).$$

It is also easy to see that $\varepsilon(I) = 0$ (since $\varepsilon_C = \varepsilon_D \circ f$). Hence, I is a coideal. The injectivity of $f|_{\operatorname{Prim}(C)}$ yields $I \cap \operatorname{Prim}(C) = 0$. Part (a) therefore gives $I = 0$, so f is injective. □

It is natural to wonder whether the gradedness of f in the former fact can be replaced by a weaker property, such as filteredness (i.e., the image of the n-th degree component of C must be contained in the sum of the 0-th, 1-st, ..., n-th degree components of D). In this note, we shall give a counterexample showing that it cannot:

Theorem 1.2. *There exist connected graded $\mathbb{Z}$-coalgebras C and D and a surjective filtered*[2] *coalgebra morphism*

$$f : C \twoheadrightarrow D$$

[2] A linear map $f : C \to D$ between two graded modules C and D is said to be *filtered* if and only if each $n \geq 0$ satisfies $f(C_0 + C_1 + \cdots + C_n) \subseteq D_0 + D_1 + \cdots + D_n$, where M_k denotes the k-th degree component of a graded module M.

such that the restriction

$$f|_{\mathrm{Prim}(C)} : \mathrm{Prim}(C) \longrightarrow \mathrm{Prim}(D)$$

is injective, whereas f itself is not injective.

In particular, there exists a connected graded $\mathbb{Z}$-coalgebra C and a nonzero coideal $I \subseteq C$ such that

$$I \cap \mathrm{Prim}(C) = 0.$$

We shall construct these C, D, f and I in what follows. The construction rests on a simple module-theoretic phenomenon: an injective map can have zero tensor square when the modules are not flat.

2 The module-theoretic ingredient

Fix a prime number p, and set

$$a_n = 2^{n-1} \qquad \text{for every } n \geq 1.$$

Thus

$$a_{n+1} = 2a_n \qquad \text{for every } n \geq 1. \tag{5}$$

Define the two (isomorphic) $\mathbb{Z}$-modules

$$L = \bigoplus_{n\geq 1} (\mathbb{Z}/p^{a_n}\mathbb{Z})e_n \qquad \text{and} \qquad N = \bigoplus_{n\geq 1} (\mathbb{Z}/p^{a_n}\mathbb{Z})u_n.$$

Lemma 2.1. *There exist homomorphisms*

$$\alpha : L \longrightarrow N \qquad \textit{and} \qquad \beta : L \otimes_{\mathbb{Z}} L \longrightarrow N$$

with the following properties:

1. *the map α is injective;*
2. *the map $\alpha \otimes \alpha : L \otimes L \to N \otimes N$ is zero;*
3. *the map β is surjective.*

Proof. Define α on the distinguished generators by

$$\alpha(e_n) = p^{a_n} u_{n+1} \qquad \text{for every } n \geq 1. \tag{6}$$

The element u_{n+1} has order $p^{a_{n+1}} = p^{2a_n}$, by (5). Hence $p^{a_n}u_{n+1}$ has order p^{a_n}, just like e_n does. Thus the restriction of α to every direct summand $(\mathbb{Z}/p^{a_n}\mathbb{Z})e_n$ is injective. Since the images of distinct summands belong to distinct direct summands of N, the map α is injective.

We next prove that $\alpha \otimes \alpha = 0$. It is enough to evaluate this map on the tensors $e_m \otimes e_n$. Assume without loss of generality that $m \leq n$. Then

$$(\alpha \otimes \alpha)(e_m \otimes e_n) = p^{a_m + a_n} u_{m+1} \otimes u_{n+1}. \tag{7}$$

But it is well-known that $(\mathbb{Z}/u\mathbb{Z})\otimes_{\mathbb{Z}}(\mathbb{Z}/v\mathbb{Z}) \cong \mathbb{Z}/\gcd(u,v)\,\mathbb{Z}$ for any integers u and v. Thus, the tensor $u_{m+1}\otimes u_{n+1}$ has order $\gcd(p^{a_{m+1}}, p^{a_{n+1}}) = p^{\min(a_{m+1},a_{n+1})} = p^{a_{m+1}} = p^{2a_m}$. Since $a_n \geq a_m$, we have $a_m + a_n \geq 2a_m$. Therefore the right hand side of (7) is zero.

Finally, define the map $\beta : L\otimes_{\mathbb{Z}} L \longrightarrow N$ by

$$\beta(e_m\otimes e_n) = \begin{cases} u_n, & \text{if } m = n,\\ 0, & \text{if } m \neq n. \end{cases} \tag{8}$$

This is well-defined. Indeed, the (m,n)-summand of $L\otimes L$ is isomorphic to

$$(\mathbb{Z}/p^{a_m}\mathbb{Z})\otimes(\mathbb{Z}/p^{a_n}\mathbb{Z}) \cong \mathbb{Z}/p^{\min(a_m,a_n)}\mathbb{Z},$$

and when $m = n$, the prescribed image u_n has exactly order p^{a_n}. The map β is surjective because $u_n = \beta(e_n\otimes e_n)$ for every $n \geq 1$. $\square$

3 The coalgebras

We now use the maps α and β from Lemma 2.1. As a $\mathbb{Z}$-module, let

$$C = \mathbb{Z}1 \oplus L \oplus (L\otimes L). \tag{9}$$

To avoid confusing the last direct summand of C with the tensor product $C\otimes C$, we denote by

$$[x \mid y] \in C$$

the element of the last summand corresponding to $x\otimes y \in L\otimes L$.

Define a coproduct on C by

$$\begin{aligned} \Delta_C(1) &= 1\otimes 1, && && (10)\\ \Delta_C(x) &= 1\otimes x + x\otimes 1 && \text{for every } x\in L, && (11)\\ \Delta_C([x\mid y]) &= 1\otimes[x\mid y] + x\otimes y + [x\mid y]\otimes 1 && \text{for every } x,y\in L. && (12) \end{aligned}$$

Let the counit $\varepsilon_C : C\to\mathbb{Z}$ be the projection onto the first summand in (9).

The formulas above make C into a (coassociative, counital) coalgebra. Indeed, Δ_C is the well-known deconcatenation coproduct of the shuffle bialgebra on L, truncated after degree 2. Alternatively, coassociativity in degree 2 follows by expanding both $(\Delta_C\otimes\mathrm{id})\Delta_C([x\mid y])$ and $(\mathrm{id}\otimes\Delta_C)\Delta_C([x\mid y])$; both expansions are

$$1\otimes 1\otimes[x\mid y] + 1\otimes x\otimes y + x\otimes 1\otimes y + x\otimes y\otimes 1 + 1\otimes[x\mid y]\otimes 1 + [x\mid y]\otimes 1\otimes 1.$$

Give C the grading

$$C_0 = \mathbb{Z}1, \qquad C_1 = L, \qquad C_2 = L\otimes L$$

and $C_k = 0$ for all $k > 2$. Thus C is a connected graded coalgebra.

Let

$$D = \mathbb{Z}1\oplus N, \tag{13}$$

and make every element of N primitive:

$$\begin{aligned}
&\Delta_D(1) = 1 \otimes 1, \\
&\Delta_D(v) = 1 \otimes v + v \otimes 1 && \text{for every } v \in N.
\end{aligned}$$

The counit ε_D is the projection onto $\mathbb{Z}1$. We grade D by

$$D_0 = \mathbb{Z}1 \qquad \text{and} \qquad D_1 = N$$

and $D_k = 0$ for all $k > 1$. Thus, D is a connected graded coalgebra.

Define a $\mathbb{Z}$-linear map $f : C \to D$ by

$$f(1) = 1, \tag{14}$$

$$f(x) = \alpha(x) \qquad \text{for every } x \in L, \tag{15}$$

$$f([x \mid y]) = \beta(x \otimes y) \qquad \text{for every } x, y \in L. \tag{16}$$

Proposition 3.1. *The map $f : C \to D$ is a surjective filtered coalgebra morphism.*

Proof. The map preserves counits by construction. It also preserves coproducts on 1 and on L, since every element of $\alpha(L)$ is primitive in D. For $x, y \in L$, formulas (12) and (16) give

$$\begin{aligned}
(f \otimes f)\Delta_C([x \mid y]) &= 1 \otimes \beta(x \otimes y) + \alpha(x) \otimes \alpha(y) + \beta(x \otimes y) \otimes 1 \\
&= 1 \otimes \beta(x \otimes y) + \beta(x \otimes y) \otimes 1 \qquad (\text{since } \alpha \otimes \alpha = 0) \\
&= \Delta_D(\beta(x \otimes y)) \\
&= \Delta_D(f([x \mid y])).
\end{aligned}$$

Hence f is a coalgebra morphism.

The map f is filtered since $f(C_0) \subseteq D_0$, $f(C_1) \subseteq D_1$ and $f(C_2) \subseteq D_1$. Finally, f is surjective because its restriction to the summand $L \otimes L$ is the surjection $\beta : L \otimes L \twoheadrightarrow N$. $\square$

4 Primitive elements and failure of injectivity

Proposition 4.1. *We have*

$$\mathrm{Prim}(C) = L.$$

Consequently, the restriction

$$f|_{\mathrm{Prim}(C)} : \mathrm{Prim}(C) \longrightarrow \mathrm{Prim}(D)$$

is injective.

Proof. Every element of L is primitive by (11), so $L \subseteq \mathrm{Prim}(C)$.

Conversely, let $c \in \mathrm{Prim}(C)$. Since primitive elements belong to the kernel of the counit, we may write uniquely

$$c = x + z \qquad \text{with } x \in L \text{ and } z \in L \otimes L.$$

By (11) and (12), the difference $\Delta(c) - 1 \otimes c - c \otimes 1$ is the image of z under the natural map

$$L \otimes L \longrightarrow C \otimes C \tag{17}$$

that places the first factor in the degree-1 summand of the first copy of C and the second factor in the degree-1 summand of the second copy. The map (17) is injective: it is split by applying the projection $C \to L$ to both tensor factors. But $\Delta(c) - 1 \otimes c - c \otimes 1 = 0$, since c is primitive. Hence $z = 0$, and therefore $c = x \in L$. Thus $\mathrm{Prim}(C) = L$.

Under this identification, the restriction of f to $\mathrm{Prim}(C)$ is exactly $\alpha : L \to N$, which is injective by Lemma 2.1. □

Proposition 4.2. *The map f is not injective.*

Proof. For every $n \geq 1$, define

$$w_n = e_n - p^{a_n}[e_{n+1} \mid e_{n+1}] \in C. \tag{18}$$

Using (6) and (8), we obtain

$$f(w_n) = \alpha(e_n) - p^{a_n}\beta(e_{n+1} \otimes e_{n+1}) = p^{a_n}u_{n+1} - p^{a_n}u_{n+1} = 0.$$

On the other hand, $w_n \neq 0$, because its component in the direct summand L is $e_n \neq 0$. Thus f is not injective. □

Proof of Theorem 1.2. Propositions 3.1, 4.1, and 4.2 prove the first assertion.

Let $I = \ker f$. Since f is a surjective coalgebra morphism, I is a coideal. More explicitly, identifying D with C/I, right exactness of tensor products gives

$$\ker(f \otimes f) = \mathrm{im}(I \otimes C \longrightarrow C \otimes C) + \mathrm{im}(C \otimes I \longrightarrow C \otimes C).$$

For $x \in I$, we have

$$(f \otimes f)\Delta_C(x) = \Delta_D(f(x)) = 0,$$

so $\Delta_C(x)$ belongs to the displayed sum. Also $\varepsilon_C(x) = \varepsilon_D(f(x)) = 0$. Thus I is a coideal.

The coideal I is nonzero by Proposition 4.2. By Proposition 4.1, the restriction of f to $\mathrm{Prim}(C)$ is injective, so

$$I \cap \mathrm{Prim}(C) = 0.$$

Finally, the connected gradings of C and D were exhibited above. □

5 Remarks

Remark 5.1 (No ambiguity about subcoalgebras)**.** *The source is defined as the direct sum*

$$C = \mathbb{Z}1 \oplus L \oplus (L \otimes L).$$

In particular, the map $L \otimes L \to C \otimes C$ used in the reduced coproduct is split injective. Hence the construction does not exploit any ambiguity caused by the possible failure of $S \otimes S \to C \otimes C$ to be injective for an arbitrary submodule $S \subseteq C$.

References

[GR26] Darij Grinberg, Victor Reiner, *Hopf algebras in combinatorics*, arXiv:1409.8356v7.