Generalized Whitney formulas for broken circuits in ambigraphs and matroids*

Darij Grinberg

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Abstract. We explore several generalizations of Whitney's theorem – a classical formula for the chromatic polynomial of a graph. Following Stanley, we replace the chromatic polynomial by the chromatic symmetric function. Following Dohmen and Trinks, we exclude not all but only an (arbitrarily selected) set of broken circuits, or even weigh these broken circuits with weight monomials instead of excluding them. Following Crew and Spirkl, we put weights on the vertices of the graph. Following Gebhard and Sagan, we lift the chromatic symmetric function to noncommuting variables. In addition, we replace the graph by an "ambigraph", an apparently new concept that includes both hypergraphs and multigraphs as particular cases.

We show that Whitney's formula endures all these generalizations, and a fairly simple sign-reversing involution can be used to prove it in each setting. Furthermore, if we restrict ourselves to the chromatic polynomial, then the graph can be replaced by a matroid.

We discuss an application to transitive digraphs (i.e., posets), and reprove an alternating-sum identity by Dahlberg and van Willigenburg.

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[*] This article was formerly titled "A note on non-broken-circuit sets and the chromatic polynomial".

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The purpose of this paper is to demonstrate several generalizations of Whitney's Broken-Circuit theorem [Whitne32, §7] – a classical formula for the chromatic polynomial of a graph (V, E) as an alternating sum over subsets of E that contain no broken circuits. We shall generalize this formula in the following directions:

- Instead of summing over the sets that contain no broken circuits, we can sum over the sets that are "$\mathfrak{K}$-free" (i.e., contain no element of $\mathfrak{K}$ as a subset), where $\mathfrak{K}$ is some fixed set of broken circuits (in particular, $\mathfrak{K}$ can be $\varnothing$, yielding another well-known formula for the chromatic polynomial). In other words, instead of excluding all broken circuits, we can choose to exclude any given set of broken circuits.

  This generalization has already been proposed by Dohmen and Trinks in [DohTri14, §3.1]; however, we give a new and self-contained proof that does not rely on Whitney's original formula.

- Even more generally, instead of summing over $\mathfrak{K}$-free subsets, we can form a weighted sum over all subsets, where the weight depends on the broken circuits contained in the subset.

- We can replace the graph by an *ambigraph*: a more general notion in which the edges are replaced by packages of edges ("edgeries"), and a proper coloring has to leave at least one edge in each such package dichromatic (i.e., color its two endpoints differently). The concept of ambigraph generalizes both multigraphs and hypergraphs. We will discuss this concept in Sections 5 and 6.

- Analogous (and more general) results hold for Stanley's chromatic symmetric functions [Stanle95] along with two of their more recent variants: the weighted chromatic symmetric functions of Crew and Spirkl [CreSpi19] and the noncommutative chromatic symmetric functions of Gebhard and Sagan [GebSag01]. The latter variants will be studied (and generalized to ambigraphs) in Section 6.

- Analogous (and more general) results hold for matroids instead of graphs. These will be discussed in Section 8.

Note that, to my knowledge, the last two generalizations cannot be combined: Unlike graphs, matroids do not seem to have a well-defined notion of a chromatic symmetric function.

We will explore these generalizations in the work that follows. We shall also use them to prove an apparently new formula for the chromatic polynomial of a graph obtained from a transitive digraph by forgetting the orientations of the edges (Proposition 4.5). This latter formula was suggested to me as a conjecture by Alexander Postnikov, during a discussion on hyperplane arrangements on a space with a bilinear form; it is this formula which gave rise to this whole paper. The topic of hyperplane arrangements, however, will not be breached here.

As a further application, we will generalize and reprove an alternating-sum identity for chromatic polynomials found by Dahlberg and van Willigenburg (Section 7), as well as an analogous identity for characteristic polynomials of matroids (Subsection 8.4).

Acknowledgments

I thank Alexander Postnikov and Richard P. Stanley for discussions on hyperplane arrangements that led to the results described here.

1. Definitions and a main result

1.1. Graphs and colorings

We begin by recalling some basic features of finite graphs. Let us start with the definition of a graph that we shall be using:

Definition 1.1. **(a)** If V is any set, then $\mathcal{P}(V)$ will denote the powerset of V. This is the set of all subsets of V.

(b) If V is any set, then $\dbinom{V}{2}$ will denote the set of all 2-element subsets of V. In other words, if V is any set, then we set

$$\begin{aligned} \binom{V}{2} &= \{S \in \mathcal{P}(V) \mid |S| = 2\} \\ &= \{\{s,t\} \mid s \in V,\ t \in V,\ s \neq t\}. \end{aligned}$$

(c) A *graph* means a pair (V, E), where V is a set, and where E is a subset of $\dbinom{V}{2}$. A graph (V, E) is said to be *finite* if the set V is finite. If $G = (V, E)$ is a graph, then the elements of V are called the *vertices* of the graph G, while the elements of E are called the *edges* of the graph G. If e is an edge of a graph G, then the two elements of e are called the *endpoints* of the edge e. If $e = \{s,t\}$ is an edge of a graph G, then we say that the edge e *connects the vertices* s *and* t of G.

Comparing our definition of a graph with some of the other definitions used in the literature, we thus observe that our graphs are undirected (i.e., their edges are sets, not pairs), loopless (i.e., the two endpoints of an edge must always be distinct), edge-unlabelled (i.e., their edges are just 2-element sets of vertices, rather than objects with “their own identity”), and do not have multiple edges (or, more precisely, there is no notion of several edges connecting two vertices, since the edges form a set, nor a multiset, and do not have labels). Such graphs are commonly known as *simple graphs*.

Definition 1.2. Let $G = (V, E)$ be a graph. Let X be a set.

(a) An *X-coloring* of G is defined to mean a map $V \to X$.

(b) An X-coloring f of G is said to be *proper* if every edge $\{s, t\} \in E$ satisfies $f(s) \neq f(t)$.

If f is an X-coloring of a graph $G = (V, E)$, then the value $f(v)$ for a given vertex $v \in V$ is called the *color* of this vertex v under the coloring f. We shall not use this terminology here, but we are mentioning it since it allows for a rather intuitive mental model and explains the word "coloring". An X-coloring of G is then proper if and only if each edge of G has two endpoints of different colors.

1.2. Symmetric functions

We shall now briefly introduce the notion of symmetric functions. We shall not use any nontrivial results about symmetric functions; we will merely need some notations.[1]

In the following, $\mathbb{N}$ means the set $\{0, 1, 2, \ldots\}$. Also, $\mathbb{N}_+$ shall mean the set $\{1, 2, 3, \ldots\}$.

A *partition* will mean a sequence $(\lambda_1, \lambda_2, \lambda_3, \ldots) \in \mathbb{N}^\infty$ of nonnegative integers such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$ and such that all sufficiently high integers $i \geq 1$ satisfy $\lambda_i = 0$. If $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ is a partition, and if a positive integer n is such that all integers $i \geq n$ satisfy $\lambda_i = 0$, then we shall identify the partition λ with the finite sequence $(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$. Thus, for example, the sequences $(3, 1)$ and $(3, 1, 0)$ and the partition $(3, 1, 0, 0, 0, \ldots)$ are all identified. Every weakly decreasing finite list of positive integers thus is identified with a unique partition.

Let $\mathbf{k}$ be a commutative ring with unity. We shall keep $\mathbf{k}$ fixed throughout the paper. The reader will not be missing out on anything if she assumes that $\mathbf{k} = \mathbb{Z}$.

We consider the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of (commutative) power series in countably many distinct indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. It is a topological $\mathbf{k}$-algebra[2]. A power series $P \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *bounded-degree* if

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[1] For an introduction to symmetric functions, see any of [Stanle99, Chapter 7], [Martin22, Chapter 9] and [GriRei14, Chapter 2] (and a variety of other texts).

[2] See [GriRei14, Section 2.6] or [Grinbe16, §2] for the definition of its topology. This topology makes sure that a sequence $(P_n)_{n \in \mathbb{N}}$ of power series converges to some power series P if and only if, for every monomial $\mathfrak{m}$, all sufficiently high $n \in \mathbb{N}$ satisfy

$$(\text{the } \mathfrak{m}\text{-coefficient of } P_n) = (\text{the } \mathfrak{m}\text{-coefficient of } P)$$

(where the meaning of "sufficiently high" can depend on the $\mathfrak{m}$).

there exists an $N \in \mathbb{N}$ such that every monomial of degree $> N$ appears with coefficient 0 in P. A power series $P \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *symmetric* if and only if P is invariant under any permutation of the indeterminates. We let Λ be the subset of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ consisting of all symmetric bounded-degree power series $P \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$. This subset Λ is a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$, and is called the *$\mathbf{k}$-algebra of symmetric functions* over $\mathbf{k}$.

We shall now define the few families of symmetric functions that we will be concerned with in this work. The first are the *power-sum symmetric functions*:

Definition 1.3. Let n be a positive integer. We define a power series $p_n \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$p_n = x_1^n + x_2^n + x_3^n + \cdots = \sum_{j \geq 1} x_j^n. \tag{1}$$

This power series p_n lies in Λ, and is called the *n-th power-sum symmetric function*.

We also set $p_0 = 1 \in \Lambda$. Thus, p_n is defined not only for all positive integers n, but also for all $n \in \mathbb{N}$.

Definition 1.4. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ be a partition. We define a power series $p_\lambda \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$p_\lambda = \prod_{i \geq 1} p_{\lambda_i}.$$

This is well-defined, because the infinite product $\prod\limits_{i \geq 1} p_{\lambda_i}$ converges (indeed, all but finitely many of its factors are 1 (because every sufficiently high integer i satisfies $\lambda_i = 0$ and thus $p_{\lambda_i} = p_0 = 1$)).

We notice that every partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ (written as a finite list of nonnegative integers) satisfies

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}. \tag{2}$$

1.3. Chromatic symmetric functions

The next symmetric functions we introduce are the actual subject of this work; they are the *chromatic symmetric functions* and have been introduced by Stanley in [Stanle95, Definition 2.1]:

Definition 1.5. Let $G = (V, E)$ be a finite graph.

(a) For every $\mathbb{N}_+$-coloring $f : V \to \mathbb{N}_+$ of G, we let $\mathbf{x}_f$ denote the monomial $\prod\limits_{v \in V} x_{f(v)}$ in the indeterminates $x_1, x_2, x_3, \ldots$.

(b) We define a power series $X_G \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$X_G = \sum_{\substack{f : V \to \mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_f .$$

This power series X_G is called the *chromatic symmetric function* of G.

We have $X_G \in \Lambda$ for every finite graph $G = (V, E)$; this will follow from Theorem 1.11 further below (but is also rather obvious).

We remark that X_G is denoted by $\Psi[G]$ in [GriRei14, §7.3.3].

1.4. Connected components

We shall now recall the notion of connected components of a graph. This notion is a particular case of the notion of a quotient set by an equivalence relation; thus, we shall briefly recall the latter first.

A *binary relation* on a set X means a subset of $X \times X$. If $\sim$ is a binary relation on a set X, and if x and y are two elements of X, then one says that *x is related to y by $\sim$* if and only if (x, y) belongs to the set $\sim$. An *equivalence relation* is a binary relation (on a set) that is reflexive, symmetric and transitive.

Definition 1.6. Let X be a set. Let $\sim$ be an equivalence relation on X. We shall write $\sim$ infix (that is, for any $x \in X$ and $y \in X$, we shall abbreviate "$(x, y) \in \sim$" as "$x \sim y$"). For every $x \in X$, we let $[x]_\sim$ be the subset $\{y \in X \mid x \sim y\}$ of X; this subset $[x]_\sim$ is called the *$\sim$-equivalence class* of x (or the *equivalence class of x with respect to the relation $\sim$*). A *$\sim$-equivalence class* means a subset of X which is the $\sim$-equivalence class of some $x \in X$. It is well-known that any two $\sim$-equivalence classes are either identical or disjoint, and that X is the union of all $\sim$-equivalence classes.

We let $X/(\sim)$ denote the set of all $\sim$-equivalence classes. This set $X/(\sim)$ is called the *quotient of the set X by the equivalence relation $\sim$*. We define a map $\pi_X : X \to X/(\sim)$ by setting

$$(\pi_X(x) = [x]_\sim \qquad \text{for every } x \in X).$$

Thus, the map π_X sends every $x \in X$ to its $\sim$-equivalence class.

Let us introduce a few more notations:

Definition 1.7. Let X and Y be two sets.

(a) Then, Y^X denotes the set of all maps $X \to Y$.

(b) Let $\sim$ be any binary relation on X. We shall write $\sim$ infix (that is, for any $x \in X$ and $y \in X$, we shall abbreviate "$(x, y) \in \sim$" as "$x \sim y$"). Then, we let $Y^X_\sim$ denote the subset

$$\left\{ g \in Y^X \ \mid \ g(x) = g(y) \text{ for any } x \in X \text{ and } y \in X \text{ satisfying } x \sim y \right\}$$

of Y^X.

The map π_X defined in Definition 1.6 has an important universal property:

Proposition 1.8. Let X be a set. Let $\sim$ be an equivalence relation on X.

(a) The map $\pi_X : X \to X/(\sim)$ is surjective and belongs to $(X/(\sim))^X_\sim$.

(b) Let Y be any set. Then, the map

$$Y^{X/(\sim)} \to Y^X_\sim, \qquad f \mapsto f \circ \pi_X$$

is a bijection.

Proof of Proposition 1.8. **(a)** The map π_X is surjective[3].

The definition of $(X/(\sim))^X_\sim$ shows that

$$\begin{aligned} &(X/(\sim))^X_\sim \\ &= \left\{ g \in (X/(\sim))^X \ \mid \ g(x) = g(y) \text{ for any } x \in X \text{ and } y \in X \text{ satisfying } x \sim y \right\}. \end{aligned} \tag{3}$$

Now, we claim that

$$\pi_X(x) = \pi_X(y) \qquad \text{for any } x \in X \text{ and } y \in X \text{ satisfying } x \sim y. \tag{4}$$

Proof of (4): Let $x \in X$ and $y \in X$ be any two elements satisfying $x \sim y$. We shall show that $\pi_X(x) = \pi_X(y)$.

Recall that $\pi_X(x) = [x]_\sim$. Thus, $\pi_X(x)$ is the $\sim$-equivalence class of x (since $[x]_\sim$ is the $\sim$-equivalence class of x). The same argument (applied to y instead of x) shows that $\pi_X(y)$ is the $\sim$-equivalence class of y.

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[3] *Proof.* Let $u \in X/(\sim)$. Thus, u is a $\sim$-equivalence class (since $X/(\sim)$ is the set of all $\sim$-equivalence classes). In other words, there exists some $x \in X$ such that u is the $\sim$-equivalence class of x (by the definition of a "$\sim$-equivalence class"). Consider this x.

We know that $[x]_\sim$ is the $\sim$-equivalence class of x. In other words, $[x]_\sim$ is u (since u is the $\sim$-equivalence class of x). Thus, $[x]_\sim = u$. Now, the definition of π_X yields $\pi_X(x) = [x]_\sim = u$. Thus, $u = \pi_X\left(\underbrace{x}_{\in X}\right) \in \pi_X(X)$.

Now, let us forget that we fixed u. We thus have shown that $u \in \pi_X(X)$ for every $u \in X/(\sim)$. In other words, $X/(\sim) \subseteq \pi_X(X)$. In other words, the map π_X is surjective. Qed.

We have $x \sim y$. Thus, the elements x and y of X lie in the same $\sim$-equivalence class. In other words, the $\sim$-equivalence class of x is the $\sim$-equivalence class of y. In other words, $\pi_X(x)$ is $\pi_X(y)$ (since $\pi_X(x)$ is the $\sim$-equivalence class of x, and since $\pi_X(y)$ is the $\sim$-equivalence class of y). In other words, $\pi_X(x) = \pi_X(y)$. This proves (4).

Now, π_X is a map $g \in (X/(\sim))^X$ which satisfies $g(x) = g(y)$ for any $x \in X$ and $y \in X$ satisfying $x \sim y$ (according to (4)). In other words,

$$\begin{aligned}\pi_X &\in \left\{ g \in (X/(\sim))^X \mid g(x) = g(y) \text{ for any } x \in X \text{ and } y \in X \text{ satisfying } x \sim y \right\} \\ &= (X/(\sim))^X_\sim \qquad \text{(by (3))}.\end{aligned}$$

This completes the proof of Proposition 1.8 **(a)**.

(b) We have

$$f \circ \pi_X \in Y^X_\sim \qquad \text{for every } f \in Y^{X/(\sim)} \tag{5}$$

[4]. Hence, we can define a map $\Phi : Y^{X/(\sim)} \to Y^X_\sim$ by

$$\left(\Phi(f) = f \circ \pi_X \qquad \text{for every } f \in Y^{X/(\sim)} \right). \tag{6}$$

Consider this map Φ. We shall now show that the map Φ is a bijection.

The map Φ is injective[5]. We shall now show that the map Φ is surjective.

Indeed, let $h \in Y^X_\sim$.

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[4] *Proof of (5):* Let $f \in Y^{X/(\sim)}$. Then, clearly, $f \circ \pi_X \in Y^X$ (since $\pi_X \in (X/(\sim))^X$). Now, let $x \in X$ and $y \in X$ be any two elements satisfying $x \sim y$. We shall show that $(f \circ \pi_X)(x) = (f \circ \pi_X)(y)$.

From (4), we obtain $\pi_X(x) = \pi_X(y)$. Now, $(f \circ \pi_X)(x) = f\left(\underbrace{\pi_X(x)}_{=\pi_X(y)}\right) = f(\pi_X(y)) = (f \circ \pi_X)(y)$.

Now, let us forget that we fixed x and y. We thus have shown that $(f \circ \pi_X)(x) = (f \circ \pi_X)(y)$ for any $x \in X$ and $y \in X$ satisfying $x \sim y$. Thus, $f \circ \pi_X$ is a map $g \in Y^X$ which satisfies $g(x) = g(y)$ for any $x \in X$ and $y \in X$ satisfying $x \sim y$. In other words,

$$\begin{aligned}f \circ \pi_X &\in \left\{ g \in Y^X \mid g(x) = g(y) \text{ for any } x \in X \text{ and } y \in X \text{ satisfying } x \sim y \right\} \\ &= Y^X_\sim\end{aligned}$$

(because $Y^X_\sim$ was defined to be $\{g \in Y^X \mid g(x) = g(y) \text{ for any } x \in X \text{ and } y \in X \text{ satisfying } x \sim y\}$). This proves (5).

[5] *Proof.* Let f and g be two elements of $Y^{X/(\sim)}$ such that $\Phi(f) = \Phi(g)$. We shall show that $f = g$.

Let $u \in X/(\sim)$. Thus, $u \in X/(\sim) = \pi_X(X)$ (since the map π_X is surjective). In other words, there exists some $x \in X$ such that $u = \pi_X(x)$. Consider this x.

Now, the definition of Φ yields $\Phi(f) = f \circ \pi_X$. Hence, $\left(\underbrace{\Phi(f)}_{=f\circ\pi_X}\right)(x) = (f \circ \pi_X)(x) =$

We shall now define a map $f \in Y^{X/(\sim)}$ as follows: Let $u \in X/(\sim)$. Thus, $u \in X/(\sim) = \pi_X(X)$ (since the map π_X is surjective). Thus, there exists some $x \in X$ satisfying $u = \pi_X(x)$. Pick such an x. Then, $h(x)$ is independent on the choice of x [6]. Hence, we can set $f(u) = h(x)$.

Thus, we have defined a map $f \in Y^{X/(\sim)}$. This map has the following property: If $u \in X/(\sim)$, and if $x \in X$ is such that $u = \pi_X(x)$, then

$$f(u) = h(x) \tag{8}$$

(because this is how $f(u)$ was defined).

Now, $f \circ \pi_X = h$ [7]. The definition of Φ now yields $\Phi(f) = f \circ \pi_X = h$. Hence, $h = \Phi\left(\underbrace{f}_{\in Y^{X/(\sim)}}\right) \in \Phi\left(Y^{X/(\sim)}\right)$.

$f\left(\underbrace{\pi_X(x)}_{=u}\right) = f(u)$. The same argument (but applied to g instead of f) yields $(\Phi(g))(x) = g(u)$. Now, $f(u) = \left(\underbrace{\Phi(f)}_{=\Phi(g)}\right)(x) = (\Phi(g))(x) = g(u)$.

Let us now forget that we fixed u. We thus have shown that $f(u) = g(u)$ for every $u \in X/(\sim)$. In other words, $f = g$.

Now, let us forget that we fixed f and g. We thus have proven that if f and g are two elements of $Y^{X/(\sim)}$ such that $\Phi(f) = \Phi(g)$, then $f = g$. In other words, the map Φ is injective. Qed.

[6] *Proof.* Let x_1 and x_2 be any two $x \in X$ satisfying $u = \pi_X(x)$. We shall show that $h(x_1) = h(x_2)$.

Indeed, x_1 is an $x \in X$ satisfying $u = \pi_X(x)$. In other words, x_1 is an element of X and satisfies $u = \pi_X(x_1)$. But the definition of π_X shows that $\pi_X(x_1) = [x_1]_\sim$. Thus, $u = \pi_X(x_1) = [x_1]_\sim$. In other words, u is $[x_1]_\sim$. In other words, u is the $\sim$-equivalence class of x_1 (since $[x_1]_\sim$ is the $\sim$-equivalence class of x_1 (by the definition of $[x_1]_\sim$)).

The same argument (applied to x_2 instead of x_1) shows that u is the $\sim$-equivalence class of x_2. Thus, the $\sim$-equivalence classes of x_1 and x_2 are both u. Hence, these two $\sim$-equivalence classes are identical. In other words, x_1 and x_2 belong to the same $\sim$-equivalence class. In other words, $x_1 \sim x_2$.

But

$$h \in Y^X_\sim = \left\{ g \in Y^X \mid g(x) = g(y) \text{ for any } x \in X \text{ and } y \in X \text{ satisfying } x \sim y \right\}$$

(by the definition of $Y^X_\sim$). In other words, h is a $g \in Y^X$ which satisfies $g(x) = g(y)$ for any $x \in X$ and $y \in X$ satisfying $x \sim y$. In other words, h is an element of Y^X and satisfies

$$h(x) = h(y) \text{ for any } x \in X \text{ and } y \in X \text{ satisfying } x \sim y. \tag{7}$$

Now, we can apply (7) to $x = x_1$ and $y = x_2$ (since $x_1 \sim x_2$). As a result, we obtain $h(x_1) = h(x_2)$.

Let us now forget that we fixed x_1 and x_2. We thus have shown that if x_1 and x_2 are any two $x \in X$ satisfying $u = \pi_X(x)$, then $h(x_1) = h(x_2)$. In other words, $h(x)$ is independent on the choice of x (when x is chosen as an element of X satisfying $u = \pi_X(x)$). Qed.

[7] *Proof.* Every $x \in X$ satisfies $(f \circ \pi_X)(x) = f(\pi_X(x)) = h(x)$ (by (8), applied to $u = \pi_X(x)$). In other words, $f \circ \pi_X = h$, qed.

Now, let us forget that we fixed h. We thus have proven that $h \in \Phi\left(Y^{X/(\sim)}\right)$ for every $h \in Y^X_\sim$. In other words, the map Φ is surjective.

Thus, we know that the map Φ is injective and surjective. In other words, Φ is bijective. In other words, Φ is a bijection. Since Φ is the map

$$Y^{X/(\sim)} \to Y^X_\sim, \qquad f \mapsto f \circ \pi_X$$

(because Φ is a map $Y^{X/(\sim)} \to Y^X_\sim$ and satisfies (6)), this shows that the map

$$Y^{X/(\sim)} \to Y^X_\sim, \qquad f \mapsto f \circ \pi_X$$

is a bijection. This proves Proposition 1.8 **(b)**. □

We shall now recall what connected components are:

Definition 1.9. Let $G = (V, E)$ be a finite graph. Let u and v be two elements of V (that is, two vertices of G). A *walk* from u to v in G will mean a sequence $(w_0, w_1, \ldots, w_k)$ of elements of V such that $w_0 = u$ and $w_k = v$ and

$$(\{w_i, w_{i+1}\} \in E \qquad \text{for every } i \in \{0, 1, \ldots, k-1\}).$$

We say that u and v are *connected (in G)* if there exists a walk from u to v in G.

Definition 1.10. Let $G = (V, E)$ be a graph.

(a) We define a binary relation $\sim_G$ (written infix) on the set V as follows: Given $u \in V$ and $v \in V$, we set $u \sim_G v$ if and only if u and v are connected (in G). It is well-known that this relation $\sim_G$ is an equivalence relation. The $\sim_G$-equivalence classes are called the *connected components* of G.

(b) Assume that the graph G is finite. We let $\lambda(G)$ denote the list of the sizes of all connected components of G, in weakly decreasing order. (Each connected component should contribute only one entry to the list.) We view $\lambda(G)$ as a partition (since $\lambda(G)$ is a weakly decreasing finite list of positive integers).

Now, we can state a formula for chromatic symmetric functions:

Theorem 1.11. Let $G = (V, E)$ be a finite graph. Then,

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda(V,F)}.$$

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\lambda(V, F)$ is understood according to Definition 1.10 **(b)**.)

This theorem is not new; it appears, e.g., in [Stanle95, Theorem 2.5]. We shall show a far-reaching generalization of it (Theorem 1.15) soon.

1.5. Circuits and broken circuits

Let us now define the notions of cycles and circuits of a graph:

Definition 1.12. Let $G = (V, E)$ be a graph. A *cycle* of G denotes a list $(v_1, v_2, \ldots, v_{m+1})$ of elements of V with the following properties:

- We have $m > 2$.
- We have $v_{m+1} = v_1$.
- The vertices $v_1, v_2, \ldots, v_m$ are pairwise distinct.
- We have $\{v_i, v_{i+1}\} \in E$ for every $i \in \{1, 2, \ldots, m\}$.

If $(v_1, v_2, \ldots, v_{m+1})$ is a cycle of G, then the set $\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_{m+1}\}\}$ is called a *circuit* of G.

For instance, if $(1, 3, 5, 7, 1)$ is a cycle of a graph G, then the corresponding circuit is $\{\{1, 3\}, \{3, 5\}, \{5, 7\}, \{7, 1\}\}$.

Definition 1.13. Let $G = (V, E)$ be a graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. We shall refer to ℓ as the *labeling function*. For every edge e of G, we shall refer to $\ell(e)$ as the *label* of e.

A *broken circuit* of G means a subset of E having the form $C \setminus \{e\}$, where C is a circuit of G, and where e is the unique edge in C having maximum label (among the edges in C). Of course, the notion of a broken circuit of G depends on the function ℓ; however, we suppress the mention of ℓ in our notation, since we will not consider situations where two different ℓ's coexist.

Thus, if G is a graph with a labeling function ℓ, then any circuit C of G gives rise to a broken circuit provided that among the edges in C, only one attains the maximum label. (If more than one of the edges of C attains the maximum label, then C does not give rise to a broken circuit.) Notice that two different circuits may give rise to one and the same broken circuit.

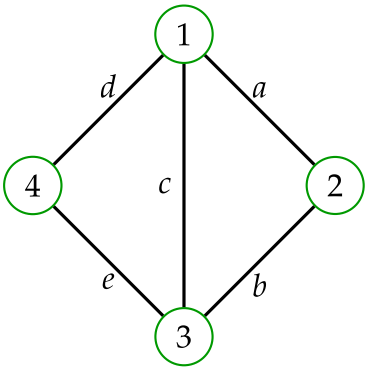
For instance, if $\{a, b, c, d\}$ is a circuit of a graph G such that $\ell(a) \leq \ell(b) \leq \ell(c) < \ell(d)$, then it gives rise to the broken circuit $\{a, b, c\}$, since its unique edge having maximum label is d. On the other hand, a circuit of the form $\{a, b, c, d\}$ with $\ell(a) \leq \ell(b) \leq \ell(c) = \ell(d)$ (and $c \neq d$) does not give rise to any broken circuit, since its edge with maximum label is not unique.

The notion of a broken circuit always depends on a labeling function $\ell : E \to X$. Any time we speak about broken circuits, we shall tacitly understand that the function $\ell : E \to X$ is used as the labeling function.

Example 1.14. Let G be the graph (V, E), where $V = \{1, 2, 3, 4\}$ and $E = \{a, b, c, d, e\}$ with

$$a = \{1, 2\}, \qquad b = \{2, 3\}, \qquad c = \{1, 3\}, \qquad d = \{1, 4\}, \qquad e = \{3, 4\}.$$

According to the standard conventions of graph theory, this graph G can be drawn as follows:



.

Let $\ell : E \to X$ be a labeling function satisfying $\ell(a) < \ell(b) < \ell(c) < \ell(d) < \ell(e)$. Then, the circuits of G are

$$\{a, b, c\}, \qquad \{a, b, d, e\}, \qquad \{c, d, e\}.$$

The broken circuits of G are therefore

$$\{a, b\}, \qquad \{a, b, d\}, \qquad \{c, d\}.$$

1.6. The main results

We are now ready to state one of our main results:

Theorem 1.15. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) p_{\lambda(V,F)}.$$

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\lambda(V, F)$ is understood according to Definition 1.10 **(b)**.)

Before we come to the proof of this result, let us explore some of its particular cases. First, a definition is in order:

Definition 1.16. Let E be a set. Let $\mathfrak{K}$ be a subset of the powerset of E (that is, a set of subsets of E). A subset F of E is said to be *$\mathfrak{K}$-free* if F contains no $K \in \mathfrak{K}$ as a subset. (For instance, if $\mathfrak{K} = \varnothing$, then every subset F of E is $\mathfrak{K}$-free.)

Here is a slightly more substantial example: If $E = \{1,2,3,4\}$ and $\mathfrak{K} = \{\{1,2\}, \{2,3\}\}$, then the subset $\{1,3\}$ of E is $\mathfrak{K}$-free whereas the subset $\{2,3,4\}$ is not (since it contains $\{2,3\} \in \mathfrak{K}$ as a subset).

Corollary 1.17. Let $G = (V,E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$X_G = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda(V,F)}.$$

Corollary 1.18. Let $G = (V,E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Then,

$$X_G = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} p_{\lambda(V,F)}.$$

Corollary 1.18 appears in [Stanle95, Theorem 2.9], at least in the particular case in which ℓ is supposed to be injective.

Example 1.19. Let $G = (V,E)$ be the graph from Example 1.14, and let $\ell : E \to X$ be a labeling function as in Example 1.14. Then, the subsets of E that contain no broken circuits of G as subsets are the 18 sets

$$\begin{aligned} &\varnothing, \quad \{a\}, \quad \{b\}, \quad \{c\}, \quad \{d\}, \quad \{e\}, \quad \{a,c\}, \\ &\{a,d\}, \quad \{a,e\}, \quad \{b,c\}, \quad \{b,d\}, \quad \{b,e\}, \quad \{c,e\}, \\ &\{d,e\}, \quad \{a,c,e\}, \quad \{a,d,e\}, \quad \{b,c,e\}, \quad \{b,d,e\}. \end{aligned}$$

Thus, the sum on the right hand side of Corollary 1.18 has 18 addends. In contrast, the sums on the right hand sides of Theorem 1.15 and of Theorem 1.11 have 32 addends. The number of addends in the sum on the right hand side of Corollary 1.17 depends on the choice of $\mathfrak{K}$.

Let us now see how Theorem 1.11, Corollary 1.17 and Corollary 1.18 can be derived from Theorem 1.15:

Proof of Corollary 1.17 using Theorem 1.15. We can apply Theorem 1.15 to 0 instead of a_K. As a result, we obtain

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} 0 \right) p_{\lambda(V,F)}. \tag{9}$$

Now, if F is any subset of E, then

$$\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0 = \begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} \tag{10}$$

[8].

---

[8] *Proof of (10):* Let F be any subset of E. Recall that the set F is $\mathfrak{K}$-free if and only if F contains no $K\in\mathfrak{K}$ as a subset (by the definition of "$\mathfrak{K}$-free"). Taking the contrapositive of this equivalence statement, we obtain the following: The set F is **not** $\mathfrak{K}$-free if and only if F contains some $K\in\mathfrak{K}$ as a subset.

We are in one of the following two cases:

Case 1: The set F is $\mathfrak{K}$-free.

Case 2: The set F is not $\mathfrak{K}$-free.

Let us first consider Case 1. In this case, the set F is $\mathfrak{K}$-free. In other words, F contains no $K\in\mathfrak{K}$ as a subset (because the set F is $\mathfrak{K}$-free if and only if F contains no $K\in\mathfrak{K}$ as a subset (by the definition of "$\mathfrak{K}$-free")). In other words, there exists no $K\in\mathfrak{K}$ such that F contains K as a subset. In other words, there exists no $K\in\mathfrak{K}$ such that $K\subseteq F$. Hence, the product $\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0$ is empty. Thus, $\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0 = (\text{empty product}) = 1$. Comparing this with

$$\begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} = 1 \qquad (\text{since } F \text{ is } \mathfrak{K}\text{-free}),$$

we obtain $\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0 = \begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases}$. Thus, (10) is proven in Case 1.

Let us now consider Case 2. In this case, the set F is **not** $\mathfrak{K}$-free. In other words, F contains some $K\in\mathfrak{K}$ as a subset (since the set F is **not** $\mathfrak{K}$-free if and only if F contains some $K\in\mathfrak{K}$ as a subset). Let L be such a K. Thus, L is an element of $\mathfrak{K}$, and the set F contains L as a subset.

Now, $L\subseteq F$ (since F contains L as a subset). Hence, the product $\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0$ has at least one factor (namely, the factor for $K=L$). This factor is clearly 0. Therefore, the whole product $\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0$ equals 0 (because if a product contains a factor which is 0, then the whole product must equal 0). In other words, $\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0 = 0$. Comparing this with

$$\begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} = 0 \qquad (\text{since } F \text{ is } \textbf{not } \mathfrak{K}\text{-free}),$$

we obtain $\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0 = \begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases}$. Thus, (10) is proven in Case 2.

We now have proven (10) in each of the two Cases 1 and 2. Since these two Cases cover all possibilities, this shows that (10) always holds.

Thus, (9) becomes

$$
\begin{aligned}
X_G &= \sum_{F \subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq F}} 0 \right)}_{\substack{= \begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} \\ \text{(by (10))}}} p_{\lambda(V,F)} \\
&= \sum_{F \subseteq E} (-1)^{|F|} \begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} p_{\lambda(V,F)} \\
&= \sum_{\substack{F \subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} \underbrace{\begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases}}_{\substack{=1 \\ \text{(since } F \text{ is } \mathfrak{K}\text{-free)}}} p_{\lambda(V,F)} \\
&\qquad + \sum_{\substack{F \subseteq E;\\ F \text{ is not } \mathfrak{K}\text{-free}}} (-1)^{|F|} \underbrace{\begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases}}_{\substack{=0 \\ \text{(since } F \text{ is not } \mathfrak{K}\text{-free)}}} p_{\lambda(V,F)} \\
&\qquad\qquad (\text{since each subset } F \text{ of } E \text{ either is } \mathfrak{K}\text{-free or is not}) \\
&= \sum_{\substack{F \subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda(V,F)} + \underbrace{\sum_{\substack{F \subseteq E;\\ F \text{ is not } \mathfrak{K}\text{-free}}} (-1)^{|F|} 0 p_{\lambda(V,F)}}_{=0} \\
&= \sum_{\substack{F \subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda(V,F)}.
\end{aligned}
$$

This proves Corollary 1.17. ☐

Proof of Corollary 1.18 using Corollary 1.17. Let $\mathfrak{K}$ be the set of all broken circuits of G. Thus, the elements of $\mathfrak{K}$ are the broken circuits of G.

Now, for every subset F of E, we have the following equivalence of statements:

$$
\begin{aligned}
& (F \text{ is } \mathfrak{K}\text{-free}) \\
\Longleftrightarrow\ & (F \text{ contains no } K \in \mathfrak{K} \text{ as a subset}) \\
& \qquad \left(\begin{array}{c} \text{because } F \text{ is } \mathfrak{K}\text{-free if and only if } F \text{ contains no } K \in \mathfrak{K} \text{ as a subset} \\ \text{(by the definition of ``}\mathfrak{K}\text{-free'')} \end{array} \right) \\
\Longleftrightarrow\ & (F \text{ contains no element of } \mathfrak{K} \text{ as a subset}) \\
\Longleftrightarrow\ & (F \text{ contains no broken circuit of } G \text{ as a subset})
\end{aligned}
$$

(since the elements of $\mathfrak{K}$ are the broken circuits of G). Hence,

$$\sum_{\substack{F\subseteq E;\\ F\text{ is }\mathfrak{K}\text{-free}}} = \sum_{\substack{F\subseteq E;\\ F\text{ contains no broken}\\ \text{circuit of }G\text{ as a subset}}}$$

(an equality between summation signs). Now, Corollary 1.17 yields

$$X_G = \underbrace{\sum_{\substack{F\subseteq E;\\ F\text{ is }\mathfrak{K}\text{-free}}}}_{=\sum_{\substack{F\subseteq E;\\ F\text{ contains no broken}\\ \text{circuit of }G\text{ as a subset}}}} (-1)^{|F|} p_{\lambda(V,F)} = \sum_{\substack{F\subseteq E;\\ F\text{ contains no broken}\\ \text{circuit of }G\text{ as a subset}}} (-1)^{|F|} p_{\lambda(V,F)}.$$

This proves Corollary 1.18. ☐

Proof of Theorem 1.11 using Theorem 1.15. Let X be the totally ordered set $\{1\}$ (equipped with the only possible order on this set). Let $\ell : E \to X$ be the function sending each $e \in E$ to $1 \in X$. Let $\mathfrak{K}$ be the empty set. Clearly, $\mathfrak{K}$ is a set of broken circuits of G. Theorem 1.15 (applied to 0 instead of a_K) yields

$$\begin{aligned} X_G &= \sum_{F\subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0\right)}_{\substack{=(\text{empty product})\\ (\text{since }\mathfrak{K}\text{ is the empty set})}} p_{\lambda(V,F)} \\ &= \sum_{F\subseteq E} (-1)^{|F|} \underbrace{(\text{empty product})}_{=1} p_{\lambda(V,F)} = \sum_{F\subseteq E} (-1)^{|F|} p_{\lambda(V,F)}. \end{aligned}$$

This proves Theorem 1.11. ☐

2. Proof of Theorem 1.15

We shall now prepare for the proof of Theorem 1.15 with some notations and some lemmas. Our proof will imitate [BlaSag86, proof of Whitney's theorem]. We note that Theorem 1.15 can also be easily obtained as a consequence of [DohTri14, §2 and §3.1] using Theorem 1.11, but our proof has the advantage of not relying on Theorem 1.11 (so that it leads to a new proof of Theorem 1.11).

2.1. Eqs f and basic lemmas

We introduce a simple notion that measures "how non-proper" a given coloring of a graph is:[9]

---

[9] If V is a set, then V^2 denotes the Cartesian product $V \times V$, that is, the set of all ordered pairs of elements of V.

Definition 2.1. Let V and X be two sets. Let $f : V \to X$ be a map. We let $\operatorname{Eqs} f$ denote the subset

$$\left\{ \{s,t\} \mid (s,t) \in V^2,\ s \neq t \text{ and } f(s) = f(t) \right\}$$

of $\binom{V}{2}$. (This is well-defined, because any two elements s and t of V satisfying $s \neq t$ clearly satisfy $\{s,t\} \in \binom{V}{2}$.)

Example 2.2. Let $V = \{1,2,3,4,5\}$ and $X = \{1,2,3\}$, and let $f : V \to X$ be the map that sends the three numbers $1,2,3$ to 1 and the remaining two numbers $4,5$ to 2. Then,

$$\operatorname{Eqs} f = \{\{1,2\},\ \{1,3\},\ \{2,3\},\ \{4,5\}\}.$$

We shall now state some first properties of this notion:

Lemma 2.3. Let $G = (V,E)$ be a graph. Let X be a set. Let $f : V \to X$ be a map. Then, the X-coloring f of G is proper if and only if $E \cap \operatorname{Eqs} f = \varnothing$.

Proof of Lemma 2.3. The definition of $\operatorname{Eqs} f$ shows that

$$\begin{aligned} \operatorname{Eqs} f &= \left\{ \{s,t\} \mid (s,t) \in V^2,\ s \neq t \text{ and } f(s) = f(t) \right\} \\ &= \left\{ \{x,y\} \mid (x,y) \in V^2,\ x \neq y \text{ and } f(x) = f(y) \right\} \qquad (11) \end{aligned}$$

(here, we renamed the index (s,t) as (x,y)).

We shall first prove the logical implication

$$(\text{the } X\text{-coloring } f \text{ of } G \text{ is proper}) \implies (E \cap \operatorname{Eqs} f = \varnothing). \qquad (12)$$

Proof of (12): Assume that the X-coloring f of G is proper. We must show that $E \cap \operatorname{Eqs} f = \varnothing$.

Recall that the X-coloring f of G is proper if and only if every edge $\{s,t\} \in E$ satisfies $f(s) \neq f(t)$ (by the definition of "proper"). Thus,

$$\text{every edge } \{s,t\} \in E \text{ satisfies } f(s) \neq f(t) \qquad (13)$$

(since the X-coloring f of G is proper).

Now, let $e \in E \cap \operatorname{Eqs} f$. Thus,

$$e \in E \cap \operatorname{Eqs} f \subseteq \operatorname{Eqs} f = \left\{ \{s,t\} \mid (s,t) \in V^2,\ s \neq t \text{ and } f(s) = f(t) \right\}$$

(by the definition of $\operatorname{Eqs} f$). In other words, $e = \{s,t\}$ for some $(s,t) \in V^2$ satisfying $s \neq t$ and $f(s) = f(t)$. Consider this (s,t). We have $\{s,t\} = e \in E \cap \operatorname{Eqs} f \subseteq E$ and therefore $f(s) \neq f(t)$ (by (13)). This contradicts $f(s) = f(t)$.

Now, let us forget that we fixed e. We thus have found a contradiction for every $e \in E \cap \operatorname{Eqs} f$. Therefore, no $e \in E \cap \operatorname{Eqs} f$ exists. In other words, the set $E \cap \operatorname{Eqs} f$ is empty. In other words, $E \cap \operatorname{Eqs} f = \varnothing$. Thus, the implication (12) is proven.

Now, we shall prove the implication

$$(E \cap \operatorname{Eqs} f = \varnothing) \implies (\text{the } X\text{-coloring } f \text{ of } G \text{ is proper}). \tag{14}$$

Proof of (14): Assume that $E \cap \operatorname{Eqs} f = \varnothing$. We have to show that the X-coloring f of G is proper.

Let $\{s,t\} \in E$ be an edge. We shall now show that $f(s) \neq f(t)$.

Indeed, assume the contrary. Thus, $f(s) = f(t)$. Now, $\{s,t\} \in E \subseteq \dbinom{V}{2}$. In other words, $\{s,t\}$ is a 2-element subset of V (since $\dbinom{V}{2}$ is the set of all 2-element subsets of V). Thus, $|\{s,t\}| = 2$, so that $s \neq t$. Also, $\{s,t\}$ is a subset of V; thus, $s \in V$ and $t \in V$. Hence, $(s,t) \in V^2$. So we know that $\{s,t\}$ has the form $\{x,y\}$ for some $(x,y) \in V^2$ satisfying $x \neq y$ and $f(x) = f(y)$ (namely, this (x,y) is (s,t)). In other words,

$$\{s,t\} \in \left\{ \{x,y\} \mid (x,y) \in V^2,\ x \neq y \text{ and } f(x) = f(y) \right\} = \operatorname{Eqs} f$$

(by (11)). Combining this with $\{s,t\} \in E$, we obtain $\{s,t\} \in E \cap \operatorname{Eqs} f = \varnothing$. Thus, the set $\varnothing$ has an element (namely, $\{s,t\}$). This contradicts the fact that the set $\varnothing$ is empty. Thus, we have obtained a contradiction. This shows that our assumption was wrong. Hence, $f(s) \neq f(t)$ is proven.

Let us now forget that we fixed $\{s,t\}$. We thus have shown that every edge $\{s,t\} \in E$ satisfies $f(s) \neq f(t)$. Therefore, the X-coloring f of G is proper (since the X-coloring f of G is proper if and only if every edge $\{s,t\} \in E$ satisfies $f(s) \neq f(t)$ (by the definition of "proper")). This proves the implication (14).

Now we have proven the two implications (12) and (14). Combining these two implications, we obtain the equivalence

$$(\text{the } X\text{-coloring } f \text{ of } G \text{ is proper}) \iff (E \cap \operatorname{Eqs} f = \varnothing).$$

This proves Lemma 2.3. □

Lemma 2.4. Let $G = (V,E)$ be a graph. Let X be a set. Let $f : V \to X$ be a map. Let C be a circuit of G. Let $e \in C$ be such that $C \setminus \{e\} \subseteq \operatorname{Eqs} f$. Then, $e \in E \cap \operatorname{Eqs} f$.

Proof of Lemma 2.4. The set C is a circuit of G. In other words, the set C has the form $\{\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_m,v_{m+1}\}\}$, where $(v_1,v_2,\ldots,v_{m+1})$ is a cycle of G

(by the definition of a "circuit"). Consider this cycle $(v_1, v_2, \ldots, v_{m+1})$. We thus have

$$C = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_{m+1}\}\}. \tag{15}$$

The list $(v_1, v_2, \ldots, v_{m+1})$ is a cycle of G. According to the definition of a "cycle", this means that this list is a list of elements of V satisfying the following four properties:

- We have $m > 2$.
- We have $v_{m+1} = v_1$.
- The vertices $v_1, v_2, \ldots, v_m$ are pairwise distinct.
- We have $\{v_i, v_{i+1}\} \in E$ for every $i \in \{1, 2, \ldots, m\}$.

Thus, $(v_1, v_2, \ldots, v_{m+1})$ is a list of elements of V satisfying the four properties that we have just mentioned. In particular, $v_{m+1} = v_1$. Also,

$$\{v_i, v_{i+1}\} \in E \qquad \text{for every } i \in \{1, 2, \ldots, m\}. \tag{16}$$

Any two distinct elements p and q of $\{1, 2, \ldots, m\}$ satisfy

$$\{v_p, v_{p+1}\} \neq \{v_q, v_{q+1}\} \tag{17}$$

[10].

---

[10] *Proof of (17):* Let p and q be two distinct elements of $\{1, 2, \ldots, m\}$. We must prove (17).

Assume the contrary. Thus, $\{v_p, v_{p+1}\} = \{v_q, v_{q+1}\}$.

The vertices $v_1, v_2, \ldots, v_m$ are pairwise distinct. In other words, any two distinct elements a and b of $\{1, 2, \ldots, m\}$ satisfy $v_a \neq v_b$. Applying this to $a = p$ and $b = q$, we obtain $v_p \neq v_q$. Combining $v_p \in \{v_p, v_{p+1}\} = \{v_q, v_{q+1}\}$ with $v_p \neq v_q$, we obtain $v_p \in \{v_q, v_{q+1}\} \setminus \{v_q\} \subseteq \{v_{q+1}\}$. Thus, $v_p = v_{q+1}$. The same argument (with p and q replaced by q and p) yields $v_q = v_{p+1}$.

We have $p \neq q$ (since p and q are distinct). Thus, we can WLOG assume that $p < q$ (since otherwise, we can simply switch p with q). Assume this. From $q \in \{1, 2, \ldots, m\}$, we obtain $q \leq m$, so that $p < q \leq m$. Since p and m are integers, this shows that $p \leq m - 1$. Thus, $p + 1 \leq m$. Hence, $p + 1 \in \{1, 2, \ldots, m\}$.

Recall again that any two distinct elements a and b of $\{1, 2, \ldots, m\}$ satisfy $v_a \neq v_b$. In other words, if two elements a and b of $\{1, 2, \ldots, m\}$ satisfy $v_a = v_b$, then $a = b$. Applying this to $a = q$ and $b = p + 1$, we obtain $q = p + 1$ (since $q \in \{1, 2, \ldots, m\}$ and $p + 1 \in \{1, 2, \ldots, m\}$ and $v_q = v_{p+1}$). The same argument (but with the roles of p and q switched) shows that $p = q + 1$ if $q + 1 \in \{1, 2, \ldots, m\}$. Since $p = q + 1$ is impossible (because $q + 1 > q = p + 1 > p$), we thus conclude that $q + 1 \in \{1, 2, \ldots, m\}$ is impossible as well. Thus, we have $q + 1 \notin \{1, 2, \ldots, m\}$. In other words, $q \notin \{0, 1, \ldots, m - 1\}$.

Combining $q \in \{1, 2, \ldots, m\}$ with $q \notin \{0, 1, \ldots, m - 1\}$, we find $q \in \{1, 2, \ldots, m\} \setminus \{0, 1, \ldots, m - 1\} = \{m\}$. In other words, $q = m$. Comparing this with $q = p + 1$, we obtain $p + 1 = m$, so that $p = m - 1$. Hence, $v_p = v_{m-1}$, so that $v_{m-1} = v_p = v_{q+1} = v_{m+1}$ (since $q = m$). Therefore, $v_{m-1} = v_{m+1} = v_1$.

However, $m > 2$, so that $m - 1 > 1$ and thus $m - 1 \neq 1$. In other words, $m - 1$ and 1 are distinct. Recall again that any two distinct elements a and b of $\{1, 2, \ldots, m\}$ satisfy $v_a \neq v_b$. Applying this to $a = m - 1$ and $b = 1$, we obtain $v_{m-1} \neq v_1$ (since $m - 1$ and 1 are distinct). This contradicts $v_{m-1} = v_1$.

We thus have found a contradiction. This contradiction proves that our assumption was wrong. Hence, (17) is proven.

We have $e \in C = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_{m+1}\}\}$. Thus, $e = \{v_i, v_{i+1}\}$ for some $i \in \{1, 2, \ldots, m\}$. Consider this i.

Now, we have

$$f(v_j) = f(v_{j+1}) \qquad \text{for every } j \in \{1, 2, \ldots, m\} \setminus \{i\} \tag{18}$$

[11]. Hence,

$$f(v_1) = f(v_i) \tag{19}$$

[12]. Also,

$$f(v_{i+1}) = f(v_{m+1}) \tag{20}$$

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[11] *Proof of (18):* Let $j \in \{1, 2, \ldots, m\} \setminus \{i\}$. Thus, $j \in \{1, 2, \ldots, m\}$ and $j \notin \{i\}$. From $j \notin \{i\}$, we obtain $j \neq i$. Therefore, the two elements j and i of $\{1, 2, \ldots, m\}$ are distinct. Thus, (17) (applied to $p = j$ and $q = i$) shows that $\{v_j, v_{j+1}\} \neq \{v_i, v_{i+1}\} = e$. But from $j \in \{1, 2, \ldots, m\}$, we obtain

$$\begin{aligned}\{v_j, v_{j+1}\} &\in \{\{v_k, v_{k+1}\} \mid k \in \{1, 2, \ldots, m\}\} \\ &= \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_{m+1}\}\} = C\end{aligned}$$

(by (15)). Combining this with $\{v_j, v_{j+1}\} \neq e$, we obtain

$$\begin{aligned}\{v_j, v_{j+1}\} &\in C \setminus \{e\} \subseteq \operatorname{Eqs} f \\ &= \left\{\{s, t\} \mid (s, t) \in V^2,\ s \neq t \text{ and } f(s) = f(t)\right\}.\end{aligned}$$

In other words, $\{v_j, v_{j+1}\}$ has the form $\{s, t\}$ for some $(s, t) \in V^2$ satisfying $s \neq t$ and $f(s) = f(t)$. Consider this (s, t). Thus, $\{v_j, v_{j+1}\} = \{s, t\}$.

We have $f(s) = f(t)$. Therefore, set $g = f(s) = f(t)$. We have

$$f(\{s, t\}) = \left\{\underbrace{f(s)}_{=g}, \underbrace{f(t)}_{=g}\right\} = \{g, g\} = \{g\}.$$

Now, $v_j \in \{v_j, v_{j+1}\} = \{s, t\}$, and thus $f\left(\underbrace{v_j}_{\in\{s,t\}}\right) \in f(\{s, t\}) = \{g\}$. In other words, $f(v_j) = g$. Also, $v_{j+1} \in \{v_j, v_{j+1}\} = \{s, t\}$, and thus $f\left(\underbrace{v_{j+1}}_{\in\{s,t\}}\right) \in f(\{s, t\}) = \{g\}$. In other words, $f(v_{j+1}) = g$. Comparing this with $f(v_j) = g$, we obtain $f(v_j) = f(v_{j+1})$. This proves (18).

[12] *Proof of (19):* Let $j \in \{1, 2, \ldots, i-1\}$. Thus, $j \in \{1, 2, \ldots, i-1\} \subseteq \{1, 2, \ldots, m\}$. Combining this with $j \neq i$ (since $j < i$ (since $j \in \{1, 2, \ldots, i-1\}$)), we obtain $j \in \{1, 2, \ldots, m\} \setminus \{i\}$. Hence, $f(v_j) = f(v_{j+1})$ (by (18)).

Now, let us forget that we fixed j. We thus have proven that $f(v_j) = f(v_{j+1})$ for every $j \in \{1, 2, \ldots, i-1\}$. In other words, $f(v_1) = f(v_2) = \cdots = f(v_i)$. Hence, $f(v_1) = f(v_i)$. This proves (19).

[13]. Now, (19) yields

$$f(v_i) = f\left(\underbrace{v_1}_{=v_{m+1}}\right) = f(v_{m+1}) = f(v_{i+1}) \qquad (\text{by (20)}).$$

Moreover, v_i and v_{i+1} are elements of V (since $(v_1, v_2, \ldots, v_{m+1})$ is a list of elements of V). In other words, $v_i \in V$ and $v_{i+1} \in V$. Hence, $(v_i, v_{i+1}) \in V^2$.

Furthermore, $v_i \neq v_{i+1}$ [14].

Now, the definition of $\operatorname{Eqs} f$ shows that

$$\operatorname{Eqs} f = \left\{\{s,t\} \mid (s,t) \in V^2,\ s \neq t \text{ and } f(s) = f(t)\right\}. \tag{21}$$

However, we have $(v_i, v_{i+1}) \in V^2$, $v_i \neq v_{i+1}$ and $f(v_i) = f(v_{i+1})$. Hence, the set $\{v_i, v_{i+1}\}$ has the form $\{s,t\}$ for some $(s,t) \in V^2$ satisfying $s \neq t$ and $f(s) = f(t)$ (namely, for $(s,t) = (v_i, v_{i+1})$). Thus,

$$\{v_i, v_{i+1}\} \in \left\{\{s,t\} \mid (s,t) \in V^2,\ s \neq t \text{ and } f(s) = f(t)\right\} = \operatorname{Eqs} f$$

(by (21)). Thus, $e = \{v_i, v_{i+1}\} \in \operatorname{Eqs} f$.

But $e = \{v_i, v_{i+1}\} \in E$ (by (16)). Combining this with $e \in \operatorname{Eqs} f$, we obtain $e \in E \cap \operatorname{Eqs} f$. This proves Lemma 2.4. □

Lemma 2.5. Let (V, B) be a finite graph. Let $\sim$ denote the equivalence relation $\sim_{(V,B)}$ (defined as in Definition 1.10 **(a)**).

Let Y be a set. A set $Y^V_\sim$ is defined (according to Definition 1.7 **(b)**). Let $f : V \to Y$ be any map. Then, we have the following logical equivalence of statements:

$$(B \subseteq \operatorname{Eqs} f) \iff \left(f \in Y^V_\sim\right).$$

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[13] *Proof of (20):* Let $j \in \{i+1, i+2, \ldots, m\}$. Thus, $j \in \{i+1, i+2, \ldots, m\} \subseteq \{1, 2, \ldots, m\}$. Combining this with $j \neq i$ (since $j > i$ (since $j \in \{i+1, i+2, \ldots, m\}$)), we obtain $j \in \{1, 2, \ldots, m\} \setminus \{i\}$. Hence, $f(v_j) = f(v_{j+1})$ (by (18)).

Now, let us forget that we fixed j. We thus have proven that $f(v_j) = f(v_{j+1})$ for every $j \in \{i+1, i+2, \ldots, m\}$. In other words, $f(v_{i+1}) = f(v_{i+2}) = \cdots = f(v_{m+1})$. Hence, $f(v_{i+1}) = f(v_{m+1})$. This proves (20).

[14] *Proof.* Assume the contrary. Thus, $v_i = v_{i+1}$.

Let us first assume (for the sake of contradiction) that $i = m$. Thus, $v_i = v_m$. Also, from $i = m$, we obtain $v_{i+1} = v_{m+1} = v_1$. Hence, $v_m = v_i = v_{i+1} = v_1$.

The vertices $v_1, v_2, \ldots, v_m$ are pairwise distinct. In other words, any two distinct elements a and b of $\{1, 2, \ldots, m\}$ satisfy $v_a \neq v_b$. Applying this to $a = m$ and $b = 1$, we obtain $v_m \neq v_1$ (since m and 1 are distinct (since $m > 2 > 1$)). This contradicts $v_m = v_1$.

This contradiction proves that our assumption (that $i = m$) was wrong. Hence, we cannot have $i = m$. We thus have $i \neq m$. Combined with $i \in \{1, 2, \ldots, m\}$, this yields $i \in \{1, 2, \ldots, m\} \setminus \{m\} \subseteq \{1, 2, \ldots, m-1\}$. Thus, $i + 1 \in \{2, 3, \ldots, m\} \subseteq \{1, 2, \ldots, m\}$.

Now, recall that any two distinct elements a and b of $\{1, 2, \ldots, m\}$ satisfy $v_a \neq v_b$. We can apply this to $a = i$ and $b = i + 1$ (since $i + 1 \in \{1, 2, \ldots, m\}$). Thus, we obtain $v_i \neq v_{i+1}$. This contradicts $v_i = v_{i+1}$. This contradiction shows that our assumption was wrong. Qed.

Proof of Lemma 2.5. We have

$$Y_{\sim}^{V} = \left\{ g \in Y^{V} \mid g\left(x\right) = g\left(y\right) \text{ for any } x \in V \text{ and } y \in V \text{ satisfying } x \sim y \right\} \tag{22}$$

(by the definition of $Y_{\sim}^{V}$).

The definition of $\operatorname{Eqs} f$ shows that

$$\begin{aligned} \operatorname{Eqs} f &= \left\{ \left\{s,t\right\} \mid \left(s,t\right) \in V^{2},\ s \neq t \text{ and } f\left(s\right) = f\left(t\right) \right\} \\ &= \left\{ \left\{x,y\right\} \mid \left(x,y\right) \in V^{2},\ x \neq y \text{ and } f\left(x\right) = f\left(y\right) \right\} \end{aligned} \tag{23}$$

(here, we renamed the index (s,t) as (x,y)).

We shall now show the following logical implication:

$$\left(B \subseteq \operatorname{Eqs} f\right) \implies \left(f \in Y_{\sim}^{V}\right) \tag{24}$$

Proof of (24): Assume that $B \subseteq \operatorname{Eqs} f$. We must show that $f \in Y_{\sim}^{V}$.

Let $x \in V$ and $y \in V$ be such that $x \sim y$. We have $x \sim y$. In other words, $x \sim_{(V,B)} y$ (since $\sim$ is the equivalence relation $\sim_{(V,B)}$). In other words, x and y are connected in the graph (V,B) (since $x \sim_{(V,B)} y$ holds if and only if x and y are connected in the graph (V,B) (by the definition of the relation $\sim_{(V,B)}$)). In other words, there exists a walk from x to y in (V,B) (since x and y are connected in (V,B) if and only if there exists a walk from x to y in (V,B) (by the definition of "connected")). Let $\mathfrak{w}$ be this walk. Thus, $\mathfrak{w}$ is a walk from x to y in (V,B). In other words, $\mathfrak{w}$ is a sequence $(w_0, w_1, \ldots, w_k)$ of elements of V such that $w_0 = x$ and $w_k = y$ and

$$\left(\left\{w_i, w_{i+1}\right\} \in B \qquad \text{for every } i \in \left\{0,1,\ldots,k-1\right\}\right) \tag{25}$$

(since a walk from x to y in (V,B) is the same as a sequence $(w_0, w_1, \ldots, w_k)$ of elements of V such that $w_0 = x$ and $w_k = y$ and
$(\{w_i, w_{i+1}\} \in B \qquad \text{for every } i \in \{0,1,\ldots,k-1\})$ (by the definition of a "walk")). Consider this sequence $(w_0, w_1, \ldots, w_k)$.

For every $i \in \{0,1,\ldots,k-1\}$, we have $f(w_i) = f(w_{i+1})$ [15]. In other words, $f(w_0) = f(w_1) = \cdots = f(w_k)$. Thus, $f(w_0) = f(w_k)$. This rewrites as $f(x) = f(y)$ (since $w_0 = x$ and $w_k = y$).

---

[15] *Proof.* Let $i \in \{0,1,\ldots,k-1\}$. Thus, (25) shows that

$$\left\{w_i, w_{i+1}\right\} \in B \subseteq \operatorname{Eqs} f = \left\{ \left\{s,t\right\} \mid \left(s,t\right) \in V^{2},\ s \neq t \text{ and } f\left(s\right) = f\left(t\right) \right\}$$

(by the definition of $\operatorname{Eqs} f$). In other words, $\{w_i, w_{i+1}\} = \{s,t\}$ for some $(s,t) \in V^2$ satisfying $s \neq t$ and $f(s) = f(t)$. Consider this (s,t).

We have $f(s) = f(t)$. Therefore, set $g = f(s) = f(t)$. Then, $f(\{s,t\}) = \left\{ \underbrace{f(s)}_{=g}, \underbrace{f(t)}_{=g} \right\} =$

Now, let us forget that we fixed x and y. We thus have shown that $f(x) = f(y)$ for any $x \in V$ and $y \in V$ satisfying $x \sim y$. Hence, f is a $g \in Y^V$ which satisfies $g(x) = g(y)$ for any $x \in V$ and $y \in V$ satisfying $x \sim y$. In other words,

$$\begin{aligned} f &\in \left\{ g \in Y^V \mid g(x) = g(y) \text{ for any } x \in V \text{ and } y \in V \text{ satisfying } x \sim y \right\} \\ &= Y^V_\sim \qquad (\text{by (22)}). \end{aligned}$$

Thus, the implication (24) is proven.

Next, we shall show the following logical implication:

$$\left(f \in Y^V_\sim \right) \implies (B \subseteq \operatorname{Eqs} f). \tag{26}$$

Proof of (26): Assume that $f \in Y^V_\sim$. We must show that $B \subseteq \operatorname{Eqs} f$.

Since (V, B) is a graph, we must have $B \subseteq \binom{V}{2}$.

Let $e \in B$. Then, $e \in B \subseteq \binom{V}{2}$. In other words, e is a 2-element subset of V (since $\binom{V}{2}$ is the set of all 2-element subsets of V). In other words, $e = \{s, t\}$ for two distinct elements s and t of V. Consider these s and t.

We have $\{s, t\} = e \in B$. Thus, $s \sim t$ [16].

$\{g, g\} = \{g\}$. Now, $f\left(\underbrace{w_i}_{\in \{w_i, w_{i+1}\} = \{s,t\}}\right) \in f(\{s,t\}) = \{g\}$, so that $f(w_i) = g$. Also, $f\left(\underbrace{w_{i+1}}_{\in \{w_i, w_{i+1}\} = \{s,t\}}\right) \in f(\{s,t\}) = \{g\}$, so that $f(w_{i+1}) = g$. Hence, $f(w_i) = g = f(w_{i+1})$, qed.

[16] *Proof.* We have $s \in V$ and $t \in V$. Thus, (s, t) is a sequence of elements of V. Let us denote this sequence (s, t) by $(p_0, p_1, \ldots, p_\ell)$. Thus, $\ell = 1$, $p_0 = s$ and $p_1 = t$.

We have $\ell = 1$ and thus $p_\ell = p_1 = t$.

Let $i \in \{0, 1, \ldots, \ell - 1\}$. Thus, $i \geq 0$ and $i \leq \underbrace{\ell}_{=1} - 1 = 1 - 1 = 0$. Combining $i \leq 0$ and $i \geq 0$, we obtain $i = 0$, so that $p_i = p_0 = s$ and $p_{i+1} = p_{0+1} = p_1 = t$. Now, $\left\{ \underbrace{p_i}_{=s}, \underbrace{p_{i+1}}_{=t} \right\} = \{s, t\} \in B$.

Now, let us forget that we fixed i. We thus have shown that $\{p_i, p_{i+1}\} \in B$ for every $i \in \{0, 1, \ldots, \ell - 1\}$. Thus, the sequence $(p_0, p_1, \ldots, p_\ell)$ is a sequence of elements of V satisfying $p_0 = s$, $p_\ell = t$ and $(\{p_i, p_{i+1}\} \in B \quad \text{for every } i \in \{0, 1, \ldots, \ell - 1\})$. In other words, the sequence $(p_0, p_1, \ldots, p_\ell)$ is a sequence $(w_0, w_1, \ldots, w_k)$ of elements of V such that $w_0 = s$ and $w_k = t$ and $(\{w_i, w_{i+1}\} \in B \quad \text{for every } i \in \{0, 1, \ldots, k-1\})$. In other words, the sequence $(p_0, p_1, \ldots, p_\ell)$ is walk from s to t in (V, B) (since a walk from s to t in (V, B) is the same as a sequence $(w_0, w_1, \ldots, w_k)$ of elements of V such that $w_0 = s$ and $w_k = t$ and $(\{w_i, w_{i+1}\} \in B \quad \text{for every } i \in \{0, 1, \ldots, k-1\})$ (by the definition of a "walk")). Hence, But $f \in Y^V_\sim = \left\{ g \in Y^V \ \mid \ g(x) = g(y) \text{ for any } x \in V \text{ and } y \in V \text{ satisfying } x \sim y \right\}$ (by (22)). In other words, f is a $g \in Y^V$ such that $g(x) = g(y)$ for any $x \in V$ and $y \in V$ satisfying $x \sim y$. In other words, f is an element of Y^V and satisfies

$$f(x) = f(y) \qquad \text{for any } x \in V \text{ and } y \in V \text{ satisfying } x \sim y. \tag{27}$$

Applying (27) to $x = s$ and $y = t$, we obtain $f(s) = f(t)$ (since $s \sim t$).

We have $s \in V$ and $t \in V$. Thus, $(s,t) \in V^2$. Also, s and t are distinct; thus, $s \neq t$. Hence, (s,t) is an element of V^2 satisfying $s \neq t$ and $f(s) = f(t)$. In other words, (s,t) is an $(x,y) \in V^2$ satisfying $x \neq y$ and $f(x) = f(y)$. Therefore, $\{s,t\}$ has the form $\{x,y\}$ for some $(x,y) \in V^2$ satisfying $x \neq y$ and $f(x) = f(y)$ (namely, for $(x,y) = (s,t)$). In other words,

$$\{s,t\} \in \left\{ \{x,y\} \ \mid \ (x,y) \in V^2, \ x \neq y \text{ and } f(x) = f(y) \right\} = \operatorname{Eqs} f$$

(by (23)). Thus, $e = \{s,t\} \in \operatorname{Eqs} f$.

Now, let us forget that we fixed e. We thus have shown that $e \in \operatorname{Eqs} f$ for every $e \in B$. In other words, $B \subseteq \operatorname{Eqs} f$. This proves the implication (26).

Now, we can combine the two implications (24) and (26). As a result, we obtain the equivalence $(B \subseteq \operatorname{Eqs} f) \iff \left(f \in Y^V_\sim\right)$. This proves Lemma 2.5. □

The next lemma is a fundamental fact about counting:

Lemma 2.6. Let W be a finite set. Let $(C_1, C_2, \ldots, C_k)$ be a list of all elements of W which contains each of these elements exactly once. Let Y be any set. Then, the map

$$\begin{aligned} & Y^W \to Y^k, \\ & \quad f \mapsto (f(C_1), f(C_2), \ldots, f(C_k)) \end{aligned}$$

is a bijection.

Proof of Lemma 2.6. Let Φ denote the map

$$\begin{aligned} & Y^W \to Y^k, \\ & \quad f \mapsto (f(C_1), f(C_2), \ldots, f(C_k)). \end{aligned}$$

We shall show that the map Φ is a bijection.

First, we notice that the map Φ is injective[17]. Now, let $s \in Y^k$ be arbitrary. Let us write s in the form $(y_1, y_2, \ldots, y_k)$. Thus, $s = (y_1, y_2, \ldots, y_k)$.

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there exists a walk from s to t in (V,B). In other words, s and t are connected in the graph (V,B) (since s and t are connected in (V,B) if and only if there exists a walk from s to t in (V,B) (by the definition of "connected")). In other words, $s \sim_{(V,B)} t$ (since $s \sim_{(V,B)} t$ holds if and only if s and t are connected in the graph (V,B) (by the definition of the relation $\sim_{(V,B)}$)). In other words, $s \sim t$ (since $\sim$ is the equivalence relation $\sim_{(V,B)}$). Qed.

[17] *Proof.* Let f and g be two elements of Y^W such that $\Phi(f) = \Phi(g)$. We shall show that $f = g$.

We now define a map $f : W \to Y$ as follows: Let $w \in W$. Then, there exists a unique $i \in \{1, 2, \ldots, k\}$ such that $w = C_i$ (since $(C_1, C_2, \ldots, C_k)$ is a list of all elements of W which contains each of these elements exactly once). Consider this i. Then, we set $f(w) = y_i$.

Thus, we have defined a map $f : W \to Y$. It is clear that if $w \in W$, and if $i \in \{1, 2, \ldots, k\}$ is such that $w = C_i$, then

$$f(w) = y_i \tag{28}$$

(by the definition of $f(w)$).

Now, every $i \in \{1, 2, \ldots, k\}$ satisfies $f(C_i) = y_i$ (by (28), applied to $w = C_i$). Hence, $(f(C_1), f(C_2), \ldots, f(C_k)) = (y_1, y_2, \ldots, y_k)$. Now, the definition of Φ yields $\Phi(f) = (f(C_1), f(C_2), \ldots, f(C_k)) = (y_1, y_2, \ldots, y_k) = s$. Thus, $s = \Phi\left(\underbrace{f}_{\in Y^W}\right) \in \Phi\left(Y^W\right)$.

Now, let us forget that we fixed s. We thus have proven that $s \in \Phi\left(Y^W\right)$ for every $s \in Y^k$. In other words, $Y^k \subseteq \Phi\left(Y^W\right)$. In other words, the map Φ is surjective.

Since the map Φ is both injective and surjective, we see that the map Φ is

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Let $w \in W$. Recall that $(C_1, C_2, \ldots, C_k)$ is a list of all elements of W. Thus, $W = \{C_1, C_2, \ldots, C_k\}$. Now, $w \in W = \{C_1, C_2, \ldots, C_k\}$. Hence, there exists some $i \in \{1, 2, \ldots, k\}$ such that $w = C_i$. Consider this i.

Now, the definition of $\Phi(f)$ yields $\Phi(f) = (f(C_1), f(C_2), \ldots, f(C_k))$. Hence,

$$\begin{aligned}&\left(\text{the } i\text{-th entry of } \underbrace{\Phi(f)}_{=(f(C_1), f(C_2), \ldots, f(C_k))}\right)\\ &= \left(\text{the } i\text{-th entry of } (f(C_1), f(C_2), \ldots, f(C_k))\right) = f\left(\underbrace{C_i}_{=w}\right) = f(w).\end{aligned}$$

The same argument (applied to g instead of f) yields

$$(\text{the } i\text{-th entry of } \Phi(g)) = g(w).$$

Hence, $f(w) = \left(\text{the } i\text{-th entry of } \underbrace{\Phi(f)}_{=\Phi(g)}\right) = (\text{the } i\text{-th entry of } \Phi(g)) = g(w)$.

Now, let us forget that we fixed w. We thus have proven that $f(w) = g(w)$ for every $w \in W$. In other words, $f = g$.

Let us now forget that we fixed f and g. We thus have shown that if f and g are two elements of Y^W such that $\Phi(f) = \Phi(g)$, then $f = g$. In other words, the map Φ is injective. Qed.

bijective. In other words, the map Φ is a bijection. In other words, the map

$$\begin{aligned} Y^W &\to Y^k, \\ f &\mapsto (f(C_1), f(C_2), \ldots, f(C_k)) \end{aligned}$$

is a bijection (since this map is Φ). Lemma 2.6 is thus proven. $\square$

Lemma 2.7. Let (V, B) be a finite graph. Then,

$$\sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} \mathbf{x}_f = p_{\lambda(V,B)}.$$

(Here, $\mathbf{x}_f$ is defined as in Definition 1.5 **(a)**, and the expression $\lambda(V, B)$ is understood according to Definition 1.10 **(b)**.)

Proof of Lemma 2.7. Let $\sim$ denote the equivalence relation $\sim_{(V,B)}$ (defined as in Definition 1.10 **(a)**). The connected components of (V, B) are the $\sim_{(V,B)}$-equivalence classes (because this is how the connected components of (V, B) are defined). In other words, the connected components of (V, B) are the $\sim$-equivalence classes (since $\sim$ is the relation $\sim_{(V,B)}$). In other words, the connected components of (V, B) are the elements of $V/(\sim)$ (since the elements of $V/(\sim)$ are the $\sim$-equivalence classes (by the definition of $V/(\sim)$)).

A set $(\mathbb{N}_+)^V_\sim$ is defined (according to Definition 1.7 **(b)**).

Proposition 1.8 **(b)** (applied to $X = V$ and $Y = \mathbb{N}_+$) shows that the map[18]

$$(\mathbb{N}_+)^{V/(\sim)} \to (\mathbb{N}_+)^V_\sim, \qquad f \mapsto f \circ \pi_V$$

is a bijection.

For every map $f : V \to \mathbb{N}_+$, we have the following equivalence:

$$(B \subseteq \operatorname{Eqs} f) \iff \left(f \in (\mathbb{N}_+)^V_\sim\right) \tag{29}$$

(according to Lemma 2.5, applied to $Y = \mathbb{N}_+$). Thus, we have the following equality of summation signs:

$$\sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} = \sum_{\substack{f:V\to\mathbb{N}_+;\\ f\in(\mathbb{N}_+)^V_\sim}} = \sum_{\substack{f\in(\mathbb{N}_+)^V;\\ f\in(\mathbb{N}_+)^V_\sim}} = \sum_{f\in(\mathbb{N}_+)^V_\sim}$$

(since $(\mathbb{N}_+)^V_\sim$ is a subset of $(\mathbb{N}_+)^V$). Hence,

$$\underbrace{\sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}}}_{=\sum_{f\in(\mathbb{N}_+)^V_\sim}} \mathbf{x}_f = \sum_{f\in(\mathbb{N}_+)^V_\sim} \mathbf{x}_f = \sum_{f\in(\mathbb{N}_+)^{V/(\sim)}} \mathbf{x}_{f\circ\pi_V} \tag{30}$$

---

[18] Here, the map π_V is defined as in Definition 1.6.

(here, we have substituted $f \circ \pi_V$ for f in the sum, since the map $(\mathbb{N}_+)^{V/(\sim)} \to (\mathbb{N}_+)^V_\sim$, $f \mapsto f \circ \pi_V$ is a bijection).

Now, let $(C_1, C_2, \ldots, C_k)$ be a list of all connected components of (V, B), ordered such that $|C_1| \geq |C_2| \geq \cdots \geq |C_k|$. [19] Then, $(|C_1|, |C_2|, \ldots, |C_k|)$ is the list of the sizes of all connected components of (V, B), in weakly decreasing order (since $|C_1| \geq |C_2| \geq \cdots \geq |C_k|$). In other words, $(|C_1|, |C_2|, \ldots, |C_k|)$ is $\lambda(V, B)$ (since $\lambda(V, B)$ is the list of the sizes of all connected components of (V, B), in weakly decreasing order (by the definition of $\lambda(V, B)$)). In other words, $\lambda(V, B) = (|C_1|, |C_2|, \ldots, |C_k|)$. Thus, (2) (applied to $\lambda(V, B)$ and $|C_i|$ instead of λ and λ_i) shows that

$$p_{\lambda(V,B)} = p_{|C_1|} p_{|C_2|} \cdots p_{|C_k|} = \prod_{i=1}^{k} p_{|C_i|}. \tag{31}$$

However, for every $i \in \{1, 2, \ldots, k\}$, we have

$$p_{|C_i|} = \sum_{s \in \mathbb{N}_+} x_s^{|C_i|} \tag{32}$$

[20]. Hence, (31) becomes

$$p_{\lambda(V,B)} = \prod_{i=1}^{k} \underbrace{p_{|C_i|}}_{\substack{= \sum_{s \in \mathbb{N}_+} x_s^{|C_i|} \\ \text{(by (32))}}} = \prod_{i=1}^{k} \sum_{s \in \mathbb{N}_+} x_s^{|C_i|}$$

$$= \sum_{(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k} \prod_{i=1}^{k} x_{s_i}^{|C_i|} \tag{33}$$

(by the product rule).

Recall that $(C_1, C_2, \ldots, C_k)$ is a list of all connected components of (V, B). In other words, $(C_1, C_2, \ldots, C_k)$ is a list of all elements of $V/(\sim)$ (since the elements of $V/(\sim)$ are the connected components of (V, B)). Moreover, every element of $V/(\sim)$ appears exactly once in this list $(C_1, C_2, \ldots, C_k)$ (since the entries of the list $(C_1, C_2, \ldots, C_k)$ are pairwise distinct[21]). Thus, $(C_1, C_2, \ldots, C_k)$ is a list of all

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[19] Every connected component of (V, B) should appear exactly once in this list.

[20] *Proof.* Let $i \in \{1, 2, \ldots, k\}$. Then, C_i is a connected component of (V, B) (since $(C_1, C_2, \ldots, C_k)$ is a list of all connected components of (V, B)). Hence, C_i is a nonempty subset of V (since every connected component of (V, B) is a nonempty subset of V). Hence, $|C_i|$ is a positive integer. Thus, (1) (applied to $n = |C_i|$) shows that $p_{|C_i|} = \underbrace{\sum_{j \geq 1}}_{= \sum_{j \in \mathbb{N}_+}} x_j^{|C_i|} = \sum_{j \in \mathbb{N}_+} x_j^{|C_i|} = \sum_{s \in \mathbb{N}_+} x_s^{|C_i|}$ (here, we have renamed the summation index j as s). Qed.

[21] since every connected component of (V, B) appears exactly once in this list

elements of $V/(\sim)$, and contains each of these elements exactly once. Hence, the map

$$\begin{aligned} (\mathbb{N}_+)^{V/(\sim)} &\to (\mathbb{N}_+)^k, \\ f &\mapsto (f(C_1), f(C_2), \ldots, f(C_k)) \end{aligned}$$

is a bijection (by Lemma 2.6, applied to $W = V/(\sim)$ and $Y = \mathbb{N}_+$).

For every $\gamma \in V/(\sim)$, we have

$$\pi_V^{-1}(\gamma) = \gamma \tag{34}$$

[22].

Also, the map $\{1, 2, \ldots, k\} \to V/(\sim)$, $i \mapsto C_i$ is a bijection (since $(C_1, C_2, \ldots, C_k)$ is a list of all elements of $V/(\sim)$, and contains each of these elements exactly once).

We have

$$\mathbf{x}_{f\circ\pi_V} = \prod_{i=1}^{k} x_{f(C_i)}^{|C_i|} \qquad \text{for every } f \in (\mathbb{N}_+)^{V/(\sim)} \tag{35}$$

[23].

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[22] *Proof of (34):* Let $\gamma \in V/(\sim)$. Thus, γ is an element of $V/(\sim)$. In other words, γ is an $\sim$-equivalence class (since the elements of $V/(\sim)$ are the $\sim$-equivalence classes). In particular, $\gamma \subseteq V$.

Let $x \in \gamma$. Then, $x \in \gamma \subseteq V$. The $\sim$-equivalence class of x must be γ (since x lies in the $\sim$-equivalence class γ (since $x \in \gamma$)). In other words, $[x]_\sim$ must be γ (since $[x]_\sim$ is the $\sim$-equivalence class of x). In other words, $[x]_\sim = \gamma$. Now, the definition of π_V yields $\pi_V(x) = [x]_\sim = \gamma$. Hence, $x \in \pi_V^{-1}(\gamma)$.

Let us now forget that we fixed x. We thus have shown that $x \in \pi_V^{-1}(\gamma)$ for every $x \in \gamma$. In other words, $\gamma \subseteq \pi_V^{-1}(\gamma)$.

Let now $y \in \pi_V^{-1}(\gamma)$. Thus, $y \in V$ and $\pi_V(y) = \gamma$. The definition of π_V yields $\pi_V(y) = [y]_\sim$. Thus, $\gamma = \pi_V(y) = [y]_\sim$. Hence, γ is the $\sim$-equivalence class of y (since $[y]_\sim$ is the $\sim$-equivalence class of y). Consequently, y must belong to γ. In other words, $y \in \gamma$.

Let us now forget that we fixed y. We thus have shown that $y \in \gamma$ for every $y \in \pi_V^{-1}(\gamma)$. In other words, $\pi_V^{-1}(\gamma) \subseteq \gamma$. Combining this with $\gamma \subseteq \pi_V^{-1}(\gamma)$, we obtain $\pi_V^{-1}(\gamma) = \gamma$. This proves (34).

[23] *Proof of (35):* Let $f \in (\mathbb{N}_+)^{V/(\sim)}$. Then, the definition of $\mathbf{x}_{f\circ\pi_V}$ yields

$$\begin{aligned} \mathbf{x}_{f\circ\pi_V} &= \prod_{v\in V} x_{(f\circ\pi_V)(v)} = \prod_{\gamma\in V/(\sim)} \underbrace{\prod_{\substack{v\in V;\\ \pi_V(v)=\gamma}}}_{\substack{= \prod_{v\in\pi_V^{-1}(\gamma)} = \prod_{v\in\gamma} \\ \text{(since } \pi_V^{-1}(\gamma)=\gamma \\ \text{(by (34)))}}} \underbrace{x_{(f\circ\pi_V)(v)}}_{\substack{=x_{f(\gamma)} \\ \text{(since } (f\circ\pi_V)(v)=f(\pi_V(v))=f(\gamma) \\ \text{(since } \pi_V(v)=\gamma\text{))}}} \\ &\qquad \left(\begin{array}{c}\text{because for every } v \in V \text{, there exists a unique } \gamma \in V/(\sim) \\ \text{such that } \pi_V(v) = \gamma \text{ (since } \pi_V \text{ is a map } V \to V/(\sim)\text{)}\end{array}\right) \\ &= \prod_{\gamma\in V/(\sim)} \underbrace{\prod_{v\in\gamma} x_{f(\gamma)}}_{=x_{f(\gamma)}^{|\gamma|}} = \prod_{\gamma\in V/(\sim)} x_{f(\gamma)}^{|\gamma|} = \prod_{i\in\{1,2,\ldots,k\}} x_{f(C_i)}^{|C_i|} \end{aligned}$$

Now, (30) becomes

$$\sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} \mathbf{x}_f = \sum_{f\in(\mathbb{N}_+)^{V/(\sim)}} \underbrace{\mathbf{x}_{f\circ\pi_V}}_{\substack{=\prod_{i=1}^{k} x_{f(C_i)}^{|C_i|}\\ \text{(by (35))}}} = \sum_{f\in(\mathbb{N}_+)^{V/(\sim)}} \prod_{i=1}^{k} x_{f(C_i)}^{|C_i|}$$
$$= \sum_{(s_1,s_2,\ldots,s_k)\in(\mathbb{N}_+)^k} \prod_{i=1}^{k} x_{s_i}^{|C_i|}$$

(here, we have substituted $(s_1,s_2,\ldots,s_k)$ for $(f(C_1),f(C_2),\ldots,f(C_k))$ in the sum, since the map $(\mathbb{N}_+)^{V/(\sim)}\to(\mathbb{N}_+)^k$, $f\mapsto(f(C_1),f(C_2),\ldots,f(C_k))$ is a bijection). Comparing this with (33), we obtain $\sum\limits_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} \mathbf{x}_f = p_{\lambda(V,B)}$. This proves Lemma 2.7. ☐

Lemma 2.8. Let $G=(V,E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E\to X$ be a labeling function. Let K be a broken circuit of G. Then, $K\neq\varnothing$.

Proof of Lemma 2.8. The set K is a broken circuit of G. In other words, the set K is a subset of E having the form $C\setminus\{e\}$, where C is a circuit of G, and where e is the unique edge in C having maximum label (among the edges in C) [24]. Consider this C and this e. Thus, we have the following facts:

- The set C is a circuit of G.
- The element e is the unique edge in C having maximum label (among the edges in C).
- We have $K=C\setminus\{e\}$.

(here, we have substituted C_i for γ in the product, since the map $\{1,2,\ldots,k\}\to V/(\sim)$, $i\mapsto C_i$ is a bijection). Thus,

$$\mathbf{x}_{f\circ\pi_V} = \underbrace{\prod_{i\in\{1,2,\ldots,k\}}}_{=\prod_{i=1}^{k}} x_{f(C_i)}^{|C_i|} = \prod_{i=1}^{k} x_{f(C_i)}^{|C_i|}.$$

This proves (35).

[24] because a broken circuit of G is the same as a subset of E having the form $C\setminus\{e\}$, where C is a circuit of G, and where e is the unique edge in C having maximum label (among the edges in C) (by the definition of a "broken circuit")

Now, assume (for the sake of contradiction) that $K = \varnothing$. Consider the map $\operatorname{id} : V \to V$. Then, $\operatorname{Eqs}\operatorname{id} = \varnothing$ [25]. But $C \setminus \{e\} = K = \varnothing \subseteq \varnothing = \operatorname{Eqs}\operatorname{id}$. Hence, Lemma 2.4 (applied to V and id instead of X and f) yields $e \in E \cap \underbrace{\operatorname{Eqs}\operatorname{id}}_{=\varnothing} = E \cap \varnothing = \varnothing$. Thus, the set $\varnothing$ has at least one element (namely, e). This contradicts the fact that this set $\varnothing$ is empty. This contradiction shows that our assumption (that $K = \varnothing$) was wrong. Hence, we cannot have $K = \varnothing$. We thus have $K \neq \varnothing$. This proves Lemma 2.8. ☐

2.2. Alternating sums

We shall now come to less simple lemmas.

Definition 2.9. We shall use the so-called *Iverson bracket notation*: If $\mathcal{S}$ is any logical statement, then $[\mathcal{S}]$ shall mean the integer $\begin{cases} 1, & \text{if } \mathcal{S} \text{ is true;} \\ 0, & \text{if } \mathcal{S} \text{ is false} \end{cases}$.

The following lemma is probably the most crucial one in this paper:

Lemma 2.10. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$.

Let Y be any set. Let $f : V \to Y$ be any map. Then,

$$\sum_{B \subseteq E \cap \operatorname{Eqs} f} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K = [E \cap \operatorname{Eqs} f = \varnothing].$$

Proof of Lemma 2.10. If $E \cap \operatorname{Eqs} f = \varnothing$, then Lemma 2.10 holds[26]. Thus, we can WLOG assume that we don't have $E \cap \operatorname{Eqs} f = \varnothing$. Assume this.

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[25] *Proof.* Let $f \in \operatorname{Eqs}\operatorname{id}$. Thus,

$$f \in \operatorname{Eqs}\operatorname{id} = \left\{ \{s, t\} \mid (s, t) \in V^2,\ s \neq t \text{ and } \operatorname{id}(s) = \operatorname{id}(t) \right\}$$

(by the definition of $\operatorname{Eqs}\operatorname{id}$). In other words, f has the form $\{s, t\}$ for some $(s, t) \in V^2$ satisfying $s \neq t$ and $\operatorname{id}(s) = \operatorname{id}(t)$. Consider this (s, t). We have $s = \operatorname{id}(s) = \operatorname{id}(t) = t$; but this contradicts $s \neq t$.

Now, let us forget that we fixed f. We thus have obtained a contradiction for every $f \in \operatorname{Eqs}\operatorname{id}$. Thus, there exists no $f \in \operatorname{Eqs}\operatorname{id}$. In other words, $\operatorname{Eqs}\operatorname{id}$ is the empty set. Thus, $\operatorname{Eqs}\operatorname{id} = \varnothing$, qed.

[26] *Proof.* Assume that $E \cap \operatorname{Eqs} f = \varnothing$. We need to check that Lemma 2.10 holds.

We have

$$\sum_{B \subseteq E \cap \operatorname{Eqs} f} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K = \sum_{B \subseteq \varnothing} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K \qquad (\text{since } E \cap \operatorname{Eqs} f = \varnothing).$$

We don't have $E\cap\operatorname{Eqs} f=\varnothing$. Thus, we have $[E\cap\operatorname{Eqs} f=\varnothing]=0$.

We know that V is finite (since the graph (V,E) is finite). Thus, E is finite (since $E\subseteq\dbinom{V}{2}$), and therefore the set $E\cap\operatorname{Eqs} f$ is also finite.

We have $E\cap\operatorname{Eqs} f\neq\varnothing$ (since we don't have $E\cap\operatorname{Eqs} f=\varnothing$). Thus, $E\cap\operatorname{Eqs} f$ is a nonempty finite set. Hence, there exists some $d\in E\cap\operatorname{Eqs} f$ with maximum $\ell(d)$ (among all $d\in E\cap\operatorname{Eqs} f$). Pick such a d. (If there are several such d, then it does not matter which one we pick.)

We have chosen d to be the element of $E\cap\operatorname{Eqs} f$ with maximum $\ell(d)$ (among all $d\in E\cap\operatorname{Eqs} f$). Thus,

$$\ell(d)\geq\ell(e)\qquad\text{for every } e\in E\cap\operatorname{Eqs} f. \tag{36}$$

As usual, we let $\mathcal{P}(S)$ denote the powerset of any set S. We now define two subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{P}(E\cap\operatorname{Eqs} f)$ as follows:

$$\mathcal{U}=\{F\in\mathcal{P}(E\cap\operatorname{Eqs} f)\ \mid\ d\notin F\};$$
$$\mathcal{V}=\{F\in\mathcal{P}(E\cap\operatorname{Eqs} f)\ \mid\ d\in F\}.$$

Every $B\in\mathcal{U}$ satisfies $B\cup\{d\}\in\mathcal{V}$ [27]. Thus, we can define a map $\Phi:\mathcal{U}\to\mathcal{V}$

But the only subset B of $\varnothing$ is the set $\varnothing$. Thus, the only addend of the sum $\sum\limits_{B\subseteq\varnothing}(-1)^{|B|}\prod\limits_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K$ is the addend for $B=\varnothing$. Hence,

$$\sum_{B\subseteq\varnothing}(-1)^{|B|}\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K=\underbrace{(-1)^{|\varnothing|}}_{\substack{=1\\ (\text{since }|\varnothing|=0\text{ is even})}}\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq\varnothing}}a_K=\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq\varnothing}}a_K.$$

But let $K\in\mathfrak{K}$ be such that $K\subseteq\varnothing$. Then, K is an element of $\mathfrak{K}$, and thus a broken circuit of G (since $\mathfrak{K}$ is a set of broken circuits of G). Hence, Lemma 2.8 shows that $K\neq\varnothing$. But from $K\subseteq\varnothing$, we obtain $K=\varnothing$; this contradicts $K\neq\varnothing$.

Now, let us forget that we fixed K. Thus, we have obtained a contradiction for every $K\in\mathfrak{K}$ satisfying $K\subseteq\varnothing$. Hence, there exists no $K\in\mathfrak{K}$ satisfying $K\subseteq\varnothing$. Therefore, the product $\prod\limits_{\substack{K\in\mathfrak{K};\\ K\subseteq\varnothing}}a_K$ is empty, and thus equals 1. In other words, $\prod\limits_{\substack{K\in\mathfrak{K};\\ K\subseteq\varnothing}}a_K=1$.

Now,

$$\sum_{B\subseteq E\cap\operatorname{Eqs} f}(-1)^{|B|}\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K=\sum_{B\subseteq\varnothing}(-1)^{|B|}\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K=\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq\varnothing}}a_K=1=[E\cap\operatorname{Eqs} f=\varnothing]$$

(since $[E\cap\operatorname{Eqs} f=\varnothing]=1$ (since $E\cap\operatorname{Eqs} f=\varnothing$ holds)). Thus, Lemma 2.10 holds, qed.

[27] *Proof.* Let $B\in\mathcal{U}$. Thus, $B\in\mathcal{U}=\{F\in\mathcal{P}(E\cap\operatorname{Eqs} f)\ \mid\ d\notin F\}$. In other words, B is an element F of $\mathcal{P}(E\cap\operatorname{Eqs} f)$ satisfying $d\notin F$. In other words, B is an element of $\mathcal{P}(E\cap\operatorname{Eqs} f)$ and satisfies $d\notin B$. We have $B\in\mathcal{P}(E\cap\operatorname{Eqs} f)$; in other words, B is a subset of $E\cap\operatorname{Eqs} f$. Also, $\{d\}\subseteq E\cap\operatorname{Eqs} f$ (since $d\in E\cap\operatorname{Eqs} f$). Thus, both B and $\{d\}$ are subsets of $E\cap\operatorname{Eqs} f$. Hence, their union $B\cup\{d\}$ is a subset of $E\cap\operatorname{Eqs} f$. In other words, $B\cup\{d\}\in\mathcal{P}(E\cap\operatorname{Eqs} f)$. Also, $d\in\{d\}\subseteq B\cup\{d\}$. Hence, $B\cup\{d\}$ is an element of $\mathcal{P}(E\cap\operatorname{Eqs} f)$ and satisfies $d\in B\cup\{d\}$. In other words, $B\cup\{d\}$ is an element F of $\mathcal{P}(E\cap\operatorname{Eqs} f)$ satisfying $d\in F$. In other words, $B\cup\{d\}\in\{F\in\mathcal{P}(E\cap\operatorname{Eqs} f)\ \mid\ d\in F\}=\mathcal{V}$, qed.

by

$$\left(\Phi\left(B\right) = B \cup \left\{d\right\} \qquad \text{for every } B \in \mathcal{U}\right).$$

Consider this map Φ.

Every $B \in \mathcal{V}$ satisfies $B \setminus \left\{d\right\} \in \mathcal{U}$ [28]. Thus, we can define a map $\Psi : \mathcal{V} \to \mathcal{U}$ by

$$\left(\Psi\left(B\right) = B \setminus \left\{d\right\} \qquad \text{for every } B \in \mathcal{V}\right).$$

Consider this map Ψ.

We have $\Phi \circ \Psi = \operatorname{id}$ [29] and $\Psi \circ \Phi = \operatorname{id}$ [30]. Thus, the maps Φ and Ψ are mutually inverse. Hence, the map Φ is a bijection.

Moreover, for every $B \in \mathcal{U}$ and every $K \in \mathfrak{K}$, we have the following logical equivalence:

$$\left(K \subseteq B\right) \iff \left(K \subseteq \Phi\left(B\right)\right). \tag{37}$$

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[28] *Proof.* Let $B \in \mathcal{V}$. Thus, $B \in \mathcal{V} = \left\{F \in \mathcal{P}\left(E \cap \operatorname{Eqs} f\right) \mid d \in F\right\}$. In other words, B is an element F of $\mathcal{P}\left(E \cap \operatorname{Eqs} f\right)$ satisfying $d \in F$. In other words, B is an element of $\mathcal{P}\left(E \cap \operatorname{Eqs} f\right)$ and satisfies $d \in B$. We have $B \in \mathcal{P}\left(E \cap \operatorname{Eqs} f\right)$; in other words, B is a subset of $E \cap \operatorname{Eqs} f$. Hence, $B \setminus \left\{d\right\}$ is a subset of $E \cap \operatorname{Eqs} f$ (since $B \setminus \left\{d\right\} \subseteq B$). In other words, $B \setminus \left\{d\right\} \in \mathcal{P}\left(E \cap \operatorname{Eqs} f\right)$. Also, $d \notin B \setminus \left\{d\right\}$ (since $d \in \left\{d\right\}$). Hence, $B \setminus \left\{d\right\}$ is an element of $\mathcal{P}\left(E \cap \operatorname{Eqs} f\right)$ and satisfies $d \notin B \setminus \left\{d\right\}$. In other words, $B \setminus \left\{d\right\}$ is an element F of $\mathcal{P}\left(E \cap \operatorname{Eqs} f\right)$ satisfying $d \notin F$. In other words, $B \setminus \left\{d\right\} \in \left\{F \in \mathcal{P}\left(E \cap \operatorname{Eqs} f\right) \mid d \notin F\right\} = \mathcal{U}$, qed.

[29] *Proof.* Let $B \in \mathcal{V}$. We have

$$\left(\Phi \circ \Psi\right)\left(B\right) = \Phi\left(\underbrace{\Psi\left(B\right)}_{\substack{= B \setminus \left\{d\right\} \\ \text{(by the definition of } \Psi\text{)}}}\right) = \Phi\left(B \setminus \left\{d\right\}\right) = \left(B \setminus \left\{d\right\}\right) \cup \left\{d\right\}$$

(by the definition of Φ).

We have $B \in \mathcal{V} = \left\{F \in \mathcal{P}\left(E \cap \operatorname{Eqs} f\right) \mid d \in F\right\}$. In other words, B is an element F of $\mathcal{P}\left(E \cap \operatorname{Eqs} f\right)$ satisfying $d \in F$. In other words, B is an element of $\mathcal{P}\left(E \cap \operatorname{Eqs} f\right)$ and satisfies $d \in B$. From $d \in B$, we obtain $\left\{d\right\} \subseteq B$. Now, $\left(\Phi \circ \Psi\right)\left(B\right) = \left(B \setminus \left\{d\right\}\right) \cup \left\{d\right\} = B$ (since $\left\{d\right\} \subseteq B$). Thus, $\left(\Phi \circ \Psi\right)\left(B\right) = B = \operatorname{id}\left(B\right)$.

Now, let us forget that we fixed B. We thus have proven that $\left(\Phi \circ \Psi\right)\left(B\right) = \operatorname{id}\left(B\right)$ for every $B \in \mathcal{V}$. In other words, $\Phi \circ \Psi = \operatorname{id}$, qed.

[30] *Proof.* Let $B \in \mathcal{U}$. We have

$$\left(\Psi \circ \Phi\right)\left(B\right) = \Psi\left(\underbrace{\Phi\left(B\right)}_{\substack{= B \cup \left\{d\right\} \\ \text{(by the definition of } \Phi\text{)}}}\right) = \Psi\left(B \cup \left\{d\right\}\right) = \left(B \cup \left\{d\right\}\right) \setminus \left\{d\right\}$$

(by the definition of Ψ).

We have $B \in \mathcal{U} = \left\{F \in \mathcal{P}\left(E \cap \operatorname{Eqs} f\right) \mid d \notin F\right\}$. In other words, B is an element F of $\mathcal{P}\left(E \cap \operatorname{Eqs} f\right)$ satisfying $d \notin F$. In other words, B is an element of $\mathcal{P}\left(E \cap \operatorname{Eqs} f\right)$ and satisfies $d \notin B$. From $d \notin B$, we see that the sets $\left\{d\right\}$ and B are disjoint. Now, $\left(\Psi \circ \Phi\right)\left(B\right) = \left(B \cup \left\{d\right\}\right) \setminus \left\{d\right\} = B$ (since the sets $\left\{d\right\}$ and B are disjoint). Thus, $\left(\Psi \circ \Phi\right)\left(B\right) = B = \operatorname{id}\left(B\right)$.

Now, let us forget that we fixed B. We thus have proven that $\left(\Psi \circ \Phi\right)\left(B\right) = \operatorname{id}\left(B\right)$ for every $B \in \mathcal{U}$. In other words, $\Psi \circ \Phi = \operatorname{id}$, qed.

Proof of (37): Let $B \in \mathcal{U}$ and $K \in \mathfrak{K}$. We need to prove the logical equivalence (37).

The definition of Φ yields $\Phi(B) = B \cup \{d\} \supseteq B$, so that $B \subseteq \Phi(B)$.

We have $B \in \mathcal{U} = \{F \in \mathcal{P}(E \cap \operatorname{Eqs} f) \mid d \notin F\}$. In other words, B is an element F of $\mathcal{P}(E \cap \operatorname{Eqs} f)$ satisfying $d \notin F$. In other words, B is an element of $\mathcal{P}(E \cap \operatorname{Eqs} f)$ and satisfies $d \notin B$. We have $B \in \mathcal{P}(E \cap \operatorname{Eqs} f)$; in other words, B is a subset of $E \cap \operatorname{Eqs} f$.

Also, $\Phi(B) \in \mathcal{V} = \{F \in \mathcal{P}(E \cap \operatorname{Eqs} f) \mid d \in F\} \subseteq \mathcal{P}(E \cap \operatorname{Eqs} f)$. In other words, $\Phi(B) \subseteq E \cap \operatorname{Eqs} f$.

We have $K \in \mathfrak{K}$. Thus, K is a broken circuit of G (since $\mathfrak{K}$ is a set of broken circuits of G). In other words, K is a subset of E having the form $C \setminus \{e\}$, where C is a circuit of G, and where e is the unique edge in C having maximum label (among the edges in C) [31]. Consider this C and this e. Thus, we have the following facts:

- The set C is a circuit of G.
- The element e is the unique edge in C having maximum label (among the edges in C).
- We have $K = C \setminus \{e\}$.

The element e is the unique edge in C having maximum label (among the edges in C). Thus, the only edge in C whose label is greater or equal to the label of e is e itself. In other words, if e' is any edge in C satisfying $\ell(e') \geq \ell(e)$, then

$$e' = e. \tag{38}$$

Let us now assume that $K \subseteq \Phi(B)$. Thus, $K \subseteq \Phi(B) = B \cup \{d\}$ (by the definition of Φ). Hence, $\underbrace{K}_{\subseteq B \cup \{d\}} \setminus \{d\} \subseteq (B \cup \{d\}) \setminus \{d\} \subseteq B$.

We shall now prove that $K \subseteq B$.

Indeed, assume the contrary. Thus, $K \not\subseteq B$. If we had $d \notin K$, then we would have $K \setminus \{d\} = K$ and therefore $K = K \setminus \{d\} \subseteq B$; this would contradict $K \not\subseteq B$. Hence, we cannot have $d \notin K$. We thus must have $d \in K$. Hence, $d \in K = C \setminus \{e\}$. Hence, $d \in C$ and $d \notin \{e\}$. From $d \notin \{e\}$, we obtain $d \neq e$.

But $C \setminus \{e\} = K \subseteq \Phi(B) \subseteq E \cap \operatorname{Eqs} f \subseteq \operatorname{Eqs} f$. Hence, Lemma 2.4 (applied to Y instead of X) shows that $e \in E \cap \operatorname{Eqs} f$. Thus, (36) shows that $\ell(d) \geq \ell(e)$.

Also, $d \in C$. In other words, d is an edge in C. Since $\ell(d) \geq \ell(e)$, we can therefore apply (38) to $e' = d$. We thus obtain $d = e$. This contradicts $d \neq e$. This contradiction proves that our assumption was wrong. Hence, $K \subseteq B$ is proven.

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[31] because a broken circuit of G is the same as a subset of E having the form $C \setminus \{e\}$, where C is a circuit of G, and where e is the unique edge in C having maximum label (among the edges in C) (by the definition of a "broken circuit")

Now, let us forget that we assumed that $K \subseteq \Phi(B)$. We thus have proven that $K \subseteq B$ under the assumption that $K \subseteq \Phi(B)$. In other words, we have proven the implication

$$(K \subseteq \Phi(B)) \implies (K \subseteq B). \tag{39}$$

On the other hand, if $K \subseteq B$, then $K \subseteq B \subseteq \Phi(B)$. Hence, the implication

$$(K \subseteq B) \implies (K \subseteq \Phi(B))$$

holds. Combining this implication with (39), we obtain the logical equivalence $(K \subseteq B) \iff (K \subseteq \Phi(B))$. Thus, (37) is proven.

Also, every $B \in \mathcal{U}$ satisfies

$$(-1)^{|B|} = -(-1)^{|\Phi(B)|} \tag{40}$$

[32]. In other words, every $B \in \mathcal{U}$ satisfies

$$(-1)^{|\Phi(B)|} = -(-1)^{|B|}. \tag{41}$$

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[32] *Proof of (40):* Let $B \in \mathcal{U}$. We have $B \in \mathcal{U} = \{F \in \mathcal{P}(E \cap \operatorname{Eqs} f) \mid d \notin F\}$. In other words, B is an element F of $\mathcal{P}(E \cap \operatorname{Eqs} f)$ satisfying $d \notin F$. In other words, B is an element of $\mathcal{P}(E \cap \operatorname{Eqs} f)$ and satisfies $d \notin B$. From $d \notin B$, we see that $|B \cup \{d\}| = |B| + 1$. Now, the definition of Φ yields $\Phi(B) = B \cup \{d\}$. Hence, $|\Phi(B)| = |B \cup \{d\}| = |B| + 1$. Thus, $(-1)^{|\Phi(B)|} = (-1)^{|B|+1} = -(-1)^{|B|}$. Therefore, $(-1)^{|B|} = -(-1)^{|\Phi(B)|}$. This proves (40).

Now,

$$\begin{aligned}
&\sum_{B\subseteq E\cap\operatorname{Eqs} f} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \\
&= \underbrace{\sum_{\substack{B\subseteq E\cap\operatorname{Eqs} f;\\ d\in B}}}_{\substack{=\sum_{B\in\{F\in\mathcal{P}(E\cap\operatorname{Eqs} f)\ \mid\ d\in F\}} = \sum_{B\in\mathcal{V}} \\ (\text{since } \{F\in\mathcal{P}(E\cap\operatorname{Eqs} f)\ \mid\ d\in F\}=\mathcal{V})}} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K + \underbrace{\sum_{\substack{B\subseteq E\cap\operatorname{Eqs} f;\\ d\notin B}}}_{\substack{=\sum_{B\in\{F\in\mathcal{P}(E\cap\operatorname{Eqs} f)\ \mid\ d\notin F\}} = \sum_{B\in\mathcal{U}} \\ (\text{since } \{F\in\mathcal{P}(E\cap\operatorname{Eqs} f)\ \mid\ d\notin F\}=\mathcal{U})}} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \\
&\qquad \left(\begin{array}{c}\text{here, we have split the sum into two parts:}\\ \text{one containing all addends with } d\in B,\\ \text{and one containing all addends with } d\notin B\end{array}\right) \\
&= \underbrace{\sum_{B\in\mathcal{V}} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K}_{\substack{=\sum_{B\in\mathcal{U}} (-1)^{|\Phi(B)|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq\Phi(B)}} a_K \\ (\text{here, we have} \\ \text{substituted } \Phi(B) \text{ for } B \text{ in the sum,} \\ \text{since the map } \Phi:\mathcal{U}\to\mathcal{V} \text{ is a bijection})}} + \sum_{B\in\mathcal{U}} \underbrace{(-1)^{|B|}}_{\substack{=-(-1)^{|\Phi(B)|} \\ (\text{by } (40))}} \underbrace{\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}}_{\substack{=\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq\Phi(B)}} \\ (\text{because for every } K\in\mathfrak{K}, \\ \text{the condition } (K\subseteq B) \\ \text{is equivalent to } (K\subseteq\Phi(B)) \\ (\text{by } (37)))}} a_K \\
&= \sum_{B\in\mathcal{U}} (-1)^{|\Phi(B)|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq\Phi(B)}} a_K + \sum_{B\in\mathcal{U}} \left(-(-1)^{|\Phi(B)|}\right) \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq\Phi(B)}} a_K \\
&= \sum_{B\in\mathcal{U}} (-1)^{|\Phi(B)|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq\Phi(B)}} a_K - \sum_{B\in\mathcal{U}} (-1)^{|\Phi(B)|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq\Phi(B)}} a_K \\
&= 0 = [E\cap\operatorname{Eqs} f=\varnothing] \qquad (\text{since } [E\cap\operatorname{Eqs} f=\varnothing]=0).
\end{aligned} \tag{42}$$

This proves Lemma 2.10. □

We now finally proceed to the proof of Theorem 1.15:

Proof of Theorem 1.15. We have

$$X_G = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a}\\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_f \tag{43}$$

(by the definition of X_G). Now, if $f : V \to \mathbb{N}_+$ is a map, then we have the following logical equivalence:

$$(\text{the } \mathbb{N}_+\text{-coloring } f \text{ of } G \text{ is proper}) \iff (E\cap\operatorname{Eqs} f=\varnothing) \tag{44}$$

(because the $\mathbb{N}_+$-coloring f of G is proper if and only if $E \cap \operatorname{Eqs} f = \varnothing$ [33]). Now,

$$\begin{aligned}
&\sum_{f:V\to\mathbb{N}_+} \left[\underbrace{E \cap \operatorname{Eqs} f = \varnothing}_{\substack{\iff (\text{the } \mathbb{N}_+\text{-coloring } f \text{ of } G \text{ is proper}) \\ (\text{by (44)})}} \right] \mathbf{x}_f \\
&= \sum_{f:V\to\mathbb{N}_+} \left[\underbrace{\text{the } \mathbb{N}_+\text{-coloring } f \text{ of } G \text{ is proper}}_{\iff (f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G)} \right] \mathbf{x}_f \\
&= \sum_{f:V\to\mathbb{N}_+} \left[f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G \right] \mathbf{x}_f \\
&= \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \underbrace{\left[f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G \right]}_{\substack{=1 \\ (\text{since } f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G)}} \mathbf{x}_f \\
&\qquad + \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is not a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \underbrace{\left[f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G \right]}_{\substack{=0 \\ (\text{since } f \text{ is not a proper } \mathbb{N}_+\text{-coloring of } G)}} \mathbf{x}_f \\
&\qquad\qquad (\text{since each } f : V \to \mathbb{N}_+ \text{ is either a proper } \mathbb{N}_+\text{-coloring of } G \text{ or not}) \\
&= \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_f + \underbrace{\sum_{\substack{f:V\to\mathbb{N}_+ \text{ is not a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} 0 \mathbf{x}_f}_{=0} = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_f \\
&= X_G
\end{aligned} \tag{45}$$

(by (43)).

However, for every $f : V \to \mathbb{N}_+$, we have

$$\sum_{B \subseteq E \cap \operatorname{Eqs} f} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K = [E \cap \operatorname{Eqs} f = \varnothing] \tag{46}$$

(by Lemma 2.10 (applied to $\mathbb{N}_+$ instead of Y)).

For every $f : V \to \mathbb{N}_+$, we have

$$\{F \subseteq E \mid F \subseteq \operatorname{Eqs} f\} = \mathcal{P}(E \cap \operatorname{Eqs} f) \tag{47}$$

[34].

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[33] by Lemma 2.3 (applied to $\mathbb{N}_+$ instead of X)

[34] *Proof of (47):* Let $f : V \to \mathbb{N}_+$.

Let $B \in \{F \subseteq E \mid F \subseteq \operatorname{Eqs} f\}$. Thus, B is a subset F of E satisfying $F \subseteq \operatorname{Eqs} f$. In other words, B is a subset of E and satisfies $B \subseteq \operatorname{Eqs} f$. Since B is a subset of E, we have $B \subseteq E$. Combining this with $B \subseteq \operatorname{Eqs} f$, we obtain $B \subseteq E \cap \operatorname{Eqs} f$. In other words, $B \in \mathcal{P}(E \cap \operatorname{Eqs} f)$.

For every $f : V \to \mathbb{N}_+$, we have

$$\begin{aligned}
&\underbrace{\sum_{\substack{B\subseteq E;\\ B\subseteq \operatorname{Eqs} f}}}_{\substack{=\sum_{B\in\{F\subseteq E\ \mid\ F\subseteq \operatorname{Eqs} f\}} = \sum_{B\in\mathcal{P}(E\cap \operatorname{Eqs} f)} \\ \text{(because } \{F\subseteq E\ \mid\ F\subseteq \operatorname{Eqs} f\}=\mathcal{P}(E\cap \operatorname{Eqs} f) \\ \text{(by (47)))}}} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \\
&= \underbrace{\sum_{B\in\mathcal{P}(E\cap \operatorname{Eqs} f)}}_{=\sum_{B\subseteq E\cap \operatorname{Eqs} f}} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K = \sum_{B\subseteq E\cap \operatorname{Eqs} f} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \\
&= [E\cap \operatorname{Eqs} f = \varnothing] \qquad (49)
\end{aligned}$$

(by (46)).

Let us now forget that we fixed B. We thus have proven that every $B \in \{F \subseteq E \mid F \subseteq \operatorname{Eqs} f\}$ satisfies $B \in \mathcal{P}(E \cap \operatorname{Eqs} f)$. In other words,

$$\{F \subseteq E \mid F \subseteq \operatorname{Eqs} f\} \subseteq \mathcal{P}(E \cap \operatorname{Eqs} f). \qquad (48)$$

On the other hand, let $C \in \mathcal{P}(E \cap \operatorname{Eqs} f)$. Thus, C is a subset of $E \cap \operatorname{Eqs} f$. Hence, $C \subseteq E \cap \operatorname{Eqs} f \subseteq E$, so that C is a subset of E. Also, $C \subseteq E \cap \operatorname{Eqs} f \subseteq \operatorname{Eqs} f$. Thus, C is a subset of E and satisfies $C \subseteq \operatorname{Eqs} f$. In other words, C is a subset F of E satisfying $F \subseteq \operatorname{Eqs} f$. In other words, $C \in \{F \subseteq E \mid F \subseteq \operatorname{Eqs} f\}$.

Let us now forget that we fixed C. We thus have proven that every $C \in \mathcal{P}(E \cap \operatorname{Eqs} f)$ satisfies $C \in \{F \subseteq E \mid F \subseteq \operatorname{Eqs} f\}$. In other words,

$$\mathcal{P}(E \cap \operatorname{Eqs} f) \subseteq \{F \subseteq E \mid F \subseteq \operatorname{Eqs} f\}.$$

Combining this inclusion with (48), we obtain $\{F \subseteq E \mid F \subseteq \operatorname{Eqs} f\} = \mathcal{P}(E \cap \operatorname{Eqs} f)$. This proves (47).

Now, (45) yields

$$\begin{aligned}
X_G &= \sum_{f:V\to\mathbb{N}_+} \underbrace{[E\cap \operatorname{Eqs} f = \varnothing]}_{\substack{= \sum\limits_{\substack{B\subseteq E;\\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \prod\limits_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \\ \text{(by (49))}}} \mathbf{x}_f \\
&= \sum_{f:V\to\mathbb{N}_+} \left(\sum_{\substack{B\subseteq E;\\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \mathbf{x}_f = \underbrace{\sum_{f:V\to\mathbb{N}_+} \sum_{\substack{B\subseteq E;\\ B\subseteq \operatorname{Eqs} f}}}_{= \sum\limits_{B\subseteq E} \sum\limits_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \mathbf{x}_f \\
&= \sum_{B\subseteq E} \sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \mathbf{x}_f \\
&= \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \underbrace{\sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} \mathbf{x}_f}_{\substack{= p_{\lambda(V,B)} \\ \text{(by Lemma 2.7} \\ \text{(since } (V,B) \text{ is a finite graph} \\ \text{(since } V \text{ is a finite set and } B\subseteq E\subseteq \binom{V}{2}\text{)))}}} \\
&= \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) p_{\lambda(V,B)} = \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} a_K \right) p_{\lambda(V,F)}
\end{aligned}$$

(here, we have renamed the summation index B as F). This proves Theorem 1.15. ☐

Thus, Theorem 1.15 is proven; as we know, this entails the correctness of Theorem 1.11, Corollary 1.17 and Corollary 1.18.

3. The chromatic polynomial

3.1. Definition

We have so far studied the chromatic symmetric function. We shall now apply the above results to the chromatic polynomial. The definition of the chromatic polynomial rests upon the following fact:

Theorem 3.1. Let $G = (V, E)$ be a finite graph. Then, there exists a unique polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies

$$P(q) = (\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G).$$

Definition 3.2. Let $G = (V, E)$ be a finite graph. Theorem 3.1 shows that there exists a polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies $P(q) = (\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G)$. This polynomial P is called the *chromatic polynomial* of G, and will be denoted by χ_G.

We shall later prove Theorem 3.1 (as a consequence of something stronger that we show). First, we shall state some formulas for the chromatic polynomial which are analogues of results proven before for the chromatic symmetric function.

3.2. Formulas for χ_G

Before we state several formulas for χ_G, we need to introduce one more notation:

Definition 3.3. Let G be a finite graph. We let $\operatorname{conn} G$ denote the number of connected components of G.

The following results are analogues of Theorem 1.11, Theorem 1.15, Corollary 1.17 and Corollary 1.18, respectively:

Theorem 3.4. Let $G = (V, E)$ be a finite graph. Then,

$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} x^{\operatorname{conn}(V,F)}.$$

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\operatorname{conn}(V, F)$ is understood according to Definition 3.3.)

Theorem 3.5. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) x^{\operatorname{conn}(V,F)}.$$

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\operatorname{conn}(V, F)$ is understood according to Definition 3.3.)

Corollary 3.6. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$\chi_G = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} x^{\operatorname{conn}(V,F)}.$$

Corollary 3.7. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Then,

$$\chi_G = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} x^{\operatorname{conn}(V,F)}.$$

Except for Theorem 3.5, these results are not new; in fact, Corollary 3.6 is a particular case of [DohTri14, (12)], and of course we can obtain Corollary 3.7 and Theorem 3.4 as particular cases of Corollary 3.6.

3.3. Proofs

Nevertheless, for the sake of completeness, we shall give proofs of all the five results above (Theorem 3.1, Theorem 3.4, Theorem 3.5, Corollary 3.6 and Corollary 3.7).

There are two approaches to these results (except for Theorem 3.1): One is to prove them similarly to how we proved the analogous results about X_G; the other is to derive them from the latter. We shall take the first approach, since it yields a proof of the classical Theorem 3.1 "for free". We begin with an analogue of Lemma 2.7:

Lemma 3.8. Let (V, B) be a finite graph. Let $q \in \mathbb{N}$. Then,

$$\sum_{\substack{f : V \to \{1,2,\ldots,q\}; \\ B \subseteq \operatorname{Eqs} f}} 1 = q^{\operatorname{conn}(V,B)}.$$

(Here, the expression $\operatorname{conn}(V, B)$ is understood according to Definition 1.10 **(b)**.)

One way to prove Lemma 3.8 is to evaluate the equality given by Lemma 2.7 at $x_k = \begin{cases} 1, & \text{if } k \leq q; \\ 0, & \text{if } k > q \end{cases}$. Another proof can be obtained by mimicking our proof of Lemma 2.7:

Proof of Lemma 3.8. Let $\sim$ denote the equivalence relation $\sim_{(V,B)}$ (defined as in Definition 1.10 **(a)**). The connected components of (V,B) are the $\sim_{(V,B)}$-equivalence classes (because this is how the connected components of (V,B) are defined). In other words, the connected components of (V,B) are the $\sim$-equivalence classes (since $\sim$ is the relation $\sim_{(V,B)}$). In other words, the connected components of (V,B) are the elements of $V/(\sim)$ (since the elements of $V/(\sim)$ are the $\sim$-equivalence classes (by the definition of $V/(\sim)$)).

Let Y be the set $\{1,2,\ldots,q\}$. Thus, $|Y|=q$. A set $Y^V_\sim$ is defined (according to Definition 1.7 **(b)**).

Proposition 1.8 **(b)** (applied to $X=V$) shows that the map

$$Y^{V/(\sim)} \to Y^V_\sim, \qquad f \mapsto f \circ \pi_V$$

is a bijection. Thus, there exists a bijection $Y^{V/(\sim)} \to Y^V_\sim$ (namely, this map). Hence, $\left|Y^V_\sim\right| = \left|Y^{V/(\sim)}\right|$.

For every map $f : V \to Y$, we have the following equivalence:

$$(B \subseteq \operatorname{Eqs} f) \iff \left(f \in Y^V_\sim\right) \tag{50}$$

(according to Lemma 2.5). Thus, we have the following equality of summation signs:

$$\sum_{\substack{f:V\to Y;\\ B\subseteq \operatorname{Eqs} f}} = \sum_{\substack{f:V\to Y;\\ f\in Y^V_\sim}} = \sum_{\substack{f\in Y^V;\\ f\in Y^V_\sim}} = \sum_{f\in Y^V_\sim}$$

(since $Y^V_\sim$ is a subset of Y^V). Hence,

$$\underbrace{\sum_{\substack{f:V\to Y;\\ B\subseteq \operatorname{Eqs} f}}}_{=\sum_{f\in Y^V_\sim}} 1 = \sum_{f\in Y^V_\sim} 1 = \left|Y^V_\sim\right| \cdot 1 = \left|Y^V_\sim\right| = \left|Y^{V/(\sim)}\right|. \tag{51}$$

Now, let $(C_1, C_2, \ldots, C_k)$ be a list of all connected components of (V,B). [35] Thus, k is the number of connected components of (V,B). In other words, k is $\operatorname{conn}(V,B)$ (since $\operatorname{conn}(V,B)$ is the number of connected components of (V,B) (by the definition of $\operatorname{conn}(V,B)$)). In other words, $k = \operatorname{conn}(V,B)$.

Recall that $(C_1, C_2, \ldots, C_k)$ is a list of all connected components of (V,B). In other words, $(C_1, C_2, \ldots, C_k)$ is a list of all elements of $V/(\sim)$ (since the elements of $V/(\sim)$ are the connected components of (V,B)). Moreover, every element of $V/(\sim)$ appears exactly once in this list $(C_1, C_2, \ldots, C_k)$ (since the entries of the list $(C_1, C_2, \ldots, C_k)$ are pairwise distinct[36]). Thus, $(C_1, C_2, \ldots, C_k)$ is a list of all

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[35] Every connected component of (V,B) should appear exactly once in this list.

[36] since every connected component of (V,B) appears exactly once in this list

elements of $V/(\sim)$, and contains each of these elements exactly once. Hence, the map

$$\begin{aligned} Y^{V/(\sim)} &\to Y^k, \\ f &\mapsto (f(C_1), f(C_2), \ldots, f(C_k)) \end{aligned}$$

is a bijection (by Lemma 2.6, applied to $W = V/(\sim)$). Hence, there exists a bijection $Y^{V/(\sim)} \to Y^k$ (namely, this map). Thus, $\left|Y^k\right| = \left|Y^{V/(\sim)}\right|$.

Comparing this with (51), we obtain

$$\begin{aligned} \sum_{\substack{f:V\to Y;\\ B\subseteq \operatorname{Eqs} f}} 1 &= \left|Y^k\right| = |Y|^k = q^k \qquad (\text{since } |Y| = q) \\ &= q^{\operatorname{conn}(V,B)} \qquad (\text{since } k = \operatorname{conn}(V,B)). \end{aligned}$$

This proves Lemma 3.8. ☐

We shall now show a weaker version of Theorem 3.5 (as a stepping stone to the actual theorem):

> **Lemma 3.9.** Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Let $q \in \mathbb{N}$. Then,
>
> $$\begin{aligned} &(\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G) \\ &= \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} a_K \right) q^{\operatorname{conn}(V,F)}. \end{aligned}$$
>
> (Here, of course, the pair (V, F) is regarded as a graph, and the expression $\operatorname{conn}(V, F)$ is understood according to Definition 3.3.)

Proof of Lemma 3.9. Let $Q = \{1, 2, \ldots, q\}$. If $f : V \to Q$ is a map, then we have the following logical equivalence:

$$(\text{the } Q\text{-coloring } f \text{ of } G \text{ is proper}) \iff (E \cap \operatorname{Eqs} f = \varnothing) \tag{52}$$

(because the Q-coloring f of G is proper if and only if $E \cap \operatorname{Eqs} f = \varnothing$ [37]).

---

[37] by Lemma 2.3 (applied to Q instead of X)

Now,[38]

$$\begin{aligned}
&\sum_{f:V\to Q}\left[\underbrace{E\cap\operatorname{Eqs}f=\varnothing}_{\substack{\Longleftrightarrow\text{ (the }Q\text{-coloring }f\text{ of }G\text{ is proper)}\\ \text{(by (52))}}}\right]\\
&=\sum_{f:V\to Q}\left[\underbrace{\text{the }Q\text{-coloring }f\text{ of }G\text{ is proper}}_{\Longleftrightarrow\text{ (}f\text{ is a proper }Q\text{-coloring of }G\text{)}}\right]\\
&=\sum_{f:V\to Q}\left[f\text{ is a proper }Q\text{-coloring of }G\right]\\
&=\sum_{\substack{f:V\to Q\text{ is a}\\ \text{proper }Q\text{-coloring of }G}}\underbrace{\left[f\text{ is a proper }Q\text{-coloring of }G\right]}_{\substack{=1\\ \text{(since }f\text{ is a proper }Q\text{-coloring of }G\text{)}}}\\
&\qquad+\sum_{\substack{f:V\to Q\text{ is not a}\\ \text{proper }Q\text{-coloring of }G}}\underbrace{\left[f\text{ is a proper }Q\text{-coloring of }G\right]}_{\substack{=0\\ \text{(since }f\text{ is not a proper }Q\text{-coloring of }G\text{)}}}\\
&=\sum_{\substack{f:V\to Q\text{ is a}\\ \text{proper }Q\text{-coloring of }G}}1+\underbrace{\sum_{\substack{f:V\to Q\text{ is not a}\\ \text{proper }Q\text{-coloring of }G}}0}_{=0}=\sum_{\substack{f:V\to Q\text{ is a}\\ \text{proper }Q\text{-coloring of }G}}1\\
&=(\text{the number of all proper }Q\text{-colorings of }G)\cdot 1\\
&=(\text{the number of all proper }Q\text{-colorings of }G)\\
&=(\text{the number of all proper }\{1,2,\ldots,q\}\text{-colorings of }G)
\end{aligned}\tag{53}$$

(since $Q=\{1,2,\ldots,q\}$).

However, for every $f:V\to Q$, we have

$$\sum_{B\subseteq E\cap\operatorname{Eqs}f}(-1)^{|B|}\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K=[E\cap\operatorname{Eqs}f=\varnothing]\tag{54}$$

(by Lemma 2.10 (applied to Q instead of Y)).

For every $f:V\to Q$, we have

$$\{F\subseteq E\ \mid\ F\subseteq\operatorname{Eqs}f\}=\mathcal{P}(E\cap\operatorname{Eqs}f)\tag{55}$$

[39].

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[38] We are again using the Iverson bracket notation, as defined in Definition 2.9.

[39] *Proof of (55):* Let $f:V\to Q$.

Let $B\in\{F\subseteq E\ \mid\ F\subseteq\operatorname{Eqs}f\}$. Thus, B is a subset F of E satisfying $F\subseteq\operatorname{Eqs}f$. In other words, B is a subset of E and satisfies $B\subseteq\operatorname{Eqs}f$. Since B is a subset of E, we have $B\subseteq E$. Combining this with $B\subseteq\operatorname{Eqs}f$, we obtain $B\subseteq E\cap\operatorname{Eqs}f$. In other words, $B\in\mathcal{P}(E\cap\operatorname{Eqs}f)$.

For every $f : V \to Q$, we have

$$\begin{aligned}
&\underbrace{\sum_{\substack{B\subseteq E;\\ B\subseteq \operatorname{Eqs} f}}}_{\substack{=\sum_{B\in\{F\subseteq E \ \mid\ F\subseteq \operatorname{Eqs} f\}} = \sum_{B\in\mathcal{P}(E\cap\operatorname{Eqs} f)} \\ \text{(because } \{F\subseteq E \ \mid\ F\subseteq \operatorname{Eqs} f\}=\mathcal{P}(E\cap\operatorname{Eqs} f) \\ \text{(by (55)))}}} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \\
&= \underbrace{\sum_{B\in\mathcal{P}(E\cap\operatorname{Eqs} f)}}_{=\sum_{B\subseteq E\cap\operatorname{Eqs} f}} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K = \sum_{B\subseteq E\cap\operatorname{Eqs} f} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \\
&= [E\cap\operatorname{Eqs} f = \varnothing] \qquad (57)
\end{aligned}$$

(by (54)).

Let us now forget that we fixed B. We thus have proven that every $B \in \{F \subseteq E \mid F \subseteq \operatorname{Eqs} f\}$ satisfies $B \in \mathcal{P}(E \cap \operatorname{Eqs} f)$. In other words,

$$\{F \subseteq E \mid F \subseteq \operatorname{Eqs} f\} \subseteq \mathcal{P}(E \cap \operatorname{Eqs} f). \qquad (56)$$

On the other hand, let $C \in \mathcal{P}(E \cap \operatorname{Eqs} f)$. Thus, C is a subset of $E \cap \operatorname{Eqs} f$. Hence, $C \subseteq E \cap \operatorname{Eqs} f \subseteq E$, so that C is a subset of E. Also, $C \subseteq E \cap \operatorname{Eqs} f \subseteq \operatorname{Eqs} f$. Thus, C is a subset of E and satisfies $C \subseteq \operatorname{Eqs} f$. In other words, C is a subset F of E satisfying $F \subseteq \operatorname{Eqs} f$. In other words, $C \in \{F \subseteq E \mid F \subseteq \operatorname{Eqs} f\}$.

Let us now forget that we fixed C. We thus have proven that every $C \in \mathcal{P}(E \cap \operatorname{Eqs} f)$ satisfies $C \in \{F \subseteq E \mid F \subseteq \operatorname{Eqs} f\}$. In other words,

$$\mathcal{P}(E \cap \operatorname{Eqs} f) \subseteq \{F \subseteq E \mid F \subseteq \operatorname{Eqs} f\}.$$

Combining this inclusion with (56), we obtain $\{F \subseteq E \mid F \subseteq \operatorname{Eqs} f\} = \mathcal{P}(E \cap \operatorname{Eqs} f)$. This proves (55).

Now, (53) yields

$$\begin{aligned}
&(\text{the number of all proper } \{1,2,\ldots,q\}\text{-colorings of } G) \\
&= \sum_{f:V\to Q} \underbrace{[E \cap \operatorname{Eqs} f = \varnothing]}_{\substack{= \sum\limits_{\substack{B\subseteq E;\\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \prod\limits_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \\ \text{(by (57))}}} \\
&= \sum_{f:V\to Q} \left(\sum_{\substack{B\subseteq E;\\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) = \underbrace{\sum_{f:V\to Q} \sum_{\substack{B\subseteq E;\\ B\subseteq \operatorname{Eqs} f}}}_{= \sum\limits_{B\subseteq E} \sum\limits_{\substack{f:V\to Q;\\ B\subseteq \operatorname{Eqs} f}}} (-1)^{|B|} \underbrace{\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right)}_{= \left(\prod\limits_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) 1} \\
&= \sum_{B\subseteq E} \sum_{\substack{f:V\to Q;\\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) 1 = \sum_{B\subseteq E} \sum_{\substack{f:V\to \{1,2,\ldots,q\};\\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) 1 \\
&\qquad (\text{since } Q = \{1,2,\ldots,q\}) \\
&= \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \underbrace{\sum_{\substack{f:V\to \{1,2,\ldots,q\};\\ B\subseteq \operatorname{Eqs} f}} 1}_{\substack{= q^{\operatorname{conn}(V,B)} \\ \text{(by Lemma 3.8} \\ \text{(since } (V,B) \text{ is a finite graph} \\ \text{(since } V \text{ is a finite set and } B\subseteq E\subseteq \binom{V}{2}\text{)))}}} \\
&= \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) q^{\operatorname{conn}(V,B)} = \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} a_K \right) q^{\operatorname{conn}(V,F)}
\end{aligned}$$

(here, we have renamed the summation index B as F). This proves Lemma 3.9. ☐

From Lemma 3.9, we obtain the following consequence:

Lemma 3.10. Let $G = (V, E)$ be a finite graph. Let $q \in \mathbb{N}$. Then,

$$\begin{aligned}
&(\text{the number of all proper } \{1,2,\ldots,q\}\text{-colorings of } G) \\
&= \sum_{F\subseteq E} (-1)^{|F|} q^{\operatorname{conn}(V,F)}.
\end{aligned}$$

> (Here, of course, the pair (V, F) is regarded as a graph, and the expression $\operatorname{conn}(V, F)$ is understood according to Definition 3.3.)

Proof of Lemma 3.10. Let X be the totally ordered set $\{1\}$ (equipped with the only possible order on this set). Let $\ell : E \to X$ be the function sending each $e \in E$ to $1 \in X$. Let $\mathfrak{K}$ be the empty set. Clearly, $\mathfrak{K}$ is a set of broken circuits of G. Lemma 3.9 (applied to 0 instead of a_K) yields

$$\begin{aligned}
&(\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G) \\
&= \sum_{F \subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq F}} 0 \right)}_{\substack{=(\text{empty product})\\ (\text{since } \mathfrak{K} \text{ is the empty set})}} q^{\operatorname{conn}(V,F)} \\
&= \sum_{F \subseteq E} (-1)^{|F|} \underbrace{(\text{empty product})}_{=1} q^{\operatorname{conn}(V,F)} = \sum_{F \subseteq E} (-1)^{|F|} q^{\operatorname{conn}(V,F)}.
\end{aligned}$$

This proves Lemma 3.10. ☐

Next, we recall a classical fact about polynomials over fields: Namely, if a polynomial (in one variable) over a field has infinitely many roots, then this polynomial is 0. Let us state this more formally:

> **Proposition 3.11.** Let K be a field. Let $P \in K[x]$ be a polynomial over K. Assume that there are infinitely many $\lambda \in K$ satisfying $P(\lambda) = 0$. Then, $P = 0$.

We shall use the following two consequences of this proposition:

> **Corollary 3.12.** Let R be an integral domain. Assume that the canonical ring homomorphism from the ring $\mathbb{Z}$ to the ring R is injective. Let $P \in R[x]$ be a polynomial over R. Assume that $P(q \cdot 1_R) = 0$ for every $q \in \mathbb{N}$ (where 1_R denotes the unity of R). Then, $P = 0$.

Proof of Corollary 3.12. Let K denote the fraction field of the integral domain R. Then, there is a canonical injective ring homomorphism $R \to K$. We use this homomorphism to regard R as a subring of K. Consequently, $R[x]$ will be regarded as a subring of $K[x]$. In particular, the polynomial $P \in R[x]$ will thus be regarded as a polynomial in $K[x]$.

Let $\iota : \mathbb{Z} \to R$ be the canonical ring homomorphism from the ring $\mathbb{Z}$ to the ring R. (Thus, ι sends every $q \in \mathbb{Z}$ to $q \cdot 1_R \in R$.)

We have assumed that the canonical ring homomorphism from the ring $\mathbb{Z}$ to the ring R is injective. In other words, the map ι is injective (since the map ι is the canonical ring homomorphism from the ring $\mathbb{Z}$ to the ring R). Hence,

$\left|\iota(\mathbb{N})\right| = |\mathbb{N}| = \infty$. Moreover, every $\lambda \in \iota(\mathbb{N})$ satisfies $P(\lambda) = 0$ [40]. Hence, there are infinitely many $\lambda \in K$ satisfying $P(\lambda) = 0$ (because there are infinitely many $\lambda \in \iota(\mathbb{N})$ (since $\left|\iota(\mathbb{N})\right| = \infty$), and because they all are elements of K (since $\iota(\mathbb{N}) \subseteq R \subseteq K$)). Hence, Proposition 3.11 shows that $P = 0$. This proves Corollary 3.12. □

Corollary 3.13. Let R be an integral domain such that $\mathbb{Z}$ is a subring of R. Let $P_1 \in R[x]$ and $P_2 \in R[x]$ be two polynomials over R. Assume that every $q \in \mathbb{N}$ satisfies

$$P_1(q) = P_2(q). \tag{58}$$

Then, $P_1 = P_2$.

Proof of Corollary 3.13. We have assumed that $\mathbb{Z}$ is a subring of R. Hence, the canonical ring homomorphism from the ring $\mathbb{Z}$ to the ring R is just the inclusion map $\mathbb{Z} \to R$, and thus is injective.

Every $q \in \mathbb{N}$ satisfies

$$(P_1 - P_2)\left(\underbrace{q \cdot 1_R}_{=q}\right) = (P_1 - P_2)(q) = \underbrace{P_1(q)}_{\substack{=P_2(q) \\ \text{(by (58))}}} - P_2(q) = P_2(q) - P_2(q) = 0.$$

In other words, we have $(P_1 - P_2)(q \cdot 1_R) = 0$ for every $q \in \mathbb{N}$. Hence, Corollary 3.12 (applied to $P = P_1 - P_2$) yields that $P_1 - P_2 = 0$. In other words, $P_1 = P_2$. This proves Corollary 3.13. □

We can now prove the classical Theorem 3.1:

Proof of Theorem 3.1. We need to show that there exists a unique polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies

$$P(q) = (\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G). \tag{59}$$

Let us first show that there exists at most one such polynomial. Indeed, let P_1 and P_2 be two polynomials $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies (59). We shall show that $P_1 = P_2$.

We know that P_1 is a polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies (59). In other words, P_1 is a polynomial in $\mathbb{Z}[x]$ and every $q \in \mathbb{N}$ satisfies

$$P_1(q) = (\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G). \tag{60}$$

---

[40] *Proof.* Let $\lambda \in \iota(\mathbb{N})$. Thus, there exists some $h \in \mathbb{N}$ such that $\lambda = \iota(h)$. Consider this h.

We have assumed that $P(q \cdot 1_R) = 0$ for every $q \in \mathbb{N}$. Applying this to $q = h$, we obtain $P(h \cdot 1_R) = 0$. But $\lambda = \iota(h) = h \cdot 1_R$ (by the definition of ι). Hence, $P\left(\underbrace{\lambda}_{=h \cdot 1_R}\right) = P(h \cdot 1_R) = 0$, qed.

The same argument (applied to P_2 instead of P_1) shows that P_2 is a polynomial in $\mathbb{Z}[x]$ and every $q \in \mathbb{N}$ satisfies

$$P_2(q) = (\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G). \tag{61}$$

The ring $\mathbb{Z}$ is clearly a subring of $\mathbb{Z}$. Furthermore, every $q \in \mathbb{N}$ satisfies

$$\begin{aligned} P_1(q) &= (\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G) \\ &\qquad\qquad (\text{by (60)}) \\ &= P_2(q) \qquad (\text{by (61)}). \end{aligned} \tag{62}$$

Hence, Corollary 3.13 (applied to $\mathbb{Z}$ instead of R) shows that $P_1 = P_2$.

Now, let us forget that we fixed P_1 and P_2. We thus have showed that if P_1 and P_2 are two polynomials $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies (59), then $P_1 = P_2$. In other words, there exists **at most one** polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies (59).

Let us now prove that there exists at least one such polynomial. Indeed, define a polynomial $Q \in \mathbb{Z}[x]$ by

$$Q = \sum_{F \subseteq E} (-1)^{|F|} x^{\operatorname{conn}(V,F)}. \tag{63}$$

Then, every $q \in \mathbb{N}$ satisfies

$$\begin{aligned} Q(q) &= \sum_{F \subseteq E} (-1)^{|F|} q^{\operatorname{conn}(V,F)} \qquad (\text{here, we have substituted } q \text{ for } x \text{ in (63)}) \\ &= (\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G) \end{aligned}$$

(by Lemma 3.10). In other words, every $q \in \mathbb{N}$ satisfies (59) for $P = Q$. Thus, Q is a polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies (59). Hence, there exists **at least** one polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies (59) (namely, $P = Q$). Consequently, there exists **exactly** one polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies (59) (because we have already shown that there exists **at most** one such polynomial). This proves Theorem 3.1. ☐

Next, it is the turn of Theorem 3.5:

Proof of Theorem 3.5. Let R be the polynomial ring $\mathbb{Z}[y_K \mid K \in \mathfrak{K}]$, where y_K is a new indeterminate for each $K \in \mathfrak{K}$. Clearly, R is an integral domain (since R is a polynomial ring over $\mathbb{Z}$).

Moreover, $\mathbb{Z}$ is a subring of R (since R is a polynomial ring over $\mathbb{Z}$). We therefore regard $\mathbb{Z}[x]$ as a subring of $R[x]$.

The chromatic polynomial χ_G is the unique polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies $P(q) = (\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G)$ (because this is how χ_G is defined). Thus, χ_G is a polynomial in $\mathbb{Z}[x]$ and has the property that every $q \in \mathbb{N}$ satisfies

$$\chi_G(q) = (\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G). \tag{64}$$

Lemma 3.9 (applied to R and y_K instead of $\mathbf{k}$ and a_K) yields that

$$\begin{aligned}&\left(\text{the number of all proper } \{1,2,\ldots,q\}\text{-colorings of } G\right)\\&=\sum_{F\subseteq E}(-1)^{|F|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}}y_K\right)q^{\operatorname{conn}(V,F)}\end{aligned}$$

for every $q\in\mathbb{N}$. Thus, for every $q\in\mathbb{N}$, we have

$$\begin{aligned}&\sum_{F\subseteq E}(-1)^{|F|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}}y_K\right)q^{\operatorname{conn}(V,F)}\\&=\left(\text{the number of all proper } \{1,2,\ldots,q\}\text{-colorings of } G\right)\\&=\chi_G(q)\qquad\left(\text{by (64)}\right).\end{aligned}\tag{65}$$

We have $\chi_G\in\mathbb{Z}[x]\subseteq R[x]$ (since $\mathbb{Z}[x]$ is a subring of $R[x]$).

On the other hand, let us define a polynomial $\widetilde{P}\in R[x]$ by

$$\widetilde{P}=\sum_{F\subseteq E}(-1)^{|F|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}}y_K\right)x^{\operatorname{conn}(V,F)}.\tag{66}$$

Every $q\in\mathbb{N}$ satisfies

$$\begin{aligned}\widetilde{P}(q)&=\sum_{F\subseteq E}(-1)^{|F|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}}y_K\right)q^{\operatorname{conn}(V,F)}\\&\qquad\qquad\left(\text{here, we have substituted } q \text{ for } x \text{ in (66)}\right)\\&=\chi_G(q)\qquad\left(\text{by (65)}\right).\end{aligned}\tag{67}$$

Thus, Corollary 3.13 (applied to $P_1=\widetilde{P}$ and $P_2=\chi_G$) shows that $\widetilde{P}=\chi_G$. In other words, $\chi_G=\widetilde{P}$.

Let $\mathbf{a}$ denote the family $(a_K)_{K\in\mathfrak{K}}\in\mathbf{k}^{\mathfrak{K}}$ of elements of $\mathbf{k}$.

Now, recall that R is the polynomial ring $\mathbb{Z}[y_K \mid K\in\mathfrak{K}]$. Hence, R satisfies the following universal property (the well-known universal property of a polynomial ring): For any commutative $\mathbb{Z}$-algebra B and any family $\mathbf{b}=(b_K)_{K\in\mathfrak{K}}\in B^{\mathfrak{K}}$ of elements of B, there exists a unique $\mathbb{Z}$-algebra homomorphism $\psi:R\to B$ satisfying

$$\left(\psi(y_K)=b_K\qquad\text{for every } K\in\mathfrak{K}\right).$$

This $\mathbb{Z}$-algebra homomorphism ψ is denoted by $\operatorname{ev}_{\mathbf{b}}$, and is called the *evaluation homomorphism* at the family $\mathbf{b}$.

Thus we have constructed a $\mathbb{Z}$-algebra homomorphism $\operatorname{ev}_{\mathbf{b}} : R \to B$ for every commutative $\mathbb{Z}$-algebra B and every family $\mathbf{b} = (b_K)_{K \in \mathfrak{K}} \in B^{\mathfrak{K}}$ of elements of B. Applying this construction to $B = \mathbf{k}$, $\mathbf{b} = \mathbf{a}$ and $b_K = a_K$, we obtain a $\mathbb{Z}$-algebra homomorphism $\operatorname{ev}_{\mathbf{a}} : R \to \mathbf{k}$. This homomorphism, in turn, induces a $\mathbb{Z}[x]$-algebra homomorphism $\operatorname{ev}_{\mathbf{a}}[x] : R[x] \to \mathbf{k}[x]$ satisfying $(\operatorname{ev}_{\mathbf{a}}[x])(x) = x$. [41]. Consider this homomorphism $\operatorname{ev}_{\mathbf{a}}[x]$.

Recall that $\mathbf{a} = (a_K)_{K \in \mathfrak{K}}$. Hence, $\operatorname{ev}_{\mathbf{a}}$ is the unique $\mathbb{Z}$-algebra homomorphism $\psi : R \to \mathbf{k}$ satisfying

$$(\psi(y_K) = a_K \qquad \text{for every } K \in \mathfrak{K})$$

(by the definition of $\operatorname{ev}_{\mathbf{a}}$). Thus, $\operatorname{ev}_{\mathbf{a}}$ is a $\mathbb{Z}$-algebra homomorphism and satisfies

$$\operatorname{ev}_{\mathbf{a}}(y_K) = a_K \qquad \text{for every } K \in \mathfrak{K}. \tag{68}$$

The construction of $\operatorname{ev}_{\mathbf{a}}[x]$ shows that

$$(\operatorname{ev}_{\mathbf{a}}[x])(u) = \operatorname{ev}_{\mathbf{a}}(u) \qquad \text{for every } u \in R. \tag{69}$$

Now, for every $K \in \mathfrak{K}$, we have

$$\begin{aligned}(\operatorname{ev}_{\mathbf{a}}[x])(y_K) &= \operatorname{ev}_{\mathbf{a}}(y_K) \qquad (\text{by (69), applied to } u = y_K) \\ &= a_K \qquad (\text{by (68)}).\end{aligned} \tag{70}$$

By its definition, the homomorphism $\operatorname{ev}_{\mathbf{a}}[x]$ preserves $\mathbb{Z}[x]$ (or, rather, sends every element of $\mathbb{Z}[x] \subseteq R[x]$ to the corresponding element of $\mathbb{Z}[x] \subseteq \mathbf{k}[x]$). In other words, $(\operatorname{ev}_{\mathbf{a}}[x])(Q) = Q$ for every $Q \in \mathbb{Z}[x]$. Applying this to $Q = \chi_G$, we obtain

$$(\operatorname{ev}_{\mathbf{a}}[x])(\chi_G) = \chi_G$$

---

[41] This homomorphism $\operatorname{ev}_{\mathbf{a}}[x]$ is explicitly given by the formula

$$(\operatorname{ev}_{\mathbf{a}}[x])\left(\sum_{i=0}^{\infty} r_i x^i\right) = \sum_{i=0}^{\infty} \operatorname{ev}_{\mathbf{a}}(r_i) \cdot x^i$$

for every sequence $(r_0, r_1, r_2, \ldots) \in R^{\infty}$ of elements of R which satisfies $r_i = 0$ for all sufficiently high i.

(since $\chi_G \in \mathbb{Z}[x]$). Thus,

$$\begin{aligned}
\chi_G &= (\operatorname{ev}_{\mathbf{a}}[x]) \left(\underbrace{\chi_G}_{=\widetilde{P}=\sum\limits_{F\subseteq E}(-1)^{|F|}\left(\prod\limits_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} y_K\right) x^{\operatorname{conn}(V,F)}} \right) \\
&= (\operatorname{ev}_{\mathbf{a}}[x]) \left(\sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} y_K \right) x^{\operatorname{conn}(V,F)} \right) \\
&= \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} \underbrace{(\operatorname{ev}_{\mathbf{a}}[x])(y_K)}_{\substack{=a_K \\ \text{(by (70))}}} \right) \left(\underbrace{(\operatorname{ev}_{\mathbf{a}}[x])(x)}_{=x} \right)^{\operatorname{conn}(V,F)} \\
&\qquad \left(\text{since } \operatorname{ev}_{\mathbf{a}}[x] \text{ is a } \mathbb{Z}\text{-algebra homomorphism}\right) \\
&= \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} a_K \right) x^{\operatorname{conn}(V,F)}.
\end{aligned}$$

This proves Theorem 3.5. □

Now that Theorem 3.5 is proven, we can derive Theorem 3.4, Corollary 3.6 and Corollary 3.7 from it in the same way as we have derived Theorem 1.11, Corollary 1.17 and Corollary 1.18 from Theorem 1.15. Here are the details:

Proof of Corollary 3.6. We can apply Theorem 3.5 to 0 instead of a_K. As a result, we obtain

$$\chi_G = \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0 \right) x^{\operatorname{conn}(V,F)}. \tag{71}$$

Now, if F is any subset of E, then

$$\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0 = \begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} \tag{72}$$

[42]. Thus, (71) becomes

$$\chi_G = \sum_{F\subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0\right)}_{\substack{=\begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free};\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases}\\ \text{(by (72))}}} x^{\operatorname{conn}(V,F)}$$

$$= \sum_{F\subseteq E} (-1)^{|F|} \begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free};\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases} x^{\operatorname{conn}(V,F)}$$

$$= \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} \underbrace{\begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free};\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases}}_{\substack{=1\\ \text{(since } F \text{ is } \mathfrak{K}\text{-free)}}} x^{\operatorname{conn}(V,F)}$$

$$+ \sum_{\substack{F\subseteq E;\\ F \text{ is not } \mathfrak{K}\text{-free}}} (-1)^{|F|} \underbrace{\begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free};\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases}}_{\substack{=0\\ \text{(since } F \text{ is not } \mathfrak{K}\text{-free)}}} x^{\operatorname{conn}(V,F)}$$

$$= \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} x^{\operatorname{conn}(V,F)} + \underbrace{\sum_{\substack{F\subseteq E;\\ F \text{ is not } \mathfrak{K}\text{-free}}} (-1)^{|F|} 0 x^{\operatorname{conn}(V,F)}}_{=0}$$

$$= \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} x^{\operatorname{conn}(V,F)}.$$

This proves Corollary 3.6. ☐

Proof of Corollary 3.7. Let $\mathfrak{K}$ be the set of all broken circuits of G. Thus, the elements of $\mathfrak{K}$ are the broken circuits of G.

Now, for every subset F of E, we have the following equivalence of statements:

$$\begin{aligned}
& (F \text{ is } \mathfrak{K}\text{-free}) \\
\Longleftrightarrow\ & (F \text{ contains no } K \in \mathfrak{K} \text{ as a subset}) \\
& \qquad \left(\begin{array}{c}\text{because } F \text{ is } \mathfrak{K}\text{-free if and only if } F \text{ contains no } K \in \mathfrak{K} \text{ as a subset} \\ \text{(by the definition of ``}\mathfrak{K}\text{-free'')}\end{array}\right) \\
\Longleftrightarrow\ & (F \text{ contains no element of } \mathfrak{K} \text{ as a subset}) \\
\Longleftrightarrow\ & (F \text{ contains no broken circuit of } G \text{ as a subset})
\end{aligned}$$

---

[42] *Proof of (72):* The equality (72) has already been proven in our proof of Corollary 1.17.

(since the elements of $\mathfrak{K}$ are the broken circuits of G). Hence, $\sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} = \sum_{\substack{F\subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}}$ (an equality between summation signs). Now, Corollary 3.6 yields

$$\chi_G = \underbrace{\sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}}}_{=\sum_{\substack{F\subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}}} (-1)^{|F|} x^{\operatorname{conn}(V,F)} = \sum_{\substack{F\subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} x^{\operatorname{conn}(V,F)}.$$

This proves Corollary 3.7. ☐

Proof of Theorem 3.4. Let X be the totally ordered set $\{1\}$ (equipped with the only possible order on this set). Let $\ell : E \to X$ be the function sending each $e \in E$ to $1 \in X$. Let $\mathfrak{K}$ be the empty set. Clearly, $\mathfrak{K}$ is a set of broken circuits of G. Theorem 3.5 (applied to 0 instead of a_K) yields

$$\chi_G = \sum_{F\subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0\right)}_{\substack{=(\text{empty product})\\ (\text{since } \mathfrak{K} \text{ is the empty set})}} x^{\operatorname{conn}(V,F)}$$
$$= \sum_{F\subseteq E} (-1)^{|F|} \underbrace{(\text{empty product})}_{=1} x^{\operatorname{conn}(V,F)} = \sum_{F\subseteq E} (-1)^{|F|} x^{\operatorname{conn}(V,F)}.$$

This proves Theorem 3.4. ☐

3.4. Special case: Whitney's Broken-Circuit Theorem

Corollary 3.7 is commonly stated in the following simplified (if less general) form:

> **Corollary 3.14.** Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be an injective labeling function. Then,
>
> $$\chi_G = \sum_{\substack{F\subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} x^{|V|-|F|}.$$

Corollary 3.14 is known as *Whitney's Broken-Circuit theorem* (see, e.g., [BlaSag86]). In his original 1932 paper [Whitne32, §7], Whitney stated its claim as "the $x^{|V|-i}$-coefficient of χ_G is $(-1)^i$ times the number of i-element subsets of E that contain no broken circuit as a subset", which is easily seen to be equivalent to our formulation.

Notice that ℓ is required to be injective in Corollary 3.14; the purpose of this requirement is to ensure that every circuit of G has a unique edge e with maximum $\ell(e)$, and thus induces a broken circuit of G. The proof of Corollary 3.14 relies on the following standard result:

Lemma 3.15. Let (V, F) be a finite graph. Assume that (V, F) has no circuits. Then, $\operatorname{conn}(V, F) = |V| - |F|$.

(A graph that has no circuits is commonly known as a *forest*.)

Lemma 3.15 is both extremely elementary and well-known; for example, it appears in [Bona17, Proposition 10.6], in [Bollob79, §I.2, Corollary 6] and in [Grinbe21, Theorem 6.3.15 **(e)**]. Let us now see how it entails Corollary 3.14:

Proof of Corollary 3.14. We first claim that if F is a subset of E such that F contains no broken circuit of G as a subset, then

$$\operatorname{conn}(V, F) = |V| - |F|. \tag{73}$$

Proof of (73): Let F be a subset of E such that F contains no broken circuit of G as a subset. We shall now show that the graph (V, F) has no circuits.

Indeed, assume the contrary (for the sake of contradiction). Thus, the graph (V, F) has a circuit. In other words, there exists a circuit D of the graph (V, F). Consider this D.

The set D is a circuit of (V, F). In other words, the set D has the form $\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_{m+1}\}\}$, where $(v_1, v_2, \ldots, v_{m+1})$ is a cycle of (V, F) (by the definition of a "circuit"). Consider this cycle $(v_1, v_2, \ldots, v_{m+1})$. We thus have

$$D = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_{m+1}\}\}. \tag{74}$$

The list $(v_1, v_2, \ldots, v_{m+1})$ is a cycle of (V, F). According to the definition of a "cycle", this means that this list is a list of elements of V satisfying the following four properties:

- We have $m > 2$.
- We have $v_{m+1} = v_1$.
- The vertices $v_1, v_2, \ldots, v_m$ are pairwise distinct.
- We have $\{v_i, v_{i+1}\} \in F$ for every $i \in \{1, 2, \ldots, m\}$.

Thus, $(v_1, v_2, \ldots, v_{m+1})$ is a list of elements of V satisfying the four properties that we have just mentioned. Notice that

$$\begin{aligned} D &= \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_{m+1}\}\} \\ &= \{\{v_i, v_{i+1}\} \mid i \in \{1, 2, \ldots, m\}\} \subseteq F \end{aligned}$$

(since $\{v_i, v_{i+1}\} \in F$ for every $i \in \{1, 2, \ldots, m\}$).

Now, we have $\{v_i, v_{i+1}\} \in F \subseteq E$ for every $i \in \{1, 2, \ldots, m\}$. Hence, $(v_1, v_2, \ldots, v_{m+1})$ is a list of elements of V satisfying the following four properties:

- We have $m > 2$.
- We have $v_{m+1} = v_1$.
- The vertices $v_1, v_2, \ldots, v_m$ are pairwise distinct.
- We have $\{v_i, v_{i+1}\} \in E$ for every $i \in \{1, 2, \ldots, m\}$.

According to the definition of a "cycle", this means that the list $(v_1, v_2, \ldots, v_{m+1})$ is a cycle of (V, E).

Thus, we conclude that the list $(v_1, v_2, \ldots, v_{m+1})$ is a cycle of (V, E). In other words, the list $(v_1, v_2, \ldots, v_{m+1})$ is a cycle of G (since $G = (V, E)$). Thus, the set $\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_{m+1}\}\}$ is a circuit of G (by the definition of a "circuit"). In view of (74), this rewrites as follows: The set D is a circuit of G.

No two distinct edges in D have the same label[43].

From $m > 2 > 1$, we obtain

$$\{v_1, v_2\} \in \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_{m+1}\}\} = D$$

(by (74)). Hence, the set D is nonempty (since it contains $\{v_1, v_2\}$). Thus, D is a nonempty finite set. Hence, there exists an edge in D having maximum label. Let f be this edge. Clearly, $D \setminus \{f\} \subseteq D \subseteq F \subseteq E$. Thus, $D \setminus \{f\}$ is a subset of E.

The edge f is an edge in D having maximum label. Since no other edge in D has the same label as f (because no two distinct edges in D have the same label), this shows that the edge f is the **unique** edge in D having maximum label. Therefore, $D \setminus \{f\}$ is a subset of E having the form $C \setminus \{e\}$, where C is a circuit of G, and where e is the unique edge in C having maximum label (among the edges in C) [44]. In other words, $D \setminus \{f\}$ is a broken circuit of G (since $D \setminus \{f\}$ is a broken circuit of G if and only if $D \setminus \{f\}$ is a subset of E having the form $C \setminus \{e\}$, where C is a circuit of G, and where e is the unique edge in C having maximum label (among the edges in C) [45]). This broken circuit $D \setminus \{f\}$

---

[43] *Proof.* Assume the contrary. Thus, two distinct edges in D have the same label. In other words, there exist two distinct edges e and e' in D such that e and e' have the same label. Consider these e and e'.

The edges e and e' have the same label. In other words, the label of e equals the label of e'. In other words, $\ell(e)$ equals $\ell(e')$ (since the label of e is $\ell(e)$ (by the definition of "label"), whereas the label of e' is $\ell(e')$ (by the definition of "label")). In other words, $\ell(e) = \ell(e')$. Since the map ℓ is injective, this shows that $e = e'$. This contradicts the assumption that e and e' are distinct. This contradiction proves that our assumption was wrong. Qed.

[44] Namely, $D \setminus \{f\}$ has this form for $C = D$ and $e = f$.

[45] by the definition of a "broken circuit"

satisfies $D \setminus \{f\} \subseteq F$. Thus, there exists a broken circuit B of G such that $B \subseteq F$ (namely, $B = D \setminus \{f\}$).

But the set F contains no broken circuit of G as a subset. In other words, there exists no broken circuit B of G such that $B \subseteq F$. This contradicts the fact that there exists a broken circuit B of G such that $B \subseteq F$. This contradiction proves that our assumption was wrong. Hence, the graph (V, F) has no circuits. Lemma 3.15 thus shows that $\operatorname{conn}(V, F) = |V| - |F|$. This proves (73).

Now, Corollary 3.7 yields

$$\chi_G = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} \underbrace{x^{\operatorname{conn}(V,F)}}_{\substack{= x^{|V|-|F|} \\ (\text{since } \operatorname{conn}(V,F) = |V|-|F| \\ (\text{by (73)}))}} = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} x^{|V|-|F|}.$$

This proves Corollary 3.14. □

4. Application: Transitive directed graphs

We shall now see an application of Corollary 3.6 to graphs which are obtained from certain directed graphs by "forgetting the directions of the edges". Let us first introduce the notations involved:

> **Definition 4.1.** **(a)** A *digraph* means a pair (V, A), where V is a set, and where A is a subset of $V^2 = V \times V$. Digraphs are also called *directed graphs*. A digraph (V, A) is said to be *finite* if the set V is finite. If $D = (V, A)$ is a digraph, then the elements of V are called the *vertices* of the digraph D, while the elements of A are called the *arcs* (or the *directed edges*) of the digraph D. If $a = (v, w)$ is an arc of a digraph D, then v is called the *source* of a, whereas w is called the *target* of a.
>
> **(b)** A digraph (V, A) is said to be *loopless* if every $v \in V$ satisfies $(v, v) \notin A$. (In other words, a digraph is loopless if and only if it has no arc whose source and target are identical.)
>
> **(c)** A digraph (V, A) is said to be *transitive* if it has the following property: For any $u \in V$, $v \in V$ and $w \in V$ satisfying $(u, v) \in A$ and $(v, w) \in A$, we have $(u, w) \in A$.
>
> **(d)** A digraph (V, A) is said to be 2-*step-free* if there exist no three elements u, v and w of V satisfying $(u, v) \in A$ and $(v, w) \in A$.
>
> **(e)** Let $D = (V, A)$ be a loopless digraph. Define a map $\operatorname{set} : A \to \binom{V}{2}$ by setting
>
> $$(\operatorname{set}(v, w) = \{v, w\} \qquad \text{for every } (v, w) \in A).$$

(It is easy to see that set is well-defined, because (V, A) is loopless.) The graph $(V, \operatorname{set} A)$ will be denoted by $\underline{D}$. (Here, $\operatorname{set} A$ means the subset $\{\operatorname{set} a \mid a \in A\}$ of $\binom{V}{2}$.)

Example 4.2. **(a)** The digraph $D = (V, A)$ with $V = \{1, 2, 3\}$ and $A = \{(1, 2), (2, 1), (2, 3), (3, 3)\}$ is not loopless (since the vertex $v = 3$ does not satisfy $(v, v) \notin A$).

(b) The digraph $D = (V, A)$ with $V = \{1, 2, 3\}$ and $A = \{(1, 2), (2, 1), (2, 3)\}$ is loopless. The corresponding (undirected) graph $\underline{D}$ is $\underline{D} = (V, \operatorname{set} A)$ with $\operatorname{set} A = \{\{1, 2\}, \{2, 3\}\}$. (Note that the two distinct arcs $(1, 2)$ and $(2, 1)$ of D yield the same edge $\{1, 2\}$ of $\underline{D}$.) Note that this digraph D is not transitive, because the three vertices $u = 1$, $v = 2$ and $w = 1$ satisfy $(u, v) \in A$ and $(v, w) \in A$ but don't satisfy $(u, w) \in A$.

(c) The digraph $D = (V, A)$ with $V = \{1, 2, 3, 4\}$ and $A = \{(1, 2), (2, 3), (1, 3), (3, 4)\}$ is not transitive, since the three vertices $u = 2$, $v = 3$ and $w = 4$ satisfy $(u, v) \in A$ and $(v, w) \in A$ but don't satisfy $(u, w) \in A$.

(d) The digraph $D = (V, A)$ with $V = \{1, 2, 3, 4\}$ and $A = \{(1, 2), (2, 3), (1, 3), (4, 2), (4, 3)\}$ is loopless and transitive. It is not 2-step-free, since the three elements $u = 1$, $v = 2$ and $w = 3$ satisfy $(u, v) \in A$ and $(v, w) \in A$.

(e) The digraph $D = (V, A)$ with $V = \{1, 2, 3, 4\}$ and $A = \{(1, 3), (2, 3), (1, 4), (2, 4)\}$ is loopless, transitive and 2-step-free. (Actually, any 2-step-free digraph is transitive, for vacuous reasons.)

Remark 4.3. A transitive loopless digraph cannot have any (directed) cycles. We omit the easy proof of this fact, as we will not use it in what follows, but it illuminates some of the arguments below.

Remark 4.4. A transitive loopless digraph is more or less the same as a poset (i.e., partially ordered set). Indeed:

- If (V, A) is a transitive loopless digraph, then we can equip the set V with a (strict) partial order $<$ defined by

  $$(u < v) \iff ((u, v) \in A),$$

  which turns V into a poset.

- Conversely, if V is a poset, then we obtain a transitive loopless digraph (V, A) by setting $A := \{(u, v) \in V^2 \mid u < v\}$.

We find the language of digraphs to be more convenient, but the reader should be aware of the possibility of restating everything in terms of posets.

We can now state our application of Corollary 3.6, answering a question suggested by Alexander Postnikov:

Proposition 4.5. Let $D = (V, A)$ be a finite transitive loopless digraph. Then,

$$\chi_{\underline{D}} = \sum_{\substack{F \subseteq A; \\ \text{the digraph } (V,F) \text{ is 2-step-free}}} (-1)^{|F|} x^{\operatorname{conn}(V, \operatorname{set} F)}.$$

(Here, $\operatorname{set} F$ means the subset $\{\operatorname{set} f \mid f \in F\}$ of $\dbinom{V}{2}$.)

Note that the graph $(V, \operatorname{set} F)$ in Proposition 4.5 can also be rewritten as $\underline{(V, F)}$. We prepare for the proof of this proposition by stating two simple lemmas:

Lemma 4.6. Let $G = (V, E)$ be a graph. Let u, v and w be three elements of V such that $\{u, v\} \in E$, $\{v, w\} \in E$ and $\{u, w\} \in E$. Let C be the set $\{\{u, v\}, \{v, w\}, \{u, w\}\}$. Then, the set C is a circuit of G and satisfies $C \setminus \{\{u, w\}\} = \{\{u, v\}, \{v, w\}\}$.

Proof of Lemma 4.6. We have $E \subseteq \dbinom{V}{2}$ (since (V, E) is a graph). For any $a \in V$ and $b \in V$ satisfying $\{a, b\} \in E$, we have

$$a \neq b \tag{75}$$

[46].

Now, (75) (applied to $a = u$ and $b = v$) yields $u \neq v$ (since $\{u, v\} \in E$). Also, (75) (applied to $a = v$ and $b = w$) yields $v \neq w$ (since $\{v, w\} \in E$). Also, (75) (applied to $a = u$ and $b = w$) yields $u \neq w$ (since $\{u, w\} \in E$).

Now, set $m = 3$. Thus, $m + 1 = 4$. Now, u, v, w and u are elements of V. Hence, $(u, v, w, u) \in V^4 = V^{m+1}$ (since $4 = m + 1$). In other words, (u, v, w, u) is an $(m+1)$-tuple of elements of V. Denote this $(m+1)$-tuple by $(v_1, v_2, \ldots, v_{m+1})$. Hence, $(v_1, v_2, \ldots, v_{m+1}) = (u, v, w, u)$. Therefore, $v_1 = u$, $v_2 = v$, $v_3 = w$ and $v_4 = u$. Now, we have $m = 3 > 2 > 1$ and

$$\begin{aligned} v_{m+1} &= v_4 \qquad (\text{since } m + 1 = 4) \\ &= u = v_1. \end{aligned}$$

---

[46] *Proof of (75):* Let $a \in V$ and $b \in V$ be such that $\{a, b\} \in E$. We need to prove that $a \neq b$.

Assume the contrary (for the sake of contradiction). Thus, $a = b$. Hence, $\{a, b\} = \{b, b\} = \{b\}$, so that $|\{a, b\}| = |\{b\}| = 1$.

But $\{a, b\} \in E \subseteq \dbinom{V}{2}$. In other words, $\{a, b\}$ is a 2-element subset of V (since $\dbinom{V}{2}$ is the set of all 2-element subsets of V). Thus, $|\{a, b\}| = 2$. This contradicts $|\{a, b\}| = 1 < 2$. This contradiction proves that our assumption was wrong. Hence, $a \neq b$ holds. This proves (75).

Also, the vertices $v_1, v_2, \ldots, v_m$ are pairwise distinct[47]. Finally, we have $\{v_i, v_{i+1}\} \in E$ for every $i \in \{1, 2, \ldots, m\}$ [48].

Altogether, we now know that $(v_1, v_2, \ldots, v_{m+1})$ is a list of elements of V satisfying the following four properties:

- We have $m > 2$.
- We have $v_{m+1} = v_1$.
- The vertices $v_1, v_2, \ldots, v_m$ are pairwise distinct.
- We have $\{v_i, v_{i+1}\} \in E$ for every $i \in \{1, 2, \ldots, m\}$.

According to the definition of a "cycle", this means that the list $(v_1, v_2, \ldots, v_{m+1})$ is a cycle of (V, E).

Thus, we conclude that the list $(v_1, v_2, \ldots, v_{m+1})$ is a cycle of (V, E). In other words, the list $(v_1, v_2, \ldots, v_{m+1})$ is a cycle of G (since $G = (V, E)$). Thus, the

---

[47] *Proof.* We have $m = 3$, and thus $(v_1, v_2, \ldots, v_m) = \left(\underbrace{v_1}_{=u}, \underbrace{v_2}_{=v}, \underbrace{v_3}_{=w} \right) = (u, v, w)$.

The vertices u, v, w are pairwise distinct (since $u \neq v$, $u \neq w$ and $v \neq w$). Since $(u, v, w) = (v_1, v_2, \ldots, v_m)$, this rewrites as follows: The vertices $v_1, v_2, \ldots, v_m$ are pairwise distinct. Qed.

[48] *Proof.* Let $i \in \{1, 2, \ldots, m\}$. We want to prove that $\{v_i, v_{i+1}\} \in E$. We have

$$i \in \{1, 2, \ldots, m\} = \{1, 2, \ldots, 3\} \qquad (\text{since } m = 3)$$
$$= \{1, 2, 3\}.$$

In other words, $i = 1$ or $i = 2$ or $i = 3$. We are thus in one of the following three cases:

Case 1: We have $i = 1$.

Case 2: We have $i = 2$.

Case 3: We have $i = 3$.

Let us first consider Case 1. In this case, we have $i = 1$. Hence, $v_i = v_1 = u$ and $v_{i+1} = v_{1+1} = v_2 = v$. Now, $\left\{ \underbrace{v_i}_{=u}, \underbrace{v_{i+1}}_{=v} \right\} = \{u, v\} \in E$. Thus, $\{v_i, v_{i+1}\} \in E$ holds in Case 1.

Let us first consider Case 2. In this case, we have $i = 2$. Hence, $v_i = v_2 = v$ and $v_{i+1} = v_{2+1} = v_3 = w$. Now, $\left\{ \underbrace{v_i}_{=v}, \underbrace{v_{i+1}}_{=w} \right\} = \{v, w\} \in E$. Thus, $\{v_i, v_{i+1}\} \in E$ holds in Case 2.

Let us first consider Case 3. In this case, we have $i = 3$. Hence, $v_i = v_3 = w$ and $v_{i+1} = v_{3+1} = v_4 = u$. Now, $\left\{ \underbrace{v_i}_{=w}, \underbrace{v_{i+1}}_{=u} \right\} = \{w, u\} = \{u, w\} \in E$. Thus, $\{v_i, v_{i+1}\} \in E$ holds in Case 3.

We have now proven $\{v_i, v_{i+1}\} \in E$ in each of the three Cases 1, 2 and 3. Thus, $\{v_i, v_{i+1}\} \in E$ always holds (since the Cases 1, 2 and 3 cover all possibilities). This completes our proof.

set $\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_{m+1}\}\}$ is a circuit of G (by the definition of a "circuit"). Since

$$\begin{aligned}
&\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_{m+1}\}\} \\
&= \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_3, v_{3+1}\}\} \qquad (\text{since } m = 3) \\
&= \left\{ \left\{ \underbrace{v_1}_{=u}, \underbrace{v_2}_{=v} \right\}, \left\{ \underbrace{v_2}_{=v}, \underbrace{v_3}_{=w} \right\}, \left\{ \underbrace{v_3}_{=w}, \underbrace{v_4}_{=u} \right\} \right\} \\
&= \left\{ \{u, v\}, \{v, w\}, \underbrace{\{w, u\}}_{=\{u,w\}} \right\} = \{\{u, v\}, \{v, w\}, \{u, w\}\} = C
\end{aligned}$$

(because $C = \{\{u, v\}, \{v, w\}, \{u, w\}\}$), this rewrites as follows: The set C is a circuit of G.

It remains to prove that $C \setminus \{\{u, w\}\} = \{\{u, v\}, \{v, w\}\}$. Indeed, we have $\{u, w\} \notin \{\{u, v\}, \{v, w\}\}$ [49]. Now,

$$C = \{\{u, v\}, \{v, w\}, \{u, w\}\} = \{\{u, v\}, \{v, w\}\} \cup \{\{u, w\}\}.$$

Hence,

$$\begin{aligned}
&\underbrace{C}_{=\{\{u,v\},\{v,w\}\} \cup \{\{u,w\}\}} \setminus \{\{u, w\}\} \\
&= (\{\{u, v\}, \{v, w\}\} \cup \{\{u, w\}\}) \setminus \{\{u, w\}\} \\
&= \{\{u, v\}, \{v, w\}\} \qquad (\text{since } \{u, w\} \notin \{\{u, v\}, \{v, w\}\}).
\end{aligned}$$

This completes the proof of Lemma 4.6. □

Lemma 4.7. Let $D = (V, A)$ be a finite transitive loopless digraph. Let $E = \operatorname{set} A$. Every $a \in A$ satisfies $\operatorname{set} \underbrace{a}_{\in A} \in \operatorname{set} A = E$. Hence, we can define a map $\pi : A \to E$ by

$$(\pi(a) = \operatorname{set} a \qquad \text{for every } a \in A).$$

---

[49] *Proof.* Assume the contrary. Thus, $\{u, w\} \in \{\{u, v\}, \{v, w\}\}$. In other words, $\{u, w\}$ equals either $\{u, v\}$ or $\{v, w\}$. In other words, we must be in one of the following two cases:

Case 1: We have $\{u, w\} = \{u, v\}$.

Case 2: We have $\{u, w\} = \{v, w\}$.

Let us first consider Case 1. In this case, we have $\{u, w\} = \{u, v\}$. Hence, $w \in \{u, w\} = \{u, v\}$. Combining this with $w \neq u$ (since $u \neq w$), we obtain $w \in \{u, v\} \setminus \{u\} \subseteq \{v\}$. In other words, $w = v$. Hence, $v = w$. But this contradicts $v \neq w$. Thus, we have found a contradiction in Case 1.

Let us now consider Case 2. In this case, we have $\{u, w\} = \{v, w\}$. Hence, $u \in \{u, w\} = \{v, w\}$. Combining this with $u \neq v$, we obtain $u \in \{v, w\} \setminus \{v\} \subseteq \{w\}$. In other words, $u = w$. This contradicts $u \neq w$. Thus, we have found a contradiction in Case 2.

We have now found a contradiction in each of the two Cases 1 and 2. Since these two Cases cover all possibilities, we thus always have a contradiction. This contradiction shows that our assumption was wrong, qed.

Consider this map π.

(a) The map $\pi : A \to E$ is bijective. Thus, an inverse map $\pi^{-1} : E \to A$ is well-defined.

(b) Let Z be the set $\left\{(i,k,j) \in V^3 \mid (i,k) \in A \text{ and } (k,j) \in A\right\}$. Define a set $\mathfrak{K}$ by

$$\mathfrak{K} = \{\{\{i,k\},\{k,j\}\} \mid (i,k,j) \in Z\}.$$

Then, $\mathfrak{K} \subseteq \mathcal{P}(E)$.

(c) Let G be the graph (V,E). For every $(i,j) \in V^2$, define a subset $\operatorname{inter}(i,j)$ of V by

$$\operatorname{inter}(i,j) = \{k \in V \mid (i,k) \in A \text{ and } (k,j) \in A\}.$$

Define a map $\ell' : A \to \mathbb{N}$ by setting

$$\left(\ell'(i,j) = |\operatorname{inter}(i,j)| \qquad \text{for every } (i,j) \in A\right).$$

Recall that a map $\pi^{-1} : E \to A$ is well-defined (by Lemma 4.7 **(a)**). Define a map $\ell : E \to \mathbb{N}$ by $\ell = \ell' \circ \pi^{-1}$. We shall refer to ℓ as the *labeling function*. For every edge e of G, we shall refer to $\ell(e)$ as the *label* of e. If u, v and w are three elements of V satisfying $(u,v) \in A$ and $(v,w) \in A$, then

$$\{u,w\} \in E, \qquad \{u,v\} \in E, \qquad \{u,w\} \in E, \tag{76}$$

$$\ell(\{u,w\}) > \ell(\{u,v\}) \qquad \text{and} \qquad \ell(\{u,w\}) > \ell(\{v,w\}). \tag{77}$$

(d) Consider the set $\mathfrak{K}$ defined in Lemma 4.7 **(b)**. Consider the labeling function ℓ defined in Lemma 4.7 **(c)**. Definition 1.13 (applied to $X = \mathbb{N}$) shows that the notion of a broken circuit of G is well-defined (since a labeling function $\ell : E \to \mathbb{N}$ is given).

Every element of $\mathfrak{K}$ is a broken circuit of G.

(e) Consider the set $\mathfrak{K}$ defined in Lemma 4.7 **(b)**. Let F be a subset of A. Then, we have the following logical equivalence:

$$(\text{the digraph } (V,F) \text{ is 2-step-free}) \iff (\text{the set } \pi(F) \text{ is } \mathfrak{K}\text{-free}).$$

Proof of Lemma 4.7. We have $A \subseteq V^2$ (since (V,A) is a digraph). The set V is finite (since the digraph (V,A) is finite).

Recall that the digraph (V,A) is transitive if and only if, for any $u \in V$, $v \in V$ and $w \in V$ satisfying $(u,v) \in A$ and $(v,w) \in A$, we have $(u,w) \in A$ (by the definition of "transitive"). Thus, for any $u \in V$, $v \in V$ and $w \in V$ satisfying $(u,v) \in A$ and $(v,w) \in A$, we have

$$(u,w) \in A \tag{78}$$

(because the digraph (V,A) is transitive).

Recall that the digraph (V,A) is loopless if and only if every $v \in V$ satisfies

$(v,v) \notin A$. Thus, every $v \in V$ satisfies

$$(v,v) \notin A \tag{79}$$

(since the digraph (V,A) is loopless).

(a) We have

$$\pi(A) = \left\{ \underbrace{\pi(a)}_{\substack{=\operatorname{set} a \\ \text{(by the definition of } \pi)}} \mid a \in A \right\} = \{\operatorname{set} a \mid a \in A\} = \operatorname{set} A = E.$$

Thus, the map π is surjective.

Furthermore, if a and b are two elements of A such that $\pi(a) = \pi(b)$, then $a = b$ [50]. In other words, the map π is injective. Now, the map π is bijective

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[50] *Proof.* Let a and b be two elements of A such that $\pi(a) = \pi(b)$. We must show that $a = b$.

Assume the contrary. Thus, $a \neq b$.

We have $a \in A \subseteq V^2$. Thus, we can write a in the form $a = (i,j)$ with $i \in V$ and $j \in V$. Consider these i and j.

We have $b \in A \subseteq V^2$. Thus, we can write b in the form $b = (i',j')$ with $i' \in V$ and $j' \in V$. Consider these i' and j'.

Applying (79) to $v = i$, we obtain $(i,i) \notin A$. If we had $i = j$, then we would have $\left(i, \underbrace{i}_{=j}\right) = (i,j) = a \in A$, which would contradict $(i,i) \notin A$. Thus, we cannot have $i = j$. Hence, we have $i \neq j$, so that $j \neq i$.

Applying the map π to both sides of the equality $a = (i,j)$, we obtain

$$\begin{aligned} \pi(a) = \pi(i,j) &= \operatorname{set}(i,j) \qquad \text{(by the definition of } \pi) \\ &= \{i,j\} \qquad \text{(by the definition of the map set)}. \end{aligned}$$

The same argument (but applied to b, i' and j' instead of a, i and j) shows that $\pi(b) = \{i',j'\}$. Hence, $\{i,j\} = \pi(a) = \pi(b) = \{i',j'\}$.

We have $i \in \{i,j\} = \{i',j'\}$. In other words, either $i = i'$ or $i = j'$. In other words, we are in one of the following two cases:

Case 1: We have $i = i'$.

Case 2: We have $i = j'$.

Let us first consider Case 1. In this case, we have $i = i'$. Now, $j \in \{i,j\} = \{i',j'\}$. Combining this with $j \neq i = i'$, we obtain $j \in \{i',j'\} \setminus \{i'\} \subseteq \{j'\}$. Thus, $j = j'$. Now, $a = \left(\underbrace{i}_{=i'}, \underbrace{j}_{=j'}\right) = (i',j') = b$; this contradicts $a \neq b$. Hence, we have found a contradiction in Case 1.

Let us now consider Case 2. In this case, we have $i = j'$. Now, $j \in \{i,j\} = \{i',j'\}$. Combining this with $j \neq i = j'$, we obtain $j \in \{i',j'\} \setminus \{j'\} \subseteq \{i'\}$. Thus, $j = i'$. Hence, $\left(\underbrace{j}_{=i'}, \underbrace{i}_{=j'}\right) = (i',j') = b \in A$. Also, $(i,j) = a \in A$. Hence, (78) (applied to $u = i$, $v = j$ and $w = i$) yields $(i,i) \in A$. This contradicts $(i,i) \notin A$. Thus, we have found a contradiction in (since π is surjective and injective). Thus, an inverse map $\pi^{-1} : E \to A$ is well-defined. This proves Lemma 4.7 **(a)**.

(b) The definition of Z yields

$$\begin{aligned} Z &= \left\{ (i,k,j) \in V^3 \ \mid \ (i,k) \in A \text{ and } (k,j) \in A \right\} \\ &= \left\{ (u,v,w) \in V^3 \ \mid \ (u,v) \in A \text{ and } (v,w) \in A \right\} \end{aligned}$$

(here, we have renamed the index (i,k,j) as (u,v,w)). Now, let $K \in \mathfrak{K}$. We shall prove that $K \in \mathcal{P}(E)$.

We have $K \in \mathfrak{K} = \{\{\{i,k\},\{k,j\}\} \mid (i,k,j) \in Z\}$. Hence, $K = \{\{i,k\},\{k,j\}\}$ for some $(i,k,j) \in Z$. Consider this (i,k,j). We have

$$(i,k,j) \in Z = \left\{ (u,v,w) \in V^3 \ \mid \ (u,v) \in A \text{ and } (v,w) \in A \right\}.$$

In other words, (i,k,j) is an $(u,v,w) \in V^3$ satisfying $(u,v) \in A$ and $(v,w) \in A$. In other words, (i,k,j) is an element of V^3 and satisfies $(i,k) \in A$ and $(k,j) \in A$.

Now, $(i,k) \in A$. The definition of π therefore yields $\pi(i,k) = \operatorname{set}(i,k) = \{i,k\}$ (by the definition of the map set). Hence, $\{i,k\} = \pi \underbrace{(i,k)}_{\in A} \in \pi(A) \subseteq E$.

Also, $(k,j) \in A$. The definition of π therefore yields $\pi(k,j) = \operatorname{set}(k,j) = \{k,j\}$ (by the definition of the map set). Hence, $\{k,j\} = \pi \underbrace{(k,j)}_{\in A} \in \pi(A) \subseteq E$.

Now, we know that both $\{i,k\}$ and $\{k,j\}$ belong to E. Hence, $\{\{i,k\},\{k,j\}\} \subseteq E$. Hence, $K = \{\{i,k\},\{k,j\}\} \subseteq E$, so that $K \in \mathcal{P}(E)$.

Now, forget that we fixed K. We thus have shown that $K \in \mathcal{P}(E)$ for every $K \in \mathfrak{K}$. In other words, $\mathfrak{K} \subseteq \mathcal{P}(E)$. This proves Lemma 4.7 **(b)**.

(c) We only need to prove that if u, v and w are three elements of V satisfying $(u,v) \in A$ and $(v,w) \in A$, then (76) and (77) hold.

So let u, v and w be three elements of V satisfying $(u,v) \in A$ and $(v,w) \in A$. We must prove (76) and (77).

The definition of $\operatorname{inter}(u,v)$ yields

$$\operatorname{inter}(u,v) = \{k \in V \ \mid \ (u,k) \in A \text{ and } (k,v) \in A\}.$$

The definition of $\operatorname{inter}(v,w)$ yields

$$\operatorname{inter}(v,w) = \{k \in V \ \mid \ (v,k) \in A \text{ and } (k,w) \in A\}.$$

The definition of $\operatorname{inter}(u,w)$ yields

$$\operatorname{inter}(u,w) = \{k \in V \ \mid \ (u,k) \in A \text{ and } (k,w) \in A\}.$$

Case 2.

We have now found a contradiction in each of the two Cases 1 and 2. Hence, we always have a contradiction (since Cases 1 and 2 cover all possibilities). Thus, our assumption was wrong. Hence, $a = b$ is proven. Qed.

We have $v \in \operatorname{inter}(u,w)$ [51].

All three sets $\operatorname{inter}(u,v)$, $\operatorname{inter}(v,w)$ and $\operatorname{inter}(u,w)$ are subsets of the finite set V, and thus are finite.

Now, $\operatorname{inter}(u,v)$ is a proper subset of $\operatorname{inter}(u,w)$ [52]. Hence,

$$\left|\operatorname{inter}(u,v)\right| < \left|\operatorname{inter}(u,w)\right| \tag{80}$$

(since $\operatorname{inter}(u,v)$ and $\operatorname{inter}(u,w)$ are finite sets).

---

[51] *Proof.* We know that v is an element of V and satisfies $(u,v) \in A$ and $(v,w) \in A$. In other words, v is an element k of V satisfying $(u,k) \in A$ and $(k,v) \in A$. Hence,

$$v \in \{k \in V \mid (u,k) \in A \text{ and } (k,w) \in A\} = \operatorname{inter}(u,w).$$

Qed.

[52] *Proof.* Let $g \in \operatorname{inter}(u,v)$. Thus,

$$g \in \operatorname{inter}(u,v) = \{k \in V \mid (u,k) \in A \text{ and } (k,v) \in A\}.$$

In other words, g is an element k of V satisfying $(u,k) \in A$ and $(k,v) \in A$. In other words, g is an element of V and satisfies $(u,g) \in A$ and $(g,v) \in A$.

Now, (78) (applied to g instead of u) yields $(g,w) \in A$ (since $(g,v) \in A$ and $(v,w) \in A$). Hence, we now know that g is an element of V and satisfies $(u,g) \in A$ and $(g,w) \in A$. In other words, g is an element k of V satisfying $(u,k) \in A$ and $(k,w) \in A$. Hence,

$$g \in \{k \in V \mid (u,k) \in A \text{ and } (k,w) \in A\} = \operatorname{inter}(u,w).$$

Now, forget that we fixed g. We therefore have proven that $g \in \operatorname{inter}(u,w)$ for every $g \in \operatorname{inter}(u,v)$. In other words, $\operatorname{inter}(u,v) \subseteq \operatorname{inter}(u,w)$.

Next, we shall prove that $\operatorname{inter}(u,v) \neq \operatorname{inter}(u,w)$. Indeed, assume the contrary. Thus, $\operatorname{inter}(u,v) = \operatorname{inter}(u,w)$. Now,

$$v \in \operatorname{inter}(u,w) = \operatorname{inter}(u,v) = \{k \in V \mid (u,k) \in A \text{ and } (k,v) \in A\}.$$

In other words, v is an element k of V satisfying $(u,k) \in A$ and $(k,v) \in A$. In other words, v is an element of V and satisfies $(u,v) \in A$ and $(v,v) \in A$. But (79) yields $(v,v) \notin A$. This contradicts $(v,v) \in A$. This contradiction proves that our assumption was false. Hence, $\operatorname{inter}(u,v) \neq \operatorname{inter}(u,w)$ is proven. Combining this with $\operatorname{inter}(u,v) \subseteq \operatorname{inter}(u,w)$, we conclude that $\operatorname{inter}(u,v)$ is a proper subset of $\operatorname{inter}(u,w)$. Qed.

Furthermore, $\operatorname{inter}(v,w)$ is a proper subset of $\operatorname{inter}(u,w)$ [53]. Hence,

$$\left|\operatorname{inter}(v,w)\right| < \left|\operatorname{inter}(u,w)\right| \tag{81}$$

(since $\operatorname{inter}(v,w)$ and $\operatorname{inter}(u,w)$ are finite sets).

From (78), we obtain $(u,w) \in A$. The definition of π thus shows that

$$\pi(u,w) = \operatorname{set}(u,w) = \{u,w\} \qquad \text{(by the definition of the map set)}.$$

Hence, $\{u,w\} = \pi \underbrace{(u,w)}_{\in A} \in \pi(A) \subseteq E$. Thus, $\ell(\{u,w\})$ is well-defined.

Also, recall that $(u,v) \in A$. The definition of π thus shows that

$$\pi(u,v) = \operatorname{set}(u,v) = \{u,v\} \qquad \text{(by the definition of the map set)}.$$

Hence, $\{u,v\} = \pi \underbrace{(u,v)}_{\in A} \in \pi(A) \subseteq E$. Thus, $\ell(\{u,v\})$ is well-defined.

Also, recall that $(v,w) \in A$. The definition of π thus shows that

$$\pi(v,w) = \operatorname{set}(v,w) = \{v,w\} \qquad \text{(by the definition of the map set)}.$$

Hence, $\{v,w\} = \pi \underbrace{(v,w)}_{\in A} \in \pi(A) \subseteq E$. Thus, $\ell(\{v,w\})$ is well-defined. We have also proven (76) by now.

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[53] *Proof.* Let $g \in \operatorname{inter}(v,w)$. Thus,

$$g \in \operatorname{inter}(v,w) = \{k \in V \mid (v,k) \in A \text{ and } (k,w) \in A\}.$$

In other words, g is an element k of V satisfying $(v,k) \in A$ and $(k,w) \in A$. In other words, g is an element of V and satisfies $(v,g) \in A$ and $(g,w) \in A$.

Now, (78) (applied to g instead of w) yields $(u,g) \in A$ (since $(u,v) \in A$ and $(v,g) \in A$). Hence, we now know that g is an element of V and satisfies $(u,g) \in A$ and $(g,w) \in A$. In other words, g is an element k of V satisfying $(u,k) \in A$ and $(k,w) \in A$. Hence,

$$g \in \{k \in V \mid (u,k) \in A \text{ and } (k,w) \in A\} = \operatorname{inter}(u,w).$$

Now, forget that we fixed g. We therefore have proven that $g \in \operatorname{inter}(u,w)$ for every $g \in \operatorname{inter}(v,w)$. In other words, $\operatorname{inter}(v,w) \subseteq \operatorname{inter}(u,w)$.

Next, we shall prove that $\operatorname{inter}(v,w) \neq \operatorname{inter}(u,w)$. Indeed, assume the contrary. Thus, $\operatorname{inter}(v,w) = \operatorname{inter}(u,w)$. Now,

$$v \in \operatorname{inter}(u,w) = \operatorname{inter}(v,w) = \{k \in V \mid (v,k) \in A \text{ and } (k,w) \in A\}.$$

In other words, v is an element k of V satisfying $(v,k) \in A$ and $(k,w) \in A$. In other words, v is an element of V and satisfies $(v,v) \in A$ and $(v,w) \in A$. But (79) yields $(v,v) \notin A$. This contradicts $(v,v) \in A$. This contradiction proves that our assumption was false. Hence, $\operatorname{inter}(v,w) \neq \operatorname{inter}(u,w)$ is proven. Combining this with $\operatorname{inter}(v,w) \subseteq \operatorname{inter}(u,w)$, we conclude that $\operatorname{inter}(v,w)$ is a proper subset of $\operatorname{inter}(u,w)$. Qed.

Now,

$$\begin{aligned}\underbrace{\ell}_{=\ell'\circ\pi^{-1}}\left(\{u,w\}\right) &= \left(\ell'\circ\pi^{-1}\right)\left(\{u,w\}\right) = \ell'\left(\underbrace{\pi^{-1}\left(\{u,w\}\right)}_{\substack{=(u,w)\\ (\text{since } \{u,w\}=\pi(u,w))}}\right) = \ell'\left(u,w\right)\\ &= \left|\operatorname{inter}\left(u,w\right)\right| \qquad \left(\text{by the definition of } \ell'\right).\end{aligned}$$

Also,

$$\begin{aligned}\underbrace{\ell}_{=\ell'\circ\pi^{-1}}\left(\{u,v\}\right) &= \left(\ell'\circ\pi^{-1}\right)\left(\{u,v\}\right) = \ell'\left(\underbrace{\pi^{-1}\left(\{u,v\}\right)}_{\substack{=(u,v)\\ (\text{since } \{u,v\}=\pi(u,v))}}\right) = \ell'\left(u,v\right)\\ &= \left|\operatorname{inter}\left(u,v\right)\right| \qquad \left(\text{by the definition of } \ell'\right),\end{aligned}$$

and

$$\begin{aligned}\underbrace{\ell}_{=\ell'\circ\pi^{-1}}\left(\{v,w\}\right) &= \left(\ell'\circ\pi^{-1}\right)\left(\{v,w\}\right) = \ell'\left(\underbrace{\pi^{-1}\left(\{v,w\}\right)}_{\substack{=(v,w)\\ (\text{since } \{v,w\}=\pi(v,w))}}\right) = \ell'\left(v,w\right)\\ &= \left|\operatorname{inter}\left(v,w\right)\right| \qquad \left(\text{by the definition of } \ell'\right).\end{aligned}$$

Now,

$$\begin{aligned}\ell\left(\{u,w\}\right) &= \left|\operatorname{inter}\left(u,w\right)\right| > \left|\operatorname{inter}\left(u,v\right)\right| \qquad \left(\text{by (80)}\right)\\ &= \ell\left(\{u,v\}\right)\end{aligned}$$

and

$$\begin{aligned}\ell\left(\{u,w\}\right) &= \left|\operatorname{inter}\left(u,w\right)\right| > \left|\operatorname{inter}\left(v,w\right)\right| \qquad \left(\text{by (81)}\right)\\ &= \ell\left(\{v,w\}\right).\end{aligned}$$

Thus, (77) holds. This proves Lemma 4.7 **(c)**.

(d) Let $K \in \mathfrak{K}$. We shall show that K is a broken circuit of G.

We have

$$\begin{aligned}K \in \mathfrak{K} &= \left\{\left\{\{i,k\},\{k,j\}\right\} \ \mid\ (i,k,j)\in Z\right\}\\ &= \left\{\left\{\{u,v\},\{v,w\}\right\} \ \mid\ (u,v,w)\in Z\right\}\end{aligned}$$

(here, we renamed the index (i,k,j) as (u,v,w)). Hence, $K = \left\{\{u,v\},\{v,w\}\right\}$ for some $(u,v,w)\in Z$. Consider this (u,v,w). We have

$$(u,v,w)\in Z = \left\{(i,k,j)\in V^3 \ \mid\ (i,k)\in A \text{ and } (k,j)\in A\right\}.$$

In other words, (u, v, w) is an $(i, k, j) \in V^3$ satisfying $(i, k) \in A$ and $(k, j) \in A$. In other words, (u, v, w) is an element of V^3 and satisfies $(u, v) \in A$ and $(v, w) \in A$. Hence, Lemma 4.7 **(c)** shows that we have

$$\begin{array}{ccc} \{u,w\} \in E, & \{u,v\} \in E, & \{u,w\} \in E, \\ \ell(\{u,w\}) > \ell(\{u,v\}) & \text{and} & \ell(\{u,w\}) > \ell(\{v,w\}). \end{array}$$

Let D be the set $\{\{u,v\}, \{v,w\}, \{u,w\}\}$. Now, Lemma 4.6 (applied to $C = D$) shows that the set D is a circuit of G and satisfies $D \setminus \{\{u,w\}\} = \{\{u,v\}, \{v,w\}\}$. Hence, $K = \{\{u,v\}, \{v,w\}\} = D \setminus \{\{u,w\}\}$. Moreover, the edge $\{u,w\}$ is the unique edge in D having maximum label (among the edges in D) [54].

Also, $K = \{\{u,v\}, \{v,w\}\} = \underbrace{\{\{u,v\}\}}_{\substack{\subseteq E \\ (\text{since } \{u,v\} \in E)}} \cup \underbrace{\{\{v,w\}\}}_{\substack{\subseteq E \\ (\text{since } \{v,w\} \in E)}} \subseteq E \cup E = E$. Thus, K is a subset of E.

Altogether, we have now shown that K is a subset of E and satisfies $K = D \setminus \{\{u,w\}\}$; we also know that D is a circuit of G, and that $\{u,w\}$ is the unique edge in D having maximum label (among the edges in D). Hence, K is a subset of E having the form $C \setminus \{e\}$, where C is a circuit of G, and where e is the unique edge in C having maximum label (among the edges in C) [55]. In other words, K is a broken circuit of G (since K is a broken circuit of G if and only if K is a subset of E having the form $C \setminus \{e\}$, where C is a circuit of G, and where e is the unique edge in C having maximum label (among the edges in C) [56])).

Now, forget that we fixed K. We thus have shown that every $K \in \mathfrak{K}$ is a broken circuit of G. In other words, every element of $\mathfrak{K}$ is a broken circuit of G. This proves Lemma 4.7 **(d)**.

---

[54] *Proof.* We have $D = \{\{u,v\}, \{v,w\}, \{u,w\}\}$. Now, $\{u,w\} \in \{\{u,v\}, \{v,w\}, \{u,w\}\} = D$. Thus, $\{u,w\}$ is an edge in D.

Now, we need to show that the edge $\{u,w\}$ is the unique edge in D having maximum label (among the edges in D). Indeed, assume the contrary (for the sake of contradiction). Thus, $\{u,w\}$ is **not** the unique edge in D having maximum label (among the edges in D). In other words, there exists an edge $e \in D$ distinct from $\{u,w\}$ such that the label of e is greater or equal to the label of $\{u,w\}$ (because we already know that $\{u,w\}$ is an edge in D). Consider such an e.

The label of e is greater or equal to the label of $\{u,w\}$. In other words, $\ell(e)$ is greater or equal to $\ell(\{u,w\})$ (since the label of e is $\ell(e)$ (by the definition of the "label") and since the label of $\{u,w\}$ is $\ell(\{u,w\})$ (by the definition of the "label")). In other words, $\ell(e) \geq \ell(\{u,w\})$.

We have $e \in D$ and $e \notin \{u,w\}$ (since e is distinct from $\{u,w\}$). Thus, $e \in D \setminus \{\{u,w\}\} = \{\{u,v\}, \{v,w\}\}$.

But $\ell(e) \geq \ell(\{u,w\}) > \ell(\{u,v\})$, so that $\ell(e) \neq \ell(\{u,v\})$ and thus $e \neq \{u,v\}$. Also, $\ell(e) \geq \ell(\{u,w\}) > \ell(\{v,w\})$, so that $\ell(e) \neq \ell(\{v,w\})$ and thus $e \neq \{v,w\}$. Thus, e equals neither of the two edges $\{u,v\}$ and $\{v,w\}$ (since $e \neq \{u,v\}$ and $e \neq \{v,w\}$). In other words, $e \notin \{\{u,v\}, \{v,w\}\}$. This contradicts $e \in \{\{u,v\}, \{v,w\}\}$. This contradiction shows that our assumption was wrong. Hence, we have shown that the edge $\{u,w\}$ is the unique edge in D having maximum label (among the edges in D). Qed.

[55] Namely, K has this form for $C = D$ and $e = \{u,w\}$.

[56] by the definition of a "broken circuit"

(e) The map $\pi : A \to E$ is bijective (by Lemma 4.7 **(a)**). In other words, the map π is surjective and injective.

We first observe a simple fact: If u and v are two elements of V such that $(u,v) \in A$ and $\{u,v\} \in \pi(F)$, then

$$(u,v) \in F \tag{82}$$

[57].

Also, $\pi\left(\underbrace{F}_{\subseteq A}\right) \subseteq \pi(A) \subseteq E$. Thus, $\pi(F)$ is a subset of E.

Let us now prove the implication

$$(\text{the digraph } (V,F) \text{ is 2-step-free}) \implies (\text{the set } \pi(F) \text{ is } \mathfrak{K}\text{-free}). \tag{83}$$

Proof of (83): Assume that the digraph (V,F) is 2-step-free. We shall show that the set $\pi(F)$ is $\mathfrak{K}$-free.

Indeed, let $K \in \mathfrak{K}$ be such that $K \subseteq \pi(F)$. We shall obtain a contradiction.

The digraph (V,F) is 2-step-free if and only if there exist no three elements u, v and w of V satisfying $(u,v) \in F$ and $(v,w) \in F$ (by the definition of "2-step-free"). Therefore, there exist no three elements u, v and w of V satisfying $(u,v) \in F$ and $(v,w) \in F$ (since we know that the digraph (V,F) is 2-step-free). In other words, if u, v and w are three elements of V, then

$$((u,v) \in F \text{ and } (v,w) \in F) \text{ is false.} \tag{84}$$

But

$$\begin{aligned} K \in \mathfrak{K} &= \{\{\{i,k\},\{k,j\}\} \mid (i,k,j) \in Z\} \\ &= \{\{\{u,v\},\{v,w\}\} \mid (u,v,w) \in Z\} \end{aligned}$$

(here, we renamed the index (i,k,j) as (u,v,w)). Hence, $K = \{\{u,v\},\{v,w\}\}$ for some $(u,v,w) \in Z$. Consider this (u,v,w). We have

$$(u,v,w) \in Z = \left\{(i,k,j) \in V^3 \mid (i,k) \in A \text{ and } (k,j) \in A\right\}.$$

In other words, (u,v,w) is an $(i,k,j) \in V^3$ satisfying $(i,k) \in A$ and $(k,j) \in A$. In other words, (u,v,w) is an element of V^3 and satisfies $(u,v) \in A$ and $(v,w) \in A$.

---

[57] *Proof of (82):* Let u and v be two elements of V such that $(u,v) \in A$ and $\{u,v\} \in \pi(F)$. We must show that $(u,v) \in F$.

We have $\{u,v\} \in \pi(F)$. In other words, there exists some $f \in F$ such that $\{u,v\} = \pi(f)$. Consider this f.

But $(u,v) \in A$; therefore, the definition of π yields

$$\begin{aligned} \pi(u,v) &= \operatorname{set}(u,v) = \{u,v\} \qquad (\text{by the definition of the map set}) \\ &= \pi(f). \end{aligned}$$

Since π is injective, this entails that $(u,v) = f \in F$. This proves (82).

Now, $(u,v) \in A$ and $\{u,v\} \in \{\{u,v\}, \{v,w\}\} = K \subseteq \pi(F)$. Hence, (82) shows that $(u,v) \in F$.

Also, $(v,w) \in A$ and $\{v,w\} \in \{\{u,v\}, \{v,w\}\} = K \subseteq \pi(F)$. Hence, (82) (applied to v and w instead of u and v) shows that $(v,w) \in F$. Thus, we have $((u,v) \in F \text{ and } (v,w) \in F)$.

But (84) shows that $((u,v) \in F \text{ and } (v,w) \in F)$ is false. This contradicts the fact that we have $((u,v) \in F \text{ and } (v,w) \in F)$.

Now, let us forget that we fixed K. We thus have obtained a contradiction for each $K \in \mathfrak{K}$ satisfying $K \subseteq \pi(F)$. Hence, there exists no $K \in \mathfrak{K}$ satisfying $K \subseteq \pi(F)$. In other words, the set $\pi(F)$ contains no $K \in \mathfrak{K}$ as a subset.

The subset $\pi(F)$ of E is $\mathfrak{K}$-free if and only if $\pi(F)$ contains no $K \in \mathfrak{K}$ as a subset (by the definition of "$\mathfrak{K}$-free"). Thus, the subset $\pi(F)$ of E is $\mathfrak{K}$-free (since $\pi(F)$ contains no $K \in \mathfrak{K}$ as a subset).

Now, let us forget that we assumed that the digraph (V,F) is 2-step-free. We thus have shown that the set $\pi(F)$ is $\mathfrak{K}$-free if the digraph (V,F) is 2-step-free. In other words, we have proven the implication (83).

Next, we shall prove the following implication:

$$(\text{the set } \pi(F) \text{ is } \mathfrak{K}\text{-free}) \Longrightarrow (\text{the digraph } (V,F) \text{ is 2-step-free}). \tag{85}$$

Proof of (85): Assume that the set $\pi(F)$ is $\mathfrak{K}$-free. We shall show that the digraph (V,F) is 2-step-free.

Let u, v and w be three elements of V satisfying $(u,v) \in F$ and $(v,w) \in F$. We shall derive a contradiction.

Clearly, $(u,v) \in F \subseteq A$ and $(v,w) \in F \subseteq A$.

The subset $\pi(F)$ of E is $\mathfrak{K}$-free if and only if $\pi(F)$ contains no $K \in \mathfrak{K}$ as a subset (by the definition of "$\mathfrak{K}$-free"). Thus, $\pi(F)$ contains no $K \in \mathfrak{K}$ as a subset (since the subset $\pi(F)$ of E is $\mathfrak{K}$-free). In other words, there exists no $K \in \mathfrak{K}$ such that $K \subseteq \pi(F)$.

But (u,v,w) is an element of V^3 and satisfies $(u,v) \in A$ and $(v,w) \in A$. In other words, (u,v,w) is an $(i,k,j) \in V^3$ satisfying $(i,k) \in A$ and $(k,j) \in A$. Hence,

$$(u,v,w) \in \left\{(i,k,j) \in V^3 \mid (i,k) \in A \text{ and } (k,j) \in A\right\} = Z.$$

Thus, $\{\{u,v\}, \{v,w\}\}$ is a set of the form $\{\{i,k\}, \{k,j\}\}$ for some $(i,k,j) \in Z$ (namely, for $(i,k,j) = (u,v,w)$). Hence,

$$\{\{u,v\}, \{v,w\}\} \in \{\{\{i,k\}, \{k,j\}\} \mid (i,k,j) \in Z\} = \mathfrak{K}. \tag{86}$$

But $\{u,v\} \in \pi(F)$ [58] and $\{v,w\} \in \pi(F)$ [59]. Thus, both $\{u,v\}$ and $\{v,w\}$ belong to the set $\pi(F)$. Therefore,

$$\{\{u,v\},\{v,w\}\} \subseteq \pi(F).$$

Combining this with (86), we see that the set $\{\{u,v\},\{v,w\}\} \in \mathfrak{K}$ satisfies $\{\{u,v\},\{v,w\}\} \subseteq \pi(F)$. Thus, there exists an $K \in \mathfrak{K}$ such that $K \subseteq \pi(F)$ (namely, $K = \{\{u,v\},\{v,w\}\}$). This contradicts the fact that there exists no $K \in \mathfrak{K}$ such that $K \subseteq \pi(F)$.

Now, forget that we fixed u, v and w. We thus have obtained a contradiction for every three elements u, v and w of V satisfying $(u,v) \in F$ and $(v,w) \in F$. Hence, there exist no three elements u, v and w of V satisfying $(u,v) \in F$ and $(v,w) \in F$.

But the digraph (V,F) is 2-step-free if and only if there exist no three elements u, v and w of V satisfying $(u,v) \in F$ and $(v,w) \in F$ (by the definition of "2-step-free"). Thus, the digraph (V,F) is 2-step-free (since there exist no three elements u, v and w of V satisfying $(u,v) \in F$ and $(v,w) \in F$).

Now, let us forget that we have assumed that the set $\pi(F)$ is $\mathfrak{K}$-free. We thus have shown that if the set $\pi(F)$ is $\mathfrak{K}$-free, then the digraph (V,F) is 2-step-free. In other words, we have proven the implication (85).

Now, by combining the two implications (83) and (85), we obtain the logical equivalence

$$(\text{the digraph } (V,F) \text{ is 2-step-free}) \iff (\text{the set } \pi(F) \text{ is } \mathfrak{K}\text{-free}).$$

This proves Lemma 4.7 **(e)**. ☐

Proof of Proposition 4.5. First, let us introduce a general notation: If X and Y are two sets, and if $f : X \to Y$ is any map, then we define a map $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ by

$$((\mathcal{P}(f))(Z) = f(Z) \qquad \text{for every } Z \in \mathcal{P}(X)).$$

If the map $f : X \to Y$ is bijective, then

$$\text{the map } \mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y) \text{ is bijective as well} \tag{87}$$

---

[58] *Proof.* We have $(u,v) \in A$; therefore, the definition of π yields

$$\pi(u,v) = \operatorname{set}(u,v) = \{u,v\} \qquad (\text{by the definition of the map set}).$$

Hence, $\{u,v\} = \pi\underbrace{(u,v)}_{\in F} \in \pi(F)$, qed.

[59] *Proof.* We have $(v,w) \in A$; therefore, the definition of π yields

$$\pi(v,w) = \operatorname{set}(v,w) = \{v,w\} \qquad (\text{by the definition of the map set}).$$

Hence, $\{v,w\} = \pi\underbrace{(v,w)}_{\in F} \in \pi(F)$, qed.

(and, in fact, its inverse is $\mathcal{P}\left(f^{-1}\right)$).

Let $E = \operatorname{set} A$ and $G = \underline{D}$. The definition of $\underline{D}$ shows that $\underline{D} = \left(V, \underbrace{\operatorname{set} A}_{=E}\right) = (V, E)$. Thus, $G = \underline{D} = (V, E)$.

Define the map $\pi : A \to E$ as in Lemma 4.7. Lemma 4.7 **(a)** shows that this map $\pi : A \to E$ is bijective. Hence, the map $\mathcal{P}(\pi) : \mathcal{P}(A) \to \mathcal{P}(E)$ is bijective (according to (87), applied to $X = A$, $Y = E$ and $f = \pi$). We notice that every $F \in \mathcal{P}(A)$ satisfies

$$\begin{aligned}(\mathcal{P}(\pi))(F) &= \pi(F) \qquad \text{(by the definition of the map } \mathcal{P}(\pi)\text{)} \qquad (88)\\ &= \left\{ \underbrace{\pi(a)}_{\substack{=\operatorname{set} a \\ \text{(by the definition of } \pi\text{)}}} \mid a \in F \right\} = \{\operatorname{set} a \mid a \in F\} \\ &= \operatorname{set} F \qquad (89)\end{aligned}$$

and

$$\left| \underbrace{(\mathcal{P}(\pi))(F)}_{\substack{=\pi(F) \\ \text{(by (88))}}} \right| = |\pi(F)| = |F| \qquad (90)$$

(since the map π is injective (since the map π is bijective)).

Define two sets Z and $\mathfrak{K}$ as in Lemma 4.7 **(b)**. Define a map $\ell : E \to \mathbb{N}$ as in Lemma 4.7 **(c)**. Definition 1.13 (applied to $X = \mathbb{N}$) shows that the notion of a broken circuit of G is well-defined (since a labeling function $\ell : E \to \mathbb{N}$ is given).

Lemma 4.7 **(d)** shows that every element of $\mathfrak{K}$ is a broken circuit of G. Thus, $\mathfrak{K}$ is a set of broken circuits of G (not necessarily containing all of them). Hence, Corollary 3.6 (applied to $X = \mathbb{N}$) shows that

$$\begin{aligned}
\chi_G &= \underbrace{\sum_{\substack{F \subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}}}_{= \sum_{\substack{F \in \mathcal{P}(E);\\ F \text{ is } \mathfrak{K}\text{-free}}}} (-1)^{|F|} x^{\operatorname{conn}(V,F)} = \sum_{\substack{F \in \mathcal{P}(E);\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} x^{\operatorname{conn}(V,F)} \\
&= \underbrace{\sum_{\substack{F \in \mathcal{P}(A);\\ (\mathcal{P}(\pi))(F) \text{ is } \mathfrak{K}\text{-free}}}}_{\substack{= \sum_{\substack{F \in \mathcal{P}(A);\\ \pi(F) \text{ is } \mathfrak{K}\text{-free}}} \\ \text{(since every } F \in \mathcal{P}(A) \text{ satisfies} \\ (\mathcal{P}(\pi))(F) = \pi(F) \\ \text{(by (88)))}}} \underbrace{(-1)^{|(\mathcal{P}(\pi))(F)|}}_{\substack{=(-1)^{|F|} \\ \text{(since } |(\mathcal{P}(\pi))(F)| = |F| \\ \text{(by (90)))}}} \underbrace{x^{\operatorname{conn}(V,(\mathcal{P}(\pi))(F))}}_{\substack{= x^{\operatorname{conn}(V, \operatorname{set} F)} \\ \text{(since } (\mathcal{P}(\pi))(F) = \operatorname{set} F \\ \text{(by (89)))}}} \\
&\qquad \left(\begin{array}{c} \text{here, we have substituted } (\mathcal{P}(\pi))(F) \text{ for } F \text{ in the sum,} \\ \text{since the map } \mathcal{P}(\pi) : \mathcal{P}(A) \to \mathcal{P}(E) \text{ is bijective} \end{array}\right) \\
&= \underbrace{\sum_{\substack{F \in \mathcal{P}(A);\\ \pi(F) \text{ is } \mathfrak{K}\text{-free}}}}_{= \sum_{\substack{F \subseteq A;\\ \pi(F) \text{ is } \mathfrak{K}\text{-free}}} = \sum_{\substack{F \subseteq A;\\ \text{the set } \pi(F) \text{ is } \mathfrak{K}\text{-free}}}} (-1)^{|F|} x^{\operatorname{conn}(V, \operatorname{set} F)} \\
&= \underbrace{\sum_{\substack{F \subseteq A;\\ \text{the set } \pi(F) \text{ is } \mathfrak{K}\text{-free}}}}_{\substack{= \sum_{\substack{F \subseteq A;\\ \text{the digraph } (V,F) \text{ is 2-step-free}}} \\ \text{(because for any subset } F \text{ of } A \text{, the condition} \\ \text{(the set } \pi(F) \text{ is } \mathfrak{K}\text{-free) is equivalent to the} \\ \text{condition (the digraph } (V,F) \text{ is 2-step-free)} \\ \text{(by Lemma 4.7 } \textbf{(e)}\text{))}}} (-1)^{|F|} x^{\operatorname{conn}(V, \operatorname{set} F)} \\
&= \sum_{\substack{F \subseteq A;\\ \text{the digraph } (V,F) \text{ is 2-step-free}}} (-1)^{|F|} x^{\operatorname{conn}(V, \operatorname{set} F)}.
\end{aligned}$$

Since $\underline{D} = G$, this rewrites as

$$\chi_{\underline{D}} = \sum_{\substack{F \subseteq A;\\ \text{the digraph } (V,F) \text{ is 2-step-free}}} (-1)^{|F|} x^{\operatorname{conn}(V, \operatorname{set} F)}.$$

This proves Proposition 4.5. □

5. Ambigraphs

5.1. Definitions of ambigraphs and proper colorings

We now move on to study various generalizations of the chromatic symmetric function.

The first generalization replaces the finite graph G by what we call an *ambigraph* (short for "ambiguous graph"). To our knowledge, this is a new notion, but it serves to unify two rather well-known concepts:

- that of a *multigraph* (see [Grinbe21, Definition 6.1.1]), which is like a graph but allows for multiple parallel edges[60];
- that of a *hypergraph* (see [Berge73, Chapter 17]), which is like a graph but allows its "edges" to have any number of endpoints instead of two.

In both of these settings, chromatic polynomials have been defined long ago (for multigraphs perhaps since the introduction of the concept[61]; for hypergraphs since Dohmen's [Dohmen95]), and it is fairly straightforward to define chromatic symmetric functions at the same levels of generality. However, we shall instead define them for *ambigraphs*, a concept which we now introduce:

Definition 5.1. **(a)** An *ambigraph* shall mean a triple $\left(V, E, \varphi\right)$, where V and E are two sets, and where $\varphi : E \to \mathcal{P}\left(\dbinom{V}{2}\right)$ is a map. (Thus, the map φ sends each $e \in E$ to a set of 2-element subsets of V.)

(b) An ambigraph $\left(V, E, \varphi\right)$ is said to be *finite* if V and E are finite.

(c) Let $G = \left(V, E, \varphi\right)$ be an ambigraph. Then, the elements of V are called the *vertices* of G, whereas the elements of E are called the *edgeries* of G. If $e \in E$ is any edgery, then the elements of $\varphi\left(e\right)$ are called the *edges* of e. Note that these edges are 2-element subsets of V.

(d) Let $G = \left(V, E, \varphi\right)$ be an ambigraph. An edgery $e \in E$ is said to be *singleton* if it has exactly one edge (i.e., if $\left|\varphi\left(e\right)\right| = 1$).

We view an ambigraph $\left(V, E, \varphi\right)$ as something akin to a graph, except that instead of having edges, it has edgeries – i.e., packages of edges. (This can be equivalently viewed as an edge-colored graph, but we eschew such an interpretation as we shall be using colors for other purposes.)

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[60] More precisely, our notion of an ambigraph generalizes *loopless* multigraphs, i.e., multigraphs with no loops. Loops would be a trivial but technically awkward distraction in the study of chromatic polynomials, so we prefer to leave them out of our notions of graphs.

[61] Authors often leave it vague whether their graphs are simple graphs or multigraphs.

Example 5.2. Let V be the set $\{1,2,3,4,5\}$. Let E be the 6-element set $\{e_1,e_2,e_3,e_4,e_5,e_6\}$. Let $\varphi : E \to \mathcal{P}\left(\dbinom{V}{2}\right)$ be the map defined as follows:

$$\begin{aligned}
\varphi(e_1) &= \{\{1,3\}, \{2,5\}\}, \\
\varphi(e_2) &= \{\{1,2\}, \{2,3\}, \{3,4\}\}, \\
\varphi(e_3) &= \{\{2,5\}\}, \\
\varphi(e_4) &= \{\{1,3\}, \{2,5\}\}, \\
\varphi(e_5) &= \{\} = \varnothing, \\
\varphi(e_6) &= \{\{2,3\}, \{3,4\}\}.
\end{aligned}$$

Let G be the triple (V,E,φ). Then, G is an ambigraph. Its edgeries are e_1,e_2,e_3,e_4,e_5,e_6. The edgery e_3 is singleton, while the other edgeries are not. The edgeries e_1 and e_4 contain the same edges, namely $\{1,3\}$ and $\{2,5\}$.

Both multigraphs and hypergraphs can now be encoded as ambigraphs:

- A multigraph can be viewed as an ambigraph whose all edgeries are singleton[62].
- A hypergraph can be encoded as an ambigraph by replacing each edge $\{v_1,v_2,\ldots,v_k\}$ with an edgery consisting of all edges $\{v_i,v_j\}$ with $i<j$. (Note that this encoding turns 1-element edges into empty edgeries[63]. Empty edgeries trivialize most of our results, but do not invalidate any of our proofs, so we have no reason to exclude them.)

We can now define X-colorings and proper X-colorings for ambigraphs:

Definition 5.3. Let $G=(V,E,\varphi)$ be an ambigraph. Let X be a set.

(a) An *X-coloring* of G is defined to mean a map $V \to X$.

(b) If $f : V \to X$ is an X-coloring of G, and if $\{s,t\}$ is a 2-element subset of V, then this subset $\{s,t\}$ is said to be *f-dichromatic* if $f(s) \neq f(t)$.

(c) An X-coloring f of G is said to be *proper* if each edgery $e \in E$ has at least one f-dichromatic edge (i.e., for each edgery $e \in E$, there exists at least one edge $\{s,t\} \in \varphi(e)$ satisfying $f(s) \neq f(t)$).

Example 5.4. Let $G=(V,E,\varphi)$ be the ambigraph from Example 5.2. Then, G has no proper X-coloring for any X, since the edgery e_5 will never have an f-dichromatic edge, no matter what f is (because e_5 has no edge to begin with).

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[62] To be more precise, this is true for *loopless* multigraphs (i.e., multigraphs that have no loops). Loops can be encoded as edgeries that have no edges.

[63] i.e., edgeries that have no edges

However, let us now modify φ by replacing $\varphi(e_5)$ by the set $\{\{1,4\},\{2,4\},\{3,4\}\}$. Then, for example, the X-coloring $f : V \to \{1,2,3,4\}$ given by

$$\begin{aligned} f(1) &= 1, & f(2) &= 2, & f(3) &= 1, \\ f(4) &= 1, & f(5) &= 3, & f(6) &= 1 \end{aligned}$$

is proper. For instance, the edgery e_1 has the f-dichromatic edge $\{2,5\}$, whereas the edgery e_2 has the two f-dichromatic edges $\{1,2\}$ and $\{2,3\}$. On the other hand, the X-coloring $f : V \to \{1,2,3,4\}$ given by

$$\begin{aligned} f(1) &= 1, & f(2) &= 2, & f(3) &= 2, \\ f(4) &= 2, & f(5) &= 3, & f(6) &= 1 \end{aligned}$$

is not proper, since the edgery e_6 has no f-dichromatic edge.

Example 5.5. Let $G = (V, E, \varphi)$ be the ambigraph with $V = \{1,2,3,4\}$ and $E = \{a, b\}$ and

$$\varphi(a) = \{\{2,3\}\} \qquad \text{and} \qquad \varphi(b) = \{\{1,2\},\{3,4\}\}.$$

Let X be a set. Then, a map $f : V \to X$ is a proper X-coloring of G if and only if it satisfies

$$f(2) \neq f(3) \qquad \text{and} \qquad (f(1) \neq f(2) \text{ or } f(3) \neq f(4)).$$

Indeed, the statement "$f(2) \neq f(3)$" is saying that the edgery a has an f-dichromatic edge, whereas the statement "$f(1) \neq f(2)$ or $f(3) \neq f(4)$" is saying that the edgery b has an f-dichromatic edge.

As Example 5.5 illustrates, the condition on an X-coloring of G to be proper is a conjunction of disjunctions of inequalities of the form $f(v) \neq f(w)$ for $(v,w) \in V^2$.

Remark 5.6. Any graph $G = (V, E)$ can be viewed as an ambigraph (V, E, φ) in a fairly obvious way: viz., by setting $\varphi(e) = \{e\}$ for each edge $e \in E$. We shall denote the latter ambigraph by G^{amb}. The proper X-colorings of this ambigraph G^{amb} are precisely the proper X-colorings of the original graph G.

Remark 5.7. Let $G = (V, E, \varphi)$ be an ambigraph, and let X be a set. If there exists an edgery $e \in E$ satisfying $\varphi(e) = \varnothing$, then there exists no proper X-coloring f of G (since the edgery e will never have an f-dichromatic edge).

5.2. The chromatic symmetric function of an ambigraph

We can now define the chromatic symmetric function of an ambigraph, by imitating Definition 1.5:

Definition 5.8. Let $G = (V, E, \varphi)$ be a finite ambigraph.

(a) For every $\mathbb{N}_+$-coloring $f : V \to \mathbb{N}_+$ of G, we let $\mathbf{x}_f$ denote the monomial $\prod\limits_{v \in V} x_{f(v)}$ in the indeterminates $x_1, x_2, x_3, \ldots$.

(b) We define a power series $X_G \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$X_G = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_f .$$

This power series X_G is called the *chromatic symmetric function* of G.

Example 5.9. Let $G = (V, E, \varphi)$ be the ambigraph from Example 5.5. Then,

$$\begin{aligned} X_G &= \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \underbrace{\mathbf{x}_f}_{=x_{f(1)}x_{f(2)}x_{f(3)}x_{f(4)}} \\ &= \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} x_{f(1)}x_{f(2)}x_{f(3)}x_{f(4)} \\ &= \sum_{\substack{f:\{1,2,3,4\}\to\mathbb{N}_+; \\ f(2)\neq f(3) \text{ and } (f(1)\neq f(2) \text{ or } f(3)\neq f(4))}} x_{f(1)}x_{f(2)}x_{f(3)}x_{f(4)} \end{aligned}$$

(since a map $f : V \to \mathbb{N}_+$ is a proper $\mathbb{N}_+$-coloring of G if and only if it satisfies $f(2) \neq f(3)$ and $(f(1) \neq f(2) \text{ or } f(3) \neq f(4))$). If we re-encode each map $f : \{1, 2, 3, 4\} \to \mathbb{N}_+$ as the 4-tuple $(i, j, k, \ell) = (f(1), f(2), f(3), f(4))$ of its values, then we can rewrite this equality as

$$X_G = \sum_{\substack{(i,j,k,\ell)\in(\mathbb{N}_+)^4; \\ j\neq k \text{ and } (i\neq j \text{ or } k\neq \ell)}} x_i x_j x_k x_\ell .$$

Remark 5.10. Let $G = (V, E, \varphi)$ be an ambigraph that has an edgery $e \in E$ satisfying $\varphi(e) = \varnothing$. Then, there exists no proper $\mathbb{N}_+$-coloring f of G (by Remark 5.7), and thus we have $X_G = 0$.

5.3. The union of a set of edgeries

An ambigraph (V, E, φ) can be transformed into a simple graph (V, E') by taking the union of some of its edgeries – i.e., by setting $E' := \bigcup\limits_{e \in F} \varphi(e)$ for some subset

F of E. Let us give this construction a name:

Definition 5.11. Let $G = (V, E, \varphi)$ be an ambigraph. Let F be a subset of E. Then, $\operatorname{union} F$ shall denote the subset $\bigcup\limits_{e \in F} \varphi(e)$ of $\dbinom{V}{2}$. Thus, we obtain a graph $(V, \operatorname{union} F)$.

Example 5.12. Let $G = (V, E, \varphi)$ be the ambigraph from Example 5.2. Then,

$$\operatorname{union} \{e_2, e_3\} = \{\{1,2\}, \{2,3\}, \{3,4\}, \{2,5\}\}$$

and

$$\operatorname{union} \{e_1, e_2, e_4\} = \{\{1,3\}, \{2,5\}, \{1,2\}, \{2,3\}, \{3,4\}\}$$

and $\operatorname{union} \{\} = \varnothing$.

We can use this notion to state our first result about ambigraphs – an analogue to Theorem 1.11. We shall prove this result at the end of the next subsection.

Theorem 5.13. Let $G = (V, E, \varphi)$ be a finite ambigraph. Then,

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda(V, \operatorname{union} F)}.$$

(Here, of course, the pair $(V, \operatorname{union} F)$ is regarded as a graph, and the expression $\lambda(V, \operatorname{union} F)$ is understood according to Definition 1.10 **(b)**.)

5.4. Circuits and broken circuits

Let us now define the notions of cycles, circuits and broken circuits of an ambigraph.

Definition 5.14. Let $G = (V, E, \varphi)$ be an ambigraph. A *cycle* of G denotes a list

$$(v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_{m+1})$$

with the following properties:

- The entries $v_1, v_2, \ldots, v_{m+1}$ at the odd positions of this list belong to V, whereas the entries $e_1, e_2, \ldots, e_m$ at its even positions belong to E.
- We have $m \geq 1$.
- We have $v_{m+1} = v_1$.
- The vertices $v_1, v_2, \ldots, v_m$ are pairwise distinct.
- The edgeries $e_1, e_2, \ldots, e_m$ are pairwise distinct.

- We have $\{v_i, v_{i+1}\} \in \varphi(e_i)$ for every $i \in \{1, 2, \ldots, m\}$.

If $(v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_{m+1})$ is a cycle of G, then the set $\{e_1, e_2, \ldots, e_m\}$ is called a *circuit* of G.

Example 5.15. Let $G = (V, E, \varphi)$ be the ambigraph from Example 5.2. Then, the tuple

$$(1, e_2, 2, e_6, 3, e_4, 1)$$

is a cycle of G (chiefly because $\{1, 2\} \in \varphi(e_2)$ and $\{2, 3\} \in \varphi(e_6)$ and $\{3, 1\} \in \varphi(e_4)$). The circuit corresponding to this cycle is $\{e_2, e_6, e_4\}$.

The tuple $(1, e_2, 2, e_6, 3, e_1, 1)$ is a cycle of G as well, and leads to the circuit $\{e_2, e_6, e_1\}$.

For comparison, the similar-looking tuple $(1, e_2, 2, e_2, 3, e_4, 1)$ is not a cycle, since its edgeries e_2, e_2, e_4 are not distinct.

Definition 5.16. Let $G = (V, E, \varphi)$ be an ambigraph. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. We shall refer to ℓ as the *labeling function*. For every edgery e of G, we shall refer to $\ell(e)$ as the *label* of e.

A *broken circuit* of G means a subset of E having the form $C \setminus \{e\}$, where C is a circuit of G, and where e is the unique singleton edgery in C having maximum label (among the singleton edgeries in C). Of course, the notion of a broken circuit of G depends on the function ℓ; however, we suppress the mention of ℓ in our notation, since we will not consider situations where two different ℓ's coexist.

Thus, if G is an ambigraph with a labeling function ℓ, then any circuit C of G gives rise to a broken circuit provided that

- at least one edgery in C is singleton, and
- among the singleton edgeries in C, only one attains the maximum label.

In all other cases, C does not give rise to a broken circuit. Notice that two different circuits may give rise to one and the same broken circuit.

Example 5.17. **(a)** Let $G = (V, E, \varphi)$ be the ambigraph from Example 5.2. Let X and $\ell : E \to X$ be arbitrary. Then, the circuit $\{e_2, e_6, e_4\}$ we found in Example 5.15 does not give rise to a broken circuit, since it contains no singleton edgery. However, the circuit $\{e_3, e_1\}$ (coming from the cycle $(2, e_3, 5, e_1, 2)$ does give rise to a broken circuit (namely, $\{e_1\}$), since its unique singleton edgery is e_3.

(b) For better examples, we can try an ambigraph having more singleton edgeries. For instance, we can choose some graph G and consider the corresponding ambigraph G^{amb} as defined in Remark 5.6. Then, the broken circuits of G^{amb} are precisely the broken circuits of G.

(c) Here is another example: Let $G = (V, E, \varphi)$ be the ambigraph with $V = \{1, 2, 3, 4\}$, $E = \{e_1, e_2, e_3\}$ and

$$\varphi(e_1) = \{\{1,2\}\}, \qquad \varphi(e_2) = \{\{2,3\}\}, \qquad \varphi(e_3) = \{\{3,4\}, \{1,3\}\}.$$

Let X and $\ell : E \to X$ be arbitrary. Then, the cycle $(1, e_1, 2, e_2, 3, e_3, 1)$ of G gives rise to the circuit $\{e_1, e_2, e_3\}$. This circuit gives rise to

- the broken circuit $\{e_2, e_3\}$ if $\ell(e_1) > \ell(e_2)$;
- the broken circuit $\{e_1, e_3\}$ if $\ell(e_1) < \ell(e_2)$;
- no broken circuit if $\ell(e_1) = \ell(e_2)$.

Note that $\ell(e_3)$ does not matter, since the edgery e_3 is not singleton.

The notion of a broken circuit always depends on a labeling function $\ell : E \to X$. Any time we speak about broken circuits, we shall tacitly understand that the function $\ell : E \to X$ is used as the labeling function.

5.5. The main results for ambigraphs

We can now generalize Theorem 1.15 to ambigraphs:

Theorem 5.18. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq F}} a_K \right) p_{\lambda(V, \operatorname{union} F)}.$$

(Here, of course, the pair $(V, \operatorname{union} F)$ is regarded as a graph, and the expression $\lambda(V, \operatorname{union} F)$ is understood according to Definition 1.10 **(b)**.)

This theorem generalizes Theorem 1.15 (in fact, the latter is easily obtained by applying the former to G^{amb} instead of G). Before we prove it, let us first explore some particular cases. Using Definition 1.16, we can obtain the following consequences of Theorem 1.15:

Corollary 5.19. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$X_G = \sum_{\substack{F \subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda(V, \operatorname{union} F)}.$$

Corollary 5.20. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Then,

$$X_G = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} p_{\lambda(V, \operatorname{union} F)}.$$

5.6. Proofs

Our proof of Theorem 5.18 is mostly similar to our above proof of Theorem 1.15, but there are some complications due to the possibility of non-singleton edgeries.

We shall use the Iverson bracket notation (Definition 2.9). We begin with a basic cancellation lemma (see, e.g., [Grinbe20, Proposition 7.8.10]):

Lemma 5.21. Let S be a finite set. Then, $\sum\limits_{I \subseteq S} (-1)^{|I|} = [S = \varnothing]$.

In Definition 2.1, we defined a set $\operatorname{Eqs} f$ for any map $f : V \to X$. This set helped us find the edges of a graph whose endpoints received the same color under a coloring f. We shall now introduce a similar notion for ambigraphs:

Definition 5.22. Let $G = (V, E, \varphi)$ be an ambigraph. Let X be a set. Let $f : V \to X$ be a map. We let $\operatorname{EQS}(G, f)$ denote the subset

$$\{e \in E \mid \varphi(e) \subseteq \operatorname{Eqs} f\}$$

of E.

Example 5.23. Let $G = (V, E, \varphi)$ be the ambigraph with $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{a, b, c\}$ and

$$\begin{aligned} \varphi(a) &= \{\{1,3\}, \{2,4\}, \{3,6\}\}, \\ \varphi(b) &= \{\{1,3\}, \{2,6\}\}, \\ \varphi(c) &= \varnothing. \end{aligned}$$

Let $X = \mathbb{N}$, and let $f : V \to X$ be the map that sends $1, 2, 3, 4, 5, 6$ to $1, 2, 1, 2, 1, 2$, respectively. Then,

$$\operatorname{Eqs} f = \{\{1,3\}, \{1,5\}, \{3,5\}, \{2,4\}, \{2,6\}, \{4,6\}\}$$

and $\operatorname{EQS}(G, f) = \{b, c\}$. Indeed, we have $b \in \operatorname{EQS}(G, f)$ since $\varphi(b) = \{\{1,3\}, \{2,6\}\} \subseteq \operatorname{Eqs} f$, and we have $c \in \operatorname{EQS}(G, f)$ since $\varphi(c) = \varnothing \subseteq \operatorname{Eqs} f$. On the other hand, $a \notin \operatorname{EQS}(G, f)$ since $\varphi(a) \not\subseteq \operatorname{Eqs} f$ (because $\{3, 6\}$ belongs to $\varphi(a)$ but not to $\operatorname{Eqs} f$).

> **Remark 5.24.** Let $G = (V, E, \varphi)$ be an ambigraph. Let X be a set. Let $f : V \to X$ be a map. The definition of $\operatorname{EQS}(G, f)$ yields
>
> $$\begin{aligned} \operatorname{EQS}(G,f) &= \{e \in E \mid \varphi(e) \subseteq \operatorname{Eqs} f\} && (91) \\ &= \{d \in E \mid \varphi(d) \subseteq \operatorname{Eqs} f\} && (92) \\ &\qquad \text{(here, we have renamed the index } e \text{ as } d\text{)} \\ &= \{d \in E \mid \text{no edge of } d \text{ is } f\text{-dichromatic}\}. && (93) \end{aligned}$$
>
> (The last equality is easy to check from the definitions.)

Proof of Remark 5.24. The equalities (91) and (92) are obvious. It remains to prove (93).

Indeed, let $d \in E$. Then, the edges of d are defined to be the elements of $\varphi(d)$. Thus, the elements of $\varphi(d)$ are precisely the edges of d. Hence, it is easy to prove the implication

$$(\varphi(d) \subseteq \operatorname{Eqs} f) \implies (\text{no edge of } d \text{ is } f\text{-dichromatic})$$

[64] and the implication

$$(\text{no edge of } d \text{ is } f\text{-dichromatic}) \implies (\varphi(d) \subseteq \operatorname{Eqs} f)$$

[65]. Combining these two implications, we obtain the equivalence

$$(\varphi(d) \subseteq \operatorname{Eqs} f) \iff (\text{no edge of } d \text{ is } f\text{-dichromatic}).$$

---

[64] *Proof.* Assume that $\varphi(d) \subseteq \operatorname{Eqs} f$. We must prove that no edge of d is f-dichromatic.

Indeed, let g be an edge of d. Thus, $g \in \varphi(d)$ (since the edges of d are the elements of $\varphi(d)$). Hence,

$$g \in \varphi(d) \subseteq \operatorname{Eqs} f = \left\{ \{s,t\} \mid (s,t) \in V^2,\ s \neq t \text{ and } f(s) = f(t) \right\}.$$

In other words, g can be written as $g = \{s,t\}$ for some pair $(s,t) \in V^2$ satisfying $s \neq t$ and $f(s) = f(t)$. Consider this pair (s,t). Then, $\{s,t\}$ is a 2-element subset of V (since $(s,t) \in V^2$ and $s \neq t$), and satisfies $f(s) = f(t)$. Hence, this subset $\{s,t\}$ is not f-dichromatic (because if it was f-dichromatic, then it would satisfy $f(s) \neq f(t)$ (by the definition of "f-dichromatic"), but this would contradict $f(s) = f(t)$). In other words, g is not f-dichromatic (since $g = \{s,t\}$).

Forget that we fixed g. We thus have shown that if g is any edge of d, then g is not f-dichromatic. In other words, no edge of d is f-dichromatic. Qed.

[65] *Proof.* Assume that no edge of d is f-dichromatic. We must prove that $\varphi(d) \subseteq \operatorname{Eqs} f$.

Indeed, let $g \in \varphi(d)$. Thus, g is an edge of d (since the edges of d are the elements of $\varphi(d)$). Hence, g is not f-dichromatic (since no edge of d is f-dichromatic).

However, $g \in \varphi(d) \subseteq \binom{V}{2}$. In other words, g is a 2-element subset of V. Thus, we can write g as $g = \{x,y\}$ for two distinct elements x and y of V. Consider these x and y. Then, $(x,y) \in V^2$ and $x \neq y$ (since x and y are distinct). Thus, $\{x,y\}$ is a 2-element subset of V.

Recall that g is not f-dichromatic. In other words, $\{x,y\}$ is not f-dichromatic (since $g = \{x,y\}$).

The 2-element subset $\{x,y\}$ of V is f-dichromatic if and only if $f(x) \neq f(y)$ (by the

Forget that we fixed d. We thus have proved the equivalence

$$(\varphi(d) \subseteq \operatorname{Eqs} f) \iff (\text{no edge of } d \text{ is } f\text{-dichromatic})$$

for each $d \in E$. Therefore,

$$\{d \in E \mid \varphi(d) \subseteq \operatorname{Eqs} f\} = \{d \in E \mid \text{no edge of } d \text{ is } f\text{-dichromatic}\}.$$

Hence, (92) can be rewritten as

$$\operatorname{EQS}(G, f) = \{d \in E \mid \text{no edge of } d \text{ is } f\text{-dichromatic}\}.$$

This proves (93). Thus, the proof of Remark 5.24 is complete. □

In analogy to Lemma 2.3, we can use $\operatorname{EQS}(G, f)$ to characterize when an X-coloring f is proper:

> **Lemma 5.25.** Let $G = (V, E, \varphi)$ be an ambigraph. Let X be a set. Let $f : V \to X$ be a map. Then, the X-coloring f of G is proper if and only if $\operatorname{EQS}(G, f) = \varnothing$.

Proof of Lemma 5.25. The definition of $\operatorname{Eqs} f$ shows that

$$\begin{aligned}\operatorname{Eqs} f &= \Big\{\{s,t\} \mid (s,t) \in V^2,\ s \neq t \text{ and } f(s) = f(t)\Big\} \\ &= \Big\{\{x,y\} \mid (x,y) \in V^2,\ x \neq y \text{ and } f(x) = f(y)\Big\} \qquad (94)\end{aligned}$$

(here, we renamed the index (s,t) as (x,y)).

The definition of $\operatorname{EQS}(G, f)$ yields $\operatorname{EQS}(G, f) = \{e \in E \mid \varphi(e) \subseteq \operatorname{Eqs} f\}$. Thus, an edgery $e \in E$ belongs to $\operatorname{EQS}(G, f)$ if and only if it satisfies $\varphi(e) \subseteq \operatorname{Eqs} f$. In other words, for any edgery $e \in E$, we have the logical equivalence

$$(e \in \operatorname{EQS}(G, f)) \iff (\varphi(e) \subseteq \operatorname{Eqs} f). \qquad (95)$$

We shall first prove the logical implication

$$(\text{the } X\text{-coloring } f \text{ of } G \text{ is proper}) \implies (\operatorname{EQS}(G, f) = \varnothing). \qquad (96)$$

Proof of (96): Assume that the X-coloring f of G is proper. We must show that $\operatorname{EQS}(G, f) = \varnothing$.

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definition of "f-dichromatic"). Hence, we don't have $f(x) \neq f(y)$ (since $\{x,y\}$ is not f-dichromatic). In other words, we have $f(x) = f(y)$.

Hence, $\{x,y\}$ is a set of the form $\{s,t\}$ for some pair $(s,t) \in V^2$ satisfying $s \neq t$ and $f(s) = f(t)$ (namely, for $(s,t) = (x,y)$). In other words,

$$g \in \Big\{\{s,t\} \mid (s,t) \in V^2,\ s \neq t \text{ and } f(s) = f(t)\Big\} = \operatorname{Eqs} f$$

(since $\operatorname{Eqs} f$ is defined to be $\big\{\{s,t\} \mid (s,t) \in V^2,\ s \neq t \text{ and } f(s) = f(t)\big\}$).

Forget that we fixed g. We thus have shown that $g \in \operatorname{Eqs} f$ for each $g \in \varphi(d)$. In other words, $\varphi(d) \subseteq \operatorname{Eqs} f$. Qed.

Recall that the X-coloring f of G is proper if and only if each edgery $e \in E$ has at least one f-dichromatic edge (by the definition of "proper"). Thus, each edgery $e \in E$ has at least one f-dichromatic edge (since the X-coloring f of G is proper).

Now, let $d \in \operatorname{EQS}(G, f)$. Thus,

$$d \in \operatorname{EQS}(G, f) = \{e \in E \mid \varphi(e) \subseteq \operatorname{Eqs} f\}$$

(by the definition of $\operatorname{EQS}(G, f)$). In other words, d is an $e \in E$ satisfying $\varphi(e) \subseteq \operatorname{Eqs} f$. In other words, $d \in E$ and $\varphi(d) \subseteq \operatorname{Eqs} f$.

However, recall that each edgery $e \in E$ has at least one f-dichromatic edge. Applying this to $e = d$, we conclude that d has at least one f-dichromatic edge. Let g be this edge. Thus, $g \in \varphi(d)$ (since g is an edge of d). Hence,

$$g \in \varphi(d) \subseteq \operatorname{Eqs} f = \Big\{\{x, y\} \mid (x, y) \in V^2,\ x \neq y \text{ and } f(x) = f(y)\Big\}$$

(by (94)). In other words, g has the form $\{x, y\}$ for some pair $(x, y) \in V^2$ satisfying $x \neq y$ and $f(x) = f(y)$. Consider this pair (x, y). Thus, $g = \{x, y\}$.

We have $f(x) = f(y)$. In other words, we don't have $f(x) \neq f(y)$. Moreover, $\{x, y\}$ is a 2-element subset of V (since $(x, y) \in V^2$ and $x \neq y$).

Now, recall that the subset $\{x, y\}$ of V is f-dichromatic if and only if $f(x) \neq f(y)$ (by the definition of "f-dichromatic"). Thus, this subset $\{x, y\}$ is not f-dichromatic (since we don't have $f(x) \neq f(y)$). In other words, g is not f-dichromatic (since $g = \{x, y\}$). This contradicts the fact that g is f-dichromatic.

Forget that we fixed d. We thus have obtained a contradiction for each $d \in \operatorname{EQS}(G, f)$. Hence, there exists no $d \in \operatorname{EQS}(G, f)$. In other words, the set $\operatorname{EQS}(G, f)$ is empty. In other words, $\operatorname{EQS}(G, f) = \varnothing$. Thus, the implication (96) is proven.

Now, we shall prove the implication

$$(\operatorname{EQS}(G, f) = \varnothing) \implies (\text{the } X\text{-coloring } f \text{ of } G \text{ is proper}). \tag{97}$$

Proof of (97): Assume that $\operatorname{EQS}(G, f) = \varnothing$. We have to show that the X-coloring f of G is proper.

Let $e \in E$ be an edgery. We shall show that e has at least one f-dichromatic edge.

Indeed, assume the contrary. Thus, e has no f-dichromatic edge. In other words, an edge of e cannot be f-dichromatic.

Now, let $g \in \varphi(e)$ be arbitrary. Thus, g is an edge of e. Hence, g is not f-dichromatic (since an edge of e cannot be f-dichromatic). But $g \in \varphi(e) \subseteq \dbinom{V}{2}$, so that g is a 2-element subset of V. In other words, we can write g as $g = \{s, t\}$ for two distinct elements s and t of V. Consider these two elements s and t. We have $s \neq t$ (since s and t are distinct).

We have $g = \{s,t\}$. Thus, $\{s,t\}$ is not f-dichromatic (since g is not f-dichromatic).

However, $\{s,t\}$ is f-dichromatic if and only if $f(s) \neq f(t)$ (by the definition of "f-dichromatic"). Hence, we cannot have $f(s) \neq f(t)$ (since g is not f-dichromatic). In other words, we have $f(s) = f(t)$. Hence, $\{s,t\}$ is a set of the form $\{x,y\}$ for some $(x,y) \in V^2$ satisfying $x \neq y$ and $f(x) = f(y)$ (namely, for $(x,y) = (s,t)$).

Now,

$$\begin{aligned} g &= \{s,t\} \\ &\in \left\{\{x,y\} \mid (x,y) \in V^2,\ x \neq y \text{ and } f(x) = f(y)\right\} \\ &\qquad \left(\begin{array}{c}\text{since } \{s,t\} \text{ is a set of the form } \{x,y\} \text{ for} \\ \text{some } (x,y) \in V^2 \text{ satisfying } x \neq y \text{ and } f(x) = f(y)\end{array}\right) \\ &= \operatorname{Eqs} f \qquad \text{(by (94))}. \end{aligned}$$

Forget that we fixed g. We thus have shown that $g \in \operatorname{Eqs} f$ for each $g \in \varphi(e)$. In other words, $\varphi(e) \subseteq \operatorname{Eqs} f$. According to the equivalence (95), this entails that $e \in \operatorname{EQS}(G,f)$ (since $e \in E$). In other words, $e \in \varnothing$ (since $\operatorname{EQS}(G,f) = \varnothing$). But this is absurd (since the empty set $\varnothing$ has no elements). This contradiction shows that our assumption was false. Hence, e has at least one f-dichromatic edge.

Forget that we fixed e. We thus have shown that each edgery $e \in E$ has at least one f-dichromatic edge. In other words, the X-coloring f of G is proper (by the definition of "proper"). This proves the implication (97).

Now we have proven the two implications (96) and (97). Combining these two implications, we obtain the equivalence

$$(\text{the } X\text{-coloring } f \text{ of } G \text{ is proper}) \iff (\operatorname{EQS}(G,f) = \varnothing).$$

This proves Lemma 5.25. ☐

The following simple lemma connects $\operatorname{EQS}(G,f)$ with the $\operatorname{union} F$ construction from Definition 5.11:

Lemma 5.26. Let $G = (V,E,\varphi)$ be an ambigraph. Let X be a set. Let $f : V \to X$ be a map. Let B be a subset of E. Then, $B \subseteq \operatorname{EQS}(G,f)$ holds if and only if $\operatorname{union} B \subseteq \operatorname{Eqs} f$.

Proof of Lemma 5.26. The definition of $\operatorname{union} B$ yields $\operatorname{union} B = \bigcup_{e \in B} \varphi(e)$. Hence, we have the following chain of logical equivalences:

$$\begin{aligned} &(\operatorname{union} B \subseteq \operatorname{Eqs} f) \\ \iff &\left(\bigcup_{e \in B} \varphi(e) \subseteq \operatorname{Eqs} f\right) \\ \iff &(\varphi(e) \subseteq \operatorname{Eqs} f \text{ for each } e \in B). \end{aligned} \tag{98}$$

The definition of $\operatorname{EQS}(G,f)$ yields $\operatorname{EQS}(G,f) = \{e \in E \mid \varphi(e) \subseteq \operatorname{Eqs} f\}$. Thus, an edgery $e \in E$ belongs to $\operatorname{EQS}(G,f)$ if and only if it satisfies $\varphi(e) \subseteq \operatorname{Eqs} f$. In other words, for any edgery $e \in E$, we have the logical equivalence

$$(e \in \operatorname{EQS}(G,f)) \iff (\varphi(e) \subseteq \operatorname{Eqs} f). \tag{99}$$

Now, we have the following chain of logical equivalences:

$$\begin{aligned}
&(B \subseteq \operatorname{EQS}(G,f)) \\
\iff &\left(\underbrace{e \in \operatorname{EQS}(G,f)}_{\substack{\iff (\varphi(e)\subseteq \operatorname{Eqs} f) \\ \text{(by (99))}}} \text{ for each } e \in B\right) \\
\iff &(\varphi(e) \subseteq \operatorname{Eqs} f \text{ for each } e \in B) \\
\iff &(\operatorname{union} B \subseteq \operatorname{Eqs} f) \qquad \text{(by (98))}.
\end{aligned}$$

In other words, $B \subseteq \operatorname{EQS}(G,f)$ holds if and only if $\operatorname{union} B \subseteq \operatorname{Eqs} f$. This proves Lemma 5.26. ☐

Next, let us show an analogue of Lemma 2.4:

> **Lemma 5.27.** Let $G = (V, E, \varphi)$ be an ambigraph. Let X be a set. Let $f : V \to X$ be a map. Let C be a circuit of G. Let $e \in C$ be a singleton edgery such that $C \setminus \{e\} \subseteq \operatorname{EQS}(G,f)$. Then, $e \in \operatorname{EQS}(G,f)$.

Proof of Lemma 5.27. We have assumed that the edgery e is singleton. In other words, e has exactly one edge (by the definition of "singleton"). In other words, $\varphi(e)$ is a 1-element set (since the edges of e are defined to be the elements of $\varphi(e)$).

The set C is a circuit of G. In other words, the set C has the form $\{e_1, e_2, \ldots, e_m\}$, where $(v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_{m+1})$ is a cycle of G (by the definition of a "circuit"). Consider this cycle $(v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_{m+1})$. We thus have

$$C = \{e_1, e_2, \ldots, e_m\}. \tag{100}$$

The list $(v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_{m+1})$ is a cycle of G. According to the definition of a "cycle", this means that this list is a list satisfying the following four properties:

- The entries $v_1, v_2, \ldots, v_{m+1}$ at the odd positions of this list belong to V, whereas the entries $e_1, e_2, \ldots, e_m$ at its even positions belong to E.
- We have $m \geq 1$.
- We have $v_{m+1} = v_1$.

- The vertices $v_1, v_2, \ldots, v_m$ are pairwise distinct.
- The edgeries $e_1, e_2, \ldots, e_m$ are pairwise distinct.
- We have $\{v_i, v_{i+1}\} \in \varphi(e_i)$ for every $i \in \{1, 2, \ldots, m\}$.

Thus, $(v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_{m+1})$ is a list satisfying the six properties that we have just mentioned. In particular, $v_{m+1} = v_1$. Also,

$$\{v_i, v_{i+1}\} \in \varphi(e_i) \qquad \text{for every } i \in \{1, 2, \ldots, m\}. \tag{101}$$

We have $e \in C = \{e_1, e_2, \ldots, e_m\}$. Thus, $e = e_i$ for some $i \in \{1, 2, \ldots, m\}$. Consider this i.

Now, we have

$$f(v_j) = f(v_{j+1}) \qquad \text{for every } j \in \{1, 2, \ldots, m\} \setminus \{i\} \tag{102}$$

[66]. Hence,

$$f(v_1) = f(v_i) \tag{103}$$

---

[66] *Proof of (102):* Let $j \in \{1, 2, \ldots, m\} \setminus \{i\}$. Thus, $j \in \{1, 2, \ldots, m\}$ and $j \notin \{i\}$. From $j \notin \{i\}$, we obtain $j \neq i$. Thus, $e_j \neq e_i$ (since the edgeries $e_1, e_2, \ldots, e_m$ are pairwise distinct). In other words, $e_j \neq e$ (since $e = e_i$).

Now, $e_j \in \{e_1, e_2, \ldots, e_m\} = C$ (by (100)). Combining this with $e_j \neq e$, we obtain

$$e_j \in C \setminus \{e\} \subseteq \operatorname{EQS}(G, f) = \{d \in E \mid \varphi(d) \subseteq \operatorname{Eqs} f\}$$

(by (92)). In other words, e_j is a $d \in E$ satisfying $\varphi(d) \subseteq \operatorname{Eqs} f$. In other words, $e_j \in E$ and $\varphi(e_j) \subseteq \operatorname{Eqs} f$.

However, (101) (applied to j instead of i) yields

$$\begin{aligned} \{v_j, v_{j+1}\} &\in \varphi(e_j) \subseteq \operatorname{Eqs} f \\ &= \left\{\{s, t\} \mid (s, t) \in V^2, \ s \neq t \text{ and } f(s) = f(t)\right\} \end{aligned}$$

(by the definition of $\operatorname{Eqs} f$). In other words, the set $\{v_j, v_{j+1}\}$ has the form $\{s, t\}$ for some $(s, t) \in V^2$ satisfying $s \neq t$ and $f(s) = f(t)$. Consider this (s, t). Thus, $\{v_j, v_{j+1}\} = \{s, t\}$.

We have $f(s) = f(t)$. Therefore, set $g = f(s) = f(t)$. We have

$$f(\{s, t\}) = \left\{\underbrace{f(s)}_{=g}, \underbrace{f(t)}_{=g}\right\} = \{g, g\} = \{g\}.$$

Now, $v_j \in \{v_j, v_{j+1}\} = \{s, t\}$, and thus $f\left(\underbrace{v_j}_{\in\{s,t\}}\right) \in f(\{s, t\}) = \{g\}$. In other words, $f(v_j) = g$. Also, $v_{j+1} \in \{v_j, v_{j+1}\} = \{s, t\}$, and thus $f\left(\underbrace{v_{j+1}}_{\in\{s,t\}}\right) \in f(\{s, t\}) = \{g\}$. In other words, $f(v_{j+1}) = g$. Comparing this with $f(v_j) = g$, we obtain $f(v_j) = f(v_{j+1})$. This proves (102).

[67]. Also,

$$f\left(v_{i+1}\right)=f\left(v_{m+1}\right) \tag{104}$$

[68]. Now, (103) yields $f\left(v_i\right)=f\left(v_{m+1}\right)=f\left(v_1\right)$ (since $v_{m+1}=v_1$), so that

$$f\left(v_i\right)=f\left(v_{m+1}\right)=f\left(v_{i+1}\right) \qquad \text{(by (104))}.$$

Moreover, v_i and v_{i+1} are elements of V (since $v_1, v_2, \ldots, v_{m+1}$ are elements of V). In other words, $v_i \in V$ and $v_{i+1} \in V$. Hence, $\left(v_i, v_{i+1}\right) \in V^2$.

Furthermore, $v_i \neq v_{i+1}$ [69].

Now, the definition of $\operatorname{Eqs} f$ shows that

$$\operatorname{Eqs} f=\left\{\{s,t\} \mid (s,t) \in V^2,\ s \neq t \text{ and } f(s)=f(t)\right\}. \tag{105}$$

But we have $\left(v_i, v_{i+1}\right) \in V^2$, $v_i \neq v_{i+1}$ and $f\left(v_i\right)=f\left(v_{i+1}\right)$. Hence, the set $\left\{v_i, v_{i+1}\right\}$ has the form $\{s,t\}$ for some $(s,t) \in V^2$ satisfying $s \neq t$ and $f(s)=f(t)$ (namely, for $(s,t)=\left(v_i, v_{i+1}\right)$). Thus,

$$\left\{v_i, v_{i+1}\right\} \in \left\{\{s,t\} \mid (s,t) \in V^2,\ s \neq t \text{ and } f(s)=f(t)\right\}=\operatorname{Eqs} f$$

(by (105)).

Now, (101) yields that $\left\{v_i, v_{i+1}\right\} \in \varphi\left(e_i\right)$. We can rewrite this as $\left\{v_i, v_{i+1}\right\} \in \varphi(e)$ (since $e=e_i$). However, we also know that $\varphi(e)$ is a 1-element set. In other words, $\varphi(e)=\{u\}$ for some element u. Consider this u. From $\left\{v_i, v_{i+1}\right\} \in \varphi(e)=\{u\}$, we obtain $\left\{v_i, v_{i+1}\right\}=u$, so that $u=\left\{v_i, v_{i+1}\right\} \in \operatorname{Eqs} f$. Hence, $\{u\} \subseteq \operatorname{Eqs} f$.

Thus, altogether, $\varphi(e)=\{u\} \subseteq \operatorname{Eqs} f$. Thus, $e \in E$ and $\varphi(e) \subseteq \operatorname{Eqs} f$. In other words, e is an element $d \in E$ satisfying $\varphi(d) \subseteq \operatorname{Eqs} f$. Hence,

$$e \in \{d \in E \mid \varphi(d) \subseteq \operatorname{Eqs} f\}=\operatorname{EQS}(G,f)$$

(by (92)). This proves Lemma 5.27. ☐

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[67] *Proof of (103):* Let $j \in \{1,2,\ldots,i-1\}$. Thus, $j \in \{1,2,\ldots,i-1\} \subseteq \{1,2,\ldots,m\}$. Combining this with $j \neq i$ (since $j<i$ (since $j \in \{1,2,\ldots,i-1\}$)), we obtain $j \in \{1,2,\ldots,m\} \setminus \{i\}$. Hence, $f\left(v_j\right)=f\left(v_{j+1}\right)$ (by (102)).

Now, let us forget that we fixed j. We thus have proven that $f\left(v_j\right)=f\left(v_{j+1}\right)$ for every $j \in \{1,2,\ldots,i-1\}$. In other words, $f\left(v_1\right)=f\left(v_2\right)=\cdots=f\left(v_i\right)$. Hence, $f\left(v_1\right)=f\left(v_i\right)$. This proves (103).

[68] *Proof of (104):* Let $j \in \{i+1,i+2,\ldots,m\}$. Thus, $j \in \{i+1,i+2,\ldots,m\} \subseteq \{1,2,\ldots,m\}$. Combining this with $j \neq i$ (since $j>i$ (since $j \in \{i+1,i+2,\ldots,m\}$)), we obtain $j \in \{1,2,\ldots,m\} \setminus \{i\}$. Hence, $f\left(v_j\right)=f\left(v_{j+1}\right)$ (by (102)).

Now, let us forget that we fixed j. We thus have proven that $f\left(v_j\right)=f\left(v_{j+1}\right)$ for every $j \in \{i+1,i+2,\ldots,m\}$. In other words, $f\left(v_{i+1}\right)=f\left(v_{i+2}\right)=\cdots=f\left(v_{m+1}\right)$. Hence, $f\left(v_{i+1}\right)=f\left(v_{m+1}\right)$. This proves (104).

[69] *Proof.* Assume the contrary. Thus, $v_i=v_{i+1}$. Thus, $\left\{v_i, v_{i+1}\right\}=\left\{v_{i+1}, v_{i+1}\right\}=\left\{v_{i+1}\right\}$, so that $\left|\left\{v_i, v_{i+1}\right\}\right|=\left|\left\{v_{i+1}\right\}\right|=1$.

However, (101) yields that $\left\{v_i, v_{i+1}\right\} \in \varphi\left(e_i\right) \subseteq \dbinom{V}{2}$. Thus, $\left\{v_i, v_{i+1}\right\}$ is a 2-element subset of V. In particular, this entails $\left|\left\{v_i, v_{i+1}\right\}\right|=2$. But this contradicts $\left|\left\{v_i, v_{i+1}\right\}\right|=1 \neq 2$. This contradiction shows that our assumption was wrong. Qed.

Our next lemma will play a role in our proof of Theorem 5.18 that is similar to the role of Lemma 2.8 in the proof of Theorem 1.15 (although it is different in its claim).

Lemma 5.28. Let $G = (V, E, \varphi)$ be an ambigraph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function.

Let Y be any set. Let $f : V \to Y$ be any map. Assume that the set $\operatorname{EQS}(G, f)$ contains no singleton edgery. Then, there exists no broken circuit K of G satisfying $K \subseteq \operatorname{EQS}(G, f)$.

Proof of Lemma 5.28. Assume the contrary. Thus, there exists some broken circuit K of G satisfying $K \subseteq \operatorname{EQS}(G, f)$.

We recall that a broken circuit of G means a subset of E having the form $C \setminus \{e\}$, where C is a circuit of G, and where e is the unique singleton edgery in C having maximum label (among the singleton edgeries in C). Thus, in particular, K is a subset of E having this form (since K is a broken circuit of G). In other words, we can write K as $K = C \setminus \{e\}$, where C is a circuit of G, and where e is the unique singleton edgery in C having maximum label (among the singleton edgeries in C). Consider these C and e.

We have $C \setminus \{e\} = K \subseteq \operatorname{EQS}(G, f)$. Thus, Lemma 5.27 (applied to Y instead of X) yields that $e \in \operatorname{EQS}(G, f)$. Thus, the set $\operatorname{EQS}(G, f)$ contains a singleton edgery (namely, e). But this contradicts the fact that the set $\operatorname{EQS}(G, f)$ contains no singleton edgery.

This contradiction shows that our assumption was false. Hence, Lemma 5.28 is proven. ☐

We are now ready to prove the keystone lemma, which of course is an analogue of Lemma 2.10:

Lemma 5.29. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$.

Let Y be any set. Let $f : V \to Y$ be any map. Then,

$$\sum_{B \subseteq \operatorname{EQS}(G,f)} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K = [\operatorname{EQS}(G, f) = \varnothing].$$

Proof of Lemma 5.29. We are in one of the following two cases:

Case 1: The set $\operatorname{EQS}(G, f)$ contains no singleton edgery.

Case 2: The set $\operatorname{EQS}(G, f)$ contains at least one singleton edgery.

Let us first consider Case 1. In this case, the set $\operatorname{EQS}(G, f)$ contains no singleton edgery. Hence, using Lemma 5.28, it is easy to see that every subset B of

$\operatorname{EQS}(G,f)$ satisfies

$$\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K = 1. \tag{106}$$

Proof of (106): Let B be a subset of $\operatorname{EQS}(G,f)$. Then, Lemma 5.28 yields that there exists no broken circuit K of G satisfying $K \subseteq \operatorname{EQS}(G,f)$. Hence, there exists no $K \in \mathfrak{K}$ satisfying $K \subseteq B$ [70]. Therefore, the product $\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K$ is empty. Thus, $\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K = (\text{empty product}) = 1$. This proves (106).

The set E is finite (since (V,E,φ) is finite). Hence, its subset $\operatorname{EQS}(G,f)$ is also finite. Now,

$$\begin{aligned}
&\sum_{B\subseteq \operatorname{EQS}(G,f)} (-1)^{|B|} \underbrace{\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K}_{\substack{=1\\ \text{(by (106))}}} \\
&= \sum_{B\subseteq \operatorname{EQS}(G,f)} (-1)^{|B|} \\
&= \sum_{I\subseteq \operatorname{EQS}(G,f)} (-1)^{|I|} \qquad \left(\begin{array}{c}\text{here, we have renamed the summation}\\ \text{index } B \text{ as } I\end{array}\right) \\
&= [\operatorname{EQS}(G,f) = \varnothing] \qquad (\text{by Lemma 5.21, applied to } S = \operatorname{EQS}(G,f)).
\end{aligned}$$

Thus, Lemma 5.29 is proved in Case 1.

Let us now consider Case 2. In this case, the set $\operatorname{EQS}(G,f)$ contains at least one singleton edgery

The set E is finite (since (V,E,φ) is finite). Hence, its subset $\operatorname{EQS}(G,f)$ is also finite. Moreover, this subset $\operatorname{EQS}(G,f)$ is nonempty (since it contains at least one singleton edgery). In other words, $\operatorname{EQS}(G,f) \neq \varnothing$. Hence, $[\operatorname{EQS}(G,f) = \varnothing] = 0$.

We know that the set $\operatorname{EQS}(G,f)$ contains at least one singleton edgery. Pick any such singleton edgery $d \in \operatorname{EQS}(G,f)$ with maximum $d \in \operatorname{EQS}(G,f)$ (among all such singleton edgeries). (If there are several such d, then it does not matter which one we pick.)

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[70] *Proof.* Assume the contrary. Thus, there exists some $K \in \mathfrak{K}$ satisfying $K \subseteq B$. Consider such a K, and denote it by L. Thus, L is a $K \in \mathfrak{K}$ satisfying $K \subseteq B$. In other words, $L \in \mathfrak{K}$ and $L \subseteq B$. From $L \in \mathfrak{K}$, we see that L is a broken circuit of G (because $\mathfrak{K}$ is a set of broken circuits of G). However, B is a subset of $\operatorname{EQS}(G,f)$; in other words, $B \subseteq \operatorname{EQS}(G,f)$. Thus, $L \subseteq B \subseteq \operatorname{EQS}(G,f)$.

Hence, L is a broken circuit K of G satisfying $K \subseteq \operatorname{EQS}(G,f)$ (since L is a broken circuit of G and since $L \subseteq \operatorname{EQS}(G,f)$). Thus, there exists a broken circuit K of G satisfying $K \subseteq \operatorname{EQS}(G,f)$ (namely, L). But this contradicts the fact that there exists no broken circuit K of G satisfying $K \subseteq \operatorname{EQS}(G,f)$. This contradiction shows that our assumption was false. Qed.

We have chosen d to be the singleton edgery in $\operatorname{EQS}(G,f)$ with maximum $\ell(d)$ (among all singleton edgeries in $\operatorname{EQS}(G,f)$). Thus, for any singleton edgery $e \in \operatorname{EQS}(G,f)$, we have

$$\ell(d) \geq \ell(e). \tag{107}$$

As usual, we let $\mathcal{P}(S)$ denote the powerset of any set S. We now define two subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{P}(\operatorname{EQS}(G,f))$ as follows:

$$\mathcal{U} = \{F \in \mathcal{P}(\operatorname{EQS}(G,f)) \mid d \notin F\};$$
$$\mathcal{V} = \{F \in \mathcal{P}(\operatorname{EQS}(G,f)) \mid d \in F\}.$$

Every $B \in \mathcal{U}$ satisfies $B \cup \{d\} \in \mathcal{V}$ [71]. Thus, we can define a map $\Phi : \mathcal{U} \to \mathcal{V}$ by

$$(\Phi(B) = B \cup \{d\} \qquad \text{for every } B \in \mathcal{U}).$$

Consider this map Φ.

Every $B \in \mathcal{V}$ satisfies $B \setminus \{d\} \in \mathcal{U}$ [72]. Thus, we can define a map $\Psi : \mathcal{V} \to \mathcal{U}$ by

$$(\Psi(B) = B \setminus \{d\} \qquad \text{for every } B \in \mathcal{V}).$$

Consider this map Ψ.

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[71] *Proof.* Let $B \in \mathcal{U}$. Thus, $B \in \mathcal{U} = \{F \in \mathcal{P}(\operatorname{EQS}(G,f)) \mid d \notin F\}$. In other words, B is an element F of $\mathcal{P}(\operatorname{EQS}(G,f))$ satisfying $d \notin F$. In other words, B is an element of $\mathcal{P}(\operatorname{EQS}(G,f))$ and satisfies $d \notin B$. We have $B \in \mathcal{P}(\operatorname{EQS}(G,f))$; in other words, B is a subset of $\operatorname{EQS}(G,f)$. Also, $\{d\} \subseteq \operatorname{EQS}(G,f)$ (since $d \in \operatorname{EQS}(G,f)$). Thus, both B and $\{d\}$ are subsets of $\operatorname{EQS}(G,f)$. Hence, their union $B \cup \{d\}$ is a subset of $\operatorname{EQS}(G,f)$. In other words, $B \cup \{d\} \in \mathcal{P}(\operatorname{EQS}(G,f))$. Also, $d \in \{d\} \subseteq B \cup \{d\}$. Hence, $B \cup \{d\}$ is an element of $\mathcal{P}(\operatorname{EQS}(G,f))$ and satisfies $d \in B \cup \{d\}$. In other words, $B \cup \{d\}$ is an element F of $\mathcal{P}(\operatorname{EQS}(G,f))$ satisfying $d \in F$. In other words, $B \cup \{d\} \in \{F \in \mathcal{P}(\operatorname{EQS}(G,f)) \mid d \in F\} = \mathcal{V}$, qed.

[72] *Proof.* Let $B \in \mathcal{V}$. Thus, $B \in \mathcal{V} = \{F \in \mathcal{P}(\operatorname{EQS}(G,f)) \mid d \in F\}$. In other words, B is an element F of $\mathcal{P}(\operatorname{EQS}(G,f))$ satisfying $d \in F$. In other words, B is an element of $\mathcal{P}(\operatorname{EQS}(G,f))$ and satisfies $d \in B$. We have $B \in \mathcal{P}(\operatorname{EQS}(G,f))$; in other words, B is a subset of $\operatorname{EQS}(G,f)$. Hence, $B \setminus \{d\}$ is a subset of $\operatorname{EQS}(G,f)$ (since $B \setminus \{d\} \subseteq B$). In other words, $B \setminus \{d\} \in \mathcal{P}(\operatorname{EQS}(G,f))$. Also, $d \notin B \setminus \{d\}$ (since $d \in \{d\}$). Hence, $B \setminus \{d\}$ is an element of $\mathcal{P}(\operatorname{EQS}(G,f))$ and satisfies $d \notin B \setminus \{d\}$. In other words, $B \setminus \{d\}$ is an element F of $\mathcal{P}(\operatorname{EQS}(G,f))$ satisfying $d \notin F$. In other words, $B \setminus \{d\} \in \{F \in \mathcal{P}(\operatorname{EQS}(G,f)) \mid d \notin F\} = \mathcal{U}$, qed.

We have $\Phi \circ \Psi = \operatorname{id}$ [73] and $\Psi \circ \Phi = \operatorname{id}$ [74]. Thus, the maps Φ and Ψ are mutually inverse. Hence, the map Φ is a bijection.

Moreover, for every $B \in \mathcal{U}$ and every $K \in \mathfrak{K}$, we have the following logical equivalence:

$$(K \subseteq B) \iff (K \subseteq \Phi(B)) . \tag{108}$$

Proof of (108): Let $B \in \mathcal{U}$ and $K \in \mathfrak{K}$. We need to prove the logical equivalence (108).

The definition of Φ yields $\Phi(B) = B \cup \{d\} \supseteq B$, so that $B \subseteq \Phi(B)$.

We have $B \in \mathcal{U} = \{F \in \mathcal{P}(\operatorname{EQS}(G,f)) \mid d \notin F\}$. In other words, B is an element F of $\mathcal{P}(\operatorname{EQS}(G,f))$ satisfying $d \notin F$. In other words, B is an element of $\mathcal{P}(\operatorname{EQS}(G,f))$ and satisfies $d \notin B$. We have $B \in \mathcal{P}(\operatorname{EQS}(G,f))$; in other words, B is a subset of $\operatorname{EQS}(G,f)$.

Also, $\Phi(B) \in \mathcal{V} = \{F \in \mathcal{P}(\operatorname{EQS}(G,f)) \mid d \in F\} \subseteq \mathcal{P}(\operatorname{EQS}(G,f))$. In other words, $\Phi(B) \subseteq \operatorname{EQS}(G,f)$.

We have $K \in \mathfrak{K}$. Thus, K is a broken circuit of G (since $\mathfrak{K}$ is a set of broken circuits of G). In other words, K is a subset of E having the form $C \setminus \{e\}$, where C is a circuit of G, and where e is the unique singleton edgery in C having

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[73] *Proof.* Let $B \in \mathcal{V}$. We have

$$(\Phi \circ \Psi)(B) = \Phi\left(\underbrace{\Psi(B)}_{\substack{=B\setminus\{d\} \\ \text{(by the definition of } \Psi)}}\right) = \Phi(B \setminus \{d\}) = (B \setminus \{d\}) \cup \{d\}$$

(by the definition of Φ).

We have $B \in \mathcal{V} = \{F \in \mathcal{P}(\operatorname{EQS}(G,f)) \mid d \in F\}$. In other words, B is an element F of $\mathcal{P}(\operatorname{EQS}(G,f))$ satisfying $d \in F$. In other words, B is an element of $\mathcal{P}(\operatorname{EQS}(G,f))$ and satisfies $d \in B$. From $d \in B$, we obtain $\{d\} \subseteq B$. Now, $(\Phi \circ \Psi)(B) = (B \setminus \{d\}) \cup \{d\} = B$ (since $\{d\} \subseteq B$). Thus, $(\Phi \circ \Psi)(B) = B = \operatorname{id}(B)$.

Now, let us forget that we fixed B. We thus have proven that $(\Phi \circ \Psi)(B) = \operatorname{id}(B)$ for every $B \in \mathcal{V}$. In other words, $\Phi \circ \Psi = \operatorname{id}$, qed.

[74] *Proof.* Let $B \in \mathcal{U}$. We have

$$(\Psi \circ \Phi)(B) = \Psi\left(\underbrace{\Phi(B)}_{\substack{=B\cup\{d\} \\ \text{(by the definition of } \Phi)}}\right) = \Psi(B \cup \{d\}) = (B \cup \{d\}) \setminus \{d\}$$

(by the definition of Ψ).

We have $B \in \mathcal{U} = \{F \in \mathcal{P}(\operatorname{EQS}(G,f)) \mid d \notin F\}$. In other words, B is an element F of $\mathcal{P}(\operatorname{EQS}(G,f))$ satisfying $d \notin F$. In other words, B is an element of $\mathcal{P}(\operatorname{EQS}(G,f))$ and satisfies $d \notin B$. Now, $(\Psi \circ \Phi)(B) = (B \cup \{d\}) \setminus \{d\} = B$ (since $d \notin B$). Thus, $(\Psi \circ \Phi)(B) = B = \operatorname{id}(B)$.

Now, let us forget that we fixed B. We thus have proven that $(\Psi \circ \Phi)(B) = \operatorname{id}(B)$ for every $B \in \mathcal{U}$. In other words, $\Psi \circ \Phi = \operatorname{id}$, qed.

maximum label (among the singleton edgeries in C) [75]. Consider this C and this e. Thus, we have the following facts:

- The set C is a circuit of G.
- The element e is the unique singleton edgery in C having maximum label (among the singleton edgeries in C).
- We have $K = C \setminus \{e\}$.

The element e is the unique singleton edgery in C having maximum label (among the singleton edgeries in C). Thus, the only singleton edgery in C whose label is greater or equal to the label of e is e itself. In other words, if e' is any singleton edgery in C satisfying $\ell(e') \geq \ell(e)$, then

$$e' = e. \tag{109}$$

Let us now assume that $K \subseteq \Phi(B)$. Thus, $K \subseteq \Phi(B) = B \cup \{d\}$. Hence, $\underbrace{K}_{\subseteq B \cup \{d\}} \setminus \{d\} \subseteq (B \cup \{d\}) \setminus \{d\} \subseteq B$.

We shall now prove that $K \subseteq B$.

Indeed, assume the contrary. Thus, $K \not\subseteq B$. If we had $d \notin K$, then we would have $K \setminus \{d\} = K$ and therefore $K = K \setminus \{d\} \subseteq B$; this would contradict $K \not\subseteq B$. Hence, we cannot have $d \notin K$. We thus must have $d \in K$. Hence, $d \in K = C \setminus \{e\}$. Hence, $d \in C$ and $d \notin \{e\}$. From $d \notin \{e\}$, we obtain $d \neq e$.

But $C \setminus \{e\} = K \subseteq \Phi(B) \subseteq \operatorname{EQS}(G, f)$. Hence, Lemma 5.27 (applied to Y instead of X) shows that $e \in \operatorname{EQS}(G, f)$. Thus, (107) shows that $\ell(d) \geq \ell(e)$ (since e is a singleton edgery).

Also, $d \in C$. Hence, d is an singleton edgery in C (since d is a singleton edgery). Since $\ell(d) \geq \ell(e)$, we can therefore apply (109) to $e' = d$. We thus obtain $d = e$. This contradicts $d \neq e$. This contradiction proves that our assumption was wrong. Hence, $K \subseteq B$ is proven.

Now, let us forget that we assumed that $K \subseteq \Phi(B)$. We thus have proven that $K \subseteq B$ under the assumption that $K \subseteq \Phi(B)$. In other words, we have proven the implication

$$(K \subseteq \Phi(B)) \implies (K \subseteq B). \tag{110}$$

On the other hand, if $K \subseteq B$, then $K \subseteq B \subseteq \Phi(B)$. Hence, the implication

$$(K \subseteq B) \implies (K \subseteq \Phi(B))$$

holds. Combining this implication with (110), we obtain the logical equivalence $(K \subseteq B) \iff (K \subseteq \Phi(B))$. Thus, (108) is proven.

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[75] because a broken circuit of G is the same as a subset of E having the form $C \setminus \{e\}$, where C is a circuit of G, and where e is the unique singleton edgery in C having maximum label (among the singleton edgeries in C) (by the definition of a "broken circuit")

Also, every $B \in \mathcal{U}$ satisfies

$$(-1)^{|B|} = -(-1)^{|\Phi(B)|} \tag{111}$$

[76].

Now,

$$\begin{aligned}
&\sum_{B\subseteq \operatorname{EQS}(G,f)} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \\
&= \underbrace{\sum_{\substack{B\subseteq \operatorname{EQS}(G,f);\\ d\in B}}}_{\substack{=\sum_{B\in\{F\in\mathcal{P}(\operatorname{EQS}(G,f)) \ \mid\ d\in F\}} = \sum_{B\in\mathcal{V}} \\ \text{(since } \{F\in\mathcal{P}(\operatorname{EQS}(G,f)) \ \mid\ d\in F\}=\mathcal{V}\text{)}}} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \\
&\qquad + \underbrace{\sum_{\substack{B\subseteq \operatorname{EQS}(G,f);\\ d\notin B}}}_{\substack{=\sum_{B\in\{F\in\mathcal{P}(\operatorname{EQS}(G,f)) \ \mid\ d\notin F\}} = \sum_{B\in\mathcal{U}} \\ \text{(since } \{F\in\mathcal{P}(\operatorname{EQS}(G,f)) \ \mid\ d\notin F\}=\mathcal{U}\text{)}}} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \\
&\qquad \left(\begin{array}{c}\text{here, we have split the sum into two parts:}\\ \text{one containing all addends with } d\in B,\\ \text{and one containing all addends with } d\notin B\end{array}\right) \\
&= \underbrace{\sum_{B\in\mathcal{V}} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K}_{\substack{=\sum_{B\in\mathcal{U}} (-1)^{|\Phi(B)|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq\Phi(B)}} a_K \\ \text{(here, we have} \\ \text{substituted } \Phi(B) \text{ for } B \text{ in the sum,} \\ \text{since the map } \Phi:\mathcal{U}\to\mathcal{V} \text{ is a bijection)}}} + \sum_{B\in\mathcal{U}} \underbrace{(-1)^{|B|}}_{\substack{=-(-1)^{|\Phi(B)|} \\ \text{(by (111))}}} \underbrace{\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}}_{\substack{=\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq\Phi(B)}} \\ \text{(because for every } K\in\mathfrak{K}, \\ \text{the condition } (K\subseteq B) \\ \text{is equivalent to } (K\subseteq\Phi(B)) \\ \text{(by (108)))}}} a_K \\
&= \sum_{B\in\mathcal{U}} (-1)^{|\Phi(B)|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq\Phi(B)}} a_K + \sum_{B\in\mathcal{U}} \left(-(-1)^{|\Phi(B)|}\right) \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq\Phi(B)}} a_K \\
&= \sum_{B\in\mathcal{U}} (-1)^{|\Phi(B)|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq\Phi(B)}} a_K - \sum_{B\in\mathcal{U}} (-1)^{|\Phi(B)|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq\Phi(B)}} a_K \\
&= 0 = [\operatorname{EQS}(G,f) = \varnothing] \qquad \left(\text{since } [\operatorname{EQS}(G,f)=\varnothing] = 0\right).
\end{aligned}$$

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[76] *Proof of (111):* Let $B \in \mathcal{U}$. We have $B \in \mathcal{U} = \{F \in \mathcal{P}(\operatorname{EQS}(G,f)) \mid d \notin F\}$. In other words, B is an element F of $\mathcal{P}(\operatorname{EQS}(G,f))$ satisfying $d \notin F$. In other words, B is an element of $\mathcal{P}(\operatorname{EQS}(G,f))$ and satisfies $d \notin B$. From $d \notin B$, we see that $|B \cup \{d\}| = |B| + 1$. Now, the definition of Φ yields $\Phi(B) = B \cup \{d\}$. Hence, $|\Phi(B)| = |B \cup \{d\}| = |B| + 1$. Thus, $(-1)^{|\Phi(B)|} = (-1)^{|B|+1} = -(-1)^{|B|}$. Therefore, $(-1)^{|B|} = -(-1)^{|\Phi(B)|}$. This proves (111).

Thus, Lemma 5.29 is proved in Case 2.

We have now proved Lemma 5.29 in both Cases 1 and 2. Since these two cases cover all possibilities, we thus have proved Lemma 5.29. $\square$

We are now ready to prove Theorem 5.18 and Corollaries 5.19 and 5.20 as well as Theorem 5.13:

Proof of Theorem 5.18. We have

$$X_G = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_f \tag{112}$$

(by the definition of X_G). Now, if $f : V \to \mathbb{N}_+$ is a map, then we have the following logical equivalence:

$$(\text{the } \mathbb{N}_+\text{-coloring } f \text{ of } G \text{ is proper}) \iff (\operatorname{EQS}(G,f) = \varnothing) \tag{113}$$

(because the $\mathbb{N}_+$-coloring f of G is proper if and only if $\operatorname{EQS}(G,f) = \varnothing$ [77]). Now,

$$\begin{aligned}
&\sum_{f:V\to\mathbb{N}_+} \Bigg[\underbrace{\operatorname{EQS}(G,f) = \varnothing}_{\substack{\iff (\text{the } \mathbb{N}_+\text{-coloring } f \text{ of } G \text{ is proper}) \\ (\text{by (113)})}} \Bigg] \mathbf{x}_f \\
&= \sum_{f:V\to\mathbb{N}_+} \Bigg[\underbrace{\text{the } \mathbb{N}_+\text{-coloring } f \text{ of } G \text{ is proper}}_{\iff (f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G)} \Bigg] \mathbf{x}_f \\
&= \sum_{f:V\to\mathbb{N}_+} [f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G]\, \mathbf{x}_f \\
&= \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \underbrace{[f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G]}_{\substack{=1 \\ (\text{since } f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G)}} \mathbf{x}_f \\
&\qquad + \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is not a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \underbrace{[f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G]}_{\substack{=0 \\ (\text{since } f \text{ is not a proper } \mathbb{N}_+\text{-coloring of } G)}} \mathbf{x}_f \\
&\qquad\qquad (\text{since each } f : V \to \mathbb{N}_+ \text{ either is a proper } \mathbb{N}_+\text{-coloring of } G \text{ or not}) \\
&= \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_f + \underbrace{\sum_{\substack{f:V\to\mathbb{N}_+ \text{ is not a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} 0\mathbf{x}_f}_{=0} = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_f \\
&= X_G
\end{aligned} \tag{114}$$

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[77] by Lemma 5.25 (applied to $\mathbb{N}_+$ instead of X)

(by (112)).

However, for every $f : V \to \mathbb{N}_+$, we have

$$\sum_{B \subseteq \operatorname{EQS}(G,f)} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K = [\operatorname{EQS}(G,f) = \varnothing] \tag{115}$$

(by Lemma 5.29 (applied to $\mathbb{N}_+$ instead of Y)).

For every $f : V \to \mathbb{N}_+$, we have

$$\{F \subseteq E \mid \operatorname{union} F \subseteq \operatorname{Eqs} f\} = \mathcal{P}(\operatorname{EQS}(G,f)) \tag{116}$$

[78].

For every $f : V \to \mathbb{N}_+$, we have

$$\begin{aligned}
&\underbrace{\sum_{\substack{B \subseteq E;\\ \operatorname{union} B \subseteq \operatorname{Eqs} f}}}_{\substack{= \sum_{B \in \{F \subseteq E \mid \operatorname{union} F \subseteq \operatorname{Eqs} f\}} = \sum_{B \in \mathcal{P}(\operatorname{EQS}(G,f))} \\ \text{(because } \{F \subseteq E \mid \operatorname{union} F \subseteq \operatorname{Eqs} f\} = \mathcal{P}(\operatorname{EQS}(G,f)) \\ \text{(by (116)))}}} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K \\
&= \underbrace{\sum_{B \in \mathcal{P}(\operatorname{EQS}(G,f))}}_{= \sum_{B \subseteq \operatorname{EQS}(G,f)}} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K = \sum_{B \subseteq \operatorname{EQS}(G,f)} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K \\
&= [\operatorname{EQS}(G,f) = \varnothing]
\end{aligned} \tag{118}$$

---

[78] *Proof of (116):* Let $f : V \to \mathbb{N}_+$.

Let $B \in \{F \subseteq E \mid \operatorname{union} F \subseteq \operatorname{Eqs} f\}$. Thus, B is a subset F of E satisfying $\operatorname{union} F \subseteq \operatorname{Eqs} f$. In other words, B is a subset of E and satisfies $\operatorname{union} B \subseteq \operatorname{Eqs} f$. However, Lemma 5.26 yields that $B \subseteq \operatorname{EQS}(G,f)$ holds if and only if $\operatorname{union} B \subseteq \operatorname{Eqs} f$. Hence, we have $B \subseteq \operatorname{EQS}(G,f)$ (since we have $\operatorname{union} B \subseteq \operatorname{Eqs} f$). In other words, $B \in \mathcal{P}(\operatorname{EQS}(G,f))$.

Let us now forget that we fixed B. We thus have proven that every $B \in \{F \subseteq E \mid \operatorname{union} F \subseteq \operatorname{Eqs} f\}$ satisfies $B \in \mathcal{P}(\operatorname{EQS}(G,f))$. In other words,

$$\{F \subseteq E \mid \operatorname{union} F \subseteq \operatorname{Eqs} f\} \subseteq \mathcal{P}(\operatorname{EQS}(G,f)). \tag{117}$$

On the other hand, let $C \in \mathcal{P}(\operatorname{EQS}(G,f))$. Thus, C is a subset of $\operatorname{EQS}(G,f)$. Hence, $C \subseteq \operatorname{EQS}(G,f) \subseteq E$, so that C is a subset of E. Hence, Lemma 5.26 (applied to $B = C$) yields that $C \subseteq \operatorname{EQS}(G,f)$ holds if and only if $\operatorname{union} C \subseteq \operatorname{Eqs} f$. Hence, we have $\operatorname{union} C \subseteq \operatorname{Eqs} f$ (since we have $C \subseteq \operatorname{EQS}(G,f)$). Thus, C is a subset of E and satisfies $\operatorname{union} C \subseteq \operatorname{Eqs} f$. In other words, C is a subset F of E satisfying $\operatorname{union} F \subseteq \operatorname{Eqs} f$. In other words, $C \in \{F \subseteq E \mid \operatorname{union} F \subseteq \operatorname{Eqs} f\}$.

Let us now forget that we fixed C. We thus have proven that every $C \in \mathcal{P}(\operatorname{EQS}(G,f))$ satisfies $C \in \{F \subseteq E \mid \operatorname{union} F \subseteq \operatorname{Eqs} f\}$. In other words,

$$\mathcal{P}(\operatorname{EQS}(G,f)) \subseteq \{F \subseteq E \mid \operatorname{union} F \subseteq \operatorname{Eqs} f\}.$$

Combining this inclusion with (117), we obtain $\{F \subseteq E \mid \operatorname{union} F \subseteq \operatorname{Eqs} f\} = \mathcal{P}(\operatorname{EQS}(G,f))$. This proves (116).

(by (115)).

Now, (114) yields

$$\begin{aligned}
X_G &= \sum_{f:V\to\mathbb{N}_+} \underbrace{[\operatorname{EQS}(G,f)=\varnothing]}_{\substack{=\sum\limits_{\substack{B\subseteq E;\\ \operatorname{union} B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \prod\limits_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \\ \text{(by (118))}}} \mathbf{x}_f \\
&= \sum_{f:V\to\mathbb{N}_+} \left(\sum_{\substack{B\subseteq E;\\ \operatorname{union} B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \mathbf{x}_f \\
&= \underbrace{\sum_{f:V\to\mathbb{N}_+} \sum_{\substack{B\subseteq E;\\ \operatorname{union} B\subseteq \operatorname{Eqs} f}}}_{=\sum\limits_{B\subseteq E} \sum\limits_{\substack{f:V\to\mathbb{N}_+;\\ \operatorname{union} B\subseteq \operatorname{Eqs} f}}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \mathbf{x}_f \\
&= \sum_{B\subseteq E} \sum_{\substack{f:V\to\mathbb{N}_+;\\ \operatorname{union} B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \mathbf{x}_f \\
&= \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \sum_{\substack{f:V\to\mathbb{N}_+;\\ \operatorname{union} B\subseteq \operatorname{Eqs} f}} \mathbf{x}_f. \qquad (119)
\end{aligned}$$

However, if B is a subset of E, then the pair $(V, \operatorname{union} B)$ is a finite graph (since V is a finite set and since $\operatorname{union} B \subseteq \dbinom{V}{2}$), and thus we have

$$\sum_{\substack{f:V\to\mathbb{N}_+;\\ \operatorname{union} B\subseteq \operatorname{Eqs} f}} \mathbf{x}_f = p_{\lambda(V,\operatorname{union} B)} \qquad (120)$$

(by Lemma 2.7, applied to $\operatorname{union} B$ instead of B).

Hence, (119) becomes

$$X_G = \sum_{B \subseteq E} (-1)^{|B|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K \right) \underbrace{\sum_{\substack{f : V \to \mathbb{N}_+; \\ \operatorname{union} B \subseteq \operatorname{Eqs} f}} \mathbf{x}_f}_{\substack{= p_{\lambda(V, \operatorname{union} B)} \\ \text{(by (120))}}}$$

$$= \sum_{B \subseteq E} (-1)^{|B|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K \right) p_{\lambda(V, \operatorname{union} B)} = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) p_{\lambda(V, \operatorname{union} F)}$$

(here, we have renamed the summation index B as F). This proves Theorem 5.18. □

Proof of Corollary 5.19. We can apply Theorem 5.18 to 0 instead of a_K. As a result, we obtain

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} 0 \right) p_{\lambda(V, \operatorname{union} F)}. \tag{121}$$

Now, if F is any subset of E, then

$$\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} 0 = \begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} \tag{122}$$

[79].

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[79] *Proof of (122):* The proof of (122) is completely analogous to the proof of (10).

Thus, (121) becomes

$$\begin{aligned}
X_G &= \sum_{F\subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0\right)}_{\substack{=\begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free};\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases}\\ \text{(by (122))}}} p_{\lambda(V,\operatorname{union} F)} \\
&= \sum_{F\subseteq E} (-1)^{|F|} \begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free};\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases} p_{\lambda(V,\operatorname{union} F)} \\
&= \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} \underbrace{\begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free};\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases}}_{\substack{=1\\ \text{(since } F \text{ is } \mathfrak{K}\text{-free)}}} p_{\lambda(V,\operatorname{union} F)} \\
&\qquad + \sum_{\substack{F\subseteq E;\\ F \text{ is not } \mathfrak{K}\text{-free}}} (-1)^{|F|} \underbrace{\begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free};\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases}}_{\substack{=0\\ \text{(since } F \text{ is not } \mathfrak{K}\text{-free)}}} p_{\lambda(V,\operatorname{union} F)} \\
&\qquad\qquad \left(\text{since each subset } F \text{ of } E \text{ either is } \mathfrak{K}\text{-free or is not}\right) \\
&= \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda(V,\operatorname{union} F)} + \underbrace{\sum_{\substack{F\subseteq E;\\ F \text{ is not } \mathfrak{K}\text{-free}}} (-1)^{|F|} 0 p_{\lambda(V,\operatorname{union} F)}}_{=0} \\
&= \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda(V,\operatorname{union} F)}.
\end{aligned}$$

This proves Corollary 5.19. ☐

Proof of Corollary 5.20. Let $\mathfrak{K}$ be the set of all broken circuits of G.

Now, just as in the proof of Corollary 1.18, we can prove the following equality:

$$\sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} = \sum_{\substack{F\subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}}$$

(an equality between summation signs). Now, Corollary 5.19 yields

$$X_G = \underbrace{\sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}}}_{=\sum_{\substack{F\subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}}} (-1)^{|F|} p_{\lambda(V,\operatorname{union} F)} = \sum_{\substack{F\subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} p_{\lambda(V,\operatorname{union} F)}.$$

This proves Corollary 5.20. ☐

Proof of Theorem 5.13. Let X be the totally ordered set $\{1\}$ (equipped with the only possible order on this set). Let $\ell : E \to X$ be the function sending each $e \in E$ to $1 \in X$. Let $\mathfrak{K}$ be the empty set. Clearly, $\mathfrak{K}$ is a set of broken circuits of G. Theorem 5.18 (applied to 0 instead of a_K) yields

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} 0 \right)}_{\substack{= (\text{empty product}) \\ (\text{since } \mathfrak{K} \text{ is the empty set})}} p_{\lambda(V, \operatorname{union} F)}$$

$$= \sum_{F \subseteq E} (-1)^{|F|} \underbrace{(\text{empty product})}_{=1} p_{\lambda(V, \operatorname{union} F)} = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda(V, \operatorname{union} F)}.$$

This proves Theorem 5.13. ☐

5.7. The chromatic polynomial

We have thus proved analogues of Theorems 1.11 and 1.15 and Corollaries 1.17 and 1.18 for ambigraphs. We can just as easily prove analogues of Theorem 3.1, Definition 3.2, Theorems 3.4 and 3.5 and Corollaries 3.6 and 3.7. Here they are, in the order in which we have just mentioned them:

Theorem 5.30. Let $G = (V, E, \varphi)$ be a finite ambigraph. Then, there exists a unique polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies

$$P(q) = (\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G).$$

Definition 5.31. Let $G = (V, E, \varphi)$ be a finite ambigraph. Theorem 5.30 shows that there exists a polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies $P(q) = (\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G)$. This polynomial P is called the *chromatic polynomial* of G, and will be denoted by χ_G.

Theorem 5.32. Let $G = (V, E, \varphi)$ be a finite ambigraph. Then,

$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} x^{\operatorname{conn}(V, \operatorname{union} F)}.$$

(Here, of course, the pair $(V, \operatorname{union} F)$ is regarded as a graph, and the expression $\operatorname{conn}(V, \operatorname{union} F)$ is understood according to Definition 3.3.)

Theorem 5.33. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq F}} a_K \right) x^{\operatorname{conn}(V, \operatorname{union} F)}.$$

(Here, of course, the pair $(V, \operatorname{union} F)$ is regarded as a graph, and the expression $\operatorname{conn}(V, \operatorname{union} F)$ is understood according to Definition 3.3.)

Corollary 5.34. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$\chi_G = \sum_{\substack{F \subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} x^{\operatorname{conn}(V, \operatorname{union} F)}.$$

Corollary 5.35. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Then,

$$\chi_G = \sum_{\substack{F \subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} x^{\operatorname{conn}(V, \operatorname{union} F)}.$$

The proofs of all these results are analogous to the proofs of the corresponding results from Section 3, so we leave them all to the reader.

One may reasonably wonder whether Corollary 3.14 has an analogue for ambigraphs as well, i.e., whether one can replace the exponent $\operatorname{conn}(V, \operatorname{union} F)$ in Corollary 5.35 by something simpler when ℓ is injective. In the case of a hypergraph, Dohmen has obtained such a result ([Dohmen95, Theorem 2.1]) under the additional condition that each cycle of G have at least one singleton edgery. Unfortunately, for ambigraphs, such a simplification does not appear possible (even under a condition like Dohmen's).

6. Weighted and noncommutative versions

In the recent decades, the chromatic symmetric function of a graph has been generalized in several directions. Two of them are the *chromatic symmetric function of a weighted graph* as defined by Crew and Spirkl ([CreSpi19, §3]), and the *noncommutative chromatic symmetric function* of Gebhard and Sagan ([GebSag01, §3]). In this section, we will recall the definitions of both of these generalizations, and extend our results to them. (The extensions will be fairly mechanical, as all the hard work has already been done.)

6.1. Weighted graphs and their chromatic symmetric functions

For us, a *weighted graph* will just mean a pair consisting of a graph $G = (V, E)$ and a weight function on V. Weight functions are defined as follows:

Definition 6.1. Let V be a set. A *weight function* on V means a function $w : V \to \mathbb{N}_+$. If $w : V \to \mathbb{N}_+$ is a weight function on V, then the *weight* of an element $v \in V$ is defined to be the positive integer $w(v) \in \mathbb{N}_+$.

Thus, a weight function on a set V just assigns a "weight" (a positive integer) to each element of V. Given such a weight function for a graph $G = (V, E)$, we can define a "weighted chromatic symmetric function":

Definition 6.2. Let $G = (V, E)$ be a finite graph. Let $w : V \to \mathbb{N}_+$ be a weight function on V.

(a) For every $\mathbb{N}_+$-coloring $f : V \to \mathbb{N}_+$ of G, we let $\mathbf{x}_{f,w}$ denote the monomial $\prod_{v\in V} x_{f(v)}^{w(v)}$ in the indeterminates $x_1, x_2, x_3, \ldots$.

(b) We define a power series $X_{G,w} \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$X_{G,w} = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_{f,w}.$$

This power series $X_{G,w}$ is called the *chromatic symmetric function* of (G, w).

This chromatic symmetric function $X_{G,w}$ was introduced by Crew and Spirkl in [CreSpi19, (1)] (where it was denoted $X_{(G,w)}$). It generalizes the original chromatic symmetric function X_G, which is obtained when all the weights $w(v)$ are 1:

Example 6.3. Let $G = (V, E)$ be a finite graph. Let $w : V \to \mathbb{N}_+$ be the weight function that sends each $v \in V$ to 1. Then, for every $\mathbb{N}_+$-coloring $f : V \to \mathbb{N}_+$ of G, we have

$$\begin{aligned} \mathbf{x}_{f,w} &= \prod_{v\in V} \underbrace{x_{f(v)}^{w(v)}}_{\substack{=x_{f(v)} \\ \text{(since } w(v)=1 \\ \text{(by the definition of } w))}} \qquad \text{(by Definition 6.2 (a))} \\ &= \prod_{v\in V} x_{f(v)} = \mathbf{x}_f \end{aligned} \qquad (123)$$

(see Definition 1.5 **(a)** for the definition of $\mathbf{x}_f$). Thus, Definition 6.2 **(b)** yields

$$X_{G,w} = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \underbrace{\mathbf{x}_{f,w}}_{\substack{=\mathbf{x}_f \\ \text{(by (123))}}} = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_f = X_G$$

(by Definition 1.5 **(b)**).

6.2. The weight of a subset and the partition $\lambda(G,w)$

To state Whitney-like formulas for $X_{G,w}$, we need to adapt the partition $\lambda(G)$ defined in Definition 1.10 **(b)** to the case of a weighted graph. This adaptation consists in replacing the size of each connected component by its *weight*. Here, the weight of a subset of V is defined as follows:

Definition 6.4. Let V be a finite set. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Let S be a subset of V. Then, the *weight* of S means the nonnegative integer $\sum_{v\in S} w(v)$. This weight is denoted by $w(S)$.

Note that if the subset S in this definition is nonempty, then its weight $w(S)$ is a positive integer, since it is defined as the nonempty sum $\sum_{v\in S} w(v)$ of the positive weights $w(v)$.

For example, $w(\{2,5,6\}) = w(2) + w(5) + w(6)$ (if $\{2,5,6\}$ is a subset of V).

Now, we can define the analogue of the partition $\lambda(G)$ for a weighted graph:

Definition 6.5. Let $G = (V,E)$ be a finite graph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. We let $\lambda(G,w)$ denote the list of the weights of all connected components of G, in weakly decreasing order. (Each connected component should contribute only one entry to the list.) We view $\lambda(G,w)$ as a partition (since $\lambda(G,w)$ is a weakly decreasing finite list of positive integers).

Example 6.6. Let $G = (V,E)$ be the finite graph with $V = \{1,2,3,4,5\}$ and $E = \{\{1,3\}, \{2,4\}, \{2,5\}\}$. Then, the connected components of G are $A = \{1,3\}$ and $B = \{2,4,5\}$.

Let $w : V \to \mathbb{N}_+$ be the weight function given by $w(1) = 6$ and $w(2) = 9$ and $w(3) = 6$ and $w(4) = 1$ and $w(5) = 3$. Then, the weights of the connected components A and B are

$$w(A) = w(\{1,3\}) = w(1) + w(3) = 6 + 6 = 12 \qquad \text{and}$$
$$w(B) = w(\{2,4,5\}) = w(2) + w(4) + w(5) = 9 + 1 + 3 = 13.$$

Hence, the partition $\lambda(G,w)$ is the list of these two weights 12 and 13, in weakly decreasing order. In other words, $\lambda(G,w) = (13,12)$.

6.3. Formulas for $X_{G,w}$

We are now ready to state analogues of Theorems 1.11 and 1.15 and Corollaries 1.17 and 1.18 for weighted graphs:

Theorem 6.7. Let $G = (V, E)$ be a finite graph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Then,

$$X_{G,w} = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda((V,F),w)}.$$

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\lambda((V, F), w)$ is understood according to Definition 6.5.)

Theorem 6.8. Let $G = (V, E)$ be a finite graph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$X_{G,w} = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq F}} a_K \right) p_{\lambda((V,F),w)}.$$

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\lambda((V, F), w)$ is understood according to Definition 6.5.)

Corollary 6.9. Let $G = (V, E)$ be a finite graph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$X_{G,w} = \sum_{\substack{F \subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda((V,F),w)}.$$

Corollary 6.10. Let $G = (V, E)$ be a finite graph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Then,

$$X_{G,w} = \sum_{\substack{F \subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} p_{\lambda((V,F),w)}.$$

Note that Theorem 6.7 is a result by Crew and Spirkl (namely, [CreSpi19, Lemma 3]).

6.4. Proofs

In this section, we shall prove Theorem 6.8, Corollary 6.9, Corollary 6.10 and Theorem 6.7. This will be fairly easy, as many of our above lemmas can be reused without any change. However, we need the following weighted version of Lemma 2.7:

> **Lemma 6.11.** Let (V,B) be a finite graph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Then,
>
> $$\sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} \mathbf{x}_{f,w} = p_{\lambda((V,B),w)}.$$
>
> (Here, $\mathbf{x}_{f,w}$ is defined as in Definition 6.2 **(a)**, and the expression $\lambda\left(\left(V,B\right),w\right)$ is understood according to Definition 6.5.)

Proof of Lemma 6.11. Let $\sim$ denote the equivalence relation $\sim_{(V,B)}$ (defined as in Definition 1.10 **(a)**). Then, the connected components of (V,B) are the elements of $V/\left(\sim\right)$. (This has already been shown in our proof of Lemma 2.7 above.)

A set $\left(\mathbb{N}_+\right)^V_\sim$ is defined (according to Definition 1.7 **(b)**).

Proposition 1.8 **(b)** (applied to $X=V$ and $Y=\mathbb{N}_+$) shows that the map[80]

$$\left(\mathbb{N}_+\right)^{V/(\sim)} \to \left(\mathbb{N}_+\right)^V_\sim, \qquad f\mapsto f\circ\pi_V$$

is a bijection.

We have the following equality of summation signs:

$$\sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} = \sum_{f\in\left(\mathbb{N}_+\right)^V_\sim}$$

(indeed, this has already been shown in our proof of Lemma 2.7 above). Hence,

$$\underbrace{\sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}}}_{=\sum_{f\in\left(\mathbb{N}_+\right)^V_\sim}} \mathbf{x}_{f,w} = \sum_{f\in\left(\mathbb{N}_+\right)^V_\sim} \mathbf{x}_{f,w} = \sum_{f\in\left(\mathbb{N}_+\right)^{V/(\sim)}} \mathbf{x}_{f\circ\pi_V,w} \tag{124}$$

(here, we have substituted $f\circ\pi_V$ for f in the sum, since the map $\left(\mathbb{N}_+\right)^{V/(\sim)} \to \left(\mathbb{N}_+\right)^V_\sim$, $f\mapsto f\circ\pi_V$ is a bijection).

Now, let $(C_1,C_2,\ldots,C_k)$ be a list of all connected components of (V,B), ordered such that $w(C_1)\geq w(C_2)\geq\cdots\geq w(C_k)$.[81] Then, $\left(w\left(C_1\right),w\left(C_2\right),\ldots,w\left(C_k\right)\right)$ is the list of the weights of all connected components of (V,B), in weakly

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[80] Here, the map π_V is defined as in Definition 1.6.

[81] Every connected component of (V,B) should appear exactly once in this list.

decreasing order (since $w(C_1) \geq w(C_2) \geq \cdots \geq w(C_k)$). In other words, $(w(C_1), w(C_2), \ldots, w(C_k))$ is $\lambda((V,B), w)$ (since $\lambda((V,B), w)$ is the list of the weights of all connected components of (V,B), in weakly decreasing order (by the definition of $\lambda((V,B), w)$)). In other words,

$$\lambda((V,B), w) = (w(C_1), w(C_2), \ldots, w(C_k)).$$

Thus, (2) (applied to $((V,B), w)$ and $w(C_i)$ instead of λ and λ_i) shows that

$$p_{\lambda((V,B),w)} = p_{w(C_1)} p_{w(C_2)} \cdots p_{w(C_k)} = \prod_{i=1}^{k} p_{w(C_i)}. \tag{125}$$

However, for every $i \in \{1, 2, \ldots, k\}$, we have

$$p_{w(C_i)} = \sum_{s \in \mathbb{N}_+} x_s^{w(C_i)} \tag{126}$$

[82]. Hence, (125) becomes

$$\begin{aligned} p_{\lambda((V,B),w)} &= \prod_{i=1}^{k} \underbrace{p_{w(C_i)}}_{\substack{= \sum_{s \in \mathbb{N}_+} x_s^{w(C_i)} \\ \text{(by (126))}}} = \prod_{i=1}^{k} \sum_{s \in \mathbb{N}_+} x_s^{w(C_i)} \\ &= \sum_{(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k} \prod_{i=1}^{k} x_{s_i}^{w(C_i)} \end{aligned} \tag{127}$$

(by the product rule).

Recall that $(C_1, C_2, \ldots, C_k)$ is a list of all connected components of (V,B). In other words, $(C_1, C_2, \ldots, C_k)$ is a list of all elements of $V/(\sim)$ (since the elements of $V/(\sim)$ are the connected components of (V,B)). Moreover, every element of $V/(\sim)$ appears exactly once in this list $(C_1, C_2, \ldots, C_k)$ (since the entries of the list $(C_1, C_2, \ldots, C_k)$ are pairwise distinct[83]). Thus, $(C_1, C_2, \ldots, C_k)$ is a list of all elements of $V/(\sim)$, and contains each of these elements exactly once. Hence, the map

$$\begin{aligned} (\mathbb{N}_+)^{V/(\sim)} &\to (\mathbb{N}_+)^k, \\ f &\mapsto (f(C_1), f(C_2), \ldots, f(C_k)) \end{aligned}$$

---

[82] *Proof.* Let $i \in \{1, 2, \ldots, k\}$. Then, C_i is a connected component of (V,B) (since $(C_1, C_2, \ldots, C_k)$ is a list of all connected components of (V,B)). Hence, C_i is a nonempty subset of V (since every connected component of (V,B) is a nonempty subset of V). Hence, $w(C_i)$ is a positive integer (since $w(S)$ is a positive integer whenever S is a nonempty subset of V). Thus, (1) (applied to $n = w(C_i)$) shows that $p_{w(C_i)} = \underbrace{\sum_{j \geq 1}}_{= \sum_{j \in \mathbb{N}_+}} x_j^{w(C_i)} = \sum_{j \in \mathbb{N}_+} x_j^{w(C_i)} = \sum_{s \in \mathbb{N}_+} x_s^{w(C_i)}$ (here, we have renamed the summation index j as s). Qed.

[83] since every connected component of (V,B) appears exactly once in this list

is a bijection (by Lemma 2.6, applied to $W = V/(\sim)$ and $Y = \mathbb{N}_+$).

For every $\gamma \in V/(\sim)$, we have

$$\pi_V^{-1}(\gamma) = \gamma. \tag{128}$$

(This has already been shown in our proof of Lemma 2.7 above.)

Also, the map $\{1, 2, \ldots, k\} \to V/(\sim)$, $i \mapsto C_i$ is a bijection (since $(C_1, C_2, \ldots, C_k)$ is a list of all elements of $V/(\sim)$, and contains each of these elements exactly once).

We have

$$\mathbf{x}_{f \circ \pi_V, w} = \prod_{i=1}^{k} x_{f(C_i)}^{w(C_i)} \qquad \text{for every } f \in (\mathbb{N}_+)^{V/(\sim)} \tag{129}$$

[84].

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[84] *Proof of (129):* Let $f \in (\mathbb{N}_+)^{V/(\sim)}$. Then, the definition of $\mathbf{x}_{f \circ \pi_V, w}$ yields

$$\begin{aligned}
\mathbf{x}_{f \circ \pi_V, w} &= \prod_{v \in V} x_{(f \circ \pi_V)(v)}^{w(v)} = \prod_{\gamma \in V/(\sim)} \underbrace{\prod_{\substack{v \in V; \\ \pi_V(v) = \gamma}}}_{\substack{= \prod_{v \in \pi_V^{-1}(\gamma)} = \prod_{v \in \gamma} \\ (\text{since } \pi_V^{-1}(\gamma) = \gamma \\ (\text{by (128)}))}} \underbrace{x_{(f \circ \pi_V)(v)}^{w(v)}}_{\substack{= x_{f(\gamma)}^{w(v)} \\ (\text{since } (f \circ \pi_V)(v) = f(\pi_V(v)) = f(\gamma) \\ (\text{since } \pi_V(v) = \gamma))}} \\
&\qquad \left(\begin{array}{c} \text{because for every } v \in V\text{, there exists a unique } \gamma \in V/(\sim) \\ \text{such that } \pi_V(v) = \gamma \text{ (since } \pi_V \text{ is a map } V \to V/(\sim)\text{)} \end{array} \right) \\
&= \prod_{\gamma \in V/(\sim)} \underbrace{\prod_{v \in \gamma} x_{f(\gamma)}^{w(v)}}_{\substack{= x_{f(\gamma)}^{\sum_{v \in \gamma} w(v)} = x_{f(\gamma)}^{w(\gamma)} \\ (\text{because } \sum_{v \in \gamma} w(v) = w(\gamma) \\ (\text{since the weight } w(\gamma) \text{ of the subset } \gamma \\ \text{is defined to be } \sum_{v \in \gamma} w(v)))}} = \prod_{\gamma \in V/(\sim)} x_{f(\gamma)}^{w(\gamma)} = \prod_{i \in \{1, 2, \ldots, k\}} x_{f(C_i)}^{w(C_i)}
\end{aligned}$$

(here, we have substituted C_i for γ in the product, since the map $\{1, 2, \ldots, k\} \to V/(\sim)$, $i \mapsto C_i$ is a bijection). Thus,

$$\mathbf{x}_{f \circ \pi_V, w} = \underbrace{\prod_{i \in \{1, 2, \ldots, k\}}}_{= \prod_{i=1}^{k}} x_{f(C_i)}^{w(C_i)} = \prod_{i=1}^{k} x_{f(C_i)}^{w(C_i)}.$$

This proves (129).

Now, (124) becomes

$$\sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} \mathbf{x}_{f,w} = \sum_{f\in(\mathbb{N}_+)^{V/(\sim)}} \underbrace{\mathbf{x}_{f\circ\pi_V,w}}_{\substack{=\prod_{i=1}^{k} x_{f(C_i)}^{w(C_i)}\\ \text{(by (129))}}} = \sum_{f\in(\mathbb{N}_+)^{V/(\sim)}} \prod_{i=1}^{k} x_{f(C_i)}^{w(C_i)}$$
$$= \sum_{(s_1,s_2,\ldots,s_k)\in(\mathbb{N}_+)^k} \prod_{i=1}^{k} x_{s_i}^{w(C_i)}$$

(here, we have substituted $(s_1, s_2, \ldots, s_k)$ for $(f(C_1), f(C_2), \ldots, f(C_k))$ in the sum, since the map $(\mathbb{N}_+)^{V/(\sim)} \to (\mathbb{N}_+)^k$, $f \mapsto (f(C_1), f(C_2), \ldots, f(C_k))$ is a bijection). Comparing this with (127), we obtain $\sum\limits_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} \mathbf{x}_{f,w} = p_{\lambda((V,B),w)}$. This proves Lemma 6.11. $\square$

It is now straightforward to adapt our above proofs of Theorem 6.8, Corollary 6.9, Corollary 6.10 and Theorem 1.11 to obtain proofs of Theorem 6.8, Corollary 6.9, Corollary 6.10 and Theorem 6.7:

Proof of Theorem 6.8. We have

$$X_{G,w} = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a}\\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_{f,w} \tag{130}$$

(by the definition of $X_{G,w}$). Now, if $f : V \to \mathbb{N}_+$ is a map, then we have the following logical equivalence:

$$\left(\text{the } \mathbb{N}_+\text{-coloring } f \text{ of } G \text{ is proper}\right) \iff \left(E \cap \operatorname{Eqs} f = \varnothing\right) \tag{131}$$

(because the $\mathbb{N}_+$-coloring f of G is proper if and only if $E \cap \operatorname{Eqs} f = \varnothing$ [85]).

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[85] by Lemma 2.3 (applied to $\mathbb{N}_+$ instead of X)

Now,

$$\begin{aligned}
&\sum_{f:V\to\mathbb{N}_+}\left[\underbrace{E\cap\operatorname{Eqs}f=\varnothing}_{\substack{\Longleftrightarrow\ (\text{the }\mathbb{N}_+\text{-coloring }f\text{ of }G\text{ is proper})\\ (\text{by (131)})}}\right]\mathbf{x}_{f,w}\\
&=\sum_{f:V\to\mathbb{N}_+}\left[\underbrace{\text{the }\mathbb{N}_+\text{-coloring }f\text{ of }G\text{ is proper}}_{\Longleftrightarrow\ (f\text{ is a proper }\mathbb{N}_+\text{-coloring of }G)}\right]\mathbf{x}_{f,w}\\
&=\sum_{f:V\to\mathbb{N}_+}[f\text{ is a proper }\mathbb{N}_+\text{-coloring of }G]\,\mathbf{x}_{f,w}\\
&=\sum_{\substack{f:V\to\mathbb{N}_+\text{ is a}\\ \text{proper }\mathbb{N}_+\text{-coloring of }G}}\underbrace{[f\text{ is a proper }\mathbb{N}_+\text{-coloring of }G]}_{\substack{=1\\ (\text{since }f\text{ is a proper }\mathbb{N}_+\text{-coloring of }G)}}\mathbf{x}_{f,w}\\
&\qquad+\sum_{\substack{f:V\to\mathbb{N}_+\text{ is not a}\\ \text{proper }\mathbb{N}_+\text{-coloring of }G}}\underbrace{[f\text{ is a proper }\mathbb{N}_+\text{-coloring of }G]}_{\substack{=0\\ (\text{since }f\text{ is not a proper }\mathbb{N}_+\text{-coloring of }G)}}\mathbf{x}_{f,w}\\
&\qquad\qquad(\text{since each }f:V\to\mathbb{N}_+\text{ is either a proper }\mathbb{N}_+\text{-coloring of }G\text{ or not})\\
&=\sum_{\substack{f:V\to\mathbb{N}_+\text{ is a}\\ \text{proper }\mathbb{N}_+\text{-coloring of }G}}\mathbf{x}_{f,w}+\underbrace{\sum_{\substack{f:V\to\mathbb{N}_+\text{ is not a}\\ \text{proper }\mathbb{N}_+\text{-coloring of }G}}0\mathbf{x}_{f,w}}_{=0}=\sum_{\substack{f:V\to\mathbb{N}_+\text{ is a}\\ \text{proper }\mathbb{N}_+\text{-coloring of }G}}\mathbf{x}_{f,w}\\
&=X_{G,w}
\end{aligned}\tag{132}$$

(by (130)).

However, for every $f:V\to\mathbb{N}_+$, we have

$$\sum_{\substack{B\subseteq E;\\ B\subseteq\operatorname{Eqs}f}}(-1)^{|B|}\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K=[E\cap\operatorname{Eqs}f=\varnothing]\tag{133}$$

(indeed, this can be shown just as in our above proof of Theorem 1.15).

Now, (132) yields

$$\begin{aligned}
X_{G,w} &= \sum_{f:V\to\mathbb{N}_+} \underbrace{[E\cap \operatorname{Eqs} f=\varnothing]}_{\substack{=\sum\limits_{\substack{B\subseteq E;\\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \prod\limits_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \\ \text{(by (133))}}} \mathbf{x}_{f,w} \\
&= \sum_{f:V\to\mathbb{N}_+} \left(\sum_{\substack{B\subseteq E;\\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \mathbf{x}_{f,w} = \underbrace{\sum_{f:V\to\mathbb{N}_+} \sum_{\substack{B\subseteq E;\\ B\subseteq \operatorname{Eqs} f}}}_{=\sum\limits_{B\subseteq E} \sum\limits_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \mathbf{x}_{f,w} \\
&= \sum_{B\subseteq E} \sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \mathbf{x}_{f,w} \\
&= \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \underbrace{\sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} \mathbf{x}_{f,w}}_{\substack{=p_{\lambda((V,B),w)} \\ \text{(by Lemma 6.11} \\ \text{(since } (V,B) \text{ is a finite graph} \\ \text{(since } V \text{ is a finite set and } B\subseteq E\subseteq \binom{V}{2}\text{)))}}} \\
&= \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) p_{\lambda((V,B),w)} = \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} a_K \right) p_{\lambda((V,F),w)}
\end{aligned}$$

(here, we have renamed the summation index B as F). This proves Theorem 6.8. ☐

Proof of Corollary 6.9. We can apply Theorem 6.8 to 0 instead of a_K. As a result, we obtain

$$X_{G,w} = \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0 \right) p_{\lambda((V,F),w)}. \tag{134}$$

Now, if F is any subset of E, then

$$\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0 = \begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} \tag{135}$$

(indeed, this was already shown in our above proof of Corollary 1.17).

Thus, (134) becomes

$$\begin{aligned}
X_{G,w} &= \sum_{F\subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0\right)}_{\substack{=\begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;}\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases}\\ \text{(by (135))}}} p_{\lambda((V,F),w)} \\
&= \sum_{F\subseteq E} (-1)^{|F|} \begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;}\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases} p_{\lambda((V,F),w)} \\
&= \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} \underbrace{\begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;}\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases}}_{\substack{=1\\ \text{(since } F \text{ is } \mathfrak{K}\text{-free)}}} p_{\lambda((V,F),w)} \\
&\qquad + \sum_{\substack{F\subseteq E;\\ F \text{ is not } \mathfrak{K}\text{-free}}} (-1)^{|F|} \underbrace{\begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;}\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases}}_{\substack{=0\\ \text{(since } F \text{ is not } \mathfrak{K}\text{-free)}}} p_{\lambda((V,F),w)} \\
&\qquad\qquad \left(\text{since each subset } F \text{ of } E \text{ either is } \mathfrak{K}\text{-free or is not}\right) \\
&= \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda((V,F),w)} + \underbrace{\sum_{\substack{F\subseteq E;\\ F \text{ is not } \mathfrak{K}\text{-free}}} (-1)^{|F|} 0 p_{\lambda((V,F),w)}}_{=0} \\
&= \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda((V,F),w)}.
\end{aligned}$$

This proves Corollary 6.9. ☐

Proof of Corollary 6.10. Let $\mathfrak{K}$ be the set of all broken circuits of G. Then, we have the following equality between summation signs:

$$\sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} = \sum_{\substack{F\subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}}$$

(indeed, this was already shown in our above proof of Corollary 1.18). Now, Corollary 6.9 yields

$$X_{G,w} = \underbrace{\sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}}}_{= \sum_{\substack{F\subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}}} (-1)^{|F|} p_{\lambda((V,F),w)} = \sum_{\substack{F\subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} p_{\lambda((V,F),w)}.$$

This proves Corollary 6.10. ☐

Proof of Theorem 6.7. Let X be the totally ordered set $\{1\}$ (equipped with the only possible order on this set). Let $\ell : E \to X$ be the function sending each $e \in E$ to $1 \in X$. Let $\mathfrak{K}$ be the empty set. Clearly, $\mathfrak{K}$ is a set of broken circuits of G. Theorem 6.8 (applied to 0 instead of a_K) yields

$$\begin{aligned} X_{G,w} &= \sum_{F\subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0 \right)}_{\substack{=(\text{empty product})\\ (\text{since } \mathfrak{K} \text{ is the empty set})}} p_{\lambda((V,F),w)} \\ &= \sum_{F\subseteq E} (-1)^{|F|} \underbrace{(\text{empty product})}_{=1} p_{\lambda((V,F),w)} = \sum_{F\subseteq E} (-1)^{|F|} p_{\lambda((V,F),w)}. \end{aligned}$$

This proves Theorem 6.7. ☐

6.5. Ambigraphs redux

Just as we have imposed weights on the vertices of a graph, we can do the same to the vertices of an ambigraph. This leads to a generalization of the chromatic symmetric function X_G we introduced in Definition 5.8:

Definition 6.12. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $w : V \to \mathbb{N}_+$ be a weight function on V.

(a) For every $\mathbb{N}_+$-coloring $f : V \to \mathbb{N}_+$ of G, we let $\mathbf{x}_{f,w}$ denote the monomial $\prod_{v\in V} x_{f(v)}^{w(v)}$ in the indeterminates $x_1, x_2, x_3, \ldots$.

(b) We define a power series $X_{G,w} \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$X_{G,w} = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a}\\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_{f,w}.$$

This power series $X_{G,w}$ is called the *chromatic symmetric function* of (G, w).

We can now state generalizations of Theorems 5.13 and 5.18 and Corollaries 5.19 and 5.20:

Theorem 6.13. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Then,

$$X_{G,w} = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda((V, \operatorname{union} F), w)}.$$

(Here, of course, the pair $(V, \operatorname{union} F)$ is regarded as a graph, and the expression $\lambda((V, \operatorname{union} F), w)$ is understood according to Definition 6.5.)

Theorem 6.14. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$X_{G,w} = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) p_{\lambda((V, \operatorname{union} F), w)}.$$

(Here, of course, the pair $(V, \operatorname{union} F)$ is regarded as a graph, and the expression $\lambda((V, \operatorname{union} F), w)$ is understood according to Definition 6.5.)

Corollary 6.15. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$X_{G,w} = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda((V, \operatorname{union} F), w)}.$$

Corollary 6.16. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Then,

$$X_{G,w} = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} p_{\lambda((V, \operatorname{union} F), w)}.$$

The proofs of these four results proceed precisely like their non-weighted counterparts, again using Lemma 6.11 instead of Lemma 2.7. For the sake of completeness, here they are in detail:

Proof of Theorem 6.14. We have

$$X_{G,w} = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_{f,w} \tag{136}$$

(by the definition of $X_{G,w}$). Now, if $f : V \to \mathbb{N}_+$ is a map, then we have the following logical equivalence:

$$(\text{the } \mathbb{N}_+\text{-coloring } f \text{ of } G \text{ is proper}) \iff (\operatorname{EQS}(G, f) = \varnothing) \tag{137}$$

(because the $\mathbb{N}_+$-coloring f of G is proper if and only if $\operatorname{EQS}(G, f) = \varnothing$ [86]). Now,

$$\begin{aligned}
&\sum_{f:V\to\mathbb{N}_+} \Bigg[\underbrace{\operatorname{EQS}(G,f) = \varnothing}_{\substack{\iff (\text{the } \mathbb{N}_+\text{-coloring } f \text{ of } G \text{ is proper}) \\ (\text{by (137)})}} \Bigg] \mathbf{x}_{f,w} \\
&= \sum_{f:V\to\mathbb{N}_+} \Bigg[\underbrace{\text{the } \mathbb{N}_+\text{-coloring } f \text{ of } G \text{ is proper}}_{\iff (f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G)} \Bigg] \mathbf{x}_{f,w} \\
&= \sum_{f:V\to\mathbb{N}_+} [f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G]\, \mathbf{x}_{f,w} \\
&= \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \underbrace{[f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G]}_{\substack{=1 \\ (\text{since } f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G)}} \mathbf{x}_{f,w} \\
&\qquad + \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is not a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \underbrace{[f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G]}_{\substack{=0 \\ (\text{since } f \text{ is not a proper } \mathbb{N}_+\text{-coloring of } G)}} \mathbf{x}_{f,w} \\
&\qquad\qquad (\text{since each } f : V \to \mathbb{N}_+ \text{ either is a proper } \mathbb{N}_+\text{-coloring of } G \text{ or not}) \\
&= \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_{f,w} + \underbrace{\sum_{\substack{f:V\to\mathbb{N}_+ \text{ is not a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} 0\mathbf{x}_{f,w}}_{=0} = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_{f,w} \\
&= X_{G,w}
\end{aligned} \tag{138}$$

(by (136)).

However, for every $f : V \to \mathbb{N}_+$, we have

$$\sum_{\substack{B\subseteq E; \\ \text{union } B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K}; \\ K\subseteq B}} a_K = [\operatorname{EQS}(G, f) = \varnothing] \tag{139}$$

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[86] by Lemma 5.25 (applied to $\mathbb{N}_+$ instead of X)

(indeed, we have already shown this in the above proof of Theorem 5.18).

Now, (138) yields

$$\begin{aligned}
X_{G,w} &= \sum_{f:V\to\mathbb{N}_+} \underbrace{[\operatorname{EQS}(G,f)=\varnothing]}_{\substack{=\sum\limits_{\substack{B\subseteq E;\\ \operatorname{union}B\subseteq\operatorname{Eqs}f}} (-1)^{|B|}\prod\limits_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K\\ \text{(by (139))}}} \mathbf{x}_{f,w}\\
&= \sum_{f:V\to\mathbb{N}_+} \left(\sum_{\substack{B\subseteq E;\\ \operatorname{union}B\subseteq\operatorname{Eqs}f}} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \mathbf{x}_{f,w}\\
&= \underbrace{\sum_{f:V\to\mathbb{N}_+} \sum_{\substack{B\subseteq E;\\ \operatorname{union}B\subseteq\operatorname{Eqs}f}}}_{=\sum\limits_{B\subseteq E}\ \sum\limits_{\substack{f:V\to\mathbb{N}_+;\\ \operatorname{union}B\subseteq\operatorname{Eqs}f}}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \mathbf{x}_{f,w}\\
&= \sum_{B\subseteq E} \sum_{\substack{f:V\to\mathbb{N}_+;\\ \operatorname{union}B\subseteq\operatorname{Eqs}f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \mathbf{x}_{f,w}\\
&= \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \sum_{\substack{f:V\to\mathbb{N}_+;\\ \operatorname{union}B\subseteq\operatorname{Eqs}f}} \mathbf{x}_{f,w}. \qquad (140)
\end{aligned}$$

However, if B is a subset of E, then the pair $(V, \operatorname{union}B)$ is a finite graph (since V is a finite set and since $\operatorname{union}B \subseteq \binom{V}{2}$), and thus we have

$$\sum_{\substack{f:V\to\mathbb{N}_+;\\ \operatorname{union}B\subseteq\operatorname{Eqs}f}} \mathbf{x}_{f,w} = p_{\lambda((V,\operatorname{union}B),w)} \qquad (141)$$

(by Lemma 6.11, applied to $\operatorname{union}B$ instead of B).

Hence, (140) becomes

$$X_{G,w} = \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \underbrace{\sum_{\substack{f:V\to\mathbb{N}_+;\\ \operatorname{union} B\subseteq \operatorname{Eqs} f}} \mathbf{x}_{f,w}}_{\substack{=p_{\lambda((V,\operatorname{union} B),w)} \\ \text{(by (141))}}}$$

$$= \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) p_{\lambda((V,\operatorname{union} B),w)}$$

$$= \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} a_K \right) p_{\lambda((V,\operatorname{union} F),w)}$$

(here, we have renamed the summation index B as F). This proves Theorem 6.14. □

Proof of Corollary 6.15. We can apply Theorem 6.14 to 0 instead of a_K. As a result, we obtain

$$X_{G,w} = \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0 \right) p_{\lambda((V,\operatorname{union} F),w)}. \tag{142}$$

Now, if F is any subset of E, then

$$\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0 = \begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} \tag{143}$$

(indeed, this was already shown in our above proof of Corollary 5.19).

Thus, (142) becomes

$$\begin{aligned}
X_{G,w} &= \sum_{F\subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0\right)}_{\substack{=\begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free};\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases}\\ \text{(by (143))}}} p_{\lambda((V,\operatorname{union} F),w)} \\
&= \sum_{F\subseteq E} (-1)^{|F|} \begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free};\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases} p_{\lambda((V,\operatorname{union} F),w)} \\
&= \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} \underbrace{\begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free};\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases}}_{\substack{=1\\ \text{(since } F \text{ is } \mathfrak{K}\text{-free)}}} p_{\lambda((V,\operatorname{union} F),w)} \\
&\qquad + \sum_{\substack{F\subseteq E;\\ F \text{ is not } \mathfrak{K}\text{-free}}} (-1)^{|F|} \underbrace{\begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free};\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases}}_{\substack{=0\\ \text{(since } F \text{ is not } \mathfrak{K}\text{-free)}}} p_{\lambda((V,\operatorname{union} F),w)} \\
&\qquad \text{(since each subset } F \text{ of } E \text{ either is } \mathfrak{K}\text{-free or is not)} \\
&= \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda((V,\operatorname{union} F),w)} + \underbrace{\sum_{\substack{F\subseteq E;\\ F \text{ is not } \mathfrak{K}\text{-free}}} (-1)^{|F|} 0 p_{\lambda((V,\operatorname{union} F),w)}}_{=0} \\
&= \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda((V,\operatorname{union} F),w)}.
\end{aligned}$$

This proves Corollary 6.15. ☐

Proof of Corollary 6.16. Let $\mathfrak{K}$ be the set of all broken circuits of G.

Now, just as in the proof of Corollary 1.18, we can prove the following equality:

$$\sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} = \sum_{\substack{F\subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}}$$

(an equality between summation signs). Now, Corollary 6.15 yields

$$\begin{aligned} X_{G,w} &= \underbrace{\sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}}}_{\substack{=\sum_{\substack{F\subseteq E;\\ F\text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}}}} (-1)^{|F|}\, p_{\lambda((V,\operatorname{union} F),w)} \\ &= \sum_{\substack{F\subseteq E;\\ F\text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|}\, p_{\lambda((V,\operatorname{union} F),w)}. \end{aligned}$$

This proves Corollary 6.16. ☐

Proof of Theorem 6.13. Let X be the totally ordered set $\{1\}$ (equipped with the only possible order on this set). Let $\ell : E \to X$ be the function sending each $e \in E$ to $1 \in X$. Let $\mathfrak{K}$ be the empty set. Clearly, $\mathfrak{K}$ is a set of broken circuits of G. Theorem 6.14 (applied to 0 instead of a_K) yields

$$\begin{aligned} X_{G,w} &= \sum_{F\subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0\right)}_{\substack{=(\text{empty product})\\ (\text{since } \mathfrak{K} \text{ is the empty set})}} p_{\lambda((V,\operatorname{union} F),w)} \\ &= \sum_{F\subseteq E} (-1)^{|F|} \underbrace{(\text{empty product})}_{=1} p_{\lambda((V,\operatorname{union} F),w)} = \sum_{F\subseteq E} (-1)^{|F|}\, p_{\lambda((V,\operatorname{union} F),w)}. \end{aligned}$$

This proves Theorem 6.13. ☐

6.6. Noncommutative chromatic symmetric functions

The *noncommutative chromatic symmetric function* Y_G of a graph G has been introduced by Gebhard and Sagan in [GebSag01, §3] as a lift of the chromatic symmetric function X_G to a noncommutative polynomial ring. In order to define it, we need to lift the monomials $\mathbf{x}_f$ to noncommutative monomials, which requires fixing a list of the vertices of G (since a noncommutative product is only defined if its factors appear in a chosen order). Unlike [GebSag01], we shall not require this list to contain each vertex of G exactly once; thus, we obtain a more general notion that refines not only Stanley's original X_G but also its weighted version $X_{G,w}$ discussed above.

We consider the $\mathbf{k}$-algebra $\mathbf{k}\left\langle\left\langle X_1, X_2, X_3, \ldots\right\rangle\right\rangle$ of noncommutative power series in countably many distinct indeterminates $X_1, X_2, X_3, \ldots$ over $\mathbf{k}$. It is a topological $\mathbf{k}$-algebra[87]. A noncommutative power series $P \in \mathbf{k}\left\langle\left\langle X_1, X_2, X_3, \ldots\right\rangle\right\rangle$ is

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[87] Its topology is defined in the same way as the topology on $\mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ (but of course the monomials are now noncommutative monomials).

said to be *bounded-degree* if there exists an $N \in \mathbb{N}$ such that every noncommutative monomial of degree $> N$ appears with coefficient 0 in P. A noncommutative power series $P \in \mathbf{k}\left\langle\left\langle X_1, X_2, X_3, \ldots\right\rangle\right\rangle$ is said to be *symmetric* if and only if P is invariant under any permutation of the indeterminates. We let Λ_{NC} be the subset of $\mathbf{k}\left\langle\left\langle X_1, X_2, X_3, \ldots\right\rangle\right\rangle$ consisting of all symmetric bounded-degree power series $P \in \mathbf{k}\left\langle\left\langle X_1, X_2, X_3, \ldots\right\rangle\right\rangle$. This subset Λ_{NC} is a $\mathbf{k}$-subalgebra of $\mathbf{k}\left\langle\left\langle X_1, X_2, X_3, \ldots\right\rangle\right\rangle$, and is called the $\mathbf{k}$*-algebra of symmetric functions in noncommutative indeterminates* over $\mathbf{k}$.

This $\mathbf{k}$-algebra Λ_{NC} is called $\Pi\left(\mathbf{x}\right)$ in [RosSag04], and should not be mistaken for the algebra NSym of noncommutative symmetric functions (which is studied, e.g., in [GriRei14, §5.4]).[88]

We can now define noncommutative chromatic symmetric functions of ambigraphs:

Definition 6.17. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be a finite list of elements of V that contains each element of V at least once.

(a) For every $\mathbb{N}_+$-coloring $f : V \to \mathbb{N}_+$ of G, we let $\mathbf{X}_{f,\mathbf{t}}$ denote the noncommutative monomial $X_{f(t_1)} X_{f(t_2)} \cdots X_{f(t_N)}$ in the indeterminates $X_1, X_2, X_3, \ldots$.

(b) We define a noncommutative power series $Y_{G,\mathbf{t}} \in \mathbf{k}\left\langle\left\langle X_1, X_2, X_3, \ldots\right\rangle\right\rangle$ by

$$Y_{G,\mathbf{t}} = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{X}_{f,\mathbf{t}}. \tag{144}$$

This power series $Y_{G,\mathbf{t}}$ is called the *noncommutative chromatic symmetric function* of $(G, \mathbf{t})$.

Remark 6.18. Why did we require the list $\mathbf{t}$ to contain each element of V at least once in Definition 6.17?

Otherwise, there could be a vertex $v \in V$ that does not appear in $\mathbf{t}$. In that case, the monomials $\mathbf{X}_{f,\mathbf{t}}$ would be independent of the color $f(v)$ of this vertex, and thus we would obtain the same monomial $\mathbf{X}_{f,\mathbf{t}}$ for infinitely many different proper $\mathbb{N}_+$-colorings f (since we could arbitrarily change the color $f(v)$ without affecting the monomial $\mathbf{X}_{f,\mathbf{t}}$). Hence, the sum on the right hand side of (144) would contain equally many identical monomials, and this would render $Y_{G,\mathbf{t}}$ undefined.

The noncommutative chromatic symmetric function $Y_{G,\mathbf{t}}$ is a lift of the weighted chromatic symmetric function $X_{G,w}$ introduced in Definition 6.12 **(b)**. This can be made precise as follows:

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[88] The latter algebra NSym can too be viewed as a subalgebra of $\mathbf{k}\left\langle\left\langle X_1, X_2, X_3, \ldots\right\rangle\right\rangle$, but it does not consist of symmetric power series (despite its name).

Remark 6.19. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be a finite list of elements of V that contains each element of V at least once. Let $w : V \to \mathbb{N}_+$ be the weight function on V that is defined by

$$\begin{aligned} w(v) &= (\text{number of times that } v \text{ appears in the list } \mathbf{t}) \\ &= (\text{number of all } i \in \{1, 2, \ldots, N\} \text{ such that } t_i = v) \qquad \text{for each } v \in V. \end{aligned}$$

Let $\pi : \mathbf{k}\langle\langle X_1, X_2, X_3, \ldots\rangle\rangle \to \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ be the topological $\mathbf{k}$-algebra homomorphism that sends the noncommuting indeterminates $X_1, X_2, X_3, \ldots$ to the respective commuting indeterminates $x_1, x_2, x_3, \ldots$. Then:

(a) For every $\mathbb{N}_+$-coloring $f : V \to \mathbb{N}_+$ of G, we have $\pi\left(\mathbf{X}_{f,\mathbf{t}}\right) = \mathbf{x}_{f,w}$. (See Definition 6.12 **(a)** and Definition 6.17 **(a)** for the meanings of $\mathbf{x}_{f,w}$ and $\mathbf{X}_{f,t}$.)

(b) We have $\pi\left(Y_{G,\mathbf{t}}\right) = X_{G,w}$.

We omit the simple proofs of these claims.

Our goal is now to state noncommutative analogues of our main theorems for ambigraphs (specifically, Theorems 6.13 and 6.14 and Corollaries 6.15 and 6.16). To do so, we need a noncommutative analogue of the power-sum symmetric functions p_λ. In the commutative case, we defined p_λ as the product $\prod\limits_{i \geq 1} p_{\lambda_i}$ (see Definition 1.4). The noncommutative case, however, will not be such a product, so we need to define it differently. This will require some preparations.

We begin by recalling the notion of a set partition:

Definition 6.20. Let X be a set.

(a) A *set partition* of X means a set $\mathbf{P}$ of disjoint nonempty subsets of X such that $\bigcup\limits_{S \in \mathbf{P}} S = X$.

(b) If $\mathbf{P}$ is a set partition of X, then the sets $S \in \mathbf{P}$ are called the *blocks* of $\mathbf{P}$.

For example:

- The set $\{\{1,3\}, \{2,4,5\}\}$ is a set partition of the set $\{1,2,3,4,5\}$, since $\{1,3\}$ and $\{2,4,5\}$ are two disjoint nonempty subsets of $\{1,2,3,4,5\}$ whose union is $\{1,3\} \cup \{2,4,5\} = \{1,2,3,4,5\}$.
- The set $\{\{1,4\}, \{2,5\}, \{3,6\}\}$ is a set partition of the set $\{1,2,3,4,5,6\}$. The blocks of this set partition are $\{1,4\}$ and $\{2,5\}$ and $\{3,6\}$.
- The set $\{\{1,2,3,4,5\}\}$ is a set partition of the set $\{1,2,3,4,5\}$. It has only one block, namely $\{1,2,3,4,5\}$.
- The set $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$ is a set partition of the set $\{1,2,3,4,5\}$. It has five blocks, namely $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$.

There is a well-known relation (actually a one-to-one correspondence) between the set partitions of a given set X and the equivalence relations on X. It can be summarized in the following theorem:

Theorem 6.21. Let X be a set.

(a) If $\sim$ is an equivalence relation on X, then the set

$$X/(\sim) = \{\text{all } \sim \text{-equivalence classes}\}$$

(see Definition 1.6 for its definition) is a set partition of X.

(b) If $\mathbf{P}$ is a set partition of X, then we can define an equivalence relation $\sim$ on the set X as follows: For any two elements a and b of X, we shall have $a \sim b$ if and only if the elements a and b belong to the same block of $\mathbf{P}$. This relation $\sim$ will be called $\sim_{\mathbf{P}}$.

(c) The maps

$$\begin{aligned} \{\text{equivalence relations on } X\} &\to \{\text{set partitions of } X\}, \\ (\sim) &\mapsto X/(\sim) \end{aligned}$$

and

$$\begin{aligned} \{\text{set partitions of } X\} &\to \{\text{equivalence relations on } X\}, \\ \mathbf{P} &\mapsto (\sim_{\mathbf{P}}) \end{aligned}$$

are mutually inverse bijections.

Example 6.22. Let X be the set $\{1,2,3,4,5,6\}$.

(a) If $\sim$ is the equivalence relation on X given by

$$(a \sim b) \iff (a \equiv b \bmod 2),$$

then the corresponding set partition $X/(\sim)$ of X is $\{\{1,3,5\}, \{2,4,6\}\}$.

(b) If $\mathbf{P}$ is the set partition $\{\{1,2\}, \{3,4,6\}, \{5\}\}$ of X, then the corresponding equivalence relation $\sim_{\mathbf{P}}$ on X is given by

$$1 \sim_{\mathbf{P}} 2, \qquad 3 \sim_{\mathbf{P}} 4 \sim_{\mathbf{P}} 6$$

and no further relations (except, of course, for the ones that follow from the relations just given by reflexivity, symmetry and transitivity).

We can now define the noncommutative analogues of the power-sum symmetric functions p_λ. These are indexed not by integer partitions λ but by set partitions $\mathbf{P}$:

Definition 6.23. Let $N \in \mathbb{N}$. Let $\mathbf{P}$ be a set partition of the set $\{1,2,\ldots,N\}$. Recall the relation $\sim_{\mathbf{P}}$ defined in Theorem 6.21 **(b)**.

Then, we define a noncommutative power series $P_{\mathbf{P}} \in \mathbf{k}\langle\langle X_1, X_2, X_3, \ldots\rangle\rangle$ by

$$P_{\mathbf{P}} = \sum_{\substack{(i_1,i_2,\ldots,i_N)\in(\mathbb{N}_+)^N; \\ i_a = i_b \text{ whenever } a \sim_{\mathbf{P}} b}} X_{i_1} X_{i_2} \cdots X_{i_N}.$$

Here, the condition "$i_a = i_b$ whenever $a \sim_{\mathbf{P}} b$" under the summation sign is shorthand for "$i_a = i_b$ for any two elements $a, b \in \{1, 2, \ldots, N\}$ that satisfy $a \sim_{\mathbf{P}} b$".

This power series $P_{\mathbf{P}}$ is called the *power-sum symmetric function in noncommutative variables* corresponding to the set partition $\mathbf{P}$. It is not hard to see that it belongs to Λ_{NC}.

Note that $P_{\mathbf{P}}$ is called $p_{\mathbf{P}}$ in [GebSag01, §2] and in [RosSag04].

Example 6.24. **(a)** If $\mathbf{P}$ is the set partition $\{\{1,3\}, \{2,4,5\}\}$ of $\{1,2,3,4,5\}$, then

$$\begin{aligned} P_{\mathbf{P}} &= \sum_{\substack{(i_1,i_2,i_3,i_4,i_5) \in (\mathbb{N}_+)^5; \\ i_a = i_b \text{ whenever } a \sim_{\mathbf{P}} b}} X_{i_1} X_{i_2} X_{i_3} X_{i_4} X_{i_5} \\ &= \sum_{\substack{(i_1,i_2,i_3,i_4,i_5) \in (\mathbb{N}_+)^5; \\ i_1 = i_3 \text{ and } i_2 = i_4 = i_5}} X_{i_1} X_{i_2} X_{i_3} X_{i_4} X_{i_5} \\ &= \sum_{(u,v) \in (\mathbb{N}_+)^2} X_u X_v X_u X_v X_v \end{aligned}$$

(here, we have substituted (u, v, u, v, v) for the summation index $(i_1, i_2, i_3, i_4, i_5)$, since the condition "$i_1 = i_3$ and $i_2 = i_4 = i_5$" is saying precisely that the 5-tuple $(i_1, i_2, i_3, i_4, i_5)$ can be written in the form (u, v, u, v, v)). This is a noncommutative power series that contains terms such as $X_4 X_7 X_4 X_7 X_7$ or $X_5 X_5 X_5 X_5 X_5$ (we are allowed to have $u = v$ in the above sum), but not terms such as $X_1 X_2 X_2 X_1 X_1$ (since the indeterminates don't commute).

(b) If $\mathbf{P}$ is the set partition $\{\{1,4\}, \{2,5\}, \{3,6\}\}$ of $\{1,2,3,4,5,6\}$, then

$$\begin{aligned} P_{\mathbf{P}} &= \sum_{\substack{(i_1,i_2,i_3,i_4,i_5,i_6) \in (\mathbb{N}_+)^6; \\ i_a = i_b \text{ whenever } a \sim_{\mathbf{P}} b}} X_{i_1} X_{i_2} X_{i_3} X_{i_4} X_{i_5} X_{i_6} \\ &= \sum_{\substack{(i_1,i_2,i_3,i_4,i_5,i_6) \in (\mathbb{N}_+)^6; \\ i_1 = i_4 \text{ and } i_2 = i_5 \text{ and } i_3 = i_6}} X_{i_1} X_{i_2} X_{i_3} X_{i_4} X_{i_5} X_{i_6} \\ &= \sum_{(u,v,w) \in (\mathbb{N}_+)^3} \underbrace{X_u X_v X_w X_u X_v X_w}_{= (X_u X_v X_w)^2} = \sum_{(u,v,w) \in (\mathbb{N}_+)^3} (X_u X_v X_w)^2 . \end{aligned}$$

Note that this cannot be simplified to $\left(\sum\limits_{u \in \mathbb{N}_+} X_u^2 \right)^3$, since the indeterminates don't commute.

(c) If $\mathbf{P}$ is the set partition $\{\{1,2,3,4\}\}$ of $\{1,2,3,4\}$, then

$$\begin{aligned} P_{\mathbf{P}} &= \sum_{\substack{(i_1,i_2,i_3,i_4)\in(\mathbb{N}_+)^4;\\ i_a=i_b \text{ whenever } a\sim_{\mathbf{P}} b}} X_{i_1}X_{i_2}X_{i_3}X_{i_4} \\ &= \sum_{\substack{(i_1,i_2,i_3,i_4)\in(\mathbb{N}_+)^4;\\ i_1=i_2=i_3=i_4}} X_{i_1}X_{i_2}X_{i_3}X_{i_4} = \sum_{u\in\mathbb{N}_+} X_uX_uX_uX_u = \sum_{u\in\mathbb{N}_+} X_u^4. \end{aligned}$$

(d) If $\mathbf{P}$ is the set partition $\varnothing$ of $\{\}$ (so we have $N=0$), then

$$\begin{aligned} P_{\mathbf{P}} &= \sum_{\substack{()\in(\mathbb{N}_+)^0;\\ i_a=i_b \text{ whenever } a\sim_{\mathbf{P}} b}} (\text{empty product}) \\ &= (\text{empty product}) \qquad \text{(since there is only one 0-tuple)} \\ &= 1. \end{aligned}$$

We shall now assign an equivalence relation to any finite graph (V,E) and any list $\mathbf{t}$ of its vertices:

Proposition 6.25. Let (V,B) be a finite graph. Then, according to Definition 1.10 **(a)**, an equivalence relation $\sim_{(V,B)}$ is defined on the set V.

Let $\mathbf{t}=(t_1,t_2,\ldots,t_N)$ be a finite list of elements of V. Let $\approx$ be the relation on the set $\{1,2,\ldots,N\}$ defined as follows: Two elements i and j of $\{1,2,\ldots,N\}$ shall satisfy $i\approx j$ if and only if $t_i\sim_{(V,B)} t_j$.

Then, this relation $\approx$ is an equivalence relation.

Proof of Proposition 6.25. This is a straightforward and easy argument, but we give it nevertheless for the sake of completeness.

We shall show that the relation $\approx$ is reflexive, symmetric and transitive.

Proof of reflexivity: Let $i\in\{1,2,\ldots,N\}$. We shall show that $i\approx i$.

The relation $\sim_{(V,B)}$ is an equivalence relation, and thus is reflexive. Hence, we have $t_i\sim_{(V,B)} t_i$. However, $i\approx i$ holds if and only if we have $t_i\sim_{(V,B)} t_i$ (by the definition of the relation $\approx$). Hence, $i\approx i$ holds (since we have $t_i\sim_{(V,B)} t_i$).

Forget that we fixed i. We thus have shown that $i\approx i$ for each $i\in\{1,2,\ldots,N\}$. In other words, the relation $\approx$ is reflexive.

Proof of symmetry: Let i and j be two elements of $\{1,2,\ldots,N\}$ satisfying $i\approx j$. We shall show that $j\approx i$.

We have $i\approx j$ if and only if we have $t_i\sim_{(V,B)} t_j$ (by the definition of the relation $\approx$). Hence, we have $t_i\sim_{(V,B)} t_j$ (since we have $i\approx j$).

The relation $\sim_{(V,B)}$ is an equivalence relation, and thus is symmetric. Hence, from $t_i\sim_{(V,B)} t_j$, we obtain $t_j\sim_{(V,B)} t_i$. However, $j\approx i$ holds if and only if we have $t_j\sim_{(V,B)} t_i$ (by the definition of the relation $\approx$). Hence, $j\approx i$ holds (since we have $t_j\sim_{(V,B)} t_i$).

Forget that we fixed i and j. We thus have shown that if i and j are two elements of $\{1,2,\ldots,N\}$ satisfying $i \approx j$, then $j \approx i$. In other words, the relation $\approx$ is symmetric.

Proof of transitivity: Let i, j and k be three elements of $\{1,2,\ldots,N\}$ satisfying $i \approx j$ and $j \approx k$. We shall show that $i \approx k$.

We have $i \approx j$ if and only if we have $t_i \sim_{(V,B)} t_j$ (by the definition of the relation $\approx$). Hence, we have $t_i \sim_{(V,B)} t_j$ (since we have $i \approx j$).

We have $j \approx k$ if and only if we have $t_j \sim_{(V,B)} t_k$ (by the definition of the relation $\approx$). Hence, we have $t_j \sim_{(V,B)} t_k$ (since we have $j \approx k$).

The relation $\sim_{(V,B)}$ is an equivalence relation, and thus is transitive. Hence, from $t_i \sim_{(V,B)} t_j$ and $t_j \sim_{(V,B)} t_k$, we obtain $t_i \sim_{(V,B)} t_k$. However, $i \approx k$ holds if and only if we have $t_i \sim_{(V,B)} t_k$ (by the definition of the relation $\approx$). Hence, $i \approx k$ holds (since we have $t_i \sim_{(V,B)} t_k$).

Forget that we fixed i, j and k. We thus have shown that if i, j and k are three elements of $\{1,2,\ldots,N\}$ satisfying $i \approx j$ and $j \approx k$, then $i \approx k$. In other words, the relation $\approx$ is transitive.

We have now shown that the relation $\approx$ is reflexive, symmetric and transitive. In other words, this relation $\approx$ is an equivalence relation (because an equivalence relation is defined to be a binary relation that is reflexive, symmetric and transitive). This proves Proposition 6.25. □

As we know from Theorem 6.21, an equivalence relation is "essentially the same as" a set partition. Thus, in particular, we can turn the equivalence relation defined in Proposition 6.25 into a set partition:

Definition 6.26. Let (V,B) be a finite graph. Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be a finite list of elements of V.

(a) Let $\approx_{(V,B,\mathbf{t})}$ be the relation $\approx$ on the set $\{1,2,\ldots,N\}$ defined in Proposition 6.25. As we know from Proposition 6.25, this relation $\approx$ is an equivalence relation. In other words, the relation $\approx_{(V,B,\mathbf{t})}$ is an equivalence relation.

(b) Therefore, Theorem 6.21 **(a)** (applied to $X = \{1,2,\ldots,N\}$ and $(\sim) = \left(\approx_{(V,B,\mathbf{t})}\right)$) shows that the set

$$\{1,2,\ldots,N\} / \left(\approx_{(V,B,\mathbf{t})}\right) = \left\{\text{all } \approx_{(V,B,\mathbf{t})} \text{-equivalence classes}\right\}$$

is a set partition of $\{1,2,\ldots,N\}$. We shall denote this set partition by $\mathbf{P}(V,B,\mathbf{t})$.

Example 6.27. Let (V,B) be the finite graph with $V = \{u,v,w,x,y\}$ and $B = \{\{u,v\}, \{v,w\}, \{x,y\}\}$. Then, the equivalence relation $\sim_{(V,B)}$ from Definition 1.10 **(a)** satisfies $u \sim_{(V,B)} v \sim_{(V,B)} w$ and $x \sim_{(V,B)} y$.

Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be the list (u,v,x,y,x,u) of elements of V (so that $N = 6$ and $t_1 = u$ and $t_2 = v$ and $t_3 = x$ and $t_4 = y$ and $t_5 = x$ and $t_6 = u$).

Then, the equivalence relation $\approx_{(V,B,\mathbf{t})}$ from Definition 6.26 **(a)** satisfies

$$1 \approx_{(V,B,\mathbf{t})} 2 \approx_{(V,B,\mathbf{t})} 6 \qquad \left(\text{since } t_1 \sim_{(V,B)} t_2 \sim_{(V,B)} t_6\right) \qquad \text{and}$$
$$3 \approx_{(V,B,\mathbf{t})} 4 \approx_{(V,B,\mathbf{t})} 5 \qquad \left(\text{since } t_3 \sim_{(V,B)} t_4 \sim_{(V,B)} t_5\right)$$

and no further relations (except for the ones that follow from the relations just given using reflexivity, symmetry and transitivity). Thus, the set partition $\mathbf{P}(V,B,\mathbf{t})$ from Definition 6.26 **(b)** is

$$\{\{1,2,6\}, \{3,4,5\}\}.$$

We now have all notations in place to state noncommutative analogues of Theorems 6.13 and 6.14 and Corollaries 6.15 and 6.16:

Theorem 6.28. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be a finite list of elements of V that contains each element of V at least once. Then,

$$Y_{G,\mathbf{t}} = \sum_{F \subseteq E} (-1)^{|F|} p_{\mathbf{P}(V, \operatorname{union} F, \mathbf{t})}.$$

Theorem 6.29. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be a finite list of elements of V that contains each element of V at least once. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$Y_{G,\mathbf{t}} = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) p_{\mathbf{P}(V, \operatorname{union} F, \mathbf{t})}.$$

(Here, of course, the pair $(V, \operatorname{union} F)$ is regarded as a graph, and the expression $\lambda((V, \operatorname{union} F), w)$ is understood according to Definition 6.5.)

Corollary 6.30. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be a finite list of elements of V that contains each element of V at least once. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$Y_{G,\mathbf{t}} = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\mathbf{P}(V, \operatorname{union} F, \mathbf{t})}.$$

Corollary 6.31. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be a finite list of elements of V that contains each element of V at least once. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Then,

$$Y_{G,\mathbf{t}} = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} P_{\mathbf{P}(V, \operatorname{union} F, \mathbf{t})}.$$

Since any graph or loopless multigraph can be viewed as an ambigraph, it is easy to see that Theorem 6.28 and Corollary 6.31 generalize [GebSag01, Theorem 3.6] and [GebSag01, Theorem 3.8], respectively.

In order to prove these four results, we proceed similarly to the commutative case, which we have studied to exhaustion. Instead of Lemma 6.11, we need the following noncommutative analogue:

Lemma 6.32. Let (V, B) be a finite graph. Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be a finite list of elements of V that contains each element of V at least once. Then,

$$\sum_{\substack{f: V \to \mathbb{N}_+; \\ B \subseteq \operatorname{Eqs} f}} \mathbf{x}_{f,\mathbf{t}} = P_{\mathbf{P}(V,B,\mathbf{t})}.$$

Proof of Lemma 6.32. The definition of $P_{\mathbf{P}(V,B,\mathbf{t})}$ yields

$$P_{\mathbf{P}(V,B,\mathbf{t})} = \sum_{\substack{(i_1, i_2, \ldots, i_N) \in (\mathbb{N}_+)^N; \\ i_a = i_b \text{ whenever } a \sim_{\mathbf{P}(V,B,\mathbf{t})} b}} X_{i_1} X_{i_2} \cdots X_{i_N} \tag{145}$$

(where the condition "$i_a = i_b$ whenever $a \sim_{\mathbf{P}(V,B,\mathbf{t})} b$" is shorthand for "$i_a = i_b$ for any two elements $a, b \in \{1, 2, \ldots, N\}$ that satisfy $a \sim_{\mathbf{P}(V,B,\mathbf{t})} b$").

However, for any two elements $a, b \in \{1, 2, \ldots, N\}$, we have the equivalence

$$\left(a \sim_{\mathbf{P}(V,B,\mathbf{t})} b\right) \iff \left(t_a \sim_{(V,B)} t_b\right) \tag{146}$$

[89].

Now, we define two sets

$$\mathcal{F} = \{g : V \to \mathbb{N}_+ \text{ is a map} \mid B \subseteq \operatorname{Eqs} g\}$$

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[89] *Proof.* Let $X = \{1, 2, \ldots, N\}$. Then, $\mathbf{P}(V, B, \mathbf{t})$ is a set partition of X (since $\mathbf{P}(V, B, \mathbf{t})$ is a set partition of $\{1, 2, \ldots, N\}$).

Let us recall how the relation $\sim_{\mathbf{P}(V,B,\mathbf{t})}$ is defined (according to Theorem 6.21 **(b)**): This relation is the relation $\sim$ on the set X such that for any two elements a and b of X, we have $a \sim b$ if and only if the elements a and b belong to the same block of $\mathbf{P}(V, B, \mathbf{t})$. Hence, for any two elements a and b of X, we have $a \sim_{\mathbf{P}(V,B,\mathbf{t})} b$ if and only if the elements a and b belong to the same block of $\mathbf{P}(V, B, \mathbf{t})$. In other words, for any two elements a and b of X, we and

$$\mathcal{I} = \left\{ (i_1, i_2, \ldots, i_N) \in (\mathbb{N}_+)^N \ \mid \ i_a = i_b \text{ whenever } a \sim_{\mathbf{P}(V,B,\mathbf{t})} b \right\}.$$

We claim the following:

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have the equivalence

$$\begin{aligned} & \left(a \sim_{\mathbf{P}(V,B,\mathbf{t})} b \right) \\ \iff & \left(\text{the elements } a \text{ and } b \text{ belong to the same block of } \mathbf{P}(V,B,\mathbf{t}) \right). \end{aligned} \tag{147}$$

Let $\approx_{(V,B,\mathbf{t})}$ be the relation $\approx$ on the set $\{1,2,\ldots,N\}$ defined in Proposition 6.25. As we know from Proposition 6.25, this relation $\approx$ is an equivalence relation. In other words, the relation $\approx_{(V,B,\mathbf{t})}$ is an equivalence relation.

Now, recall how the set partition $\mathbf{P}(V,B,\mathbf{t})$ is defined (according to Definition 6.26): It is defined by

$$\mathbf{P}(V,B,\mathbf{t}) = \{1,2,\ldots,N\} / \left(\approx_{(V,B,\mathbf{t})} \right) = \left\{ \text{all } \approx_{(V,B,\mathbf{t})} \text{-equivalence classes} \right\}.$$

Hence, the blocks of $\mathbf{P}(V,B,\mathbf{t})$ are the $\approx_{(V,B,\mathbf{t})}$-equivalence classes.

Now, let a and b be two elements of $\{1,2,\ldots,N\}$. Thus, a and b are two elements of X (since $X = \{1,2,\ldots,N\}$). Therefore, we have the following chain of equivalences:

$$\begin{aligned} & \left(a \sim_{\mathbf{P}(V,B,\mathbf{t})} b \right) \\ \iff & \left(\text{the elements } a \text{ and } b \text{ belong to the same block of } \mathbf{P}(V,B,\mathbf{t}) \right) \\ & \qquad (\text{by (147)}) \\ \iff & \left(\text{the elements } a \text{ and } b \text{ belong to the } \approx_{(V,B,\mathbf{t})} \text{-equivalence class} \right) \\ & \qquad \left(\text{since the blocks of } \mathbf{P}(V,B,\mathbf{t}) \text{ are the } \approx_{(V,B,\mathbf{t})} \text{-equivalence classes} \right) \\ \iff & \left(a \approx_{(V,B,\mathbf{t})} b \right) \end{aligned} \tag{148}$$

(because two elements of $\{1,2,\ldots,N\}$ belong to the $\approx_{(V,B,\mathbf{t})}$-equivalence class if and only if they are related by the relation $\approx_{(V,B,\mathbf{t})}$).

However, the relation $\approx_{(V,B,\mathbf{t})}$ was defined as the relation $\approx$ on the set $\{1,2,\ldots,N\}$ defined in Proposition 6.25. But the latter relation $\approx$ was defined as follows: Two elements i and j of $\{1,2,\ldots,N\}$ shall satisfy $i \approx j$ if and only if $t_i \sim_{(V,B)} t_j$.

Since we have denoted this relation $\approx$ by $\approx_{(V,B,\mathbf{t})}$, we can rewrite this fact as follows: Two elements i and j of $\{1,2,\ldots,N\}$ shall satisfy $i \approx_{(V,B,\mathbf{t})} j$ if and only if $t_i \sim_{(V,B)} t_j$. In other words, for any two elements i and j of $\{1,2,\ldots,N\}$, we have the equivalence $\left(i \approx_{(V,B,\mathbf{t})} j \right) \iff \left(t_i \sim_{(V,B)} t_j \right)$.

Applying this to $i = a$ and $j = b$, we obtain the equivalence $\left(a \approx_{(V,B,\mathbf{t})} b \right) \iff \left(t_a \sim_{(V,B)} t_b \right)$. Hence, we have the equivalence

$$\begin{aligned} \left(a \sim_{\mathbf{P}(V,B,\mathbf{t})} b \right) & \iff \left(a \approx_{(V,B,\mathbf{t})} b \right) \qquad (\text{by (148)}) \\ & \iff \left(t_a \sim_{(V,B)} t_b \right). \end{aligned}$$

This proves (146).

Claim 1: For any $f \in \mathcal{F}$, we have $\left(f\left(t_1\right), f\left(t_2\right), \ldots, f\left(t_N\right)\right) \in \mathcal{I}$.

[*Proof of Claim 1:* Let $f \in \mathcal{F}$. Then,

$$f \in \mathcal{F} = \left\{g : V \to \mathbb{N}_+ \text{ is a map} \mid B \subseteq \operatorname{Eqs} g\right\}.$$

In other words, f is a map $g : V \to \mathbb{N}_+$ satisfying $B \subseteq \operatorname{Eqs} g$. In other words, f is a map from V to $\mathbb{N}_+$ and satisfies $B \subseteq \operatorname{Eqs} f$.

Let $a, b \in \{1, 2, \ldots, N\}$ be two elements that satisfy $a \sim_{\mathbf{P}(V,B,\mathbf{t})} b$. We shall show that $f\left(t_a\right) = f\left(t_b\right)$.

From (146), we know that the statements "$a \sim_{\mathbf{P}(V,B,\mathbf{t})} b$" and "$t_a \sim_{(V,B)} t_b$" are equivalent. Hence, we have $t_a \sim_{(V,B)} t_b$ (since we have $a \sim_{\mathbf{P}(V,B,\mathbf{t})} b$).

Now, let $\sim$ denote the equivalence relation $\sim_{(V,B)}$. Thus, we have $t_a \sim t_b$ (since $t_a \sim_{(V,B)} t_b$).

Let $Y = \mathbb{N}_+$. Thus, $f : V \to Y$ is a map (since $f : V \to \mathbb{N}_+$ is a map). A set $Y^V_\sim$ is defined (according to Definition 1.7 **(b)**). Lemma 2.5 yields that we have the following logical equivalence of statements:

$$\left(B \subseteq \operatorname{Eqs} f\right) \iff \left(f \in Y^V_\sim\right).$$

Hence, we have $f \in Y^V_\sim$ (since we have $B \subseteq \operatorname{Eqs} f$). Therefore,

$$f \in Y^V_\sim = \left\{g \in Y^V \mid g\left(x\right) = g\left(y\right) \text{ for any } x \in V \text{ and } y \in V \text{ satisfying } x \sim y\right\}$$

(by the definition of $Y^V_\sim$). In other words, f is a $g \in Y^V$ satisfying

$$g\left(x\right) = g\left(y\right) \text{ for any } x \in V \text{ and } y \in V \text{ satisfying } x \sim y.$$

In other words, f is an element of Y^V and satisfies

$$f\left(x\right) = f\left(y\right) \text{ for any } x \in V \text{ and } y \in V \text{ satisfying } x \sim y. \tag{149}$$

We can apply (149) to $x = t_a$ and $y = t_b$ (since $t_a \in V$ and $t_b \in V$ and $t_a \sim t_b$). Thus, we obtain $f\left(t_a\right) = f\left(t_b\right)$.

Now, forget that we fixed a and b. We thus have shown that $f\left(t_a\right) = f\left(t_b\right)$ for any two elements $a, b \in \{1, 2, \ldots, N\}$ that satisfy $a \sim_{\mathbf{P}(V,B,\mathbf{t})} b$. In other words, we have $f\left(t_a\right) = f\left(t_b\right)$ whenever $a \sim_{\mathbf{P}(V,B,\mathbf{t})} b$.

Hence, the N-tuple $\left(f\left(t_1\right), f\left(t_2\right), \ldots, f\left(t_N\right)\right)$ is an N-tuple $\left(i_1, i_2, \ldots, i_N\right) \in \left(\mathbb{N}_+\right)^N$ satisfying $i_a = i_b$ whenever $a \sim_{\mathbf{P}(V,B,\mathbf{t})} b$. In other words,

$$\begin{aligned}\left(f\left(t_1\right), f\left(t_2\right), \ldots, f\left(t_N\right)\right) &\in \left\{\left(i_1, i_2, \ldots, i_N\right) \in \left(\mathbb{N}_+\right)^N \mid i_a = i_b \text{ whenever } a \sim_{\mathbf{P}(V,B,\mathbf{t})} b\right\} \\ &= \mathcal{I} \qquad \left(\text{by the definition of } \mathcal{I}\right).\end{aligned}$$

This proves Claim 1.]

Thanks to Claim 1, we can define a map

$$\begin{aligned}\Psi : \mathcal{F} &\to \mathcal{I}, \\ f &\mapsto \left(f\left(t_1\right), f\left(t_2\right), \ldots, f\left(t_N\right)\right).\end{aligned}$$

Consider this map Ψ. We claim the following:

Claim 2: The map Ψ is injective.

[*Proof of Claim 2:* Let f and g be two elements of $\mathcal{F}$ that satisfy $\Psi(f) = \Psi(g)$. We shall show that $f = g$.

Note that both f and g are elements of $\mathcal{F}$, and thus are maps $V \to \mathbb{N}_+$.

The definition of Ψ yields $\Psi(f) = (f(t_1), f(t_2), \ldots, f(t_N))$ and $\Psi(g) = (g(t_1), g(t_2), \ldots, g(t_N))$. Hence,

$$(f(t_1), f(t_2), \ldots, f(t_N)) = \Psi(f) = \Psi(g) = (g(t_1), g(t_2), \ldots, g(t_N)).$$

In other words,

$$f(t_i) = g(t_i) \qquad \text{for each } i \in \{1, 2, \ldots, N\}. \tag{150}$$

Now, let $v \in V$. We shall show that $f(v) = g(v)$.

The list $(t_1, t_2, \ldots, t_N)$ contains each element of V at least once (according to the hypotheses of Lemma 6.32). In particular, this list contains v at least once (since v is an element of V). In other words, there exists an $i \in \{1, 2, \ldots, N\}$ such that $t_i = v$. Consider this i. From (150), we obtain $f(t_i) = g(t_i)$. In view of $t_i = v$, we can rewrite this as $f(v) = g(v)$.

Forget that we fixed v. We thus have shown that $f(v) = g(v)$ for each $v \in V$. In other words, $f = g$ (since both f and g are maps $V \to \mathbb{N}_+$).

Forget that we fixed f and g. We thus have shown that if f and g are two elements of $\mathcal{F}$ that satisfy $\Psi(f) = \Psi(g)$, then $f = g$. In other words, the map Ψ is injective. This proves Claim 2.]

Claim 3: The map Ψ is surjective.

[*Proof of Claim 3:* Let $\mathbf{i} \in \mathcal{I}$. We shall construct an $f \in \mathcal{F}$ satisfying $\Psi(f) = \mathbf{i}$.

Indeed, $\mathbf{i} \in \mathcal{I} = \left\{ (i_1, i_2, \ldots, i_N) \in (\mathbb{N}_+)^N \mid i_a = i_b \text{ whenever } a \sim_{\mathbf{P}(V,B,\mathbf{t})} b \right\}$ (by the definition of $\mathcal{I}$). In other words, $\mathbf{i}$ has the form $\mathbf{i} = (i_1, i_2, \ldots, i_N)$ for some N-tuple $(i_1, i_2, \ldots, i_N) \in (\mathbb{N}_+)^N$ that satisfies $i_a = i_b$ whenever $a \sim_{\mathbf{P}(V,B,\mathbf{t})} b$. Consider this N-tuple $(i_1, i_2, \ldots, i_N)$.

We have $i_a = i_b$ whenever $a \sim_{\mathbf{P}(V,B,\mathbf{t})} b$. In other words, we have

$$i_a = i_b \tag{151}$$

for any two elements $a, b \in \{1, 2, \ldots, N\}$ that satisfy $a \sim_{\mathbf{P}(V,B,\mathbf{t})} b$ (since this is what "$i_a = i_b$ whenever $a \sim_{\mathbf{P}(V,B,\mathbf{t})} b$" means).

We shall now define a map $f : V \to \mathbb{N}_+$ as follows:

Let $v \in V$. The list $(t_1, t_2, \ldots, t_N)$ contains each element of V at least once (according to the hypotheses of Lemma 6.32). In particular, this list contains v at least once (since v is an element of V). In other words, there exists a $k \in \{1, 2, \ldots, N\}$ such that $t_k = v$. Pick the smallest such k, and set $f(v) := i_k$.

Thus, we have defined a positive integer $f(v) \in \mathbb{N}_+$ for each $v \in V$. In other words, we have defined a map $f : V \to \mathbb{N}_+$. According to its definition, this map can be computed as follows: If $v \in V$ is any vertex, then

$$f(v) = i_k, \tag{152}$$

where k is the smallest element of $\{1, 2, \ldots, N\}$ such that $t_k = v$.

We shall now show that $B \subseteq \operatorname{Eqs} f$.

Indeed, let $\sim$ denote the equivalence relation $\sim_{(V,B)}$. We shall first show that

$$f(x) = f(y) \text{ for any } x \in V \text{ and } y \in V \text{ satisfying } x \sim y. \tag{153}$$

[*Proof of (153):* Let $x \in V$ and $y \in V$ be such that $x \sim y$. We must show that $f(x) = f(y)$.

We know that the list $(t_1, t_2, \ldots, t_N)$ contains each element of V at least once. In particular, this list contains x at least once (since x is an element of V). In other words, there exists an $a \in \{1, 2, \ldots, N\}$ such that $t_a = x$. Pick the smallest such a. Thus, a is the smallest element of $\{1, 2, \ldots, N\}$ such that $t_a = x$. Hence, $f(x) = i_a$ (by (152), applied to $v = x$ and $k = a$).

We know that the list $(t_1, t_2, \ldots, t_N)$ contains each element of V at least once. In particular, this list contains y at least once (since y is an element of V). In other words, there exists a $b \in \{1, 2, \ldots, N\}$ such that $t_b = y$. Pick the smallest such b. Thus, b is the smallest element of $\{1, 2, \ldots, N\}$ such that $t_b = y$. Hence, $f(y) = i_b$ (by (152), applied to $v = y$ and $k = b$).

We have $x \sim y$. In other words, $x \sim_{(V,B)} y$ (since $\sim$ is the relation $\sim_{(V,B)}$). In other words, $t_a \sim_{(V,B)} t_b$ (since $t_a = x$ and $t_b = y$). However, from (146), we know that the statements "$a \sim_{\mathbf{P}(V,B,\mathbf{t})} b$" and "$t_a \sim_{(V,B)} t_b$" are equivalent. Hence, we have $a \sim_{\mathbf{P}(V,B,\mathbf{t})} b$ (since we have $t_a \sim_{(V,B)} t_b$). Thus, from (151), we conclude that $i_a = i_b$. In view of $f(x) = i_a$ and $f(y) = i_b$, we can rewrite this as $f(x) = f(y)$. Thus, (153) is proved.]

Now, let $Y = \mathbb{N}_+$. Thus, $f : V \to Y$ is a map (since $f : V \to \mathbb{N}_+$ is a map). A set $Y^V_\sim$ is defined (according to Definition 1.7 **(b)**). Its definition yields that

$$Y^V_\sim = \left\{ g \in Y^V \mid g(x) = g(y) \text{ for any } x \in V \text{ and } y \in V \text{ satisfying } x \sim y \right\}.$$

However, f is a $g \in Y^V$ satisfying

$$g(x) = g(y) \text{ for any } x \in V \text{ and } y \in V \text{ satisfying } x \sim y$$

(since (153) shows that $f(x) = f(y)$ for any $x \in V$ and $y \in V$ satisfying $x \sim y$). In other words,

$$f \in \left\{ g \in Y^V \mid g(x) = g(y) \text{ for any } x \in V \text{ and } y \in V \text{ satisfying } x \sim y \right\} \\ = Y^V_\sim$$

(since $Y^V_\sim = \{g \in Y^V \mid g(x) = g(y) \text{ for any } x \in V \text{ and } y \in V \text{ satisfying } x \sim y\}$).

However, Lemma 2.5 yields that we have the following logical equivalence of statements:

$$(B \subseteq \operatorname{Eqs} f) \iff \left(f \in Y^V_\sim\right).$$

Hence, we have $B \subseteq \operatorname{Eqs} f$ (since we have $f \in Y^V_\sim$).

Thus, we know that f is a map $V \to \mathbb{N}_+$ and satisfies $B \subseteq \operatorname{Eqs} f$. In other words, f is a map $g : V \to \mathbb{N}_+$ satisfying $B \subseteq \operatorname{Eqs} g$. In other words,

$$f \in \{g : V \to \mathbb{N}_+ \text{ is a map} \mid B \subseteq \operatorname{Eqs} g\} = \mathcal{F}$$

(by the definition of $\mathcal{F}$).

We shall now show that $\Psi(f) = \mathbf{i}$.

Indeed, the definition of Ψ yields $\Psi(f) = (f(t_1), f(t_2), \ldots, f(t_N))$.

Now, let $j \in \{1, 2, \ldots, N\}$. We shall show that $f(t_j) = i_j$.

Indeed, there exists a $k \in \{1, 2, \ldots, N\}$ such that $t_k = t_j$ (for example, $k = j$ qualifies). Pick the smallest such k. Thus, k is the smallest element of $\{1, 2, \ldots, N\}$ such that $t_k = t_j$. Hence, (152) (applied to $v = t_j$) yields $f(t_j) = i_k$. However, the relation $\sim_{(V,B)}$ is an equivalence relation, and thus is reflexive. Hence, $t_k \sim_{(V,B)} t_k$. In other words, $t_k \sim_{(V,B)} t_j$ (since $t_k = t_j$). However, (146) (applied to $a = k$ and $b = j$) shows that we have the equivalence

$$\left(k \sim_{\mathbf{P}(V,B,\mathbf{t})} j\right) \iff \left(t_k \sim_{(V,B)} t_j\right).$$

Thus, we have $k \sim_{\mathbf{P}(V,B,\mathbf{t})} j$ (since we have $t_k \sim_{(V,B)} t_j$). Hence, (151) (applied to $a = k$ and $b = j$) yields $i_k = i_j$. Hence, $f(t_j) = i_k = i_j$.

Forget that we fixed j. We thus have shown that $f(t_j) = i_j$ for each $j \in \{1, 2, \ldots, N\}$. In other words,

$$(f(t_1), f(t_2), \ldots, f(t_N)) = (i_1, i_2, \ldots, i_N).$$

In view of $\Psi(f) = (f(t_1), f(t_2), \ldots, f(t_N))$ and $\mathbf{i} = (i_1, i_2, \ldots, i_N)$, we can rewrite this as $\Psi(f) = \mathbf{i}$. Hence,

$$\mathbf{i} = \Psi\left(\underbrace{f}_{\in \mathcal{F}}\right) \in \Psi(\mathcal{F}).$$

Forget that we fixed $\mathbf{i}$. We thus have shown that $\mathbf{i} \in \Psi(\mathcal{F})$ for each $\mathbf{i} \in \mathcal{I}$. In other words, $\mathcal{I} \subseteq \Psi(\mathcal{F})$. In other words, the map Ψ is surjective. This proves Claim 3.]

We now know that the map Ψ is injective (by Claim 2) and surjective (by Claim 3). In other words, this map Ψ is bijective, i.e., is a bijection.

However, the map Ψ is the map

$$\begin{aligned} &\mathcal{F} \to \mathcal{I}, \\ &f \mapsto (f(t_1), f(t_2), \ldots, f(t_N)) \end{aligned}$$

(by its definition). Hence, the map

$$\begin{aligned} &\mathcal{F} \to \mathcal{I}, \\ &f \mapsto (f(t_1), f(t_2), \ldots, f(t_N)) \end{aligned}$$

is a bijection (since Ψ is a bijection).

Now, recall that

$$\mathcal{F} = \{g : V \to \mathbb{N}_+ \text{ is a map } \mid B \subseteq \operatorname{Eqs} g\}.$$

Hence,

$$\sum_{f\in\mathcal{F}} = \sum_{f\in\{g:V\to\mathbb{N}_+ \text{ is a map } \mid B\subseteq \operatorname{Eqs} g\}} = \sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} \tag{154}$$

(an equality between summation signs).

Furthermore, recall that

$$\mathcal{I} = \left\{(i_1, i_2, \ldots, i_N) \in (\mathbb{N}_+)^N \mid i_a = i_b \text{ whenever } a \sim_{\mathbf{P}(V,B,\mathbf{t})} b\right\}.$$

Hence,

$$\sum_{(i_1,i_2,\ldots,i_N)\in\mathcal{I}} = \sum_{\substack{(i_1,i_2,\ldots,i_N)\in(\mathbb{N}_+)^N;\\ i_a=i_b \text{ whenever } a\sim_{\mathbf{P}(V,B,\mathbf{t})} b}}$$

(an equality between summation signs). In light of this, we can rewrite (145) as follows:

$$\begin{aligned} P_{\mathbf{P}(V,B,\mathbf{t})} &= \sum_{(i_1,i_2,\ldots,i_N)\in\mathcal{I}} X_{i_1}X_{i_2}\cdots X_{i_N} \\ &= \underbrace{\sum_{f\in\mathcal{F}}}_{\substack{=\sum\limits_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}}\\ \text{(by (154))}}} X_{f(t_1)}X_{f(t_2)}\cdots X_{f(t_N)} \\ &\qquad \left(\begin{array}{c} \text{here, we have substituted } (f(t_1), f(t_2), \ldots, f(t_N)) \\ \text{for } (i_1, i_2, \ldots, i_N) \text{ in the sum, since the} \\ \text{map } \mathcal{F}\to\mathcal{I},\ f\mapsto (f(t_1), f(t_2), \ldots, f(t_N)) \\ \text{is a bijection} \end{array}\right) \\ &= \sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} \underbrace{X_{f(t_1)}X_{f(t_2)}\cdots X_{f(t_N)}}_{\substack{=\mathbf{X}_{f,\mathbf{t}}\\ \text{(since } \mathbf{X}_{f,\mathbf{t}} \text{ was defined to be } X_{f(t_1)}X_{f(t_2)}\cdots X_{f(t_N)}\\ \text{in Definition 6.17 } \textbf{(a)}\text{)}}} \\ &= \sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} \mathbf{X}_{f,\mathbf{t}}. \end{aligned}$$

This proves Lemma 6.32. □

We can now prove Theorems 6.29 and 6.28 and Corollaries 6.30 and 6.31 by making straightforward changes to the above proofs of Theorems 6.14 and 6.13 and Corollaries 6.15 and 6.16 (replacing, in particular, the use of Lemma 6.11 by a use of Lemma 6.32). Here are the details:

Proof of Theorem 6.29. We have

$$Y_{G,\mathbf{t}} = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_{f,\mathbf{t}} \tag{155}$$

(by the definition of $Y_{G,\mathbf{t}}$). Now, if $f : V \to \mathbb{N}_+$ is a map, then we have the following logical equivalence:

$$(\text{the } \mathbb{N}_+\text{-coloring } f \text{ of } G \text{ is proper}) \iff (\operatorname{EQS}(G,f) = \varnothing) \tag{156}$$

(because the $\mathbb{N}_+$-coloring f of G is proper if and only if $\operatorname{EQS}(G,f) = \varnothing$ [90]). Now,

$$\begin{aligned}
&\sum_{f:V\to\mathbb{N}_+} \left[\underbrace{\operatorname{EQS}(G,f) = \varnothing}_{\substack{\iff (\text{the } \mathbb{N}_+\text{-coloring } f \text{ of } G \text{ is proper}) \\ (\text{by (156)})}} \right] \mathbf{x}_{f,\mathbf{t}} \\
&= \sum_{f:V\to\mathbb{N}_+} \left[\underbrace{\text{the } \mathbb{N}_+\text{-coloring } f \text{ of } G \text{ is proper}}_{\iff (f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G)} \right] \mathbf{x}_{f,\mathbf{t}} \\
&= \sum_{f:V\to\mathbb{N}_+} [f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G]\, \mathbf{x}_{f,\mathbf{t}} \\
&= \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \underbrace{[f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G]}_{\substack{=1 \\ (\text{since } f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G)}} \mathbf{x}_{f,\mathbf{t}} \\
&\qquad + \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is not a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \underbrace{[f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G]}_{\substack{=0 \\ (\text{since } f \text{ is not a proper } \mathbb{N}_+\text{-coloring of } G)}} \mathbf{x}_{f,\mathbf{t}} \\
&\qquad\qquad (\text{since each } f : V \to \mathbb{N}_+ \text{ either is a proper } \mathbb{N}_+\text{-coloring of } G \text{ or not}) \\
&= \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_{f,\mathbf{t}} + \underbrace{\sum_{\substack{f:V\to\mathbb{N}_+ \text{ is not a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} 0\mathbf{x}_{f,\mathbf{t}}}_{=0} = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_{f,\mathbf{t}} \\
&= Y_{G,\mathbf{t}}
\end{aligned} \tag{157}$$

(by (155)).

---

[90] by Lemma 5.25 (applied to $\mathbb{N}_+$ instead of X)

However, for every $f : V \to \mathbb{N}_+$, we have

$$\sum_{\substack{B \subseteq E; \\ \operatorname{union} B \subseteq \operatorname{Eqs} f}} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K = [\operatorname{EQS}(G, f) = \varnothing] \tag{158}$$

(indeed, we have already shown this in the above proof of Theorem 5.18).

Now, (157) yields

$$\begin{aligned} Y_{G,\mathbf{t}} &= \sum_{f : V \to \mathbb{N}_+} \underbrace{[\operatorname{EQS}(G, f) = \varnothing]}_{\substack{= \sum\limits_{\substack{B \subseteq E; \\ \operatorname{union} B \subseteq \operatorname{Eqs} f}} (-1)^{|B|} \prod\limits_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K \\ \text{(by (158))}}} \mathbf{X}_{f,\mathbf{t}} \\ &= \sum_{f : V \to \mathbb{N}_+} \left(\sum_{\substack{B \subseteq E; \\ \operatorname{union} B \subseteq \operatorname{Eqs} f}} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K \right) \mathbf{X}_{f,\mathbf{t}} \\ &= \underbrace{\sum_{f : V \to \mathbb{N}_+} \sum_{\substack{B \subseteq E; \\ \operatorname{union} B \subseteq \operatorname{Eqs} f}}}_{= \sum\limits_{B \subseteq E} \sum\limits_{\substack{f : V \to \mathbb{N}_+; \\ \operatorname{union} B \subseteq \operatorname{Eqs} f}}} (-1)^{|B|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K \right) \mathbf{X}_{f,\mathbf{t}} \\ &= \sum_{B \subseteq E} \sum_{\substack{f : V \to \mathbb{N}_+; \\ \operatorname{union} B \subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K \right) \mathbf{X}_{f,\mathbf{t}} \\ &= \sum_{B \subseteq E} (-1)^{|B|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K \right) \sum_{\substack{f : V \to \mathbb{N}_+; \\ \operatorname{union} B \subseteq \operatorname{Eqs} f}} \mathbf{X}_{f,\mathbf{t}}. \end{aligned} \tag{159}$$

However, if B is a subset of E, then the pair $(V, \operatorname{union} B)$ is a finite graph (since V is a finite set and since $\operatorname{union} B \subseteq \dbinom{V}{2}$), and thus we have

$$\sum_{\substack{f : V \to \mathbb{N}_+; \\ \operatorname{union} B \subseteq \operatorname{Eqs} f}} \mathbf{X}_{f,\mathbf{t}} = P_{\lambda(V, \operatorname{union} B, \mathbf{t})} \tag{160}$$

(by Lemma 6.11, applied to $\operatorname{union} B$ instead of B).

Hence, (159) becomes

$$\begin{aligned} Y_{G,\mathbf{t}} &= \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \underbrace{\sum_{\substack{f:V\to\mathbb{N}_+;\\ \operatorname{union} B\subseteq \operatorname{Eqs} f}} \mathbf{x}_{f,\mathbf{t}}}_{\substack{=P_{\lambda(V,\operatorname{union} B,\mathbf{t})}\\ \text{(by (160))}}} \\ &= \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) P_{\lambda(V,\operatorname{union} B,\mathbf{t})} \\ &= \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} a_K \right) P_{\lambda(V,\operatorname{union} F,\mathbf{t})} \end{aligned}$$

(here, we have renamed the summation index B as F). This proves Theorem 6.29. ☐

Proof of Corollary 6.30. We can apply Theorem 6.29 to 0 instead of a_K. As a result, we obtain

$$Y_{G,\mathbf{t}} = \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0 \right) P_{\lambda(V,\operatorname{union} F,\mathbf{t})}. \tag{161}$$

Now, if F is any subset of E, then

$$\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0 = \begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} \tag{162}$$

(indeed, this was already shown in our above proof of Corollary 5.19).

Thus, (161) becomes

$$\begin{aligned}
Y_{G,\mathbf{t}} &= \sum_{F\subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0\right)}_{\substack{=\begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;}\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases}\\ \text{(by (162))}}} P_{\lambda(V,\operatorname{union} F,\mathbf{t})} \\
&= \sum_{F\subseteq E} (-1)^{|F|} \begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;}\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases} P_{\lambda(V,\operatorname{union} F,\mathbf{t})} \\
&= \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} \underbrace{\begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;}\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases}}_{\substack{=1\\ \text{(since } F \text{ is } \mathfrak{K}\text{-free)}}} P_{\lambda(V,\operatorname{union} F,\mathbf{t})} \\
&\qquad + \sum_{\substack{F\subseteq E;\\ F \text{ is not } \mathfrak{K}\text{-free}}} (-1)^{|F|} \underbrace{\begin{cases}1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;}\\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free}\end{cases}}_{\substack{=0\\ \text{(since } F \text{ is not } \mathfrak{K}\text{-free)}}} P_{\lambda(V,\operatorname{union} F,\mathbf{t})} \\
&\qquad\qquad \left(\text{since each subset } F \text{ of } E \text{ either is } \mathfrak{K}\text{-free or is not}\right) \\
&= \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} P_{\lambda(V,\operatorname{union} F,\mathbf{t})} + \underbrace{\sum_{\substack{F\subseteq E;\\ F \text{ is not } \mathfrak{K}\text{-free}}} (-1)^{|F|} 0 P_{\lambda(V,\operatorname{union} F,\mathbf{t})}}_{=0} \\
&= \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} P_{\lambda(V,\operatorname{union} F,\mathbf{t})}.
\end{aligned}$$

This proves Corollary 6.30. □

Proof of Corollary 6.31. Let $\mathfrak{K}$ be the set of all broken circuits of G.

Now, just as in the proof of Corollary 1.18, we can prove the following equality:

$$\sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} = \sum_{\substack{F\subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}}$$

(an equality between summation signs). Now, Corollary 6.30 yields

$$\begin{aligned} Y_{G,\mathbf{t}} &= \sum_{\substack{F\subseteq E;\\ \underbrace{F \text{ is } \mathfrak{K}\text{-free}}_{\substack{= \sum\limits_{\substack{F\subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}}}}}} (-1)^{|F|} P_{\lambda(V,\operatorname{union} F,\mathbf{t})} \\ &= \sum_{\substack{F\subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} P_{\lambda(V,\operatorname{union} F,\mathbf{t})}. \end{aligned}$$

This proves Corollary 6.31. ☐

Proof of Theorem 6.28. Let X be the totally ordered set $\{1\}$ (equipped with the only possible order on this set). Let $\ell : E \to X$ be the function sending each $e \in E$ to $1 \in X$. Let $\mathfrak{K}$ be the empty set. Clearly, $\mathfrak{K}$ is a set of broken circuits of G. Theorem 6.29 (applied to 0 instead of a_K) yields

$$\begin{aligned} Y_{G,\mathbf{t}} &= \sum_{F\subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0 \right)}_{\substack{=(\text{empty product})\\ (\text{since } \mathfrak{K} \text{ is the empty set})}} P_{\lambda(V,\operatorname{union} F,\mathbf{t})} \\ &= \sum_{F\subseteq E} (-1)^{|F|} \underbrace{(\text{empty product})}_{=1} P_{\lambda(V,\operatorname{union} F,\mathbf{t})} = \sum_{F\subseteq E} (-1)^{|F|} P_{\lambda(V,\operatorname{union} F,\mathbf{t})}. \end{aligned}$$

This proves Theorem 6.28. ☐

6.7. An abstract setting

The reader will by now have realized that we have been making the same arguments in a series of slightly different settings. In particular, the chromatic symmetric function X_G, its weighted version $X_{G,w}$ and its noncommutative version $Y_{G,\mathbf{t}}$ are all defined as sums over proper $\mathbb{N}_+$-colorings of G; they differ only in the addends being summed. We can generalize them all by allowing these addends to be arbitrary, i.e., replacing them by arbitrary elements α_f of a $\mathbb{Z}$-module M, provided that the resulting (potentially infinite) sums are still well-defined. While at that, we can also replace $\mathbb{N}_+$-colorings by Y-colorings for an arbitrary set Y. Thus, we are led to the following general setting:

Definition 6.33. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let Y be any set.

Let M be a topological $\mathbb{Z}$-module. Let $\alpha_f \in M$ be an element for each Y-coloring $f : V \to Y$. Assume that the family $(\alpha_f)_{f:V\to Y}$ of these elements is summable (so that the sum $\sum\limits_{f:V\to Y} \alpha_f$ and any of its subsums is well-defined).

Then:
(a) We define an element

$$\Xi_G := \sum_{\substack{f:V\to Y \text{ is a} \\ \text{proper } Y\text{-coloring of } G}} \alpha_f \in M.$$

(b) Furthermore, if B is a subset of $\binom{V}{2}$, then we set

$$\pi_B := \sum_{\substack{f:V\to Y; \\ B\subseteq \operatorname{Eqs} f}} \alpha_f \in M.$$

Through appropriate choices of α_f, we recover the previously defined power series X_G, $X_{G,w}$ and $Y_{G,\mathbf{t}}$:

- If $Y = \mathbb{N}_+$ and $\alpha_f = \mathbf{x}_f$, then $\Xi_G = X_G$ and $\pi_B = p_{\lambda(V, \operatorname{union} B)}$.
- If $Y = \mathbb{N}_+$ and $\alpha_f = \mathbf{x}_{f,w}$ (for a given weight function $w : V \to \mathbb{N}_+$), then $\Xi_G = X_{G,w}$ and $\pi_B = p_{\lambda((V, \operatorname{union} B), w)}$.
- If $Y = \mathbb{N}_+$ and $\alpha_f = \mathbf{X}_{f,\mathbf{t}}$ (for a given list $\mathbf{t}$ of elements of V that contains each element at least once), then $\Xi_G = Y_{G,\mathbf{t}}$ and $\pi_B = P_{\mathbf{P}(V, \operatorname{union} B, \mathbf{t})}$.

We can now state analogues of Theorems 6.28 and 6.29 and Corollaries 6.30 and 6.31 in this general context:

Theorem 6.34. Let G, V, E, φ, Y, M and α_f be as in Definition 6.33. Then, using the notations of Definition 6.33, we have

$$\Xi_G = \sum_{F\subseteq E} (-1)^{|F|} \pi_{\operatorname{union} F}.$$

Theorem 6.35. Let G, V, E, φ, Y, M and α_f be as in Definition 6.33. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then, using the notations of Definition 6.33, we have

$$\Xi_G = \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K}; \\ K\subseteq F}} a_K \right) \pi_{\operatorname{union} F}.$$

Corollary 6.36. Let G, V, E, φ, Y, M and α_f be as in Definition 6.33. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then, using the notations of Definition 6.33, we have

$$\Xi_G = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} \pi_{\operatorname{union} F}.$$

Corollary 6.37. Let G, V, E, φ, Y, M and α_f be as in Definition 6.33. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Then, using the notations of Definition 6.33, we have

$$\Xi_G = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} \pi_{\operatorname{union} F}.$$

These four results can be proved through arguments very similar to the ones we used in earlier proofs (e.g., in the proofs of Theorems 6.14 and 6.13 and Corollaries 6.15 and 6.16). In Here are the details:

Proof of Theorem 6.35. We have

$$\Xi_G = \sum_{\substack{f : V \to Y \text{ is a} \\ \text{proper } Y\text{-coloring of } G}} \alpha_f \tag{163}$$

(by the definition of Ξ_G). Now, if $f : V \to Y$ is a map, then we have the following logical equivalence:

$$(\text{the } Y\text{-coloring } f \text{ of } G \text{ is proper}) \iff (\operatorname{EQS}(G, f) = \varnothing) \tag{164}$$

(because the Y-coloring f of G is proper if and only if $\operatorname{EQS}(G, f) = \varnothing$ [91]).

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[91] by Lemma 5.25 (applied to Y instead of X)

Now,

$$\begin{aligned}
&\sum_{f:V\to Y}\left[\underbrace{\operatorname{EQS}(G,f)=\varnothing}_{\substack{\iff (\text{the } Y\text{-coloring } f \text{ of } G \text{ is proper})\\ (\text{by (164)})}}\right]\alpha_f \\
&=\sum_{f:V\to Y}\left[\underbrace{\text{the } Y\text{-coloring } f \text{ of } G \text{ is proper}}_{\iff (f \text{ is a proper } Y\text{-coloring of } G)}\right]\alpha_f \\
&=\sum_{f:V\to Y}\left[f \text{ is a proper } Y\text{-coloring of } G\right]\alpha_f \\
&=\sum_{\substack{f:V\to Y \text{ is a}\\ \text{proper } Y\text{-coloring of } G}}\underbrace{\left[f \text{ is a proper } Y\text{-coloring of } G\right]}_{\substack{=1\\ (\text{since } f \text{ is a proper } Y\text{-coloring of } G)}}\alpha_f \\
&\qquad+\sum_{\substack{f:V\to Y \text{ is not a}\\ \text{proper } Y\text{-coloring of } G}}\underbrace{\left[f \text{ is a proper } Y\text{-coloring of } G\right]}_{\substack{=0\\ (\text{since } f \text{ is not a proper } Y\text{-coloring of } G)}}\alpha_f \\
&\qquad\qquad(\text{since each } f:V\to Y \text{ either is a proper } Y\text{-coloring of } G \text{ or not}) \\
&=\sum_{\substack{f:V\to Y \text{ is a}\\ \text{proper } Y\text{-coloring of } G}}\alpha_f+\underbrace{\sum_{\substack{f:V\to Y \text{ is not a}\\ \text{proper } Y\text{-coloring of } G}}0\alpha_f}_{=0}=\sum_{\substack{f:V\to Y \text{ is a}\\ \text{proper } Y\text{-coloring of } G}}\alpha_f \\
&=\Xi_G \qquad (165)
\end{aligned}$$

(by (163)).

However, for every $f : V \to Y$, we have

$$\sum_{\substack{B\subseteq E;\\ \text{union } B\subseteq \operatorname{Eqs} f}}(-1)^{|B|}\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K=\left[\operatorname{EQS}(G,f)=\varnothing\right] \qquad (166)$$

(indeed, we have already shown this in the above proof of Theorem 5.18).

Now, (165) yields

$$\begin{aligned}
\Xi_G &= \sum_{f:V\to Y} \underbrace{[\operatorname{EQS}(G,f)=\varnothing]}_{\substack{=\sum\limits_{\substack{B\subseteq E;\\ \operatorname{union} B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \prod\limits_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \\ \text{(by (166))}}} \alpha_f \\
&= \sum_{f:V\to Y} \left(\sum_{\substack{B\subseteq E;\\ \operatorname{union} B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \alpha_f \\
&= \underbrace{\sum_{f:V\to Y} \sum_{\substack{B\subseteq E;\\ \operatorname{union} B\subseteq \operatorname{Eqs} f}}}_{=\sum\limits_{B\subseteq E} \sum\limits_{\substack{f:V\to Y;\\ \operatorname{union} B\subseteq \operatorname{Eqs} f}}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \alpha_f \\
&= \sum_{B\subseteq E} \sum_{\substack{f:V\to Y;\\ \operatorname{union} B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \alpha_f \\
&= \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \underbrace{\sum_{\substack{f:V\to Y;\\ \operatorname{union} B\subseteq \operatorname{Eqs} f}} \alpha_f}_{\substack{=\pi_{\operatorname{union} B} \\ \text{(since } \pi_{\operatorname{union} B} \text{ was defined} \\ \text{to be } \sum\limits_{\substack{f:V\to Y;\\ \operatorname{union} B\subseteq \operatorname{Eqs} f}} \alpha_f\text{)}}} \\
&= \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \pi_{\operatorname{union} B} = \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} a_K \right) \pi_{\operatorname{union} F}
\end{aligned}$$

(here, we have renamed the summation index B as F). This proves Theorem 6.35. ☐

Proof of Corollary 6.36. We can apply Theorem 6.35 to 0 instead of a_K. As a result, we obtain

$$\Xi_G = \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0 \right) \pi_{\operatorname{union} F}. \tag{167}$$

Now, if F is any subset of E, then

$$\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0 = \begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} \tag{168}$$

(indeed, this was already shown in our above proof of Corollary 5.19).

Thus, (167) becomes

$$\begin{aligned}
\Xi_G &= \sum_{F\subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0 \right)}_{\substack{= \begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} \\ \text{(by (162))}}} \pi_{\operatorname{union} F} \\
&= \sum_{F\subseteq E} (-1)^{|F|} \begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} \pi_{\operatorname{union} F} \\
&= \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} \underbrace{\begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases}}_{\substack{=1 \\ \text{(since } F \text{ is } \mathfrak{K}\text{-free)}}} \pi_{\operatorname{union} F} \\
&\qquad + \sum_{\substack{F\subseteq E;\\ F \text{ is not } \mathfrak{K}\text{-free}}} (-1)^{|F|} \underbrace{\begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases}}_{\substack{=0 \\ \text{(since } F \text{ is not } \mathfrak{K}\text{-free)}}} \pi_{\operatorname{union} F} \\
&\qquad\qquad \left(\text{since each subset } F \text{ of } E \text{ either is } \mathfrak{K}\text{-free or is not}\right) \\
&= \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} \pi_{\operatorname{union} F} + \underbrace{\sum_{\substack{F\subseteq E;\\ F \text{ is not } \mathfrak{K}\text{-free}}} (-1)^{|F|} 0 \pi_{\operatorname{union} F}}_{=0} \\
&= \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} \pi_{\operatorname{union} F}.
\end{aligned}$$

This proves Corollary 6.36. ☐

Proof of Corollary 6.37. Let $\mathfrak{K}$ be the set of all broken circuits of G.

Now, just as in the proof of Corollary 1.18, we can prove the following equality:

$$\sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} = \sum_{\substack{F\subseteq E;\\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}}$$

(an equality between summation signs). Now, Corollary 6.36 yields

$$\Xi_G = \underbrace{\sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}}}_{=\sum_{\substack{F\subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}}} (-1)^{|F|} \pi_{\operatorname{union} F} = \sum_{\substack{F\subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} \pi_{\operatorname{union} F}.$$

This proves Corollary 6.37. □

Proof of Theorem 6.34. Let X be the totally ordered set $\{1\}$ (equipped with the only possible order on this set). Let $\ell : E \to X$ be the function sending each $e \in E$ to $1 \in X$. Let $\mathfrak{K}$ be the empty set. Clearly, $\mathfrak{K}$ is a set of broken circuits of G. Theorem 6.35 (applied to 0 instead of a_K) yields

$$\Xi_G = \sum_{F\subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0\right)}_{\substack{=(\text{empty product})\\ (\text{since } \mathfrak{K} \text{ is the empty set})}} \pi_{\operatorname{union} F}$$
$$= \sum_{F\subseteq E} (-1)^{|F|} \underbrace{(\text{empty product})}_{=1} \pi_{\operatorname{union} F} = \sum_{F\subseteq E} (-1)^{|F|} \pi_{\operatorname{union} F}.$$

This proves Theorem 6.34. □

Corollary 6.37 can be used to prove certain results about list colorings (i.e., colorings of a graph or ambigraph that are not allowed to use certain colors for certain vertices); in particular, [Erey19, Lemma 3.2] follows easily from Corollary 6.37 (just turn the graph G into an ambigraph, and set α_f to be the Iverson bracket $[f(v) \neq r(v) \text{ for each } v \in V]$).

7. Application: A vanishing alternating sum

Chromatic symmetric functions of different graphs are far from being linearly independent; they satisfy several linear relations. One such relation was observed by Dahlberg and van Willigenburg in 2018 [DahWil18, Proposition 5]:

Theorem 7.1. Let $G = (V, E)$ be a graph. Let C be a circuit of G, and let $e \in C$ be arbitrary. Then,

$$\sum_{F\subseteq C\setminus\{e\}} (-1)^{|F|} X_{G\setminus F} = 0.$$

Here, whenever F is a subset of E, the notation $G \setminus F$ denotes the graph $(V, E \setminus F)$ (that is, the graph obtained from G by removing the edgeries in F).

This was extended to noncommutative chromatic symmetric functions $Y_{G,\mathbf{t}}$ by Dahlberg and van Willigenburg in [DahWil19, Proposition 3.6], and to weighted chromatic symmetric functions $X_{G,w}$ by Crew and Spirkl in [CreSpi19, Theorem 6]. Again, we shall now one-up these results by generalizing them from graphs to ambigraphs and by moving to the abstract setting of Definition 6.33. Thus, we claim the following:

Theorem 7.2. Let G, V, E, φ, Y, M and α_f be as in Definition 6.33. Let C be a circuit of G, and let $e \in C$ be a singleton edgery. Then, using the notations of Definition 6.33 **(a)**, we have

$$\sum_{F \subseteq C \setminus \{e\}} (-1)^{|F|} \Xi_{G \setminus F} = 0.$$

Here, whenever F is a subset of E, the notation $G \setminus F$ denotes the ambigraph $\left(V,\ E \setminus F,\ \varphi \mid_{E \setminus F}\right)$ (that is, the ambigraph obtained from G by removing the edges in F).

Applying this to a graph instead of an ambigraph, and setting $Y = \mathbb{N}_+$ and $\alpha_f = \mathbf{x}_f$, we recover Theorem 7.1.

Theorem 7.2 is quite easy to prove despite its generality; in fact, the beautiful sign-reversing involution argument from [DahWil19, proof of Proposition 3.6] still does the trick. However, by way of illustration, we shall now demonstrate how Theorem 7.2 can be derived from Theorem 6.34 and Corollary 6.36.

Proof of Theorem 7.2. Let us set $B := C \setminus \{e\}$. Thus, $B = C \setminus \{e\} \subseteq C \subseteq E$.

Now, we shall show the following (using the notations of Definition 6.33 **(b)**):

Claim 1: Let J be a subset of B. Then,

$$\Xi_{G \setminus J} = \sum_{\substack{F \subseteq E; \\ J \subseteq E \setminus F}} (-1)^{|F|} \pi_{\operatorname{union} F}.$$

[*Proof of Claim 1:* The definition of the ambigraph $G \setminus J$ shows that $G \setminus J = \left(V,\ E \setminus J,\ \varphi \mid_{E \setminus J}\right)$. Thus, we can apply Theorem 6.34 to $G \setminus J$, $E \setminus J$ and $\varphi \mid_{E \setminus J}$ instead of G, E and φ. As a result, we obtain

$$\Xi_{G \setminus J} = \sum_{F \subseteq E \setminus J} (-1)^{|F|} \pi_{\operatorname{union} F} \tag{169}$$

[92]. However, we have $J \subseteq B \subseteq E$. Thus, it is easy to see that

$$\mathcal{P}(E \setminus J) = \{Z \in \mathcal{P}(E) \mid J \subseteq E \setminus Z\} \tag{170}$$

---

[92] We need to be careful here: Theorem 6.34 uses the notation union F, which implicitly depends on the map φ that is part of the ambigraph $G = (V, E, \varphi)$. When we apply Theorem 6.34 to

[93].

Now, we have the following equality between summation signs:

$$\sum_{F\subseteq E\setminus J} = \sum_{F\in\mathcal{P}(E\setminus J)} = \sum_{F\in\{Z\in\mathcal{P}(E)\ \mid\ J\subseteq E\setminus Z\}} \qquad \text{(by (170))}$$
$$= \sum_{\substack{F\in\mathcal{P}(E);\\ J\subseteq E\setminus F}} = \sum_{\substack{F\subseteq E;\\ J\subseteq E\setminus F}} .$$

$G\setminus J$, $E\setminus J$ and $\varphi\mid_{E\setminus J}$ instead of G, E and φ, we are thus using this notation $\operatorname{union} F$ in a slightly different sense than it is used in Claim 1. Namely, in Claim 1, the notation $\operatorname{union} F$ means $\bigcup_{e\in F}\varphi(e)$ (since it is defined with respect to the ambigraph $G=(V,E,\varphi)$), whereas in (169) it means $\bigcup_{e\in F}\left(\varphi\mid_{E\setminus J}\right)(e)$ (since it is defined with respect to the ambigraph $G\setminus J=\left(V,\ E\setminus J,\ \varphi\mid_{E\setminus J}\right)$). Fortunately, however, these two meanings of $\operatorname{union} F$ are equivalent, since any subset F of $E\setminus J$ satisfies

$$\bigcup_{e\in F}\underbrace{\left(\varphi\mid_{E\setminus J}\right)(e)}_{=\varphi(e)} = \bigcup_{e\in F}\varphi(e).$$

Thus, we have no need to distinguish between these two meanings of $\operatorname{union} F$.

[93] *Proof:* Let $H\in\mathcal{P}(E\setminus J)$. Thus, H is a subset of $E\setminus J$. Hence, $H\subseteq E\setminus J\subseteq E$, so that $H\in\mathcal{P}(E)$. Also, from $H\subseteq E\setminus J$, we conclude that H is disjoint from J. In other words, J is disjoint from H. Combining this with $J\subseteq E$, we obtain $J\subseteq E\setminus H$.

Now, we know that $H\in\mathcal{P}(E)$ and $J\subseteq E\setminus H$. In other words, H is a $Z\in\mathcal{P}(E)$ satisfying $J\subseteq E\setminus Z$. In other words, $H\in\{Z\in\mathcal{P}(E)\ \mid\ J\subseteq E\setminus Z\}$.

Forget that we fixed H. We thus have shown that $H\in\{Z\in\mathcal{P}(E)\ \mid\ J\subseteq E\setminus Z\}$ for each $H\in\mathcal{P}(E\setminus J)$. In other words,

$$\mathcal{P}(E\setminus J)\subseteq\{Z\in\mathcal{P}(E)\ \mid\ J\subseteq E\setminus Z\}. \qquad (171)$$

On the other hand, let $U\in\{Z\in\mathcal{P}(E)\ \mid\ J\subseteq E\setminus Z\}$. Thus, U is a $Z\in\mathcal{P}(E)$ satisfying $J\subseteq E\setminus Z$. In other words, $U\in\mathcal{P}(E)$ and $J\subseteq E\setminus U$.

From $J\subseteq E\setminus U$, we conclude that J is disjoint from U. In other words, U is disjoint from J. However, from $U\in\mathcal{P}(E)$, we conclude that U is a subset of E. Thus, U is a subset of E that is disjoint from J. In other words, U is a subset of $E\setminus J$. In other words, $U\in\mathcal{P}(E\setminus J)$.

Forget that we fixed U. We thus have shown that $U\in\mathcal{P}(E\setminus J)$ for each $U\in\{Z\in\mathcal{P}(E)\ \mid\ J\subseteq E\setminus Z\}$. In other words,

$$\{Z\in\mathcal{P}(E)\ \mid\ J\subseteq E\setminus Z\}\subseteq\mathcal{P}(E\setminus J).$$

Combining this with (171), we obtain

$$\mathcal{P}(E\setminus J)=\{Z\in\mathcal{P}(E)\ \mid\ J\subseteq E\setminus Z\}.$$

This proves (170).

Hence, (169) becomes

$$\Xi_{G\setminus J} = \underbrace{\sum_{F\subseteq E\setminus J}}_{=\sum_{\substack{F\subseteq E;\\ J\subseteq E\setminus F}}} (-1)^{|F|} \pi_{\operatorname{union} F} = \sum_{\substack{F\subseteq E;\\ J\subseteq E\setminus F}} (-1)^{|F|} \pi_{\operatorname{union} F}.$$

Thus, Claim 1 is proved.]

Claim 2: We have

$$\sum_{\substack{F\subseteq E;\\ B\subseteq F}} (-1)^{|F|} \pi_{\operatorname{union} F} = 0. \qquad (172)$$

[*Proof of Claim 2:* Theorem 6.34 yields

$$\begin{aligned}\Xi_G &= \sum_{F\subseteq E} (-1)^{|F|} \pi_{\operatorname{union} F}\\ &= \sum_{\substack{F\subseteq E;\\ B\subseteq F}} (-1)^{|F|} \pi_{\operatorname{union} F} + \sum_{\substack{F\subseteq E;\\ B\not\subseteq F}} (-1)^{|F|} \pi_{\operatorname{union} F} \qquad (173)\end{aligned}$$

(since each subset F of E satisfies either $B \subseteq F$ or $B \not\subseteq F$, but not both at the same time).

Let us now find a different formula for Ξ_G. We define a function $\ell : E \to \mathbb{N}$ by setting

$$\ell(f) = [f = e] \qquad \text{for each } f \in E.$$

We shall use this function ℓ as our labeling function (where the role of the totally ordered set X is played by $\mathbb{N}$ equipped with the usual total order). It is easy to see that the edgery e is the unique singleton edgery in C having maximum label[94]. Therefore, $C \setminus \{e\}$ is a broken circuit of G (by the definition of a "broken circuit" in Definition 5.16). In other words, B is a broken circuit of G (since $B = C \setminus \{e\}$). Hence, $\{B\}$ is a set of broken circuits of G. Therefore, Corollary 6.36 (applied to $\mathfrak{K} = \{B\}$) yields

$$\Xi_G = \sum_{\substack{F\subseteq E;\\ F \text{ is } \{B\}\text{-free}}} (-1)^{|F|} \pi_{\operatorname{union} F}. \qquad (174)$$

---

[94] *Proof.* Clearly, e is a singleton edgery in C (since $e \in C$ is a singleton edgery). We shall now show that e has a larger label than any other singleton edgery in C.

Indeed, let f be any singleton edgery in C distinct from e. Then, the definition of ℓ yields $\ell(f) = [f = e] = 0$ (since we don't have $f = e$ (because f is distinct from e)). On the other hand, the definition of ℓ yields $\ell(e) = [e = e] = 1$ (since $e = e$). Hence, $\ell(e) = 1 > 0 = \ell(f)$. In other words, e has a larger label than f (since the label of e is $\ell(e)$, whereas the label of f is $\ell(f)$).

Forget that we fixed f. We thus have shown that e has a larger label than f whenever f is any singleton edgery in C distinct from e. In other words, e has a larger label than any other singleton edgery in C. Hence, the edgery e is the unique singleton edgery in C having maximum label (since e itself is a singleton edgery in C).

However, if F is a subset of E, then the condition "F is $\{B\}$-free" is equivalent to "$B \not\subseteq F$" [95]. Hence, the summation sign "$\sum\limits_{\substack{F\subseteq E;\\ F \text{ is } \{B\}\text{-free}}}$" can be rewritten as "$\sum\limits_{\substack{F\subseteq E;\\ B\not\subseteq F}}$". Therefore, we can rewrite (174) as

$$\Xi_G = \sum_{\substack{F\subseteq E;\\ B\not\subseteq F}} (-1)^{|F|} \pi_{\operatorname{union} F}. \tag{175}$$

Subtracting this equality from (173), we obtain

$$\begin{aligned}\Xi_G - \Xi_G &= \left(\sum_{\substack{F\subseteq E;\\ B\subseteq F}} (-1)^{|F|} \pi_{\operatorname{union} F} + \sum_{\substack{F\subseteq E;\\ B\not\subseteq F}} (-1)^{|F|} \pi_{\operatorname{union} F} \right) - \sum_{\substack{F\subseteq E;\\ B\not\subseteq F}} (-1)^{|F|} \pi_{\operatorname{union} F} \\ &= \sum_{\substack{F\subseteq E;\\ B\subseteq F}} (-1)^{|F|} \pi_{\operatorname{union} F}.\end{aligned}$$

Comparing this with $\Xi_G - \Xi_G = 0$, we obtain

$$\sum_{\substack{F\subseteq E;\\ B\subseteq F}} (-1)^{|F|} \pi_{\operatorname{union} F} = 0.$$

This proves Claim 2.]

---

[95] *Proof.* Let F be a subset of E. Then, we have the following chain of logical equivalences:

$$\begin{aligned} & (F \text{ is } \{B\}\text{-free}) \\ \iff & (F \text{ contains no } K \in \{B\} \text{ as a subset}) \\ & \qquad (\text{by the definition of "}\{B\}\text{-free" in Definition 1.16}) \\ \iff & (\text{there exists no } K \in \{B\} \text{ such that } F \text{ contains } K \text{ as a subset}) \\ \iff & (\text{there exists no } K \in \{B\} \text{ such that } K \subseteq F) \\ \iff & (\text{each } K \in \{B\} \text{ satisfies } K \not\subseteq F) \\ \iff & (B \not\subseteq F) \qquad (\text{since the only } K \in \{B\} \text{ is } B). \end{aligned}$$

Hence, the condition "F is $\{B\}$-free" is equivalent to "$B \not\subseteq F$". Qed.

However, from $C \setminus \{e\} = B$, we obtain

$$\sum_{F\subseteq C\setminus\{e\}} (-1)^{|F|}\,\Xi_{G\setminus F} = \sum_{F\subseteq B} (-1)^{|F|}\,\Xi_{G\setminus F} = \sum_{J\subseteq B} (-1)^{|J|} \underbrace{\Xi_{G\setminus J}}_{\substack{=\sum\limits_{\substack{F\subseteq E;\\ J\subseteq E\setminus F}} (-1)^{|F|}\pi_{\operatorname{union} F} \\ \text{(by Claim 1)}}}$$

$$\left(\begin{array}{c}\text{here, we have renamed the}\\ \text{summation index } F \text{ as } J\end{array}\right)$$

$$= \sum_{J\subseteq B} (-1)^{|J|} \sum_{\substack{F\subseteq E;\\ J\subseteq E\setminus F}} (-1)^{|F|}\,\pi_{\operatorname{union} F}$$

$$= \underbrace{\sum_{J\subseteq B}\ \sum_{\substack{F\subseteq E;\\ J\subseteq E\setminus F}}}_{=\sum\limits_{F\subseteq E}\ \sum\limits_{\substack{J\subseteq B;\\ J\subseteq E\setminus F}}} (-1)^{|J|}\,(-1)^{|F|}\,\pi_{\operatorname{union} F}$$

$$= \sum_{F\subseteq E}\ \underbrace{\sum_{\substack{J\subseteq B;\\ J\subseteq E\setminus F}}}_{=\sum\limits_{J\subseteq B\cap(E\setminus F)}} (-1)^{|J|}\,(-1)^{|F|}\,\pi_{\operatorname{union} F}$$

$$= \sum_{F\subseteq E}\ \underbrace{\sum_{J\subseteq B\cap(E\setminus F)} (-1)^{|J|}}_{\substack{=\sum\limits_{I\subseteq B\cap(E\setminus F)} (-1)^{|I|} \\ \text{(here, we have renamed} \\ \text{the summation index } J \text{ as } I\text{)}}} (-1)^{|F|}\,\pi_{\operatorname{union} F}$$

$$= \sum_{F\subseteq E}\ \underbrace{\sum_{I\subseteq B\cap(E\setminus F)} (-1)^{|I|}}_{\substack{=[B\cap(E\setminus F)=\varnothing] \\ \text{(by Lemma 5.21,} \\ \text{applied to } S=B\cap(E\setminus F)\text{)}}} (-1)^{|F|}\,\pi_{\operatorname{union} F}$$

$$= \sum_{F\subseteq E} \left[\underbrace{B\cap(E\setminus F)}_{=(B\cap E)\setminus F} = \varnothing\right] (-1)^{|F|}\,\pi_{\operatorname{union} F}$$

$$= \sum_{F\subseteq E} \left[\underbrace{(B\cap E)}_{\substack{=B \\ \text{(since } B\subseteq E\text{)}}} \setminus F = \varnothing\right] (-1)^{|F|}\,\pi_{\operatorname{union} F}$$

$$= \sum_{F \subseteq E} \underbrace{[B \setminus F = \varnothing]}_{\substack{=[B \subseteq F] \\ \text{(since the statement ``}B\setminus F=\varnothing\text{''} \\ \text{is equivalent to ``}B\subseteq F\text{'')}}} (-1)^{|F|} \pi_{\operatorname{union} F}$$

$$= \sum_{F \subseteq E} [B \subseteq F] (-1)^{|F|} \pi_{\operatorname{union} F}$$

$$= \sum_{\substack{F \subseteq E; \\ B \subseteq F}} \underbrace{[B \subseteq F]}_{\substack{=1 \\ \text{(since } B\subseteq F\text{)}}} (-1)^{|F|} \pi_{\operatorname{union} F} + \sum_{\substack{F \subseteq E; \\ \text{we don't have } B \subseteq F}} \underbrace{[B \subseteq F]}_{\substack{=0 \\ \text{(since we don't} \\ \text{have } B\subseteq F\text{)}}} (-1)^{|F|} \pi_{\operatorname{union} F}$$

$$\text{(since each subset } F \text{ of } E \text{ either satisfies } B \subseteq F \text{ or does not)}$$

$$= \sum_{\substack{F \subseteq E; \\ B \subseteq F}} (-1)^{|F|} \pi_{\operatorname{union} F} + \underbrace{\sum_{\substack{F \subseteq E; \\ \text{we don't have } B \subseteq F}} 0 (-1)^{|F|} \pi_{\operatorname{union} F}}_{=0}$$

$$= \sum_{\substack{F \subseteq E; \\ B \subseteq F}} (-1)^{|F|} \pi_{\operatorname{union} F} = 0 \qquad \text{(by (172))}.$$

This proves Theorem 7.2. □

8. The characteristic polynomial of a matroid

8.1. An introduction to matroids

We shall now present a result that can be considered as a generalization of Theorem 3.5 in a different direction than Theorem 1.15: namely, a formula for the characteristic polynomial of a matroid. Let us first recall the basic notions from the theory of matroids that will be needed to state it.

[TODO: Make the following proofs more detailed.]

First, we introduce some basic poset-related terminology:

Definition 8.1. Let P be a poset.

(a) An element v of P is said to be *maximal* (with respect to P) if and only if every $w \in P$ satisfying $w \geq v$ must satisfy $w = v$.

(b) An element v of P is said to be *minimal* (with respect to P) if and only if every $w \in P$ satisfying $w \leq v$ must satisfy $w = v$.

Definition 8.2. For any set E, we shall regard the powerset $\mathcal{P}(E)$ as a poset (with respect to inclusion). Thus, any subset $\mathcal{S}$ of $\mathcal{P}(E)$ also becomes a poset, and therefore the notions of "minimal" and "maximal" elements in $\mathcal{S}$ make sense. Beware that these notions are not related to size; i.e., a maximal element of $\mathcal{S}$ might not be a maximum-size element of $\mathcal{S}$.

Now, let us define the notion of "matroid" that we will use:

Definition 8.3. **(a)** A *matroid* means a pair $(E, \mathcal{I})$ consisting of a finite set E and a set $\mathcal{I} \subseteq \mathcal{P}(E)$ satisfying the following axioms:

- *Matroid axiom 1:* We have $\varnothing \in \mathcal{I}$.
- *Matroid axiom 2:* If $Y \in \mathcal{I}$ and $Z \in \mathcal{P}(E)$ are such that $Z \subseteq Y$, then $Z \in \mathcal{I}$.
- *Matroid axiom 3:* If $Y \in \mathcal{I}$ and $Z \in \mathcal{I}$ are such that $|Y| < |Z|$, then there exists some $x \in Z \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$.

(b) Let $(E, \mathcal{I})$ be a matroid. A subset S of E is said to be *independent* (for this matroid) if and only if $S \in \mathcal{I}$. The set E is called the *ground set* of the matroid $(E, \mathcal{I})$.

There are different definitions of a matroid in the literature; these definitions are (mostly) equivalent, but not always in the obvious way[96]. Definition 8.3 is how a matroid is defined in [Schrij13, §10.1] and in [Martin22, Definition 3.4.1] (where it is called a "(matroid) independence system"). The definition of a matroid given in Stanley's [Stanle06, Definition 3.8] is directly equivalent to Definition 8.3, with the only differences that

- Stanley replaces Matroid axiom 1 by the requirement that $\mathcal{I} \neq \varnothing$ (which is, of course, equivalent to Matroid axiom 1 as long as Matroid axiom 2 is assumed), and
- Stanley replaces Matroid axiom 3 by the requirement that for every $T \in \mathcal{P}(E)$, all maximal elements of $\mathcal{I} \cap \mathcal{P}(T)$ have the same cardinality[97] (this requirement is equivalent to Matroid axiom 3 as long as Matroid axiom 2 is assumed).

We now introduce some terminology related to matroids:

Definition 8.4. Let $M = (E, \mathcal{I})$ be a matroid.

(a) We define a function $r_M : \mathcal{P}(E) \to \mathbb{N}$ by setting

$$r_M(S) = \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq S\} \qquad \text{for every } S \subseteq E. \tag{176}$$

---

[96] Indeed, most of these definitions define a matroid as a pair (E, U) consisting of a finite set E and a subset $U \subseteq \mathcal{P}(E)$ satisfying a certain set of axioms, but these sets of axioms are not always equivalent, so they define different classes of pairs (E, U). Thus, a matroid in the sense of one definition is not necessarily a matroid in the sense of another definition. However, there are canonical bijections between one type of matroids and another (see, e.g., [Schrij13, §10.2]); these are commonly known as "cryptomorphisms".

[97] Here, we regard $\mathcal{I} \cap \mathcal{P}(T)$ as a poset with respect to inclusion (as explained in Definition 8.2). Thus, an element Y of this poset is maximal if and only if there exists no $Z \in \mathcal{I} \cap \mathcal{P}(T)$ such that Y is a proper subset of Z.

(Note that the right hand side of (176) is well-defined, because there exists at least one $Z \in \mathcal{I}$ satisfying $Z \subseteq S$ (namely, $Z = \varnothing$).) If S is a subset of E, then the nonnegative integer $r_M(S)$ is called the *rank* of S (with respect to M). It is clear that r_M is a weakly increasing function from the poset $\mathcal{P}(E)$ to $\mathbb{N}$.

(b) If $k \in \mathbb{N}$, then a *k-flat* of M means a subset of E that has rank k and is maximal among all such subsets (i.e., it is not a proper subset of any other subset having rank k). (Beware: Not all k-flats have the same size.) A *flat* of M is a subset of E which is a k-flat for some $k \in \mathbb{N}$. We let $\operatorname{Flats} M$ denote the set of all flats of M; thus, $\operatorname{Flats} M$ is a subposet of $\mathcal{P}(E)$.

(c) A *circuit* of M means a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$. (That is, a circuit of M means a subset of E which is not independent (for M) and which is minimal among such subsets.)

(d) An element e of E is said to be a *loop* (of M) if $\{e\} \notin \mathcal{I}$. The matroid M is said to be *loopless* if no loops (of M) exist.

Notice that the function that we called r_M in Definition 8.4 **(a)** is called the *rank function* of M, and is denoted by rk in Stanley's [Stanle06, Lecture 3].

One of the most classical examples of a matroid is the *graphical matroid* of a graph:

Example 8.5. Let $G = (V, E)$ be a finite graph. Define a subset $\mathcal{I}$ of $\mathcal{P}(E)$ by

$$\mathcal{I} = \{T \in \mathcal{P}(E) \mid T \text{ contains no circuit of } G \text{ as a subset}\}.$$

Then, $(E, \mathcal{I})$ is a matroid; it is called the *graphical matroid* (or the *cycle matroid*) of G. It has the following properties:

- The matroid $(E, \mathcal{I})$ is loopless.
- For each $T \in \mathcal{P}(E)$, we have

  $$r_{(E,\mathcal{I})}(T) = |V| - \operatorname{conn}(V, T)$$

  (where $\operatorname{conn}(V, T)$ is defined as in Definition 3.3).
- The circuits of the matroid $(E, \mathcal{I})$ are precisely the circuits of the graph G.
- The flats of the matroid $(E, \mathcal{I})$ are related to colorings of G. More precisely: For each set X and each X-coloring f of G, the set $E \cap \operatorname{Eqs} f$ is a flat of $(E, \mathcal{I})$. Every flat of $(E, \mathcal{I})$ can be obtained in this way when X is chosen large enough; but often, several distinct X-colorings f lead to one and the same flat $E \cap \operatorname{Eqs} f$.

We recall three basic facts that are used countless times in arguing about matroids:

Lemma 8.6. Let $M = (E, \mathcal{I})$ be a matroid. Let $T \in \mathcal{I}$. Then, $r_M(T) = |T|$.

Proof of Lemma 8.6. We have $T \in \mathcal{I}$ and $T \subseteq T$. Thus, T is a $Z \in \mathcal{I}$ satisfying $Z \subseteq T$. Therefore, $|T| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}$, so that

$$|T| \leq \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \tag{177}$$

(since any element of a set of integers is smaller or equal to the maximum of this set).

On the other hand, the definition of r_M yields

$$r_M(T) = \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}.$$

Hence, (177) rewrites as follows:

$$|T| \leq r_M(T).$$

Also,

$$\begin{aligned} r_M(T) &= \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \qquad \text{(by the definition of } r_M\text{)} \\ &\in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \end{aligned}$$

(since the maximum of any set belongs to this set). Thus, there exists a $Z \in \mathcal{I}$ satisfying $Z \subseteq T$ and $r_M(T) = |Z|$. Consider this Z. From $Z \subseteq T$, we obtain $|Z| \leq |T|$, so that $r_M(T) = |Z| \leq |T|$. Combining this with $|T| \leq r_M(T)$, we obtain $r_M(T) = |T|$. This proves Lemma 8.6. □

Lemma 8.7. Let $M = (E, \mathcal{I})$ be a matroid. Let $Q \in \mathcal{P}(E) \setminus \mathcal{I}$. Then, there exists a circuit C of M such that $C \subseteq Q$.

Proof of Lemma 8.7. We have $Q \in \mathcal{P}(E) \setminus \mathcal{I}$. Thus, there exists at least one $C \in \mathcal{P}(E) \setminus \mathcal{I}$ such that $C \subseteq Q$ (namely, $C = Q$). Thus, there also exists a **minimal** such C. Consider this minimal C. We know that C is a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$ such that $C \subseteq Q$. In other words, C is an element of $\mathcal{P}(E) \setminus \mathcal{I}$ satisfying $C \subseteq Q$, and moreover,

$$\text{every } D \in \mathcal{P}(E) \setminus \mathcal{I} \text{ satisfying } D \subseteq Q \text{ and } D \subseteq C \text{ must satisfy } D = C. \tag{178}$$

Thus, C is a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$ [98]. In other words, C is a circuit of M (by the definition of a "circuit"). This circuit C satisfies $C \subseteq Q$. Thus, we have constructed a circuit C of M satisfying $C \subseteq Q$. Lemma 8.7 is thus proven. □

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[98] *Proof.* We need to show that every $D \in \mathcal{P}(E) \setminus \mathcal{I}$ satisfying $D \subseteq C$ must satisfy $D = C$ (since we already know that $C \in \mathcal{P}(E) \setminus \mathcal{I}$).

So let $D \in \mathcal{P}(E) \setminus \mathcal{I}$ be such that $D \subseteq C$. Then, $D \subseteq C \subseteq Q$. Hence, (178) shows that $D = C$. This completes our proof.

Lemma 8.8. Let $M = (E, \mathcal{I})$ be a matroid. Let T be a subset of E. Let $S \in \mathcal{I}$ be such that $S \subseteq T$. Then, there exists an $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T$ and $|S'| = r_M(T)$.

Proof of Lemma 8.8. Clearly, there exists at least one $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T$ (namely, $S' = S$). Hence, there exists a **maximal** such S'. Let Q be such a maximal S'. Thus, Q is an element of $\mathcal{I}$ satisfying $S \subseteq Q \subseteq T$.

Recall that

$$\begin{aligned} r_M(T) &= \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \qquad (\text{by the definition of } r_M) \\ &\in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \end{aligned}$$

(since the maximum of any set must belong to this set). Hence, there exists some $Z \in \mathcal{I}$ satisfying $Z \subseteq T$ and $r_M(T) = |Z|$. Denote such a Z by W. Thus, W is an element of $\mathcal{I}$ satisfying $W \subseteq T$ and $r_M(T) = |W|$.

We have $|Q| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}$ (since $Q \in \mathcal{I}$ and $Q \subseteq T$). Since any element of a set is smaller or equal to the maximum of this set, this entails that $|Q| \leq \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} = r_M(T) = |W|$.

Now, assume (for the sake of contradiction) that $|Q| \neq |W|$. Thus, $|Q| < |W|$ (since $|Q| \leq |W|$). Hence, Matroid axiom 3 (applied to $Y = Q$ and $Z = W$) shows that there exists some $x \in W \setminus Q$ such that $Q \cup \{x\} \in \mathcal{I}$. Consider this x. We have $x \in W \setminus Q \subseteq W \subseteq T$, so that $Q \cup \{x\} \subseteq T$ (since $Q \subseteq T$). Also, $x \notin Q$ (since $x \in W \setminus Q$).

Recall that Q is a **maximal** $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T$. Thus, if some $S' \in \mathcal{I}$ satisfies $S \subseteq S' \subseteq T$ and $S' \supseteq Q$, then $S' = Q$. Applying this to $S' = Q \cup \{x\}$, we obtain $Q \cup \{x\} = Q$ (since $S \subseteq Q \subseteq Q \cup \{x\} \subseteq T$ and $Q \cup \{x\} \supseteq Q$). Thus, $x \in Q$. But this contradicts $x \notin Q$. This contradiction shows that our assumption (that $|Q| \neq |W|$) was wrong. Hence, $|Q| = |W| = r_M(T)$. Thus, there exists an $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T$ and $|S'| = r_M(T)$ (namely, $S' = Q$). This proves Lemma 8.8. □

8.2. The lattice of flats

We shall now show a lemma that can be regarded as an alternative criterion for a subset of E to be a flat:

Lemma 8.9. Let $M = (E, \mathcal{I})$ be a matroid. Let T be a subset of E. Then, the following statements are equivalent:

Statement $\mathfrak{F}_1$: The set T is a flat of M.

Statement $\mathfrak{F}_2$: If C is a circuit of M, and if $e \in C$ is such that $C \setminus \{e\} \subseteq T$, then $C \subseteq T$.

Proof of Lemma 8.9. Proof of the implication $\mathfrak{F}_1 \Longrightarrow \mathfrak{F}_2$: Assume that Statement $\mathfrak{F}_1$ holds. We must prove that Statement $\mathfrak{F}_2$ holds.

Let C be a circuit of M. Let $e \in C$ be such that $C \setminus \{e\} \subseteq T$. We must prove that $C \subseteq T$.

Assume the contrary. Thus, $C \not\subseteq T$. Combining this with $C \setminus \{e\} \subseteq T$, we obtain $e \notin T$. Hence, T is a proper subset of $T \cup \{e\}$.

We have assumed that Statement $\mathfrak{F}_1$ holds. In other words, the set T is a flat of M. In other words, there exists some $k \in \mathbb{N}$ such that T is a k-flat of M. Consider this k.

The set T is a k-flat of M, thus a subset of E that has rank k and is maximal among all such subsets. In other words, $r_M(T) = k$, but every subset S of E for which T is a proper subset of S must satisfy

$$r_M(S) \neq k. \tag{179}$$

Applying (179) to $S = T \cup \{e\}$, we obtain $r_M(T \cup \{e\}) \neq k$. Since $T \cup \{e\} \supseteq T$ (and since the function $r_M : \mathcal{P}(E) \to \mathbb{N}$ is weakly increasing), we have $r_M(T \cup \{e\}) \geq r_M(T) = k$. Combined with $r_M(T \cup \{e\}) \neq k$, this yields $r_M(T \cup \{e\}) > k = r_M(T)$.

Notice that $C \setminus \{e\}$ is a proper subset of C (since $e \in C$). The set C is a circuit of M, thus a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$ (by the definition of a "circuit"). Hence, no proper subset of C belongs to $\mathcal{P}(E) \setminus \mathcal{I}$ (because C is minimal). In other words, every proper subset of C belongs to $\mathcal{I}$. Applying this to the proper subset $C \setminus \{e\}$ of C, we conclude that $C \setminus \{e\}$ belongs to $\mathcal{I}$. Hence, Lemma 8.8 (applied to $S = C \setminus \{e\}$) shows that there exists an $S' \in \mathcal{I}$ satisfying $C \setminus \{e\} \subseteq S' \subseteq T$ and $|S'| = r_M(T)$. Denote this S' by S. Thus, S is an element of $\mathcal{I}$ satisfying $C \setminus \{e\} \subseteq S \subseteq T$ and $|S| = r_M(T)$.

Furthermore, $S \subseteq T \subseteq T \cup \{e\}$. Thus, Lemma 8.8 (applied to $T \cup \{e\}$ instead of T) shows that there exists an $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T \cup \{e\}$ and $|S'| = r_M(T \cup \{e\})$. Consider this S'.

We have $|S'| = r_M(T \cup \{e\}) > r_M(T)$. Hence, $S' \not\subseteq T$ [99]. Combining this with $S' \subseteq T \cup \{e\}$, we obtain $e \in S'$. Combining this with $C \setminus \{e\} \subseteq S \subseteq S'$, we find that $(C \setminus \{e\}) \cup \{e\} \subseteq S'$. Thus, $C = (C \setminus \{e\}) \cup \{e\} \subseteq S'$. Since $S' \in \mathcal{I}$, this entails that $C \in \mathcal{I}$ (by Matroid axiom 2). But $C \in \mathcal{P}(E) \setminus \mathcal{I}$ (since C is a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$), so that $C \notin \mathcal{I}$. This contradicts $C \in \mathcal{I}$. This contradiction shows that our assumption was wrong. Hence, $C \subseteq T$ is proven. Therefore, Statement $\mathfrak{F}_2$ holds. Thus, the implication $\mathfrak{F}_1 \Longrightarrow \mathfrak{F}_2$ is proven.

Proof of the implication $\mathfrak{F}_2 \Longrightarrow \mathfrak{F}_1$: Assume that Statement $\mathfrak{F}_2$ holds. We must prove that Statement $\mathfrak{F}_1$ holds.

Let $k = r_M(T)$. We shall show that T is a k-flat of M.

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[99] *Proof.* Assume the contrary. Thus, $S' \subseteq T$. Hence, S' is an element of $\mathcal{I}$ and satisfies $S' \subseteq T$. Thus, $|S'| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}$.

Now, the definition of r_M yields

$$r_M(T) = \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \geq |S'|$$

(since $|S'| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}$). This contradicts $|S'| > r_M(T)$. This contradiction proves that our assumption was wrong, qed.

Let W be a subset of E that has rank k and satisfies $T \subseteq W$. We shall show that $T = W$.

Indeed, assume the contrary. Thus, $T \neq W$. Combined with $T \subseteq W$, this shows that T is a proper subset of W. Thus, there exists an $e \in W \setminus T$. Consider this e. We have $e \notin T$ (since $e \in W \setminus T$).

We have

$$\begin{aligned} k = r_M(T) &= \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \qquad \text{(by the definition of } r_M\text{)} \\ &\in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \end{aligned}$$

(since the maximum of a set must belong to that set). Hence, there exists some $Z \in \mathcal{I}$ satisfying $Z \subseteq T$ and $k = |Z|$. Denote this Z by K. Thus, K is an element of $\mathcal{I}$ and satisfies $K \subseteq T$ and $k = |K|$. Notice that $e \notin T$, so that $e \notin K$ (since $K \subseteq T$).

We have $r_M(W) = k$ (since W has rank k). Hence, $K \cup \{e\} \notin \mathcal{I}$ [100]. In other words, $K \cup \{e\} \in \mathcal{P}(E) \setminus \mathcal{I}$. Hence, Lemma 8.7 (applied to $Q = K \cup \{e\}$) shows that there exists a circuit C of M such that $C \subseteq K \cup \{e\}$. Consider this C. From $C \subseteq K \cup \{e\}$, we obtain $C \setminus \{e\} \subseteq K \subseteq T$.

From $C \setminus \{e\} \subseteq K$, we conclude (using Matroid axiom 2) that $C \setminus \{e\} \in \mathcal{I}$ (since $K \in \mathcal{I}$). On the other hand, C is a circuit of M. In other words, C is a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$ (by the definition of a "circuit"). Hence, $C \in \mathcal{P}(E) \setminus \mathcal{I}$, so that $C \notin \mathcal{I}$. Hence, $e \in C$ (since otherwise, we would have $C \setminus \{e\} = C \notin \mathcal{I}$, which would contradict $C \setminus \{e\} \in \mathcal{I}$). Now, Statement $\mathfrak{F}_2$ shows that $C \subseteq T$. Hence, $e \in C \subseteq T$, which contradicts $e \notin T$.

This contradiction shows that our assumption was wrong. Hence, $T = W$ is proven.

Now, forget that we fixed W. Thus, we have shown that if W is a subset of E that has rank k and satisfies $T \subseteq W$, then $T = W$. In other words, T is a subset of E that has rank k and is maximal among all such subsets (because we already know that T has rank $r_M(T) = k$). In other words, T is a k-flat of M (by the definition of a "k-flat"). Thus, T is a flat of M. In other words, Statement $\mathfrak{F}_1$ holds. This proves the implication $\mathfrak{F}_2 \Longrightarrow \mathfrak{F}_1$.

We have now proven the implications $\mathfrak{F}_1 \Longrightarrow \mathfrak{F}_2$ and $\mathfrak{F}_2 \Longrightarrow \mathfrak{F}_1$. Together, these implications show that Statements $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are equivalent. This proves Lemma 8.9. $\square$

Corollary 8.10. Let $M = (E, \mathcal{I})$ be a matroid. Let $F_1, F_2, \ldots, F_k$ be flats of M. Then, $F_1 \cap F_2 \cap \cdots \cap F_k$ is a flat of M. (Notice that k is allowed to be 0 here; in this case, the empty intersection $F_1 \cap F_2 \cap \cdots \cap F_k$ is to be interpreted as E.)

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[100] *Proof.* Assume the contrary. Thus, $K \cup \{e\} \in \mathcal{I}$. Thus, $r_M(K \cup \{e\}) = |K \cup \{e\}|$ (by Lemma 8.6). Thus, $r_M(K \cup \{e\}) = |K \cup \{e\}| > |K|$ (since $e \notin K$).

But $K \cup \{e\} \subseteq W$ (since $K \subseteq T \subseteq W$ and $e \in W \setminus T \subseteq W$). Since the function r_M is weakly increasing, this yields $r_M(K \cup \{e\}) \leq r_M(W) = k = |K|$. This contradicts $r_M(K \cup \{e\}) > |K|$. This contradiction proves that our assumption was wrong, qed.

Proof of Corollary 8.10. Lemma 8.9 gives a necessary and sufficient criterion for a subset T of E to be a flat of M. It is easy to see that if this criterion is satisfied for $T = F_1$, for $T = F_2$, etc., and for $T = F_k$, then it is satisfied for $T = F_1 \cap F_2 \cap \cdots \cap F_k$. In other words, if $F_1, F_2, \ldots, F_k$ are flats of M, then $F_1 \cap F_2 \cap \cdots \cap F_k$ is a flat of M. [101] This proves Corollary 8.10. $\square$

Corollary 8.10 (a well-known fact, which is left to the reader to prove in [Stanle06, §3.1]) allows us to define the *closure* of a set in a matroid:

Definition 8.11. Let $M = (E, \mathcal{I})$ be a matroid. Let T be a subset of E. The *closure* of T is defined to be the intersection of all flats of M which contain T as a subset. In other words, the closure of T is defined to be $\bigcap\limits_{\substack{F \in \operatorname{Flats} M; \\ T \subseteq F}} F$. The closure of T is denoted by $\overline{T}$.

The following proposition gathers some simple properties of closures in matroids:

Proposition 8.12. Let $M = (E, \mathcal{I})$ be a matroid.

(a) If T is a subset of E, then $\overline{T}$ is a flat of M satisfying $T \subseteq \overline{T}$.

(b) If G is a flat of M, then $\overline{G} = G$.

(c) If T is a subset of E and if G is a flat of M satisfying $T \subseteq G$, then $\overline{T} \subseteq G$.

(d) If S and T are two subsets of E satisfying $S \subseteq T$, then $\overline{S} \subseteq \overline{T}$.

(e) If the matroid M is loopless, then $\overline{\varnothing} = \varnothing$.

(f) Every subset T of E satisfies $r_M(T) = r_M(\overline{T})$.

(g) If T is a subset of E and if G is a flat of M, then the conditions $(\overline{T} \subseteq G)$ and $(T \subseteq G)$ are equivalent.

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[101] Here is this argument in slightly more detail:

For every $i \in \{1, 2, \ldots, k\}$, the following statement holds: If C is a circuit of M, and if $e \in C$ is such that $C \setminus \{e\} \subseteq F_i$, then

$$C \subseteq F_i. \tag{180}$$

Proof of (180): Let $i \in \{1, 2, \ldots, k\}$. Then, the set F_i is a flat of M. In other words, Statement $\mathfrak{F}_1$ of Lemma 8.9 is satisfied for $T = F_i$. Therefore, Statement $\mathfrak{F}_2$ of Lemma 8.9 must also be satisfied for $T = F_i$ (since Lemma 8.9 shows that the Statements $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are equivalent). In other words, if C is a circuit of M, and if $e \in C$ is such that $C \setminus \{e\} \subseteq F_i$, then $C \subseteq F_i$. This proves (180).

Now, let C be a circuit of M, and let $e \in C$ be such that $C \setminus \{e\} \subseteq F_1 \cap F_2 \cap \cdots \cap F_k$. For every $i \in \{1, 2, \ldots, k\}$, we have $C \setminus \{e\} \subseteq F_1 \cap F_2 \cap \cdots \cap F_k \subseteq F_i$, and therefore $C \subseteq F_i$ (by (180)). So we have shown the inclusion $C \subseteq F_i$ for each $i \in \{1, 2, \ldots, k\}$. Combining these k inclusions, we obtain $C \subseteq F_1 \cap F_2 \cap \cdots \cap F_k$.

Now, forget that we fixed C. We thus have shown that if C is a circuit of M, and if $e \in C$ is such that $C \setminus \{e\} \subseteq F_1 \cap F_2 \cap \cdots \cap F_k$, then $C \subseteq F_1 \cap F_2 \cap \cdots \cap F_k$. In other words, Statement $\mathfrak{F}_2$ of Lemma 8.9 is satisfied for $T = F_1 \cap F_2 \cap \cdots \cap F_k$. Therefore, Statement $\mathfrak{F}_1$ of Lemma 8.9 must also be satisfied for $T = F_1 \cap F_2 \cap \cdots \cap F_k$ (since Lemma 8.9 shows that the Statements $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are equivalent). In other words, the set $F_1 \cap F_2 \cap \cdots \cap F_k$ is a flat of M. Qed.

Proof of Proposition 8.12. **(a)** The set $\operatorname{Flats} M$ is a subset of the finite set $\mathcal{P}(E)$, and thus itself finite.

Let T be a subset of E. The closure $\overline{T}$ of T is defined as $\bigcap\limits_{\substack{F \in \operatorname{Flats} M; \\ T \subseteq F}} F$. Now, Corollary 8.10 shows that any intersection of finitely many flats of M is a flat of M. Hence, $\bigcap\limits_{\substack{F \in \operatorname{Flats} M; \\ T \subseteq F}} F$ (being an intersection of finitely many flats of M [102]) is a flat of M. In other words, $\overline{T}$ is a flat of M (since $\overline{T} = \bigcap\limits_{\substack{F \in \operatorname{Flats} M; \\ T \subseteq F}} F$).

Also, $T \subseteq F$ for every $F \in \operatorname{Flats} M$ satisfying $T \subseteq F$. Hence, $T \subseteq \bigcap\limits_{\substack{F \in \operatorname{Flats} M; \\ T \subseteq F}} F = \overline{T}$. This completes the proof of Proposition 8.12 **(a)**.

(c) Let T be a subset of E, and let G be a flat of M satisfying $T \subseteq G$. Then, G is an element of $\operatorname{Flats} M$ satisfying $T \subseteq G$. Hence, G is one term in the intersection $\bigcap\limits_{\substack{F \in \operatorname{Flats} M; \\ T \subseteq F}} F$. Thus, $\bigcap\limits_{\substack{F \in \operatorname{Flats} M; \\ T \subseteq F}} F \subseteq G$. But the definition of $\overline{T}$ yields $\overline{T} = \bigcap\limits_{\substack{F \in \operatorname{Flats} M; \\ T \subseteq F}} F \subseteq G$. This proves Proposition 8.12 **(c)**.

(b) Let G be a flat of M. Proposition 8.12 **(b)** (applied to $T = G$) yields $\overline{G} \subseteq G$. But Proposition 8.12 **(a)** (applied to $T = G$) shows that $\overline{G}$ is a flat of M satisfying $G \subseteq \overline{G}$. Combining $G \subseteq \overline{G}$ with $\overline{G} \subseteq G$, we obtain $\overline{G} = G$. This proves Proposition 8.12 **(b)**.

(d) Let S and T be two subsets of E satisfying $S \subseteq T$. Proposition 8.12 **(a)** shows that $\overline{T}$ is a flat of M satisfying $T \subseteq \overline{T}$. Now, $S \subseteq T \subseteq \overline{T}$. Hence, Proposition 8.12 **(b)** (applied to S and $\overline{T}$ instead of T and G) shows $\overline{S} \subseteq \overline{T}$. This proves Proposition 8.12 **(d)**.

(e) Assume that the matroid M is loopless. In other words, no loops (of M) exist.

The definition of r_M quickly yields $r_M(\varnothing) = 0$. In other words, the set $\varnothing$ has rank 0. We shall now show that $\varnothing$ is a 0-flat of M.

Indeed, let W be a subset of E that has rank 0 and satisfies $\varnothing \subseteq W$. We shall show that $\varnothing = W$.

Assume the contrary. Thus, $\varnothing \neq W$. Hence, W has an element w. Consider this w. The element w of E is not a loop (since no loops exist). In other words, we cannot have $\{w\} \notin \mathcal{I}$ (since w is a loop if and only if $\{w\} \notin \mathcal{I}$ (by the definition of a loop)). In other words, we must have $\{w\} \in \mathcal{I}$. Clearly, $\{w\} \subseteq W$ (since $w \in W$). Thus, $\{w\}$ is a $Z \in \mathcal{I}$ satisfying $Z \subseteq W$. Thus, $|\{w\}| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq W\}$.

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[102] "Finitely many" since the set $\operatorname{Flats} M$ is finite.

But W has rank 0. In other words,

$$\begin{aligned}0 &= r_M(W) = \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq W\} \qquad \text{(by the definition of } r_M\text{)}\\ &\geq |\{w\}| \qquad \left(\text{since } |\{w\}| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq W\}\right)\\ &= 1,\end{aligned}$$

which is absurd. This contradiction shows that our assumption was wrong. Hence, $\varnothing = W$ is proven.

Let us now forget that we fixed W. We thus have proven that if W is any subset of E that has rank 0 and satisfies $\varnothing \subseteq W$, then $\varnothing = W$. Thus, $\varnothing$ is a subset of E that has rank 0 and is maximal among all such subsets (because we already know that $\varnothing$ has rank 0). In other words, $\varnothing$ is a 0-flat of M (by the definition of a "0-flat"). Thus, $\varnothing$ is a flat of M. Thus, Proposition 8.12 **(b)** (applied to $G = \varnothing$) yields $\overline{\varnothing} = \varnothing$. This proves Proposition 8.12 **(e)**.

(f) Let T be a subset of E. We have $T \subseteq \overline{T}$ (by Proposition 8.12 **(a)**), and thus $r_M(T) \leq r_M\left(\overline{T}\right)$ (since the function r_M is weakly increasing).

Let $k = r_M(T)$. Thus, there exists a $Q \in \mathcal{P}(E)$ satisfying $T \subseteq Q$ and $k = r_M(Q)$ (namely, $Q = T$). Hence, there exists a **maximal** such Q. Denote this Q by R. Thus, R is a maximal $Q \in \mathcal{P}(E)$ satisfying $T \subseteq Q$ and $k = r_M(Q)$. In particular, R is an element of $\mathcal{P}(E)$ and satisfies $T \subseteq R$ and $k = r_M(R)$.

Now, R is a subset of E (since $R \in \mathcal{P}(E)$) and has rank $r_M(R) = k$. Thus, R is a subset of E that has rank k. Furthermore, R is maximal among all such subsets[103]. Thus, R is a k-flat of M (by the definition of a "k-flat"), and therefore a flat of M. Now, Proposition 8.12 **(c)** (applied to $G = R$) shows that $\overline{T} \subseteq R$. Since the function r_M is weakly increasing, this yields $r_M\left(\overline{T}\right) \leq r_M(R) = k$. Combining this with $k = r_M(T) \leq r_M\left(\overline{T}\right)$, we obtain $r_M\left(\overline{T}\right) = k = r_M(T)$. This proves Proposition 8.12 **(f)**.

(g) Let T be a subset of E. Let G be a flat of M. Proposition 8.12 **(a)** shows that $T \subseteq \overline{T}$. Hence, if $\overline{T} \subseteq G$, then $T \subseteq \overline{T} \subseteq G$. Thus, we have proven the implication $\left(\overline{T} \subseteq G\right) \Longrightarrow (T \subseteq G)$. The reverse implication (i.e., the implication $(T \subseteq G) \Longrightarrow \left(\overline{T} \subseteq G\right)$) follows from Proposition 8.12 **(c)**. Combining these two implications, we obtain the equivalence $\left(\overline{T} \subseteq G\right) \Longleftrightarrow (T \subseteq G)$. This proves Proposition 8.12 **(g)**. □

We shall now recall a few more classical notions related to posets:

Definition 8.13. Let P be a poset.

(a) An element $p \in P$ is said to be a *global minimum* of P if every $q \in P$ satisfies $p \leq q$. Clearly, a global minimum of P is unique if it exists.

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[103] *Proof.* Let W be any subset of E that has rank k and satisfies $W \supseteq R$. We must prove that $W = R$.

We have $W \in \mathcal{P}(E)$, $T \subseteq R \subseteq W$ and $k = r_M(W)$ (since W has rank k). Thus, W is a $Q \in \mathcal{P}(E)$ satisfying $T \subseteq Q$ and $k = r_M(Q)$. But recall that R is a **maximal** such Q. Hence, if $W \supseteq R$, then $W = R$. Therefore, $W = R$ (since we know that $W \supseteq R$). Qed.

(b) An element $p \in P$ is said to be a *global maximum* of P if every $q \in P$ satisfies $p \geq q$. Clearly, a global maximum of P is unique if it exists.

(c) Let x and y be two elements of P. An *upper bound* of x and y (in P) means an element $z \in P$ satisfying $z \geq x$ and $z \geq y$. A *join* (or *least upper bound*) of x and y (in P) means an upper bound z of x and y such that every upper bound z' of x and y satisfies $z' \geq z$. In other words, a join of x and y is a global minimum of the subposet $\{w \in P \mid w \geq x \text{ and } w \geq y\}$ of P. Thus, a join of x and y is unique if it exists.

(d) Let x and y be two elements of P. A *lower bound* of x and y (in P) means an element $z \in P$ satisfying $z \leq x$ and $z \leq y$. A *meet* (or *greatest lower bound*) of x and y (in P) means a lower bound z of x and y such that every lower bound z' of x and y satisfies $z' \leq z$. In other words, a meet of x and y is a global maximum of the subposet $\{w \in P \mid w \leq x \text{ and } w \leq y\}$ of P. Thus, a meet of x and y is unique if it exists.

(e) The poset P is said to be a *lattice* if and only if it has a global minimum and a global maximum, and every two elements of P have a meet and a join.

Proposition 8.14. Let $M = (E, \mathcal{I})$ be a matroid. The subposet $\operatorname{Flats} M$ of the poset $\mathcal{P}(E)$ is a lattice.

Proof of Proposition 8.14. By the definition of a lattice, it suffices to check the following four claims:

Claim 1: The poset $\operatorname{Flats} M$ has a global minimum.

Claim 2: The poset $\operatorname{Flats} M$ has a global maximum.

Claim 3: Every two elements of $\operatorname{Flats} M$ have a meet (in $\operatorname{Flats} M$).

Claim 4: Every two elements of $\operatorname{Flats} M$ have a join (in $\operatorname{Flats} M$).

Proof of Claim 1: Applying Proposition 8.12 **(a)** to $T = \varnothing$, we see that $\overline{\varnothing}$ is a flat of M satisfying $\varnothing \subseteq \overline{\varnothing}$. In particular, $\overline{\varnothing}$ is a flat of M, so that $\overline{\varnothing} \in \operatorname{Flats} M$. If G is a flat of M, then $\overline{\varnothing} \subseteq G$ (by Proposition 8.12 **(c)**, applied to $T = \varnothing$). Hence, $\overline{\varnothing}$ is a global minimum of the poset $\operatorname{Flats} M$. Thus, the poset $\operatorname{Flats} M$ has a global minimum. This proves Claim 1.

Proof of Claim 2: Applying Proposition 8.12 **(a)** to $T = E$, we see that $\overline{E}$ is a flat of M satisfying $E \subseteq \overline{E}$. From $E \subseteq \overline{E}$, we conclude that $\overline{E} = E$. Thus, E is a flat of M (since $\overline{E}$ is a flat of M). In other words, $E \in \operatorname{Flats} M$. If G is a flat of M, then $E \supseteq G$ (obviously). Hence, E is a global maximum of the poset $\operatorname{Flats} M$. Thus, the poset $\operatorname{Flats} M$ has a global maximum. This proves Claim 2.

Proof of Claim 3: Let F and G be two elements of $\operatorname{Flats} M$. We have to prove that F and G have a meet.

We know that F and G are elements of $\operatorname{Flats} M$, thus flats of M. Hence, Corollary 8.10 shows that $F \cap G$ is a flat of M. In other words, $F \cap G \in \operatorname{Flats} M$. Clearly, $F \cap G \subseteq F$ and $F \cap G \subseteq G$; thus, $F \cap G$ is a lower bound of F and G in $\operatorname{Flats} M$. Also, every lower bound H of F and G in $\operatorname{Flats} M$ satisfies $H \subseteq F \cap G$

[104]. Hence, $F \cap G$ is a meet of F and G. Thus, F and G have a meet. This proves Claim 3.

Proof of Claim 4: Let F and G be two elements of $\operatorname{Flats} M$. We have to prove that F and G have a join.

We know that F and G are elements of $\operatorname{Flats} M$, thus flats of M. Proposition 8.12 **(a)** (applied to $T = F \cup G$) shows that $\overline{F \cup G}$ is a flat of M satisfying $F \cup G \subseteq \overline{F \cup G}$. Now, $\overline{F \cup G} \in \operatorname{Flats} M$ (since $\overline{F \cup G}$ is a flat of M). Clearly, $F \subseteq F \cup G \subseteq \overline{F \cup G}$ and $G \subseteq F \cup G \subseteq \overline{F \cup G}$; thus, $\overline{F \cup G}$ is an upper bound of F and G in $\operatorname{Flats} M$. Also, every upper bound H of F and G in $\operatorname{Flats} M$ satisfies $H \supseteq \overline{F \cup G}$ [105]. Hence, $\overline{F \cup G}$ is a join of F and G. Thus, F and G have a join. This proves Claim 4.

We have now proven all four Claims 1, 2, 3, and 4. Thus, Proposition 8.14 is proven. □

Definition 8.15. Let $M = (E, \mathcal{I})$ be a matroid. Proposition 8.14 shows that the subposet $\operatorname{Flats} M$ of the poset $\mathcal{P}(E)$ is a lattice. This subposet $\operatorname{Flats} M$ is called the *lattice of flats* of M. (Beware: It is a subposet, but not a sublattice of $\mathcal{P}(E)$, since its join is not a restriction of the join of $\mathcal{P}(E)$.)

The lattice of flats $\operatorname{Flats} M$ of a matroid M is denoted by $L(M)$ in [Stanle06, §3.2].

Next, we recall the definition of the Möbius function of a poset (see, e.g., [Stanle06, Definition 1.2] or [Martin22, §2.2]):

Definition 8.16. Let P be a poset.

(a) If x and y are two elements of P satisfying $x \leq y$, then the set $\{z \in P \mid x \leq z \leq y\}$ is denoted by $[x, y]$.

(b) A subset of P is called a *closed interval* of P if it has the form $[x, y]$ for two elements x and y of P satisfying $x \leq y$.

(c) We denote by $\operatorname{Int} P$ the set of all closed intervals of P.

(d) If $f : \operatorname{Int} P \to \mathbb{Z}$ is any map, then the image $f([x, y])$ of a closed interval $[x, y] \in \operatorname{Int} P$ under f will be abbreviated by $f(x, y)$.

(e) Assume that every closed interval of P is finite. The *Möbius function* of the poset P is defined to be the unique function $\mu : \operatorname{Int} P \to \mathbb{Z}$ having the following two properties:

- We have
$$\mu(x, x) = 1 \qquad \text{for every } x \in P. \tag{181}$$

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[104] *Proof.* Let H be a lower bound of F and G in $\operatorname{Flats} M$. Thus, $H \subseteq F$ and $H \subseteq G$. Combining these two inclusions, we obtain $H \subseteq F \cap G$, qed.

[105] *Proof.* Let H be an upper bound of F and G in $\operatorname{Flats} M$. Thus, $H \supseteq F$ and $H \supseteq G$. Combining these two inclusions, we obtain $H \supseteq F \cup G$. But $H \in \operatorname{Flats} M$; thus, H is a flat of M. Since H satisfies $F \cup G \subseteq H$, we therefore obtain $\overline{F \cup G} \subseteq H$ (by Proposition 8.12 **(c)**, applied to $F \cup G$ and H instead of T and G). In other words, $H \supseteq \overline{F \cup G}$, qed.

- We have
$$\mu(x,y) = -\sum_{\substack{z\in P;\\ x\leq z<y}} \mu(x,z) \qquad \text{for all } x,y\in P \text{ satisfying } x<y. \tag{182}$$

(It is easy to see that these two properties indeed determine μ uniquely.) This Möbius function is denoted by μ.

We can now define the characteristic polynomial of a matroid M, following [Stanle06, (22)][106]:

Definition 8.17. Let $M = (E,\mathcal{I})$ be a matroid. Let $m = r_M(E)$. The *characteristic polynomial* χ_M of the matroid M is defined to be the polynomial

$$\sum_{F\in\operatorname{Flats} M} \mu\left(\overline{\varnothing},F\right) x^{m-r_M(F)} \in \mathbb{Z}[x]$$

(where μ is the Möbius function of the lattice $\operatorname{Flats} M$). We further define a polynomial $\widetilde{\chi}_M \in \mathbb{Z}[x]$ by $\widetilde{\chi}_M = \left[\overline{\varnothing}=\varnothing\right]\chi_M$. Here, we are using the Iverson bracket notation (as in Definition 2.9). If the matroid M is loopless, then

$$\widetilde{\chi}_M = \underbrace{\left[\overline{\varnothing}=\varnothing\right]}_{\substack{=1\\ \text{(by Proposition 8.12 \textbf{(e)})}}} \chi_M = \chi_M.$$

Example 8.18. Let $G = (V,E)$ be a finite graph. Consider the graphical matroid $(E,\mathcal{I})$ defined as in Example 8.5. Then, the characteristic polynomial $\chi_{(E,\mathcal{I})}$ of this matroid is connected to the chromatic polynomial χ_G of the graph G as follows:
$$x^{\operatorname{conn} G}\cdot \chi_{(E,\mathcal{I})}(x) = \chi_G(x).$$
This equality is a classical result (see, e.g., [Zaslav87, Proposition 7.5.1]), but can also be derived from our results below (specifically, by comparing Theorem 8.21 with Theorem 3.4).

Note that Zaslavsky, in [Zaslav87, §7.2], defines the "characteristic polynomial" of a matroid M to be our $\widetilde{\chi}_M$ instead of our χ_M; but this makes no difference when M is the graphical matroid from Example 8.5, since such a matroid M is always loopless.

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[106] Our notation slightly differs from that in [Stanle06, (22)]. Namely, we use x as the indeterminate, while Stanley instead uses t. Stanley also denotes the global minimum $\overline{\varnothing}$ of $\operatorname{Flats} M$ by $\widehat{0}$.

8.3. Generalized Whitney formulas

Let us next define broken circuits of a matroid $M = (E, \mathcal{I})$. Stanley, in [Stanle06, §4.1], defines them in terms of a total ordering $\mathcal{O}$ on the set E, whereas we shall use a "labeling function" $\ell : E \to X$ instead (as in the case of graphs); our setting is slightly more general than Stanley's.

Definition 8.19. Let $M = (E, \mathcal{I})$ be a matroid. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. We shall refer to ℓ as the *labeling function*. For every $e \in E$, we shall refer to $\ell(e)$ as the *label* of e.

A *broken circuit* of M means a subset of E having the form $C \setminus \{e\}$, where C is a circuit of M, and where e is the unique element of C having maximum label (among the elements of C). Of course, the notion of a broken circuit of M depends on the function ℓ; however, we suppress the mention of ℓ in our notation, since we will not consider situations where two different ℓ's coexist.

We shall now state analogues (and, in light of Example 8.18, generalizations, although we shall not elaborate on the few minor technicalities of seeing them as such) of Theorem 3.5, Theorem 3.4, Corollary 3.6, Corollary 3.7 and Corollary 3.14:

Theorem 8.20. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of M (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$\widetilde{\chi}_M = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq F}} a_K \right) x^{m - r_M(F)}.$$

Theorem 8.21. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. Then,

$$\widetilde{\chi}_M = \sum_{F \subseteq E} (-1)^{|F|} x^{m - r_M(F)}.$$

Corollary 8.22. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of M (not necessarily containing all of them). Then,

$$\widetilde{\chi}_M = \sum_{\substack{F \subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} x^{m - r_M(F)}.$$

Corollary 8.23. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Then,

$$\widetilde{\chi}_M = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } M \text{ as a subset}}} (-1)^{|F|} x^{m - r_M(F)}.$$

Corollary 8.24. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. Let X be a totally ordered set. Let $\ell : E \to X$ be an injective labeling function. Then,

$$\widetilde{\chi}_M = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } M \text{ as a subset}}} (-1)^{|F|} x^{m - |F|}.$$

We notice that Corollary 8.24 is equivalent to [Stanle06, Theorem 4.12] (at least when M is loopless).

Before we prove these results, let us state a lemma which will serve as an analogue of Lemma 2.10:

Lemma 8.25. Let $M = (E, \mathcal{I})$ be a matroid. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of M (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$.

Let F be any flat of M. Then,

$$\sum_{B \subseteq F} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K = [F = \varnothing]. \tag{183}$$

(Again, we are using the Iverson bracket notation as in Definition 2.9.)

Proof of Lemma 8.25. Our proof will imitate the proof of Lemma 2.10 much of the time (with $E \cap \operatorname{Eqs} f$ replaced by F); thus, we will allow ourselves some more brevity.

We WLOG assume that $F \neq \varnothing$ (since otherwise, the claim is obvious[107]). Thus, $[F = \varnothing] = 0$.

---

[107] *Proof.* Assume that $F = \varnothing$. We must show that the claim is obvious.

Let us first show that no $K \in \mathfrak{K}$ satisfies $K = \varnothing$. Indeed, assume the contrary. Thus, there exists a $K \in \mathfrak{K}$ satisfying $K = \varnothing$. In other words, $\varnothing \in \mathfrak{K}$. Thus, $\varnothing$ is a broken circuit of M (since $\mathfrak{K}$ is a set of broken circuits of M). Therefore, $\varnothing$ is obtained from a circuit of M by removing one element (by the definition of a broken circuit). This latter circuit must therefore be a one-element set, i.e., it has the form $\{e\}$ for some $e \in E$. Consider this e. Thus, $\{e\}$ is a circuit of M.

But F is a flat of M. In other words, Statement $\mathfrak{F}_1$ (of Lemma 8.9) holds for $T = F$. Hence, Statement $\mathfrak{F}_2$ (of Lemma 8.9) also holds for $T = F$ (since Lemma 8.9 shows that these two

Pick any $d \in F$ with maximum $\ell(d)$ (among all $d \in F$). (This is clearly possible, since $F \neq \varnothing$.) Define two subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{P}(F)$ as follows:

$$\mathcal{U} = \{T \in \mathcal{P}(F) \mid d \notin T\};$$
$$\mathcal{V} = \{T \in \mathcal{P}(F) \mid d \in T\}.$$

Thus, we have $\mathcal{P}(F) = \mathcal{U} \cup \mathcal{V}$, and the sets $\mathcal{U}$ and $\mathcal{V}$ are disjoint. Now, we define a map $\Phi : \mathcal{U} \to \mathcal{V}$ by

$$(\Phi(B) = B \cup \{d\} \qquad \text{for every } B \in \mathcal{U}).$$

This map Φ is well-defined (because for every $B \in \mathcal{U}$, we have $B \cup \{d\} \in \mathcal{V}$ [108]) and a bijection[109]. Moreover, every $B \in \mathcal{U}$ satisfies

$$(-1)^{|\Phi(B)|} = -(-1)^{|B|} \tag{184}$$

[110].

Now, we claim that, for every $B \in \mathcal{U}$ and every $K \in \mathfrak{K}$, we have the following logical equivalence:

$$(K \subseteq B) \iff (K \subseteq \Phi(B)). \tag{185}$$

Proof of (185): Let $B \in \mathcal{U}$ and $K \in \mathfrak{K}$. We must prove the equivalence (185). The definition of Φ yields $\Phi(B) = B \cup \{d\} \supseteq B$, so that $B \subseteq \Phi(B)$. Hence, if $K \subseteq B$, then $K \subseteq B \subseteq \Phi(B)$. Therefore, the forward implication of the equivalence (185) is proven. It thus remains to prove the backward implication of this equivalence. In other words, it remains to prove that if $K \subseteq \Phi(B)$, then $K \subseteq B$. So let us assume that $K \subseteq \Phi(B)$.

We want to prove that $K \subseteq B$. Assume the contrary. Thus, $K \not\subseteq B$. We have $K \in \mathfrak{K}$. Thus, K is a broken circuit of M (since $\mathfrak{K}$ is a set of broken circuits of M).

---

statements are equivalent). Applying Statement $\mathfrak{F}_2$ to $T = F$ and $C = \{e\}$, we thus obtain $\{e\} \subseteq F$ (because $\{e\} \setminus \{e\} = \varnothing \subseteq F$). Thus, $e \in \{e\} \subseteq F = \varnothing$, which is absurd. This contradiction proves that our assumption was wrong.

Hence, we have shown that no $K \in \mathfrak{K}$ satisfies $K = \varnothing$. But from $F = \varnothing$, we see that the sum $\sum_{B \subseteq F} (-1)^{|B|} \prod_{K \in \mathfrak{K};\ K \subseteq B} a_K$ has only one addend (namely, the addend for $B = \varnothing$), and thus simplifies to

$$\underbrace{(-1)^{|\varnothing|}}_{=(-1)^0=1} \underbrace{\prod_{K \in \mathfrak{K};\ K \subseteq \varnothing}}_{=\prod_{K \in \mathfrak{K};\ K = \varnothing}} a_K = \prod_{K \in \mathfrak{K};\ K = \varnothing} a_K = (\text{empty product}) \qquad (\text{since no } K \in \mathfrak{K} \text{ satisfies } K = \varnothing)$$
$$= 1 = [F = \varnothing] \qquad (\text{since } F = \varnothing).$$

Thus, Lemma 8.25 is proven.

[108] This follows from the fact that $d \in F$.

[109] Its inverse is the map $\Psi : \mathcal{V} \to \mathcal{U}$ defined by $(\Psi(B) = B \setminus \{d\} \qquad \text{for every } B \in \mathcal{V})$.

[110] *Proof.* This is proven exactly like we proved (41).

In other words, K is a subset of E having the form $C \setminus \{e\}$, where C is a circuit of M, and where e is the unique element of C having maximum label (among the elements of C) (because this is how a broken circuit is defined). Consider these C and e. Thus, $K = C \setminus \{e\}$.

The element e is the unique element of C having maximum label (among the elements of C). Thus, if e' is any element of C satisfying $\ell(e') \geq \ell(e)$, then

$$e' = e. \tag{186}$$

But $\underbrace{K}_{\subseteq \Phi(B) = B \cup \{d\}} \setminus \{d\} \subseteq (B \cup \{d\}) \setminus \{d\} \subseteq B$.

If we had $d \notin K$, then we would have $K \setminus \{d\} = K$ and therefore $K = K \setminus \{d\} \subseteq B$; this would contradict $K \not\subseteq B$. Hence, we cannot have $d \notin K$. We thus must have $d \in K$. Hence, $d \in K = C \setminus \{e\}$. Hence, $d \in C$ and $d \neq e$.

But $C \setminus \{e\} = K \subseteq \Phi(B) \subseteq F$ (since $\Phi(B) \in \mathcal{P}(F)$). On the other hand, Statement $\mathfrak{F}_1$ (of Lemma 8.9) holds for $T = F$ (since F is a flat of M). Hence, Statement $\mathfrak{F}_2$ (of Lemma 8.9) also holds for $T = F$ (since Lemma 8.9 shows that these two statements are equivalent). Thus, from $C \setminus \{e\} \subseteq F$, we obtain $C \subseteq F$. Thus, $e \in C \subseteq F$. Consequently, $\ell(d) \geq \ell(e)$ (since d was defined to be an element of F with maximum $\ell(d)$ among all $d \in F$).

Also, $d \in C$. Since $\ell(d) \geq \ell(e)$, we can therefore apply (186) to $e' = d$. We thus obtain $d = e$. This contradicts $d \neq e$. This contradiction proves that our assumption was wrong. Hence, $K \subseteq B$ is proven. Thus, we have proven the backward implication of the equivalence (185); this completes the proof of (185).

Now, proceeding as in the proof of (42), we can show that

$$\sum_{B \subseteq F} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K = [F = \varnothing].$$

This proves Lemma 8.25. □

We shall furthermore use a classical and fundamental result on the Möbius function of any finite poset:

Proposition 8.26. Let P be a finite poset. Let μ denote the Möbius function of P.

(a) For any $x \in P$ and $y \in P$, we have

$$\sum_{\substack{z \in P; \\ x \leq z \leq y}} \mu(x,z) = [x = y]. \tag{187}$$

(b) For any $x \in P$ and $y \in P$, we have

$$\sum_{\substack{z \in P; \\ x \leq z \leq y}} \mu(z,y) = [x = y]. \tag{188}$$

(c) Let $\mathbf{k}$ be a $\mathbb{Z}$-module. Let $(\beta_x)_{x\in P}$ be a family of elements of $\mathbf{k}$. Then, every $z \in P$ satisfies

$$\beta_z = \sum_{\substack{y\in P;\\ y\leq z}} \mu(y,z) \sum_{\substack{x\in P;\\ x\leq y}} \beta_x.$$

For the sake of completeness, let us give a self-contained proof of this proposition (slicker arguments appear in the literature[111]):

Proof of Proposition 8.26. We first notice that

$$\{w \in P \mid x \leq w \leq x\} = \{x\} \tag{189}$$

for every $x \in P$ [112].

(a) Let $x \in P$ and $y \in P$. We must prove the equality (187). If we do not have $x \leq y$, then (187) holds for obvious reasons[113]. Hence, for the rest of our proof of (187), we can WLOG assume that $x \leq y$. Assume this.

We have $x \leq y$. Thus, either $x = y$ or $x < y$. In other words, we are in one of the following two cases:

Case 1: We have $x = y$.

Case 2: We have $x < y$.

Let us first consider Case 1. In this case, we have $x = y$, so that $[x = y] = 1$.

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[111] For example, Proposition 8.26 **(c)** is equivalent to the $\Longrightarrow$ implication of [Martin22, (2.3a)].

[112] *Proof of (189):* Let $x \in P$.

Clearly, x is an element of P and satisfies $x \leq x \leq x$. Thus, x is an element w of P satisfying $x \leq w \leq x$. In other words, $x \in \{w \in P \mid x \leq w \leq x\}$. Hence, $\{x\} \subseteq \{w \in P \mid x \leq w \leq x\}$.

On the other hand, let $z \in \{w \in P \mid x \leq w \leq x\}$. Thus, z is an element w of P satisfying $x \leq w \leq x$. In other words, z is an element of P and satisfies $x \leq z \leq x$. Combining $x \leq z$ with $z \leq x$, we obtain $x = z$ (since the partial order on P is antisymmetric). Hence, $z = x \in \{x\}$.

Now, forget that we fixed z. We thus have proven that $z \in \{x\}$ for every $z \in \{w \in P \mid x \leq w \leq x\}$. In other words, $\{w \in P \mid x \leq w \leq x\} \subseteq \{x\}$. Combining this with $\{x\} \subseteq \{w \in P \mid x \leq w \leq x\}$, we obtain $\{w \in P \mid x \leq w \leq x\} = \{x\}$. This proves (189).

[113] *Proof.* Assume that we do not have $x \leq y$. Then, there exists no $z \in P$ satisfying $x \leq z \leq y$ (because if such a z would exist, then it would satisfy $x \leq z \leq y$, which would contradict the fact that we do not have $x \leq y$). Therefore, the sum $\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(x,z)$ is an empty sum. Hence,

$$\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(x,z) = (\text{empty sum}) = 0.$$

But we do not have $x = y$ (because if we had $x = y$, then we would have $x \leq y$, which would contradict the fact that we do not have $x \leq y$). Thus, $[x = y] = 0$. Now, $\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(x,z) = 0 = [x = y]$. Hence, (187) is proven, qed.

On the other hand,

$$\left\{w \in P \mid x \leq w \leq \underbrace{y}_{=x}\right\} = \{w \in P \mid x \leq w \leq x\} = \{x\}$$

(by (189)). Now,

$$\underbrace{\sum_{\substack{z\in P;\\ x\leq z\leq y}}}_{\substack{=\sum\limits_{z\in\{w\in P\ \mid\ x\leq w\leq y\}} = \sum\limits_{z\in\{x\}}\\ \text{(since } \{w\in P\ \mid\ x\leq w\leq y\}=\{x\})}} \mu(x,z) = \sum_{z\in\{x\}} \mu(x,z) = \mu(x,x) = 1$$

(by (181)). Comparing this with $[x=y]=1$, we obtain $\sum\limits_{\substack{z\in P;\\ x\leq z\leq y}} \mu(x,z) = [x=y]$.

Hence, (187) is proven in Case 1.

Let us now consider Case 2. In this case, we have $x<y$. Thus, (182) yields

$$\mu(x,y) = -\underbrace{\sum_{\substack{z\in P;\\ x\leq z<y}}}_{\substack{=\sum\limits_{\substack{z\in P;\\ x\leq z \text{ and } z<y}} = \sum\limits_{\substack{z\in P;\\ x\leq z \text{ and } z\leq y \text{ and } z\neq y}}\\ \text{(since the condition } (z<y) \text{ is equivalent to}\\ \text{the condition } (z\leq y \text{ and } z\neq y))}} \mu(x,z) = -\underbrace{\sum_{\substack{z\in P;\\ x\leq z \text{ and } z\leq y \text{ and } z\neq y}}}_{=\sum\limits_{\substack{z\in P;\\ x\leq z\leq y \text{ and } z\neq y}}} \mu(x,z)$$

$$= -\sum_{\substack{z\in P;\\ x\leq z\leq y \text{ and } z\neq y}} \mu(x,z).$$

Hence, $\sum\limits_{\substack{z\in P;\\ x\leq z\leq y \text{ and } z\neq y}} \mu(x,z) = -\mu(x,y)$.

On the other hand,

$$\{w \in P \mid x \leq w \leq y \text{ and } w = y\} = \{y\}$$

[114].

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[114] *Proof.* Let $z \in \{w \in P \mid x \leq w \leq y \text{ and } w = y\}$. Thus, z is an element w of P satisfying $x \leq w \leq y$ and $w = y$. In other words, z is an element of P and satisfies $x \leq z \leq y$ and $z = y$. Now, $z = y \in \{y\}$. Let us now forget that we fixed z. We thus have proven that $z \in \{y\}$ for every $z \in \{w \in P \mid x \leq w \leq y \text{ and } w = y\}$. In other words, $\{w \in P \mid x \leq w \leq y \text{ and } w = y\} \subseteq \{y\}$.

On the other hand, y is an element of P and satisfies $x \leq y \leq y$ and $y = y$. In other words, y is an element w of P satisfying $x \leq w \leq y$ and $w = y$. Hence, $y \in \{w \in P \mid x \leq w \leq y \text{ and } w = y\}$. Thus, $\{y\} \subseteq \{w \in P \mid x \leq w \leq y \text{ and } w = y\}$. Combining this with $\{w \in P \mid x \leq w \leq y \text{ and } w = y\} \subseteq \{y\}$, we obtain $\{w \in P \mid x \leq w \leq y \text{ and } w = y\} = \{y\}$. Qed.

But every $z \in P$ satisfies either $z = y$ or $z \neq y$ (but not both). Thus,

$$\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(x,z) = \underbrace{\sum_{\substack{z\in P;\\ x\leq z\leq y \text{ and } z=y}}}_{\substack{= \sum\limits_{z\in\{w\in P \ \mid\ x\leq w\leq y \text{ and } w=y\}} = \sum\limits_{z\in\{y\}} \\ \text{(since } \{w\in P \ \mid\ x\leq w\leq y \text{ and } w=y\}=\{y\}\text{)}}} \mu(x,z) + \underbrace{\sum_{\substack{z\in P;\\ x\leq z\leq y \text{ and } z\neq y}} \mu(x,z)}_{=-\mu(x,y)}$$

$$= \underbrace{\sum_{z\in\{y\}} \mu(x,z)}_{=\mu(x,y)} + (-\mu(x,y)) = \mu(x,y) + (-\mu(x,y)) = 0.$$

Thus, (187) is proven in Case 2.

We have now proven (187) in each of the two Cases 1 and 2. Thus, (187) always holds (since Cases 1 and 2 cover all possibilities). Proposition 8.26 **(a)** is thus proven.

(b) For any two elements u and v of P, we define a subset $[u,v]$ of P by

$$[u,v] = \{w \in P \ \mid\ u \leq w \leq v\}.$$

This subset $[u,v]$ is finite (since P is finite), and thus its size $|[u,v]|$ is a nonnegative integer.

We shall now prove Proposition 8.26 **(b)** by strong induction on $|[x,y]|$:

Induction step: Let $N \in \mathbb{N}$. Assume that Proposition 8.26 **(b)** holds whenever $|[x,y]| < N$. We must now prove that Proposition 8.26 **(b)** holds whenever $|[x,y]| = N$.

We have assumed that Proposition 8.26 **(b)** holds whenever $|[x,y]| < N$. In other words, we have assumed the following claim:

> *Claim 1:* For any $x \in P$ and $y \in P$ satisfying $|[x,y]| < N$, we have
>
> $$\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y) = [x=y].$$

Now, let x and y be two elements of P satisfying $|[x,y]| = N$. We are going to prove that

$$\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y) = [x=y]. \tag{190}$$

If we do not have $x \leq y$, then (190) holds for obvious reasons[115]. Hence, for the rest of our proof of (190), we can WLOG assume that $x \leq y$. Assume this.

---

[115] *Proof.* Assume that we do not have $x \leq y$. Then, there exists no $z \in P$ satisfying $x \leq z \leq y$ (because if such a z would exist, then it would satisfy $x \leq z \leq y$, which would contradict with the fact that we do not have $x \leq y$). Therefore, the sum $\sum\limits_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y)$ is an empty sum.

If $x = y$, then (190) holds for obvious reasons[116]. Hence, for the rest of our proof of (190), we can WLOG assume that we don't have $x = y$. Assume this. Thus, we don't have $x = y$. In other words, we have $x \neq y$. Combining this with $x \leq y$, we obtain $x < y$.

Notice that $[x = y] = 0$ (since we don't have $x = y$).

For any pair $(z, t) \in P^2$, we have the following logical equivalence:

$$(x \leq z \leq y \text{ and } z \leq t \leq y) \iff (x \leq t \leq y \text{ and } x \leq z \leq t) \tag{191}$$

[117].

Hence,

$$\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y) = (\text{empty sum}) = 0.$$

But we do not have $x = y$ (because if we had $x = y$, then we would have $x \leq y$, which would contradict with the fact that we do not have $x \leq y$). Thus, $[x = y] = 0$. Now, $\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y) = 0 = [x = y]$. Hence, (190) is proven, qed.

[116] *Proof.* Assume that $x = y$. Then,

$$\left\{w \in P \mid x \leq w \leq \underbrace{y}_{=x}\right\} = \{w \in P \mid x \leq w \leq x\} = \{x\}$$

(by (189)). Now,

$$\underbrace{\sum_{\substack{z\in P;\\ x\leq z\leq y}}}_{\substack{=\sum_{z\in\{w\in P \ \mid\ x\leq w\leq y\}} = \sum_{z\in\{x\}}\\ (\text{since } \{w\in P \ \mid\ x\leq w\leq y\}=\{x\})}} \mu(z,y) = \sum_{z\in\{x\}} \mu(z,y) = \mu\left(x, \underbrace{y}_{=x}\right) = \mu(x,x) = 1$$

(by (181)). Comparing this with

$$[x = y] = 1 \qquad (\text{since } x = y),$$

we obtain $\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y) = [x = y]$. Hence, (190) is proven, qed.

[117] *Proof of (191):* Let $(z, t) \in P^2$. We shall first prove the logical implication

$$(x \leq z \leq y \text{ and } z \leq t \leq y) \implies (x \leq t \leq y \text{ and } x \leq z \leq t). \tag{192}$$

Proof of (192): Assume that $(x \leq z \leq y \text{ and } z \leq t \leq y)$ holds. We must prove that $(x \leq t \leq y \text{ and } x \leq z \leq t)$ holds. We have $x \leq z \leq t$, so that $x \leq t \leq y$. Also, $x \leq z \leq t$. Thus, $(x \leq t \leq y \text{ and } x \leq z \leq t)$ holds. This proves the implication (192).

Next, we shall prove the logical implication

$$(x \leq t \leq y \text{ and } x \leq z \leq t) \implies (x \leq z \leq y \text{ and } z \leq t \leq y). \tag{193}$$

Proof of (193): Assume that $(x \leq t \leq y \text{ and } x \leq z \leq t)$ holds. We must prove that $(x \leq z \leq y \text{ and } z \leq t \leq y)$ holds. We have $z \leq t \leq y$, thus $x \leq z \leq y$. Also, $z \leq t \leq y$. Thus, $(x \leq z \leq y \text{ and } z \leq t \leq y)$ holds. This proves the implication (193).

For every $t \in P$ satisfying $x \leq t < y$, we have

$$\left| [x,t] \right| < N \tag{194}$$

[118]. Therefore, for every $t \in P$ satisfying $x \leq t < y$, we have

$$\sum_{\substack{z \in P; \\ x \leq z \leq t}} \mu(z,t) = [x = t] \tag{195}$$

Combining the two implications (192) and (193), we obtain the logical equivalence

$$(x \leq z \leq y \text{ and } z \leq t \leq y) \iff (x \leq t \leq y \text{ and } x \leq z \leq t).$$

This proves (191).

[118] *Proof of (194):* Let $t \in P$ be such that $x \leq t < y$. We shall proceed in several steps:

- We have $[x,t] = \{w \in P \mid x \leq w \leq t\}$ (by the definition of $[x,t]$) and $[x,y] = \{w \in P \mid x \leq w \leq y\}$ (by the definition of $[x,y]$).
- Let us first prove that $[x,t] \subseteq [x,y]$.

  Indeed, let $g \in [x,t]$. Then, $g \in [x,t] = \{w \in P \mid x \leq w \leq t\}$. In other words, g is an element w of P satisfying $x \leq w \leq t$. In other words, g is an element of P and satisfies $x \leq g \leq t$. We have $g \leq t < y$, so that $g \leq y$. Combining this with $x \leq g$, we obtain $x \leq g \leq y$. Thus, g is an element of P and satisfies $x \leq g \leq y$. In other words, g is an element w of P satisfying $x \leq w \leq y$. In other words, $g \in \{w \in P \mid x \leq w \leq y\}$. Since $[x,y] = \{w \in P \mid x \leq w \leq y\}$, this rewrites as $g \in [x,y]$.

  Now, forget that we fixed g. Thus, we have shown that $g \in [x,y]$ for each $g \in [x,t]$. In other words, $[x,t] \subseteq [x,y]$.
- Now, let us prove that $y \notin [x,t]$.

  Indeed, assume the contrary. Thus, $y \in [x,t]$. Hence, $y \in [x,t] = \{w \in P \mid x \leq w \leq t\}$. In other words, y is an element w of P satisfying $x \leq w \leq t$. In other words, g is an element of P and satisfies $x \leq y \leq t$. But $y \leq t$ contradicts $t < y$ (since P is a poset). This contradiction proves that our assumption was wrong. Hence, $y \notin [x,t]$ is proven.
- Next, let us prove that $y \in [x,y]$.

  Indeed, y is an element of P and satisfies $x \leq y \leq y$. In other words, y is an element w of P satisfying $x \leq w \leq y$. In other words, $y \in \{w \in P \mid x \leq w \leq y\}$. Since $[x,y] = \{w \in P \mid x \leq w \leq y\}$, this rewrites as $y \in [x,y]$.
- Let us now prove that $[x,t] \neq [x,y]$.

  Indeed, assume the contrary. Thus, $[x,t] = [x,y]$. Now, $y \notin [x,t] = [x,y]$ contradicts $y \notin [x,y]$. This contradiction proves that our assumption was wrong. Hence, $[x,t] \neq [x,y]$ is proven.
- Combining $[x,t] \subseteq [x,y]$ with $[x,t] \neq [x,y]$, we conclude that $[x,t]$ is a proper subset of $[x,y]$. Thus, $\left|[x,t]\right| < \left|[x,y]\right|$ (since both $[x,t]$ and $[x,y]$ are finite sets). Hence, $\left|[x,t]\right| < \left|[x,y]\right| = N$. This proves (194).

[119]. Also, for every $u \in P$ and $v \in P$, we have

$$\sum_{\substack{t\in P;\\ u\leq t\leq v}} \mu\left(u,t\right) = \left[u=v\right] \tag{196}$$

[120].

Furthermore,

$$\left\{z \in P \ \mid \ x \leq z \leq y \text{ and } z = y\right\} = \left\{y\right\} \tag{197}$$

[121]. Thus,

$$\begin{aligned} &\left\{w \in P \ \mid \ x \leq w \leq y \text{ and } w = y\right\} \\ &= \left\{z \in P \ \mid \ x \leq z \leq y \text{ and } z = y\right\} \\ &\qquad\qquad \text{(here, we have renamed the index } w \text{ as } z\text{)} \\ &= \left\{y\right\}. \end{aligned} \tag{198}$$

Also,

$$\left\{z \in P \ \mid \ x \leq z \leq y \text{ and } z = x \text{ and } z \neq y\right\} = \left\{x\right\} \tag{199}$$

[122].

---

[119] *Proof of (195):* Let $t \in P$ be such that $x \leq t < y$. Then, $\left|\left[x,t\right]\right| < N$ (by (194)). Hence, Claim 1 (applied to t instead of y) shows that $\sum_{\substack{z\in P;\\ x\leq z\leq t}} \mu\left(z,t\right) = \left[x=t\right]$. This proves (195).

[120] *Proof of (196):* Let $u \in P$ and $v \in P$. Proposition 8.26 **(a)** (applied to $x = u$ and $y = v$) shows that $\sum_{\substack{z\in P;\\ u\leq z\leq v}} \mu\left(u,z\right) = \left[u=v\right]$. Now,

$$\begin{aligned} \sum_{\substack{t\in P;\\ u\leq t\leq v}} \mu\left(u,t\right) &= \sum_{\substack{z\in P;\\ u\leq z\leq v}} \mu\left(u,z\right) \qquad \text{(here, we have substituted } z \text{ for } t \text{ in the sum)} \\ &= \left[u=v\right]. \end{aligned}$$

This proves (196).

[121] *Proof of (197):* We know that y is an element of P and satisfies $x \leq y \leq y$ and $y = y$. In other words, y is an element z of P satisfying $x \leq z \leq y$ and $z = y$. In other words, $y \in \left\{z \in P \ \mid \ x \leq z \leq y \text{ and } z = y\right\}$. Thus, $\left\{y\right\} \subseteq \left\{z \in P \ \mid \ x \leq z \leq y \text{ and } z = y\right\}$.

On the other hand, let $g \in \left\{z \in P \ \mid \ x \leq z \leq y \text{ and } z = y\right\}$ be arbitrary. Thus, g is an element z of P satisfying $x \leq z \leq y$ and $z = y$. In other words, g is an element of P and satisfies $x \leq g \leq y$ and $g = y$. Thus, $g = y \in \left\{y\right\}$. Now, forget that we fixed g. We thus have shown that $g \in \left\{y\right\}$ for every $g \in \left\{z \in P \ \mid \ x \leq z \leq y \text{ and } z = y\right\}$. In other words, $\left\{z \in P \ \mid \ x \leq z \leq y \text{ and } z = y\right\} \subseteq \left\{y\right\}$. Combining this with $\left\{y\right\} \subseteq \left\{z \in P \ \mid \ x \leq z \leq y \text{ and } z = y\right\}$, we obtain

$$\left\{z \in P \ \mid \ x \leq z \leq y \text{ and } z = y\right\} = \left\{y\right\}.$$

This proves (197).

[122] *Proof of (199):* We know that x is an element of P and satisfies $x \leq x \leq y$ and $x = x$ and $x \neq y$. In other words, x is an element z of P satisfying $x \leq z \leq y$ and $z = x$ and $z \neq y$. In other words, $x \in \left\{z \in P \ \mid \ x \leq z \leq y \text{ and } z = x \text{ and } z \neq y\right\}$. Thus, $\left\{x\right\} \subseteq$

Now,

$$\begin{aligned}
&\underbrace{\sum_{\substack{(z,t)\in P^2;\\ x\leq z\leq y \text{ and } z\leq t\leq y}}}_{=\sum\limits_{\substack{z\in P;\\ x\leq z\leq y}}\ \sum\limits_{\substack{t\in P;\\ z\leq t\leq y}}} \mu(z,t) \\
&= \sum_{\substack{z\in P;\\ x\leq z\leq y}} \underbrace{\sum_{\substack{t\in P;\\ z\leq t\leq y}} \mu(z,t)}_{\substack{=[z=y]\\ \text{(by (196)}\\ \text{(applied to } u=z \text{ and } v=y\text{))}}} = \sum_{\substack{z\in P;\\ x\leq z\leq y}} [z=y] \\
&= \sum_{\substack{z\in P;\\ x\leq z\leq y \text{ and } z=y}} \underbrace{[z=y]}_{\substack{=1\\ \text{(since } z=y\text{)}}} + \sum_{\substack{z\in P;\\ x\leq z\leq y \text{ and } z\neq y}} \underbrace{[z=y]}_{\substack{=0\\ \text{(since } z=y \text{ is false}\\ \text{(since } z\neq y\text{))}}} \\
&\qquad \left(\text{since every } z\in P \text{ satisfies either } z=y \text{ or } z\neq y \text{ (but not both)}\right) \\
&= \sum_{\substack{z\in P;\\ x\leq z\leq y \text{ and } z=y}} 1 + \underbrace{\sum_{\substack{z\in P;\\ x\leq z\leq y \text{ and } z\neq y}} 0}_{=0} = \sum_{\substack{z\in P;\\ x\leq z\leq y \text{ and } z=y}} 1 \\
&= \left|\left\{z\in P \mid x\leq z\leq y \text{ and } z=y\right\}\right| \cdot 1 \\
&= \left|\underbrace{\left\{z\in P \mid x\leq z\leq y \text{ and } z=y\right\}}_{\substack{=\{y\}\\ \text{(by (197))}}}\right| = \left|\{y\}\right| = 1.
\end{aligned}$$

$\{z\in P \mid x\leq z\leq y \text{ and } z=x \text{ and } z\neq y\}$.

On the other hand, let $g \in \{z\in P \mid x\leq z\leq y \text{ and } z=x \text{ and } z\neq y\}$ be arbitrary. Thus, g is an element z of P satisfying $x\leq z\leq y$ and $z=x$ and $z\neq y$. In other words, g is an element of P and satisfies $x\leq g\leq y$ and $g=x$ and $g\neq y$. Thus, $g=x\in\{x\}$. Now, forget that we fixed g. We thus have shown that $g\in\{x\}$ for every $g \in \{z\in P \mid x\leq z\leq y \text{ and } z=x \text{ and } z\neq y\}$. In other words, $\{z\in P \mid x\leq z\leq y \text{ and } z=x \text{ and } z\neq y\} \subseteq \{x\}$. Combining this with $\{x\} \subseteq \{z\in P \mid x\leq z\leq y \text{ and } z=x \text{ and } z\neq y\}$, we obtain

$$\{z\in P \mid x\leq z\leq y \text{ and } z=x \text{ and } z\neq y\} = \{x\}.$$

This proves (199).

Hence,

$$\begin{aligned}
1 &= \underbrace{\sum_{\substack{(z,t)\in P^2;\\ x\leq z\leq y \text{ and } z\leq t\leq y}}}_{\substack{=\sum_{\substack{(z,t)\in P^2;\\ x\leq t\leq y \text{ and } x\leq z\leq t}}\\ \text{(because for any } (z,t)\in P^2\text{, the}\\ \text{condition } (x\leq z\leq y \text{ and } z\leq t\leq y)\\ \text{is equivalent to } (x\leq t\leq y \text{ and } x\leq z\leq t)\\ \text{(by (191)))}}} \mu\left(z,t\right) = \underbrace{\sum_{\substack{(z,t)\in P^2;\\ x\leq t\leq y \text{ and } x\leq z\leq t}}}_{=\sum_{\substack{t\in P;\\ x\leq t\leq y}} \sum_{\substack{z\in P;\\ x\leq z\leq t}}} \mu\left(z,t\right) \\
&= \sum_{\substack{t\in P;\\ x\leq t\leq y}} \sum_{\substack{z\in P;\\ x\leq z\leq t}} \mu\left(z,t\right) \\
&= \underbrace{\sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t=y}}}_{\substack{=\sum_{t\in\{w\in P\ \mid\ x\leq w\leq y \text{ and } w=y\}} = \sum_{t\in\{y\}}\\ \text{(since } \{w\in P\ \mid\ x\leq w\leq y \text{ and } w=y\}=\{y\}\text{)}}} \sum_{\substack{z\in P;\\ x\leq z\leq t}} \mu\left(z,t\right) \\
&\qquad + \sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t\neq y}} \underbrace{\sum_{\substack{z\in P;\\ x\leq z\leq t}} \mu\left(z,t\right)}_{\substack{=[x=t]\\ \text{(by (195)}\\ \text{(since } t<y \text{ (because } t\leq y \text{ and } t\neq y\text{) and } x\leq t\text{))}}} \\
&\qquad\qquad \left(\text{since every } t\in P \text{ satisfies either } t=y \text{ or } t\neq y \text{ (but not both)}\right) \\
&= \underbrace{\sum_{t\in\{y\}} \sum_{\substack{z\in P;\\ x\leq z\leq t}} \mu\left(z,t\right)}_{=\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y)} + \sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t\neq y}} \left[x=t\right] \\
&= \sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu\left(z,y\right) + \sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t\neq y}} \left[x=t\right].
\end{aligned}$$

Subtracting $\sum\limits_{\substack{z\in P;\\ x\leq z\leq y}} \mu\left(z,y\right)$ from both sides of this equality, we obtain

$$\begin{aligned}
1-\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu\left(z,y\right) &= \sum_{\substack{t\in P;\\ x\leq t\leq y\text{ and }t\neq y}} \left[x=t\right] \\
&= \underbrace{\sum_{\substack{t\in P;\\ x\leq t\leq y\text{ and }t=x\text{ and }t\neq y}}}_{\substack{=\sum_{t\in\left\{z\in P\ \mid\ x\leq z\leq y\text{ and }z=x\text{ and }z\neq y\right\}}=\sum_{t\in\left\{x\right\}}\\ \left(\text{since }\left\{z\in P\ \mid\ x\leq z\leq y\text{ and }z=x\text{ and }z\neq y\right\}=\left\{x\right\}\right)}} \underbrace{\left[x=t\right]}_{\substack{=1\\ \left(\text{since }x=t\right.\\ \left.\left(\text{since }t=x\right)\right)}} \\
&\qquad + \sum_{\substack{t\in P;\\ x\leq t\leq y\text{ and }t\neq x\text{ and }t\neq y}} \underbrace{\left[x=t\right]}_{\substack{=0\\ \left(\text{since }x=t\text{ is false}\right.\\ \left.\left(\text{since }x\neq t\ \left(\text{since }t\neq x\right)\right)\right)}} \\
&\qquad \left(\begin{array}{c}\text{since every }t\in P\text{ satisfies}\\ \text{either }t=x\text{ or }t\neq x\text{ (but not both)}\end{array}\right) \\
&= \sum_{t\in\left\{x\right\}} 1 + \underbrace{\sum_{\substack{t\in P;\\ x\leq t\leq y\text{ and }t\neq x\text{ and }t\neq y}} 0}_{=0} = \sum_{t\in\left\{x\right\}} 1 = 1.
\end{aligned}$$

Solving this equality for $\sum\limits_{\substack{z\in P;\\ x\leq z\leq y}} \mu\left(z,y\right)$, we obtain

$$\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu\left(z,y\right) = 1-1 = 0 = \left[x=y\right].$$

Thus, (190) is proven.

Let us now forget that we fixed x and y. We thus have proven that for any $x\in P$ and $y\in P$ satisfying $\left|\left[x,y\right]\right| = N$, we have

$$\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu\left(z,y\right) = \left[x=y\right].$$

In other words, Proposition 8.26 **(b)** holds whenever $\left|\left[x,y\right]\right| = N$. This completes the induction step. Thus, Proposition 8.26 **(b)** is proven by induction.

(c) For every $v \in P$, we have

$$\begin{aligned}
&\sum_{\substack{y\in P;\\ y\leq v}} \mu\left(y,v\right) \sum_{\substack{x\in P;\\ x\leq y}} \beta_x \\
&= \sum_{\substack{z\in P;\\ z\leq v}} \mu\left(z,v\right) \sum_{\substack{x\in P;\\ x\leq z}} \beta_x \qquad \left(\begin{array}{c}\text{here, we have renamed the summation}\\ \text{index } y \text{ as } z \text{ in the outer sum}\end{array}\right) \\
&= \underbrace{\sum_{\substack{z\in P;\\ z\leq v}} \sum_{\substack{x\in P;\\ x\leq z}}}_{=\sum_{\substack{(x,z)\in P^2;\\ z\leq v \text{ and } x\leq z}} = \sum_{x\in P} \sum_{\substack{z\in P;\\ z\leq v \text{ and } x\leq z}}} \mu\left(z,v\right) \beta_x \\
&= \sum_{x\in P} \underbrace{\sum_{\substack{z\in P;\\ z\leq v \text{ and } x\leq z}}}_{\substack{= \sum_{\substack{z\in P;\\ x\leq z \text{ and } z\leq v}} \\ = \sum_{\substack{z\in P;\\ x\leq z\leq v}}}} \mu\left(z,v\right) \beta_x = \sum_{x\in P} \sum_{\substack{z\in P;\\ x\leq z\leq v}} \mu\left(z,v\right) \beta_x = \sum_{x\in P} \underbrace{\left(\sum_{\substack{z\in P;\\ x\leq z\leq v}} \mu\left(z,v\right)\right)}_{\substack{=[x=v] \\ \text{(by Proposition 8.26 (b)} \\ \text{(applied to } y=v\text{))}}} \beta_x \\
&= \sum_{x\in P} \left[x=v\right] \beta_x = \sum_{\substack{x\in P;\\ x=v}} \underbrace{\left[x=v\right]}_{\substack{=1 \\ \text{(since } x=v\text{)}}} \beta_x + \sum_{\substack{x\in P;\\ x\neq v}} \underbrace{\left[x=v\right]}_{\substack{=0 \\ \text{(since } x=v \text{ is false} \\ \text{(since } x\neq v\text{))}}} \beta_x \\
&\qquad \left(\text{since every } x\in P \text{ satisfies either } x=v \text{ or } x\neq v \text{ (but not both)}\right) \\
&= \sum_{\substack{x\in P;\\ x=v}} \beta_x + \underbrace{\sum_{\substack{x\in P;\\ x\neq v}} 0\beta_x}_{=0} = \sum_{\substack{x\in P;\\ x=v}} \beta_x = \beta_v \qquad \left(\text{since } v\in P\right).
\end{aligned}$$

Renaming v as z in this statement, we obtain the following: For every $z \in P$, we have

$$\sum_{\substack{y\in P;\\ y\leq z}} \mu\left(y,z\right) \sum_{\substack{x\in P;\\ x\leq y}} \beta_x = \beta_z.$$

This proves Proposition 8.26 **(c)**. ☐

Proof of Theorem 8.20. If T is a subset of E, then $\overline{T}$ is a flat of M (by Proposition 8.12 **(a)**). In other words, if T is a subset of E, then $\overline{T} \in \operatorname{Flats} M$. Renaming T as B in this statement, we conclude that if B is a subset of E, then $\overline{B} \in \operatorname{Flats} M$.

For every $F \in \operatorname{Flats} M$, define an element $\beta_F \in \mathbf{k}$ by

$$\beta_F = \sum_{\substack{B\subseteq E;\\ \overline{B}=F}} \left(-1\right)^{\left|B\right|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K\right).$$

Now, using Lemma 8.25, we can easily see that

$$\sum_{\substack{G\in\operatorname{Flats}M;\\ G\subseteq F}} \beta_G = [F=\varnothing] \qquad \text{for every } F\in\operatorname{Flats}M \tag{200}$$

[123].

Let μ be the Möbius function of the lattice $\operatorname{Flats}M$. The element $\overline{\varnothing}$ is the global minimum of the poset $\operatorname{Flats}M$. [124] In particular, $\overline{\varnothing}\in\operatorname{Flats}M$ and $\overline{\varnothing}\subseteq F$. Hence, $\mu\left(\overline{\varnothing},F\right)$ is well-defined.

Now, fix $F\in\operatorname{Flats}M$. Proposition 8.26 **(c)** (applied to $P=\operatorname{Flats}M$ and $z=F$)

---

[123] *Proof of (200):* Let $F\in\operatorname{Flats}M$. Thus, F is a flat of M.

If B is a subset of E, then the statements $\left(\overline{B}\subseteq F\right)$ and $(B\subseteq F)$ are equivalent. (This follows from Proposition 8.12 **(g)**, applied to $T=B$ and $G=F$.)

Now,

$$\begin{aligned}
&\sum_{\substack{G\in\operatorname{Flats}M;\\ G\subseteq F}} \underbrace{\beta_G}_{\substack{=\sum\limits_{\substack{B\subseteq E;\\ \overline{B}=G}}(-1)^{|B|}\left(\prod\limits_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K\right)\\ \text{(by the definition of }\beta_G\text{)}}}\\
&=\underbrace{\sum_{\substack{G\in\operatorname{Flats}M;\\ G\subseteq F}}\ \sum_{\substack{B\subseteq E;\\ \overline{B}=G}}}_{\substack{=\sum\limits_{\substack{B\subseteq E;\\ \overline{B}\subseteq F}}\\ \text{(because if }B\text{ is a subset of }E\text{,}\\ \text{then }\overline{B}\in\operatorname{Flats}M\text{)}}} (-1)^{|B|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K\right)\\
&=\underbrace{\sum_{\substack{B\subseteq E;\\ \overline{B}\subseteq F}}}_{\substack{=\sum\limits_{\substack{B\subseteq E;\\ B\subseteq F}}\\ \text{(because if }B\text{ is a subset of }E\text{, then}\\ \text{the statements }\left(\overline{B}\subseteq F\right)\text{ and }(B\subseteq F)\text{ are}\\ \text{equivalent)}}} (-1)^{|B|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K\right) = \underbrace{\sum_{\substack{B\subseteq E;\\ B\subseteq F}}}_{=\sum\limits_{B\subseteq F}} (-1)^{|B|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K\right)\\
&=\sum_{B\subseteq F}(-1)^{|B|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K\right)=[F=\varnothing] \qquad \text{(by (183))}.
\end{aligned}$$

This proves (200).

[124] This was proven during our proof of Proposition 8.14.

shows that

$$\begin{aligned}
\beta_F &= \sum_{\substack{y\in\operatorname{Flats}M;\\ y\subseteq F}} \mu(y,F) \sum_{\substack{x\in\operatorname{Flats}M;\\ x\subseteq y}} \beta_x \\
&\qquad\qquad (\text{since the relation } \leq \text{ of the poset } \operatorname{Flats}M \text{ is } \subseteq) \\
&= \sum_{\substack{H\in\operatorname{Flats}M;\\ H\subseteq F}} \mu(H,F) \underbrace{\sum_{\substack{G\in\operatorname{Flats}M;\\ G\subseteq H}} \beta_G}_{\substack{=[H=\varnothing]\\ \text{(by (200), applied to}\\ H \text{ instead of } F\text{)}}} \\
&\qquad\qquad (\text{here, we renamed the summation indices } y \text{ and } x \text{ as } H \text{ and } G) \\
&= \sum_{\substack{H\in\operatorname{Flats}M;\\ H\subseteq F}} \mu(H,F)\,[H=\varnothing] \\
&= \sum_{\substack{H\in\operatorname{Flats}M;\\ H\subseteq F;\\ H=\varnothing}} \mu(H,F) \underbrace{[H=\varnothing]}_{\substack{=1\\ \text{(since } H=\varnothing\text{)}}} + \sum_{\substack{H\in\operatorname{Flats}M;\\ H\subseteq F;\\ H\neq\varnothing}} \mu(H,F) \underbrace{[H=\varnothing]}_{\substack{=0\\ \text{(since } H\neq\varnothing\text{)}}} \\
&= \underbrace{\sum_{\substack{H\in\operatorname{Flats}M;\\ H\subseteq F;\\ H=\varnothing}}}_{\substack{=\sum_{\substack{H\in\operatorname{Flats}M;\\ H=\varnothing}}\\ \text{(since the condition } H\subseteq F\\ \text{is automatically implied by}\\ \text{the condition } H=\varnothing\text{)}}} \mu(H,F) \\
&= \sum_{\substack{H\in\operatorname{Flats}M;\\ H=\varnothing}} \mu(H,F). \qquad (201)
\end{aligned}$$

Now, we shall prove that

$$\beta_F = \left[\overline{\varnothing}=\varnothing\right]\mu\left(\overline{\varnothing},F\right). \qquad (202)$$

Proof of (202): We are in one of the following two cases:

Case 1: We have $\overline{\varnothing}=\varnothing$.

Case 2: We have $\overline{\varnothing}\neq\varnothing$.

Let us consider Case 1 first. In this case, we have $\overline{\varnothing}=\varnothing$. Hence, $\varnothing=\overline{\varnothing}\in\operatorname{Flats}M$. Thus, the sum $\sum_{\substack{H\in\operatorname{Flats}M;\\ H=\varnothing}}\mu(H,F)$ has exactly one addend: namely, the addend for $H=\varnothing$. Thus, $\sum_{\substack{H\in\operatorname{Flats}M;\\ H=\varnothing}}\mu(H,F)=\mu\left(\underbrace{\varnothing}_{=\overline{\varnothing}},F\right)=\mu\left(\overline{\varnothing},F\right)$. Thus, (201) becomes $\beta_F=\sum_{\substack{H\in\operatorname{Flats}M;\\ H=\varnothing}}\mu(H,F)=\mu\left(\overline{\varnothing},F\right)$. Comparing this with

$\underbrace{\left[\overline{\varnothing}=\varnothing\right]}_{\substack{=1\\ (\text{since } \overline{\varnothing}=\varnothing)}} \mu\left(\overline{\varnothing},F\right) = \mu\left(\overline{\varnothing},F\right)$, we obtain $\beta_F = \left[\overline{\varnothing}=\varnothing\right] \mu\left(\overline{\varnothing},F\right)$. Thus, (202) is proven in Case 1.

Let us now consider Case 2. In this case, we have $\overline{\varnothing} \neq \varnothing$. Thus, there exists no $H \in \operatorname{Flats} M$ such that $H = \overline{\varnothing}$ [125]. Hence, the sum $\sum_{\substack{H\in\operatorname{Flats} M;\\ H=\overline{\varnothing}}} \mu\left(H,F\right)$ is empty. Thus, $\sum_{\substack{H\in\operatorname{Flats} M;\\ H=\overline{\varnothing}}} \mu\left(H,F\right) = \left(\text{empty sum}\right) = 0$, so that (201) becomes $\beta_F = \sum_{\substack{H\in\operatorname{Flats} M;\\ H=\overline{\varnothing}}} \mu\left(H,F\right) = 0$. Comparing this with $\underbrace{\left[\overline{\varnothing}=\varnothing\right]}_{\substack{=0\\ (\text{since } \overline{\varnothing}\neq\varnothing)}} \mu\left(\overline{\varnothing},F\right) = 0$, we obtain $\beta_F = \left[\overline{\varnothing}=\varnothing\right] \mu\left(\overline{\varnothing},F\right)$. Thus, (202) is proven in Case 2.

Now, we have proven (202) in both possible Cases 1 and 2. Thus, (202) always holds.

Now, let us forget that we fixed F. We thus have proven (202) for each $F \in \operatorname{Flats} M$.

---

[125] *Proof.* Assume the contrary. Thus, there exists some $H \in \operatorname{Flats} M$ such that $H = \overline{\varnothing}$. In other words, $\overline{\varnothing} \in \operatorname{Flats} M$. Hence, $\overline{\varnothing}$ is a flat of M. Proposition 8.12 **(b)** (applied to $G = \varnothing$) thus shows that $\overline{\varnothing} = \varnothing$. This contradicts $\overline{\varnothing} \neq \varnothing$. This contradiction proves that our assumption was wrong, qed.

Now,

$$\begin{aligned}
&\sum_{F\subseteq E}(-1)^{|F|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}}a_K\right)x^{m-r_M(F)}\\
&=\underbrace{\sum_{B\subseteq E}}_{\substack{=\sum\limits_{F\in\operatorname{Flats}M}\ \sum\limits_{\substack{B\subseteq E;\\ \overline{B}=F}}\\ \text{(because if } B \text{ is a subset of } E\text{,}\\ \text{then } \overline{B}\in\operatorname{Flats}M\text{)}}}(-1)^{|B|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K\right)\underbrace{x^{m-r_M(B)}}_{\substack{=x^{m-r_M(\overline{B})}\\ \text{(since Proposition 8.12 \textbf{(f)} (applied to } T=B\text{)}\\ \text{shows that } r_M(B)=r_M(\overline{B})\text{)}}}\\
&\qquad\qquad\text{(here, we have renamed the summation index } F \text{ as } B\text{)}\\
&=\sum_{F\in\operatorname{Flats}M}\ \sum_{\substack{B\subseteq E;\\ \overline{B}=F}}(-1)^{|B|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K\right)\underbrace{x^{m-r_M(\overline{B})}}_{\substack{=x^{m-r_M(F)}\\ \text{(since } \overline{B}=F\text{)}}}\\
&=\sum_{F\in\operatorname{Flats}M}\underbrace{\sum_{\substack{B\subseteq E;\\ \overline{B}=F}}(-1)^{|B|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K\right)}_{\substack{=\beta_F=[\overline{\varnothing}=\varnothing]\mu(\overline{\varnothing},F)\\ \text{(by (202))}}}x^{m-r_M(F)}\\
&=\sum_{F\in\operatorname{Flats}M}[\overline{\varnothing}=\varnothing]\,\mu\left(\overline{\varnothing},F\right)x^{m-r_M(F)}\\
&=[\overline{\varnothing}=\varnothing]\sum_{F\in\operatorname{Flats}M}\mu\left(\overline{\varnothing},F\right)x^{m-r_M(F)}. \qquad (203)
\end{aligned}$$

But the definition of χ_M yields $\chi_M=\sum\limits_{F\in\operatorname{Flats}M}\mu\left(\overline{\varnothing},F\right)x^{m-r_M(F)}$. The definition of $\widetilde{\chi}_M$ yields

$$\begin{aligned}
\widetilde{\chi}_M&=[\overline{\varnothing}=\varnothing]\underbrace{\chi_M}_{=\sum\limits_{F\in\operatorname{Flats}M}\mu(\overline{\varnothing},F)x^{m-r_M(F)}}=[\overline{\varnothing}=\varnothing]\sum_{F\in\operatorname{Flats}M}\mu\left(\overline{\varnothing},F\right)x^{m-r_M(F)}\\
&=\sum_{F\subseteq E}(-1)^{|F|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}}a_K\right)x^{m-r_M(F)}\qquad\text{(by (203))}.
\end{aligned}$$

This proves Theorem 8.20. ☐

Proof of Corollary 8.22. Corollary 8.22 can be derived from Theorem 8.20 in the same way as Corollary 1.17 was derived from Theorem 1.15. ☐

Proof of Theorem 8.21. Theorem 8.21 can be derived from Theorem 8.20 in the same way as Theorem 1.11 was derived from Theorem 1.15. ☐

Proof of Corollary 8.23. Corollary 8.23 follows from Corollary 8.22 when $\mathfrak{K}$ is set to be the set of **all** broken circuits of M. ☐

Proof of Corollary 8.24. If F is a subset of E such that F contains no broken circuit of M as a subset, then

$$r_M(F) = |F| \tag{204}$$

[126]. Now, Corollary 8.23 yields

$$\widetilde{\chi}_M = \sum_{\substack{F\subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } M \text{ as a subset}}} (-1)^{|F|} \underbrace{x^{m-r_M(F)}}_{\substack{=x^{m-|F|}\\ (\text{by (204)})}} = \sum_{\substack{F\subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } M \text{ as a subset}}} (-1)^{|F|} x^{m-|F|}.$$

This proves Corollary 8.24. ☐

8.4. A vanishing alternating sum for matroids

As an application of the above, we can prove an analogue of the alternating sum identity of Dahlberg and van Willigenburg (Theorem 7.1 above) for the characteristic polynomials of matroids:

Theorem 8.27. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. Let C be a circuit of M, and let $e \in C$ be arbitrary. Then,

$$\sum_{F\subseteq C\setminus\{e\}} (-1)^{|F|} x^{m-r_M(E\setminus F)} \cdot \widetilde{\chi}_{M\setminus F} = 0.$$

Here, whenever F is a subset of E, the notation $M \setminus F$ denotes the matroid $(E \setminus F, \mathcal{I} \cap \mathcal{P}(E \setminus F))$ (that is, the matroid whose ground set is $E \setminus F$ and whose independent sets are those subsets of $E \setminus F$ that are independent in M).

Proof of Theorem 8.27. This proof is rather similar to our above proof of Theorem 7.2, but using Theorem 8.21 and Corollary 8.22 instead of Theorem 6.34 and Corollary 6.36. Here is the argument in detail:

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[126] *Proof of (204):* Let F be a subset of E such that F contains no broken circuit of M as a subset.

We shall show that $F \in \mathcal{I}$. Indeed, assume the contrary. Thus, $F \notin \mathcal{I}$, so that $F \in \mathcal{P}(E) \setminus \mathcal{I}$. Hence, there exists a circuit C of M such that $C \subseteq F$ (according to Lemma 8.7, applied to $Q = F$). Consider this C. The set C is a circuit, and thus nonempty (because the empty set is in $\mathcal{I}$). Let e be the unique element of C having maximum label. (This is clearly well-defined, since the labeling function ℓ is injective.) Then, $C \setminus \{e\}$ is a broken circuit of M (by the definition of a broken circuit). Thus, F contains a broken circuit of M as a subset (since $C \setminus \{e\} \subseteq C \subseteq F$). This contradicts the fact that F contains no broken circuit of M as a subset. This contradiction shows that our assumption was wrong. Hence, $F \in \mathcal{I}$ is proven.

Thus, Lemma 8.6 (applied to $T = F$) shows that $r_M(F) = |F|$, qed.

Let us set $B := C \setminus \{e\}$. Thus, $B = C \setminus \{e\} \subseteq C \subseteq E$.
Now, we shall show the following:

Claim 1: Let J be a subset of B. Then,

$$x^{m-r_M(E\setminus J)} \cdot \widetilde{\chi}_{M\setminus J} = \sum_{\substack{F\subseteq E;\\ J\subseteq E\setminus F}} (-1)^{|F|} x^{m-r_M(F)}.$$

[*Proof of Claim 1:* First, we observe that every subset F of $E \setminus J$ satisfies

$$r_{M\setminus J}(F) = r_M(F) \tag{205}$$

[127].

Applying this to $F = E \setminus J$, we obtain $r_{M\setminus J}(E \setminus J) = r_M(E \setminus J)$. In other words, $r_M(E \setminus J) = r_{M\setminus J}(E \setminus J)$.

However, the definition of the matroid $M \setminus J$ yields $M \setminus J = (E \setminus J, \mathcal{I} \cap \mathcal{P}(E \setminus J))$. Hence, Theorem 8.21 (applied to $M \setminus J$, $E \setminus J$, $\mathcal{I} \cap \mathcal{P}(E \setminus J)$ and $r_M(E \setminus J)$ instead

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[127] *Proof.* Let F be a subset of $E \setminus J$. Then, $F \subseteq E \setminus J \subseteq E$. Hence, the definition of the rank function r_M yields

$$r_M(F) = \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq F\}. \tag{206}$$

Furthermore, the definition of the matroid $M \setminus J$ yields $M \setminus J = (E \setminus J, \mathcal{I} \cap \mathcal{P}(E \setminus J))$. Hence, the definition of its rank function $r_{M\setminus J}$ yields

$$r_{M\setminus J}(F) = \max\{|Z| \mid Z \in \mathcal{I} \cap \mathcal{P}(E \setminus J) \text{ and } Z \subseteq F\}. \tag{207}$$

However, we can make the following two observations:

- Each $Z \in \mathcal{I}$ that satisfies $Z \subseteq F$ must automatically satisfy $Z \in \mathcal{I} \cap \mathcal{P}(E \setminus J)$ (because $Z \subseteq F \subseteq E \setminus J$ yields $Z \in \mathcal{P}(E \setminus J)$, and thus we can combine $Z \in \mathcal{I}$ with $Z \in \mathcal{P}(E \setminus J)$ to obtain $Z \in \mathcal{I} \cap \mathcal{P}(E \setminus J)$). Hence, each $Z \in \mathcal{I}$ that satisfies $Z \subseteq F$ must be a $Z \in \mathcal{I} \cap \mathcal{P}(E \setminus J)$ that satisfies $Z \subseteq F$. In other words,

  $$\{Z \in \mathcal{I} \mid Z \subseteq F\} \subseteq \{Z \in \mathcal{I} \cap \mathcal{P}(E \setminus J) \mid Z \subseteq F\}.$$

  Therefore,

  $$\begin{aligned}&\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq F\}\\ &\subseteq \{|Z| \mid Z \in \mathcal{I} \cap \mathcal{P}(E \setminus J) \text{ and } Z \subseteq F\}.\end{aligned} \tag{208}$$

- Each $Z \in \mathcal{I} \cap \mathcal{P}(E \setminus J)$ that satisfies $Z \subseteq F$ must automatically satisfy $Z \in \mathcal{I}$ (because $Z \in \mathcal{I} \cap \mathcal{P}(E \setminus J) \subseteq \mathcal{I}$). Hence, each $Z \in \mathcal{I} \cap \mathcal{P}(E \setminus J)$ that satisfies $Z \subseteq F$ must be a $Z \in \mathcal{I}$ that satisfies $Z \subseteq F$. In other words,

  $$\{Z \in \mathcal{I} \cap \mathcal{P}(E \setminus J) \mid Z \subseteq F\} \subseteq \{Z \in \mathcal{I} \mid Z \subseteq F\}.$$

  Therefore,

  $$\begin{aligned}&\{|Z| \mid Z \in \mathcal{I} \cap \mathcal{P}(E \setminus J) \text{ and } Z \subseteq F\}\\ &\subseteq \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq F\}.\end{aligned} \tag{209}$$

Combining (208) with (209), we obtain

$$\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq F\} = \{|Z| \mid Z \in \mathcal{I} \cap \mathcal{P}(E \setminus J) \text{ and } Z \subseteq F\}.$$

Hence, we can rewrite (206) as

$$r_M(F) = \max\{|Z| \mid Z \in \mathcal{I} \cap \mathcal{P}(E \setminus J) \text{ and } Z \subseteq F\}.$$

Comparing this with (207), we obtain $r_{M\setminus J}(F) = r_M(F)$. This proves (205).

of M, E, $\mathcal{I}$ and m) yields

$$\begin{aligned}\widetilde{\chi}_{M\setminus J} &= \sum_{F\subseteq E\setminus J} (-1)^{|F|} \underbrace{x^{r_M(E\setminus J)-r_{M\setminus J}(F)}}_{\substack{=x^{r_M(E\setminus J)-r_M(F)} \\ \text{(since } r_{M\setminus J}(F)=r_M(F) \\ \text{(by (205))}}} \qquad \left(\text{since } r_M(E\setminus J) = r_{M\setminus J}(E\setminus J)\right) \\ &= \sum_{F\subseteq E\setminus J} (-1)^{|F|} x^{r_M(E\setminus J)-r_M(F)}. \qquad (210)\end{aligned}$$

However, we have $J \subseteq B \subseteq E$. Thus, it is easy to see that

$$\mathcal{P}(E\setminus J) = \{Z\in\mathcal{P}(E) \mid J\subseteq E\setminus Z\} \qquad (211)$$

[128].

Now, we have the following equality between summation signs:

$$\begin{aligned}\sum_{F\subseteq E\setminus J} = \sum_{F\in\mathcal{P}(E\setminus J)} &= \sum_{F\in\{Z\in\mathcal{P}(E) \mid J\subseteq E\setminus Z\}} \qquad (\text{by (211)}) \\ &= \sum_{\substack{F\in\mathcal{P}(E);\\ J\subseteq E\setminus F}} = \sum_{\substack{F\subseteq E;\\ J\subseteq E\setminus F}}.\end{aligned}$$

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[128] *Proof:* Let $H \in \mathcal{P}(E\setminus J)$. Thus, H is a subset of $E\setminus J$. Hence, $H\subseteq E\setminus J\subseteq E$, so that $H\in\mathcal{P}(E)$. Also, from $H\subseteq E\setminus J$, we conclude that H is disjoint from J. In other words, J is disjoint from H. Combining this with $J\subseteq E$, we obtain $J\subseteq E\setminus H$.

Now, we know that $H\in\mathcal{P}(E)$ and $J\subseteq E\setminus H$. In other words, H is a $Z\in\mathcal{P}(E)$ satisfying $J\subseteq E\setminus Z$. In other words, $H\in\{Z\in\mathcal{P}(E) \mid J\subseteq E\setminus Z\}$.

Forget that we fixed H. We thus have shown that $H\in\{Z\in\mathcal{P}(E) \mid J\subseteq E\setminus Z\}$ for each $H\in\mathcal{P}(E\setminus J)$. In other words,

$$\mathcal{P}(E\setminus J) \subseteq \{Z\in\mathcal{P}(E) \mid J\subseteq E\setminus Z\}. \qquad (212)$$

On the other hand, let $U\in\{Z\in\mathcal{P}(E) \mid J\subseteq E\setminus Z\}$. Thus, U is a $Z\in\mathcal{P}(E)$ satisfying $J\subseteq E\setminus Z$. In other words, $U\in\mathcal{P}(E)$ and $J\subseteq E\setminus U$.

From $J\subseteq E\setminus U$, we conclude that J is disjoint from U. In other words, U is disjoint from J. However, from $U\in\mathcal{P}(E)$, we conclude that U is a subset of E. Thus, U is a subset of E that is disjoint from J. In other words, U is a subset of $E\setminus J$. In other words, $U\in\mathcal{P}(E\setminus J)$.

Forget that we fixed U. We thus have shown that $U\in\mathcal{P}(E\setminus J)$ for each $U\in\{Z\in\mathcal{P}(E) \mid J\subseteq E\setminus Z\}$. In other words,

$$\{Z\in\mathcal{P}(E) \mid J\subseteq E\setminus Z\} \subseteq \mathcal{P}(E\setminus J).$$

Combining this with (212), we obtain

$$\mathcal{P}(E\setminus J) = \{Z\in\mathcal{P}(E) \mid J\subseteq E\setminus Z\}.$$

This proves (211).

Hence, (210) becomes

$$\widetilde{\chi}_{M\setminus J}=\underbrace{\sum_{F\subseteq E\setminus J}}_{=\sum_{\substack{F\subseteq E;\\ J\subseteq E\setminus F}}}(-1)^{|F|}x^{r_M(E\setminus J)-r_M(F)}=\sum_{\substack{F\subseteq E;\\ J\subseteq E\setminus F}}(-1)^{|F|}x^{r_M(E\setminus J)-r_M(F)}.$$

Multiplying both sides of this equality by $x^{m-r_M(E\setminus J)}$, we obtain

$$\begin{aligned}x^{m-r_M(E\setminus J)}\cdot\widetilde{\chi}_{M\setminus J}&=x^{m-r_M(E\setminus J)}\cdot\sum_{\substack{F\subseteq E;\\ J\subseteq E\setminus F}}(-1)^{|F|}x^{r_M(E\setminus J)-r_M(F)}\\&=\sum_{\substack{F\subseteq E;\\ J\subseteq E\setminus F}}(-1)^{|F|}\underbrace{x^{m-r_M(E\setminus J)}\cdot x^{r_M(E\setminus J)-r_M(F)}}_{\substack{=x^{(m-r_M(E\setminus J))+(r_M(E\setminus J)-r_M(F))}\\ =x^{m-r_M(F)}\\ \text{(since }(m-r_M(E\setminus J))+(r_M(E\setminus J)-r_M(F))=m-r_M(F))}}\\&=\sum_{\substack{F\subseteq E;\\ J\subseteq E\setminus F}}(-1)^{|F|}x^{m-r_M(F)}.\end{aligned}$$

Thus, Claim 1 is proved.]

Claim 2: We have

$$\sum_{\substack{F\subseteq E;\\ B\subseteq F}}(-1)^{|F|}x^{m-r_M(F)}=0. \tag{213}$$

[*Proof of Claim 2:* Theorem 8.21 yields

$$\begin{aligned}\widetilde{\chi}_M&=\sum_{F\subseteq E}(-1)^{|F|}x^{m-r_M(F)}\\&=\sum_{\substack{F\subseteq E;\\ B\subseteq F}}(-1)^{|F|}x^{m-r_M(F)}+\sum_{\substack{F\subseteq E;\\ B\not\subseteq F}}(-1)^{|F|}x^{m-r_M(F)}\end{aligned} \tag{214}$$

(since each subset F of E satisfies either $B\subseteq F$ or $B\not\subseteq F$, but not both at the same time).

Let us now find a different formula for $\widetilde{\chi}_M$. We define a function $\ell:E\to\mathbb{N}$ by setting

$$\ell(f)=[f=e]\qquad\text{for each }f\in E.$$

We shall use this function ℓ as our labeling function (where the role of the totally ordered set X is played by $\mathbb{N}$ equipped with the usual total order). It is easy to see that the element e is the unique element of C having maximum label[129].

---

[129] *Proof.* Clearly, e is an element of C (since $e\in C$). We shall now show that e has a larger label than any other element of C.

Therefore, $C \setminus \{e\}$ is a broken circuit of M (by the definition of a "broken circuit" in Definition 8.19). In other words, B is a broken circuit of M (since $B = C \setminus \{e\}$). Hence, $\{B\}$ is a set of broken circuits of M. Therefore, Corollary 8.22 (applied to $\mathfrak{K} = \{B\}$) yields

$$\widetilde{\chi}_M = \sum_{\substack{F \subseteq E; \\ F \text{ is } \{B\}\text{-free}}} (-1)^{|F|} x^{m - r_M(F)}. \tag{215}$$

However, if F is a subset of E, then the condition "F is $\{B\}$-free" is equivalent to "$B \not\subseteq F$" [130]. Hence, the summation sign "$\sum\limits_{\substack{F \subseteq E; \\ F \text{ is } \{B\}\text{-free}}}$" can be rewritten as "$\sum\limits_{\substack{F \subseteq E; \\ B \not\subseteq F}}$". Therefore, we can rewrite (215) as

$$\widetilde{\chi}_M = \sum_{\substack{F \subseteq E; \\ B \not\subseteq F}} (-1)^{|F|} x^{m - r_M(F)}. \tag{216}$$

Subtracting this equality from (214), we obtain

$$\begin{aligned} \widetilde{\chi}_M - \widetilde{\chi}_M &= \left(\sum_{\substack{F \subseteq E; \\ B \subseteq F}} (-1)^{|F|} x^{m - r_M(F)} + \sum_{\substack{F \subseteq E; \\ B \not\subseteq F}} (-1)^{|F|} x^{m - r_M(F)} \right) - \sum_{\substack{F \subseteq E; \\ B \not\subseteq F}} (-1)^{|F|} x^{m - r_M(F)} \\ &= \sum_{\substack{F \subseteq E; \\ B \subseteq F}} (-1)^{|F|} x^{m - r_M(F)}. \end{aligned}$$

Indeed, let f be any element of C distinct from e. Then, the definition of ℓ yields $\ell(f) = [f = e] = 0$ (since we don't have $f = e$ (because f is distinct from e)). On the other hand, the definition of ℓ yields $\ell(e) = [e = e] = 1$ (since $e = e$). Hence, $\ell(e) = 1 > 0 = \ell(f)$. In other words, e has a larger label than f (since the label of e is $\ell(e)$, whereas the label of f is $\ell(f)$).

Forget that we fixed f. We thus have shown that e has a larger label than f whenever f is any element of C distinct from e. In other words, e has a larger label than any other element of C. Hence, e is the unique element of C having maximum label (since e itself is an element of C).

[130] *Proof.* Let F be a subset of E. Then, we have the following chain of logical equivalences:

$$\begin{aligned} & (F \text{ is } \{B\}\text{-free}) \\ \Longleftrightarrow\ & (F \text{ contains no } K \in \{B\} \text{ as a subset}) \\ & \qquad \left(\text{by the definition of "} \{B\}\text{-free" in Definition 1.16}\right) \\ \Longleftrightarrow\ & (\text{there exists no } K \in \{B\} \text{ such that } F \text{ contains } K \text{ as a subset}) \\ \Longleftrightarrow\ & (\text{there exists no } K \in \{B\} \text{ such that } K \subseteq F) \\ \Longleftrightarrow\ & (\text{each } K \in \{B\} \text{ satisfies } K \not\subseteq F) \\ \Longleftrightarrow\ & (B \not\subseteq F) \qquad (\text{since the only } K \in \{B\} \text{ is } B). \end{aligned}$$

Hence, the condition "F is $\{B\}$-free" is equivalent to "$B \not\subseteq F$". Qed.

Comparing this with $\widetilde{\chi}_M - \widetilde{\chi}_M = 0$, we obtain

$$\sum_{\substack{F \subseteq E; \\ B \subseteq F}} (-1)^{|F|} x^{m - r_M(F)} = 0.$$

This proves Claim 2.]

However, from $C\setminus\{e\}=B$, we obtain

$$\begin{aligned}
&\sum_{F\subseteq C\setminus\{e\}} (-1)^{|F|}\, x^{m-r_M(E\setminus F)}\cdot \widetilde{\chi}_{M\setminus F} \\
&= \sum_{F\subseteq B} (-1)^{|F|}\, x^{m-r_M(E\setminus F)}\cdot \widetilde{\chi}_{M\setminus F} \\
&= \sum_{J\subseteq B} (-1)^{|J|}\, \underbrace{x^{m-r_M(E\setminus J)}\cdot \widetilde{\chi}_{M\setminus J}}_{\substack{=\sum\limits_{\substack{F\subseteq E;\\ J\subseteq E\setminus F}} (-1)^{|F|} x^{m-r_M(F)} \\ \text{(by Claim 1)}}} \\
&\qquad\qquad \left(\text{here, we have renamed the summation index } F \text{ as } J\right) \\
&= \sum_{J\subseteq B} (-1)^{|J|} \sum_{\substack{F\subseteq E;\\ J\subseteq E\setminus F}} (-1)^{|F|}\, x^{m-r_M(F)} \\
&= \underbrace{\sum_{J\subseteq B}\ \sum_{\substack{F\subseteq E;\\ J\subseteq E\setminus F}}}_{=\sum\limits_{F\subseteq E}\ \sum\limits_{\substack{J\subseteq B;\\ J\subseteq E\setminus F}}} (-1)^{|J|}\, (-1)^{|F|}\, x^{m-r_M(F)} \\
&= \sum_{F\subseteq E}\ \underbrace{\sum_{\substack{J\subseteq B;\\ J\subseteq E\setminus F}}}_{=\sum\limits_{J\subseteq B\cap(E\setminus F)}} (-1)^{|J|}\, (-1)^{|F|}\, x^{m-r_M(F)} \\
&= \sum_{F\subseteq E}\ \underbrace{\sum_{J\subseteq B\cap(E\setminus F)} (-1)^{|J|}}_{\substack{=\sum\limits_{I\subseteq B\cap(E\setminus F)} (-1)^{|I|} \\ \text{(here, we have renamed} \\ \text{the summation index } J \text{ as } I\text{)}}}\ (-1)^{|F|}\, x^{m-r_M(F)} \\
&= \sum_{F\subseteq E}\ \underbrace{\sum_{I\subseteq B\cap(E\setminus F)} (-1)^{|I|}}_{\substack{=[B\cap(E\setminus F)=\varnothing] \\ \text{(by Lemma 5.21,} \\ \text{applied to } S=B\cap(E\setminus F)\text{)}}}\ (-1)^{|F|}\, x^{m-r_M(F)} \\
&= \sum_{F\subseteq E} \left[\underbrace{B\cap(E\setminus F)}_{=(B\cap E)\setminus F}=\varnothing\right] (-1)^{|F|}\, x^{m-r_M(F)} \\
&= \sum_{F\subseteq E} \left[\underbrace{(B\cap E)}_{\substack{=B \\ \text{(since } B\subseteq E\text{)}}}\setminus F=\varnothing\right] (-1)^{|F|}\, x^{m-r_M(F)}
\end{aligned}$$

$$= \sum_{F \subseteq E} \underbrace{[B \setminus F = \varnothing]}_{\substack{=[B \subseteq F] \\ \text{(since the statement "}B\setminus F=\varnothing\text{"} \\ \text{is equivalent to "}B\subseteq F\text{")}}} (-1)^{|F|} x^{m-r_M(F)}$$

$$= \sum_{F \subseteq E} [B \subseteq F] (-1)^{|F|} x^{m-r_M(F)}$$

$$= \sum_{\substack{F \subseteq E; \\ B \subseteq F}} \underbrace{[B \subseteq F]}_{\substack{=1 \\ \text{(since } B\subseteq F\text{)}}} (-1)^{|F|} x^{m-r_M(F)} + \sum_{\substack{F \subseteq E; \\ \text{we don't have } B\subseteq F}} \underbrace{[B \subseteq F]}_{\substack{=0 \\ \text{(since we don't} \\ \text{have } B\subseteq F\text{)}}} (-1)^{|F|} x^{m-r_M(F)}$$

$$\left(\text{since each subset } F \text{ of } E \text{ either satisfies } B \subseteq F \text{ or does not}\right)$$

$$= \sum_{\substack{F \subseteq E; \\ B \subseteq F}} (-1)^{|F|} x^{m-r_M(F)} + \underbrace{\sum_{\substack{F \subseteq E; \\ \text{we don't have } B\subseteq F}} 0 (-1)^{|F|} x^{m-r_M(F)}}_{=0}$$

$$= \sum_{\substack{F \subseteq E; \\ B \subseteq F}} (-1)^{|F|} x^{m-r_M(F)} = 0 \qquad (\text{by } (213)).$$

This proves Theorem 8.27. □

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