The Clifford algebra and the Chevalley map - a computational approach (summary version[1])

Darij Grinberg

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1. Introduction: the Clifford algebra

One of the basic properties of the Clifford algebra gives an explicit basis for it in terms of a basis of the underlying vector space (Theorem 1 below), and another one provides a vector space isomorphism between the Clifford algebra and the exterior algebra of the same vector space (the so-called Chevalley map, Theorem 2 below). While both of these properties appear in standard literature such as [1] and [2], sadly I have never seen them proven in the generality they deserve (they hold over arbitrary commutative rings rather than just fields of characteristic 0, at least as long as we are talking about bilinear rather than quadratic forms). Besides, some proofs found in literature are sloppily written or otherwise unsatisfactory. Here we are going to present a computational proof of both of these properties, giving integral[2] recursive formulas for the vector space isomorphism between the Clifford algebra and the exterior algebra (in both directions).

Remark (added in 2016). As I now know, most of what is done in this paper is not new. In particular, its main results already appear in §9 of Chapter IX of [7][3]; they also (essentially) appear in Chapter 2 of [8][4]; the main ideas also appear in (1.7) of Chapter IV of [9][5]. Moreover, the proofs given in [7], in [8] and in [9] are essentially the same as ours. (Moreover, similar ideas and a variant of our map α^f have been used for different purposes in [10].) The results in Sections 11–14 of this paper might still be new.

First, let us define everything in maximal generality:

Definition 1. In this note, a *ring* will always mean a ring with 1. If k is a ring, a k*-algebra* will mean a (not necessarily commutative) k-algebra with 1. Sometimes we will use the word "algebra" as an abbreviation for "k-algebra". If L is a k-algebra, then a *left* L*-module* is always supposed to be a left L-module on which the unity of L acts as the identity. Whenever we use the tensor product sign $\otimes$ without an index, we mean $\otimes_k$.

[1]This is a version including all the results, but excluding the straightforward proofs. Due to the computational nature of the proofs, a reader with experience in tensor manipulations will be able to derive all the proofs on his own without any trouble. If not, he can read them up in reference [0].

[2]in the sense of: no division by k!

[3]More precisely: Our Theorem 33 is Proposition 3 in §9 of Chapter IX of [7] (and thus, our Theorem 1 is a consequence of said proposition); our Theorem 2 is a particular case (for $L = \{1, 2, \ldots, n\}$) of Théorème 1 in §9 of Chapter IX of [7].

[4]More precisely, Theorem (2.16) in Chapter 2 of [8] includes both our Theorem 1 and our Theorem 2 in the case when the k-module L is finitely generated and projective. But the proof given in [8], as far as it concerns our Theorem (2.16), does not require the "finitely generated and projective" condition.

[5]Thanks to Rainer Schulze-Pillot for making me aware of [9].

Definition 2. Let k be a commutative ring. Let L be a k-module. A *bilinear form* on L means a bilinear map $f : L \times L \to k$. A bilinear form f on L is said to be *symmetric* if it satisfies ($f(x,y) = f(y,x)$ for any $x \in L$ and $y \in L$).

Definition 3. Let k be a commutative ring. Let L be a k-module, and $f : L \times L \to k$ be a bilinear form on L. For every $i \in \mathbb{N}$, we define the so-called *i-th tensor power* $L^{\otimes i}$ of L to be the k-module $\underbrace{L \otimes L \otimes ... \otimes L}_{i \text{ times}}$.

The *tensor algebra* $\otimes L$ of L over k is defined to be the algebra $\otimes L = L^{\otimes 0} \oplus L^{\otimes 1} \oplus L^{\otimes 2} \oplus ...$, where the multiplication is given by the tensor product. Now, we define the *Clifford algebra* $\mathrm{Cl}(L, f)$ to be the factor algebra $(\otimes L) \diagup I_f$, where I_f is the two-sided ideal

$$(\otimes L) \cdot \langle v \otimes v - f(v,v) \mid v \in L \rangle \cdot (\otimes L)$$

of the algebra $\otimes L$. [6]

Remark. We denote by $\mathbf{0}$ the symmetric bilinear form on L defined by ($\mathbf{0}(x,y) = 0$ for every $x \in L$ and $y \in L$). Then, $I_{\mathbf{0}} = (\otimes L) \cdot \langle v \otimes v \mid v \in L \rangle \cdot (\otimes L)$, and thus $\mathrm{Cl}(L, \mathbf{0}) = (\otimes L) \diagup I_{\mathbf{0}}$ is the exterior algebra $\wedge L$ of the k-module L. Hence, the exterior algebra $\wedge L$ is a particular case of the Clifford algebra - namely, it is the Clifford algebra $\mathrm{Cl}(L, \mathbf{0})$.

In general, the Clifford algebra $\mathrm{Cl}(L, f)$ is not isomorphic to the exterior algebra $\wedge L$ as algebra. However, they are isomorphic as k-modules, as the following theorem states:

Theorem 1 (Chevalley map theorem): Let k be a commutative ring. Let L be a k-module, and $f : L \times L \to k$ be a bilinear form on L. Then, the k-modules $\wedge L$ and $\mathrm{Cl}(L, f)$ are isomorphic.

We are going to prove this theorem by explicitly constructing mutually inverse homomorphisms in both directions. This proof substantially differs from the proofs given in standard literature for the particular case of k being a field of characteristic 0 and L being a finite-dimensional k-vector space, which proceed by constructing the isomorphism in one direction and showing either its injectivity or its surjectivity, or proving both using the basis theorem (Theorem 2 below).[7] Using Theorem 1 we will be able to construct a basis for $\mathrm{Cl}(L, f)$ in the case when L has one:

Theorem 2 (Clifford basis theorem): Let k be a commutative ring. Let L be a free k-module with a finite basis $(e_1, e_2, ..., e_n)$, and $f : L \times L \to k$ be a bilinear form on L. Let $\varphi_f : L \to \mathrm{Cl}(L, f)$ be the k-module homomorphism defined by $\varphi_f = \mathrm{proj}_f \circ \mathrm{inj}$, where $\mathrm{inj} : L \to \otimes L$ is the canonical injection

[6] Here, whenever U is a set, and $P : U \to \otimes L$ is a map (not necessarily a linear map), we denote by $\langle P(v) \mid v \in U \rangle$ the k-submodule of $\otimes L$ generated by the elements $P(v)$ for all $v \in U$.

[7] The proof of Theorem 1 in [2] (where Theorem 1 appears as Theorem 1.2, albeit only in the case of k being a field) seems different, but I don't completely understand it; to me it seems that it has a flaw (it states that "the r-homogeneous part of φ is then of the form $\varphi_r = \sum a_i \otimes v_i \otimes v_i \otimes b_i$ (where $\deg a_i + \deg b_i = r - 2$ for each i)", which I am not sure about, because theoretically one could imagine that the representation of φ in the form $\varphi = \sum a_i \otimes (v_i \otimes v_i + q(v_i)) \otimes b_i$ involves some a_i and b_i of extremely huge degree which cancel out in the sum).

of the k-module L into its tensor algebra $\otimes L$, and where $\operatorname{proj}_f : \otimes L \to \operatorname{Cl}(L, f)$ is the canonical projection of the tensor algebra $\otimes L$ onto its factor algebra $(\otimes L) \diagup I_f = \operatorname{Cl}(L, f)$.

Then, $\left(\overrightarrow{\prod\limits_{i\in I}}\varphi_f(e_i)\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$ is a basis of the k-module $\operatorname{Cl}(L, f)$, where $\mathcal{P}(\{1, 2, ..., n\})$ denotes the power set of the set $\{1, 2, ..., n\}$.

Here, we are using the following notation:

Definition 4. Let A be a ring, and let I be a finite subset of $\mathbb{Z}$. Let a_i be an element of A for each $i \in I$. Then, we denote by $\overrightarrow{\prod\limits_{i\in I}}a_i$ the element of A defined as follows: We write the set I in the form $I = \{i_1, i_2, ..., i_\ell\}$ with $i_1 < i_2 < ... < i_\ell$ (in other words, we let i_1, i_2, ..., i_ℓ be the elements of I, written down in ascending order). Then, we define $\overrightarrow{\prod\limits_{i\in I}}a_i$ as the product $a_{i_1}a_{i_2}...a_{i_\ell}$. This product $\overrightarrow{\prod\limits_{i\in I}}a_i$ is called the *ascending product* of the elements a_i of A.

One more theorem that is often (silently) used and will follow from our considerations:

Theorem 3. Let k be a commutative ring. Let L be a k-module, and $f : L \times L \to k$ be a bilinear form on L. Let $\varphi_f : L \to \operatorname{Cl}(L, f)$ be the k-module homomorphism defined by $\varphi_f = \operatorname{proj}_f \circ \operatorname{inj}$, where $\operatorname{inj} : L \to \otimes L$ is the canonical injection of the k-module L into its tensor algebra $\otimes L$, and where $\operatorname{proj}_f : \otimes L \to \operatorname{Cl}(L, f)$ is the canonical projection of the tensor algebra $\otimes L$ onto its factor algebra $(\otimes L) \diagup I_f = \operatorname{Cl}(L, f)$. Then, the homomorphism φ_f is injective.

Theorem 2 is known in the case of k being a field and L being a finite-dimensional k-vector space; in this case, it is often proved using orthogonal decomposition of L into f-orthogonal subspaces - a tactic not available to us in the general case of k being an arbitrary commutative ring. We will have to derive Theorem 2 from Theorem 1 to prove it in this generality. Most proofs of Theorem 1 rely on Theorem 2, and Theorem 3 is usually proven using either Theorem 1 or Theorem 2.

The nature of our proof will be computational - we are going to define some k-module automorphisms of the tensor algebra $\otimes L$ by recursive formulae. During the course of the proof, we will show a lot of formulas, each of which has a more or less straightforward inductive proof (using the results proven before). The inductive proofs will be straightforward using the following three tactics:

- In order to prove an identity for every tensor $U \in \otimes L$, it is enough to prove it only for homogeneous tensors U (i. e., for tensors $U \in L^{\otimes p}$ for every $p \in \mathbb{N}$), as long as the identity is linear in U. (This is because every tensor $U \in \otimes L$ is a linear combination of elements of $L^{\otimes p}$ for different $p \in \mathbb{N}$.)

- Let $p \in \mathbb{N}$. In order to prove an identity for every tensor $U \in L^{\otimes p}$, it is enough to prove it only for tensors U of the form $u \otimes \ddot{U}$ (where $u \in L$ and $\ddot{U} \in L^{\otimes(p-1)}$), as long as the identity is linear in U. (This is because every tensor $U \in L^{\otimes p}$ is a linear combination of tensors of the form $u \otimes \ddot{U}$ for different $u \in L$ and $\ddot{U} \in L^{\otimes(p-1)}$.)

- Let $p \in \mathbb{N}$. In order to prove an identity for every tensor $U \in L^{\otimes p}$, it is enough to prove it only for tensors U of the form $\ddot{U} \otimes u$ (where $u \in L$ and $\ddot{U} \in L^{\otimes(p-1)}$), as long as the identity is linear in U. (This is because every tensor $U \in L^{\otimes p}$ is a linear combination of tensors of the form $\ddot{U} \otimes u$ for different $u \in L$ and $\ddot{U} \in L^{\otimes(p-1)}$.)

2. Left interior products on the tensor algebra

From now on, we fix a commutative ring k, and a k-module L. Let f be some bilinear form on L.

First, we define some operations of L on $\otimes L$ - the so-called *interior products*:

Definition 6. Let $f : L \times L \to k$ be a bilinear form. For every $p \in \mathbb{N}$, we are going to define a bilinear map $\overset{f}{\llcorner} : L \times (\otimes L) \to \otimes L$. We are going to use infix notation for the map $\overset{f}{\llcorner}$; this means that for every $v \in L$ and every $T \in \otimes L$, we will denote the image of (v, T) under this bilinear map by $v \overset{f}{\llcorner} T$ rather than by $\overset{f}{\llcorner} (v, T)$.

In order to define this map $\overset{f}{\llcorner}$ on $L \times (\otimes L)$, it is enough to specify the value of $v \overset{f}{\llcorner} (u_1 \otimes u_2 \otimes ... \otimes u_p)$ for every $p \in \mathbb{N}$, every $v \in L$ and every p elements u_1, u_2, ..., u_p of L (because every element T of $\otimes L$ is a k-linear combination of pure tensors of the form $u_1 \otimes u_2 \otimes ... \otimes u_p$ for various p and u_1, u_2, ..., u_p, and thus the values of $v \overset{f}{\llcorner} (u_1 \otimes u_2 \otimes ... \otimes u_p)$ determine the value of $v \overset{f}{\llcorner} T$). This we do by setting

$$v \overset{f}{\llcorner} (u_1 \otimes u_2 \otimes ... \otimes u_p) = \sum_{i=1}^{p} (-1)^{i-1} f(v, u_i) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p. \quad (1)$$

[8] In particular, we have $v \overset{f}{\llcorner} u = f(v, u)$ [9] for every $u \in L$ and $v \in L$, and we have $v \overset{f}{\llcorner} \lambda = 0$ [10] for every $v \in L$ and $\lambda \in k$.

We have two rather easy properties of our map:

[8]Here, the hat over the vector u_i means that the vector u_i is being omitted from the tensor product; in other words, $u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p$ is just another way to write $\underbrace{u_1 \otimes u_2 \otimes ... \otimes u_{i-1}}_{\text{tensor product of the first } i-1 \text{ vectors } u_\ell} \otimes \underbrace{u_{i+1} \otimes u_{i+2} \otimes ... \otimes u_p}_{\text{tensor product of the last } p-i \text{ vectors } u_\ell}$.

[9]Here, $f(v, u) \in k$ is considered as an element of $\otimes L$ by means of the canonical inclusion $k = L^{\otimes 0} \subseteq \otimes L$.

[10]Here, $\lambda \in k$ is considered as an element of $\otimes L$ by means of the canonical inclusion $k = L^{\otimes 0} \subseteq \otimes L$.

Theorem 5. The bilinear map $\overset{f}{\llcorner} : L \times (\otimes L) \to \otimes L$ is well-defined by Definition 6.

Theorem 6. If $u \in L$, $U \in \otimes L$, and $v \in L$, then

$$v \overset{f}{\llcorner} (u \otimes U) = f(v, u) U - u \otimes \left(v \overset{f}{\llcorner} U \right). \tag{2}$$

Three remarks:

- While the Definition 5 above is useful for computing the map $\overset{f}{\llcorner}$ in concrete cases, Theorem 6 gives a powerful recurrence equation for the map $\overset{f}{\llcorner}$ ("recurrence" because it reduces the computation of $v \overset{f}{\llcorner} T$ for a $(p+1)$-tensor T to the computation of $v \overset{f}{\llcorner} S$ for p-tensors S), which (together with $v \overset{f}{\llcorner} \lambda = 0$ for $v \in L$ and $\lambda \in k$) allows us to prove most properties of $\overset{f}{\llcorner}$ by induction (without having to work with summations as we would have to do if we would use Definition 5).

- In the detailed version [0] of this paper, I define the map $\overset{f}{\llcorner}$ not by the Definition 5 given above, but instead by a different definition (which is more or less Theorem 6 in disguise).

- Many authors omit the f in the notation $\overset{f}{\llcorner}$; in other words, they simply write $\llcorner$ for $\overset{f}{\llcorner}$. We, however, cannot afford using this abbreviation, since we will have to work with several different f's at once.

Now, it is time for some actually nontrivial formulas for $\overset{f}{\llcorner}$. However, "nontrivial" doesn't mean that the proofs aren't obvious inductions using the three tactics I described above.

Theorem 7. If $v \in L$ and $U \in \otimes L$, then

$$v \overset{f}{\llcorner} \left(v \overset{f}{\llcorner} U \right) = 0. \tag{3}$$

Theorem 8. If $v \in L$, $w \in L$ and $U \in \otimes L$, then

$$v \overset{f}{\llcorner} \left(w \overset{f}{\llcorner} U \right) = -w \overset{f}{\llcorner} \left(v \overset{f}{\llcorner} U \right). \tag{4}$$

Theorem 9. If $p \in \mathbb{N}$, $u \in L$, $U \in L^{\otimes p}$, and $v \in L$, then

$$v \overset{f}{\llcorner} (U \otimes u) = (-1)^p f(v, u) U + \left(v \overset{f}{\llcorner} U \right) \otimes u. \tag{5}$$

Theorem 10. If $p \in \mathbb{N}$, $v \in L$, $U \in L^{\otimes p}$, and $V \in \otimes L$, then

$$v \overset{f}{\llcorner} (U \otimes V) = (-1)^p U \otimes \left(v \overset{f}{\llcorner} V \right) + \left(v \overset{f}{\llcorner} U \right) \otimes V. \tag{6}$$

Theorem $10\frac{1}{2}$. If $p \in \mathbb{N}$, $u \in L$, $U \in \bigoplus\limits_{\substack{i\in\mathbb{N};\\ i\equiv p \bmod 2}} L^{\otimes i}$, and $v \in L$, then

$$v \overset{f}{\llcorner} (U \otimes u) = (-1)^p f(v,u) U + \left(v \overset{f}{\llcorner} U\right) \otimes u. \qquad (7)$$

Theorem $10\frac{3}{4}$. Let $f : L \times L \to k$ and $g : L \times L \to k$ be two bilinear forms. If $w \in L$ and $U \in \otimes L$, then

$$w \overset{f}{\llcorner} U + w \overset{g}{\llcorner} U = w \overset{f+g}{\llcorner} U. \qquad (8)$$

3. Right interior products on the tensor algebra

We have proven a number of properties of the interior product $\overset{f}{\llcorner}$. We are now going to introduce a very analogous construction $\overset{f}{\lrcorner}$ which works "from the right" almost the same way as $\overset{f}{\llcorner}$ works "from the left":

Definition 7. Let $f : L \times L \to k$ be a bilinear form. For every $p \in \mathbb{N}$, we are going to define a bilinear map $\overset{f}{\lrcorner} : (\otimes L) \times L \to \otimes L$. We are going to use infix notation for the map $\overset{f}{\lrcorner}$; this means that for every $v \in L$ and every $T \in \otimes L$, we will denote the image of (T, v) under this bilinear map by $T \overset{f}{\lrcorner} v$ rather than by $\overset{f}{\lrcorner}(T, v)$.

In order to define this map $\overset{f}{\lrcorner}$ on $(\otimes L) \times L$, it is enough to specify the value of $(u_1 \otimes u_2 \otimes ... \otimes u_p) \overset{f}{\lrcorner} v$ for every $p \in \mathbb{N}$, every $v \in L$ and every p elements u_1, u_2, ..., u_p of L (because every element T of $\otimes L$ is a k-linear combination of pure tensors of the form $u_1 \otimes u_2 \otimes ... \otimes u_p$ for various p and u_1, u_2, ..., u_p, and thus the values of $(u_1 \otimes u_2 \otimes ... \otimes u_p) \overset{f}{\lrcorner} v$ determine the value of $T \overset{f}{\lrcorner} v$). This we do by setting

$$(u_1 \otimes u_2 \otimes ... \otimes u_p) \overset{f}{\lrcorner} v = \sum_{i=1}^{p} (-1)^{p-i} f(u_i, v) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p. \qquad (9)$$

[11] In particular, we have $u \overset{f}{\lrcorner} v = f(v,u)$ [12] for every $u \in L$ and $v \in L$, and we have $\lambda \overset{f}{\lrcorner} v = 0$ [13] for every $v \in L$ and $\lambda \in k$.

Again, many authors omit the f in the notation $\overset{f}{\lrcorner}$, but we will not.

Everything that we have proven for $\overset{f}{\llcorner}$ has an analogue for $\overset{f}{\lrcorner}$. In fact, we can take any identity concerning $\overset{f}{\llcorner}$, and "read it from right to left" to obtain an analogous property of $\overset{f}{\lrcorner}$ [14]. This way, we get the following new theorems:

[11]Here, the hat over the vector u_i means that the same as it did in Definition 5.

[12]Here, $f(v,u) \in k$ is considered as an element of $\otimes L$ by means of the canonical inclusion $k = L^{\otimes 0} \subseteq \otimes L$.

[13]Here, $\lambda \in k$ is considered as an element of $\otimes L$ by means of the canonical inclusion $k = L^{\otimes 0} \subseteq \otimes L$.

[14]See [0] for details about how this is to be understood.

Theorem 11. The bilinear map $\overset{f}{\lrcorner} : (\otimes L) \times L \to \otimes L$ is well-defined by Definition 7.

Theorem 12. If $u \in L$, $U \in \otimes L$, and $v \in L$, then

$$(U \otimes u) \overset{f}{\lrcorner} v = f(u, v) U - \left(U \overset{f}{\lrcorner} v\right) \otimes u. \tag{10}$$

Theorem 13. If $v \in L$ and $U \in \otimes L$, then

$$\left(U \overset{f}{\lrcorner} v\right) \overset{f}{\lrcorner} v = 0. \tag{11}$$

Theorem 14. If $v \in L$, $w \in L$ and $U \in \otimes L$, then

$$\left(U \overset{f}{\lrcorner} w\right) \overset{f}{\lrcorner} v = -\left(U \overset{f}{\lrcorner} v\right) \overset{f}{\lrcorner} w. \tag{12}$$

Theorem 15. If $p \in \mathbb{N}$, $u \in L$, $U \in L^{\otimes p}$, and $v \in L$, then

$$(u \otimes U) \overset{f}{\lrcorner} v = (-1)^p f(u, v) U + u \otimes \left(U \overset{f}{\lrcorner} v\right). \tag{13}$$

Theorem 16. If $p \in \mathbb{N}$, $v \in L$, $U \in L^{\otimes p}$, and $V \in \otimes L$, then

$$(V \otimes U) \overset{f}{\lrcorner} v = (-1)^p \left(V \overset{f}{\lrcorner} v\right) \otimes U + V \otimes \left(U \overset{f}{\lrcorner} v\right). \tag{14}$$

Theorem $16\frac{1}{2}$. If $p \in \mathbb{N}$, $u \in L$, $U \in \bigoplus\limits_{\substack{i \in \mathbb{N};\\ i \equiv p \bmod 2}} L^{\otimes i}$, and $v \in L$, then

$$(u \otimes U) \overset{f}{\lrcorner} v = (-1)^p f(u, v) U + u \otimes \left(U \overset{f}{\lrcorner} v\right). \tag{15}$$

Theorem $16\frac{3}{4}$. Let $f : L \times L \to k$ and $g : L \times L \to k$ be two bilinear forms. If $w \in L$ and $U \in \otimes L$, then

$$U \overset{f}{\lrcorner} w + U \overset{g}{\lrcorner} w = U \overset{f+g}{\lrcorner} w.$$

These Theorems 11-16 are simply the results of reading Theorems 5-10 from right to left, so if we have proofs of Theorems 5-10, we automatically obtain proofs of Theorems 11-16. However, there is also an alternative way to prove Theorems 11-16 - namely, by explicitly relating the right interior product $\overset{f}{\lrcorner}$ to the left one:

Definition 8. Let $t : \otimes L \to \otimes L$ be the k-module endomorphism of $\otimes L$ defined by

$$\left(\begin{array}{c} t(u_1 \otimes u_2 \otimes ... \otimes u_p) = u_p \otimes u_{p-1} \otimes ... \otimes u_1 \\ \text{for any } p \in \mathbb{N} \text{ and any vectors } u_1, u_2, ..., u_p \text{ in } L \end{array} \right).$$

(This is obviously well-defined.)

Clearly, $t^2 = \mathrm{id}$. Hence, $t : \otimes L \to \otimes L$ is bijective. Besides,

$$t(U \otimes V) = t(V) \otimes t(U) \qquad \text{for every } U \in \otimes L \text{ and } V \in \otimes L.$$

Also, obviously, $t(u) = u$ for every $u \in L$.

Our use for the map t is now to reduce the right interior product $\overset{f}{\lrcorner}$ to the left interior product $\overset{f}{\llcorner}$. For this we need yet another definition:

Definition 9. Let $f : L \times L \to k$ be a bilinear form. Then, we define a new bilinear form $f^t : L \times L \to k$ by

$$\left(f^t(u,v) = f(v,u) \qquad \text{for every } u \in L \text{ and } v \in L\right).$$

This bilinear form f^t is called the *transpose* of the bilinear form f.

It is clear that $(f^t)^t = f$ for any bilinear form f, and that a bilinear form f is symmetric if and only if $f = f^t$.

Now, here is a way to write $\overset{f}{\lrcorner}$ in terms of $\overset{f^t}{\llcorner}$:

Theorem 17. Let $v \in L$ and $U \in \otimes L$. Then,

$$t\left(U \overset{f}{\lrcorner} v\right) = v \overset{f^t}{\llcorner} t(U)$$

and

$$t\left(v \overset{f^t}{\llcorner} U\right) = t(U) \overset{f}{\lrcorner} v.$$

4. The two operations commute

Now that we know quite a lot about each of the operations $\overset{f}{\llcorner}$ and $\overset{f}{\lrcorner}$, let us show a relation between them:

Theorem 18. Let $v \in L$, $w \in L$ and $U \in \otimes L$. Then

$$v \overset{f}{\llcorner} \left(U \overset{f}{\lrcorner} w\right) = \left(v \overset{f}{\llcorner} U\right) \overset{f}{\lrcorner} w. \tag{16}$$

More generally, if $f : L \times L \to k$ and $g : L \times L \to k$ are two bilinear forms, then

$$v \overset{f}{\llcorner} \left(U \overset{g}{\lrcorner} w\right) = \left(v \overset{f}{\llcorner} U\right) \overset{g}{\lrcorner} w. \tag{17}$$

The proof is, just as all proofs above, induction over the rank of the tensor U (after U has been assumed homogeneous).

5. The endomorphism α^f

We are now going to define an endomorphism $\alpha^f : \otimes L \to \otimes L$ which depends on the bilinear form f:

Definition 10. Let $f : L \times L \to k$ be a bilinear form. For every $p \in \mathbb{N}$, we define a k-linear map $\alpha_p^f : L^{\otimes p} \to \otimes L$ by induction over p:

Induction base: For $p = 0$, we define the map $\alpha_p^f : L^{\otimes 0} \to \otimes L$ to be the canonical inclusion of $L^{\otimes 0}$ into the tensor algebra $\otimes L = L^{\otimes 0} \oplus L^{\otimes 1} \oplus L^{\otimes 2} \oplus ....$ (In other words, we define the map $\alpha_0^f : k \to \otimes L$ by $\alpha_p^f(\lambda) = \lambda$ for every $\lambda \in k = L^{\otimes 0}$.)

Induction step: For each $p \in \mathbb{N}_+$, we define a k-linear map $\alpha_p^f : L^{\otimes p} \to \otimes L$ by

$$\left(\alpha_p^f(u \otimes U) = u \otimes \alpha_{p-1}^f(U) - u \overset{f}{\llcorner} \alpha_{p-1}^f(U) \qquad \text{for every } u \in L \text{ and } U \in L^{\otimes(p-1)}\right), \tag{18}$$

assuming that we have already defined a k-linear map $\alpha_{p-1}^f : L^{\otimes(p-1)} \to \otimes L$. (This definition is justified, because in order to define a k-linear map from $L^{\otimes p}$ to some other k-module, it is enough to define how it acts on tensors of the form $u \otimes U$ for every $u \in L$ and $U \in L^{\otimes(p-1)}$, as long as this action is bilinear with respect to u and U. This is because $L^{\otimes p} = L \otimes L^{\otimes(p-1)}$.)

This way we have defined a k-linear map $\alpha_p^f : L^{\otimes p} \to \otimes L$ for every $p \in \mathbb{N}$. We can combine these maps α_0^f, α_1^f, α_2^f, ... into one k-linear map $\alpha^f : \otimes L \to \otimes L$ (since $\otimes L = L^{\otimes 0} \oplus L^{\otimes 1} \oplus L^{\otimes 2} \oplus ...$), and the formula (18) rewrites as

$$\left(\alpha^f(u \otimes U) = u \otimes \alpha^f(U) - u \overset{f}{\llcorner} \alpha^f(U) \qquad \text{for every } u \in L \text{ and } U \in L^{\otimes(p-1)}\right). \tag{19}$$

We note that, in contrast to the map δ_v^f (which maps every homogeneous tensor from $L^{\otimes p}$ to $L^{\otimes(p-1)}$), the map α^f can map homogeneous tensors to inhomogeneous tensors.

This endomorphism α^f now turns out to have plenty of properties. But first let us first see how it evaluates on pure tensors of low rank (0, 1, 2, 3, 4):

Action of α^f on tensors of rank 0: For any $\lambda \in k$, we have $\alpha^f(\lambda) = \lambda$, where we consider λ as an element of $\otimes L$ through the canonical injection $k = L^{\otimes 0} \to \otimes L$. (In fact, $\lambda \in k = L^{\otimes 0}$ yields $\alpha^f(\lambda) = \alpha_0^f(\lambda) = \lambda$ by the definition of α_0^f).

Action of α^f on tensors of rank 1: For any $u \in L$, we have

$$\begin{aligned}\alpha^f(u) = \alpha^f(u \otimes 1) &= u \otimes \underbrace{\alpha^f(1)}_{=1 \text{ (since } 1 \in k)} - \underbrace{u \overset{f}{\llcorner} \alpha^f(1)}_{\substack{=0 \text{ (by Theorem 5 } \textbf{(a)}, \\ \text{since } \alpha^f(1) = 1 \in k)}} \qquad \text{(by (19), applied to } U = 1) \\ &= u \otimes 1 - 0 = u \otimes 1 = u. \end{aligned} \tag{20}$$

Action of α^f on tensors of rank 2: For any $u \in L$ and $v \in L$, we have

$$\alpha^f(u \otimes v) = u \otimes v - f(u, v),$$

as a simple computation shows.

Action of α^f on tensors of rank 3: For any $u \in L$, $v \in L$ and $w \in L$, we have

$$\alpha^f(u \otimes v \otimes w) = u \otimes v \otimes w - f(v, w) u + f(u, w) v - f(u, v) w,$$

as a result of a computation.

Action of α^f on tensors of rank 4*:* For any $u \in L$, $v \in L$, $w \in L$ and $x \in L$, we have

$$\begin{aligned}
&\alpha^f (u \otimes v \otimes w \otimes x) \\
&= u \otimes v \otimes w \otimes x - f(w,x)\, u \otimes v + f(v,x)\, u \otimes w - f(v,w)\, u \otimes x \\
&\qquad - f(u,v)\, w \otimes x + f(u,w)\, v \otimes x - f(u,x)\, v \otimes w \\
&\qquad + f(w,x)\, f(u,v) - f(v,x)\, f(u,w) + f(v,w)\, f(u,x)
\end{aligned}$$

as the result of a rather lengthy computation.

These formulas can be generalized to $\alpha^f (u_1 \otimes u_2 \otimes ... \otimes u_p)$ for general $p \in \mathbb{N}$. As a result, we obtain

$$\begin{aligned}
&\alpha^f (u_1 \otimes u_2 \otimes ... \otimes u_p) \\
&= \sum (-1)^{(\text{number of all bad pairs})} f(u_{i_1}, u_{j_1})\, f(u_{i_2}, u_{j_2}) ... f(u_{i_k}, u_{j_k})\, u_{r_1} \otimes u_{r_2} \otimes ... \otimes u_{r_{p-2k}}
\end{aligned}$$

for any p vectors u_1, u_2, ..., u_p in L, where the sum is over all partitions of the set $\{1, 2, ..., p\}$ into three subsets $\{i_1, i_2, ..., i_k\}$, $\{j_1, j_2, ..., j_k\}$ and $\{r_1, r_2, ..., r_{p-2k}\}$ (for various k) which satisfy $i_1 < i_2 < ... < i_k$, $j_1 < j_2 < ... < j_k$, $r_1 < r_2 < ... < r_{p-2k}$ and ($i_\ell < j_\ell$ for every $\ell \in \{1, 2, ..., k\}$). Here, a "bad pair" means a pair $(\ell, \ell') \in \{1, 2, ..., k\}^2$ satisfying $\ell \geq \ell'$ and $i_\ell < j_{\ell'}$ (so, in particular, for every $\ell \in \{1, 2, ..., k\}$, the pair (ℓ, ℓ) is bad, since $i_\ell < j_\ell$). [15] Thus we have an explicit formula for $\alpha^f (u_1 \otimes u_2 \otimes ... \otimes u_p)$, but it is extremely hard to deal with; this is the reason why I defined α^f by induction rather than by a direct formula.

We remark that the formula (19) can be slightly generalized, in the sense that U doesn't have to be a homogeneous tensor:

Theorem 19. Let $u \in L$ and $U \in \otimes L$. Then,

$$\alpha^f (u \otimes U) = u \otimes \alpha^f (U) - u \overset{f}{\llcorner} \alpha^f (U) . \tag{21}$$

Again, this is simply a consequence of (19) because every tensor $U \in \otimes L$ is a linear combination of homogeneous tensors.

Another fact is, while α^f is not necessarily homogeneous, the degrees of all the terms it spits out have the same parity as that of the original tensor:

Theorem 20. Let $U \in L^{\otimes p}$ for some $p \in \mathbb{N}$. Then,

$$\alpha^f (U) \in \bigoplus_{\substack{i \in \mathbb{N};\\ i \equiv p \bmod 2}} L^{\otimes i}. \tag{22}$$

Even a stronger assertion holds:

$$\alpha^f (U) \in \bigoplus_{\substack{i \in \{0,1,...,p\};\\ i \equiv p \bmod 2}} L^{\otimes i}. \tag{23}$$

[15] I hope I haven't made a mistake in the formula.

The proof of this fact is an induction over p; the "trick" is that the terms $u \otimes V$ and $u \overset{f}{\llcorner} V$ (for a homogeneous tensor V) are homogeneous tensors whose degrees are different but differ by 2 and thus have the same parity.

Now let us show some more interesting properties of α^f. The proofs will be again by induction akin to the proofs of Theorems 6-10 and 12-16.

First, we notice that the definition of α^f had a bias towards left tensoring: we defined the value of α_p^f on a tensor of rank p by writing this tensor as a linear combination of tensors of the form $u \otimes U$ with $u \in L$ and $U \in L^{\otimes(p-1)}$, and then by setting the value of α_p^f on each such $u \otimes U$ tensor according (18). But what if we would try to define a "right analogue" $\widetilde{\alpha}^f$ of α^f, which would be (inductively) defined by

$$\left(\widetilde{\alpha}_p^f (U \otimes u) = \widetilde{\alpha}_{p-1}^f (U) \otimes u - \alpha_{p-1}^f (U) \overset{f}{\lrcorner} u \qquad \text{for every } u \in L \text{ and } U \in L^{\otimes(p-1)}\right)$$

instead of (18) ? It turns out that this wouldn't give us anything new: This "right analogue" $\widetilde{\alpha}^f$ would be the same as α^f. This is explained by the following theorem:

Theorem 21. Let $u \in L$ and $U \in \otimes L$. Then,

$$\alpha^f (U \otimes u) = \alpha^f (U) \otimes u - \alpha^f (U) \overset{f}{\lrcorner} u. \tag{24}$$

Theorem 22. Let $u \in L$ and $U \in \otimes L$. Let $g : L \times L \to k$ be a bilinear form. Then,

$$\alpha^f \left(U \overset{g}{\lrcorner} u\right) = \alpha^f (U) \overset{g}{\lrcorner} u. \tag{25}$$

Theorem 23. Let $u \in L$ and $U \in \otimes L$. Let $g : L \times L \to k$ be a bilinear form. Then,

$$\alpha^f \left(u \overset{g}{\llcorner} U\right) = u \overset{g}{\llcorner} \alpha^f (U). \tag{26}$$

Theorem 24. We have $\alpha^f \circ t = t \circ \alpha^{f^t}$.

6. The endomorphism α^g and the ideals $I_f^{(v)}$

In Definition 3, we have introduced the two-sided ideal I_f of the algebra $\otimes L$. It was defined as

$$(\otimes L) \cdot \langle v \otimes v - f(v,v) \mid v \in L \rangle \cdot (\otimes L).$$

We will now write this ideal I_f as a sum (not a direct sum, however) of certain smaller k-modules, which we denote by $I_f^{(v)}$ and $I_f^{(v;p;q)}$ (the $I_f^{(v;p;q)}$ are an even finer subdivision of the $I_f^{(v)}$). These ideals are not really necessary for our further goals, but they help keeping our proof a bit more organized:

Definition 11. For any vector $v \in L$, let $I_f^{(v)}$ be the k-submodule

$$(\otimes L) \cdot (v \otimes v - f(v,v)) \cdot (\otimes L)$$

of the k-module $\otimes L$.

Note that the dot sign (the sign $\cdot$) in this definition stands for multiplication in the algebra $\otimes L$; in other words, it is synonymous to the tensor product sign (the sign $\otimes$).

We then have $I_f = \sum\limits_{v \in L} I_f^{(v)}$ (where the $\sum$ sign means a sum of k-modules).

Our main goal in this section is to prove the following result:

Theorem 25. Let $f : L \times L \to k$ and $g : L \times L \to k$ be two bilinear forms. Then, $\alpha^g \left(I_f \right) \subseteq I_{f+g}$.

In order to prove this theorem, we first start with an easy fact (easily following from our above formulae):

Lemma 26. If $w \in L$, $U \in \otimes L$, and $v \in L$, then

$$w \llcorner^{g} \left(v \otimes v \otimes U \right) = v \otimes v \otimes \left(w \llcorner^{g} U \right),$$

and

$$\alpha^g \left(v \otimes v \otimes U \right) = \left(v \otimes v - g \left(v, v \right) \right) \otimes \alpha^g \left(U \right).$$

As a consequence,

$$w \llcorner^{g} \left(\left(v \otimes v - f \left(v, v \right) \right) \otimes U \right) = \left(v \otimes v - f \left(v, v \right) \right) \otimes \left(w \llcorner^{g} U \right) \tag{27}$$

and

$$\alpha^g \left(\left(v \otimes v - f \left(v, v \right) \right) \otimes U \right) = \left(v \otimes v - \left(f + g \right) \left(v, v \right) \right) \otimes \alpha^g \left(U \right). \tag{28}$$

Now we are going to prove that the ideal I_f is stable under the map $w \llcorner^{g}$ for any two bilinear forms f and g and any vector w:

Theorem 27. Let $w \in L$. Let $f : L \times L \to k$ and $g : L \times L \to k$ be two bilinear forms. Then, $w \llcorner^{g} I_f \subseteq I_f$. (Here, whenever P is a k-submodule of $\otimes L$, we denote by $w \llcorner^{g} P$ the k-submodule $\left\{ w \llcorner^{g} p \mid p \in P \right\}$ of $\otimes L$. This is indeed a k-submodule, as follows from the bilinearity of $\llcorner^{g}$.)

Proof of Theorem 27. We are going to show something stronger: We will show that $w \llcorner^{g} I_f^{(v)} \subseteq I_f^{(v)}$ for every $v \in L$.

In fact, in order to show this, we must prove that $w \llcorner^{g} T \in I_f^{(v)}$ for every $T \in I_f^{(v)}$. Notice that $T \in I_f^{(v)}$ yields that T is a linear combination of tensors of the form $V \otimes \left(v \otimes v - f \left(v, v \right) \right) \otimes U$ for some $V \in \otimes L$ and $U \in \otimes L$ (by the definition of $I_f^{(v)}$). Hence, in order to prove that $w \llcorner^{g} T \in I_f^{(v)}$, it is enough to prove that $w \llcorner^{g} \left(V \otimes \left(v \otimes v - f \left(v, v \right) \right) \otimes U \right) \in I_f^{(v)}$ for every $V \in \otimes L$ and $U \in \otimes L$. So let us prove this now.

We can WLOG assume that the tensors V and U are homogeneous (because everything is linear), and we denote by $|V|$ the degree of V. Now, Theorem 10 (applied to

g, w, V and $(v\otimes v-f(v,v))\otimes U$ instead of f, v, U and W) yields

$$\begin{aligned}
& w\overset{g}{\llcorner}\left(V\otimes(v\otimes v-f(v,v))\otimes U\right)\\
&=(-1)^{|V|}V\otimes\left(\underbrace{w\overset{g}{\llcorner}\left((v\otimes v-f(v,v))\otimes U\right)}_{\substack{=(v\otimes v-f(v,v))\otimes\left(w\overset{g}{\llcorner}U\right)\\ \text{(by (27))}}}\right)+\left(w\overset{g}{\llcorner}V\right)\otimes\left((v\otimes v-f(v,v))\otimes U\right)\\
&=\underbrace{(-1)^{|V|}V\otimes(v\otimes v-f(v,v))\otimes\left(w\overset{g}{\llcorner}U\right)}_{\in I_f^{(v)}}+\underbrace{\left(w\overset{g}{\llcorner}V\right)\otimes\left((v\otimes v-f(v,v))\otimes U\right)}_{\in I_f^{(v)}}\in I_f^{(v)}.
\end{aligned}$$

As we know, this yields $w\overset{g}{\llcorner}T\in I_f^{(v)}$. Thus, Theorem 27 is proven.

As an analogue of Theorem 27, we can show:

Theorem 28. Let $w\in L$. Let $f:L\times L\to k$ and $g:L\times L\to k$ be two bilinear forms. Then, $I_f\overset{g}{\lrcorner}w\subseteq I_f$. (Here, whenever P is a k-submodule of $\otimes L$, we denote by $P\overset{g}{\lrcorner}w$ the k-submodule $\left\{p\overset{g}{\lrcorner}w \mid p\in P\right\}$ of $\otimes L$. This is indeed a k-submodule, as follows from the bilinearity of $\overset{g}{\lrcorner}$.)

We can either prove this in complete analogy to Theorem 27, or use Theorem 27 and the following triviality:

Theorem 29. We have $t(I_f)=I_f$.

Now, something more interesting: The map α^g doesn't (in general) leave I_f stable, but instead maps it to I_{f+g}:

Theorem 30. Let $w\in L$. Let $f:L\times L\to k$ and $g:L\times L\to k$ be two bilinear forms. Then, $\alpha^g(I_f)\subseteq I_{f+g}$.

Proof of Theorem 30. Again, we can do better: We can show that $\alpha^g\left(I_f^{(v)}\right)\subseteq I_{f+g}^{(v)}$ for every $v\in L$.

In order to show this, it is enough to prove that $\alpha^g\left(V\otimes(v\otimes v-f(v,v))\otimes U\right)\in I_f^{(v)}$ for every $V\in\otimes L$ and $U\in\otimes L$ (for the same reasons as in the proof of Theorem 27). So let us prove this. We can WLOG assume that the tensors V and U are homogeneous tensors, and denote by $|V|$ the degree of V. If $|V|=0$, then we are easily done using (28). So let us assume that $|V|>0$. We can WLOG assume that $V=v\otimes S$ for some $v\in L$ and $S\in L^{\otimes(|V|-1)}$ (because $V\in L^{\otimes|V|}$, and the k-module $L^{\otimes|V|}=L\otimes L^{\otimes(|V|-1)}$ is spanned by elements of the form $v\otimes S$ for some $v\in L$ and $S\in L^{\otimes(|V|-1)}$). Then,

$$\begin{aligned}
&\alpha^g\Bigl(\underbrace{V}_{=v\otimes S}\otimes(v\otimes v-f(v,v))\otimes U\Bigr)\\
&=\alpha^g\left(v\otimes S\otimes(v\otimes v-f(v,v))\otimes U\right)\\
&=v\otimes\alpha^g\left(S\otimes(v\otimes v-f(v,v))\otimes U\right)-v\overset{g}{\llcorner}\alpha^g\left(S\otimes(v\otimes v-f(v,v))\otimes U\right)
\end{aligned}$$

(by (21), applied to g, v and $S\otimes(v\otimes v - f(v,v))\otimes U$ instead of f, u and U). Therefore, if we know that $\alpha^g(S\otimes(v\otimes v - f(v,v))\otimes U)\in I_f^{(v)}$, then we can conclude that

$$\begin{aligned}
&\alpha^g\left(V\otimes(v\otimes v - f(v,v))\otimes U\right)\\
&= v\otimes \underbrace{\alpha^g\left(S\otimes(v\otimes v - f(v,v))\otimes U\right)}_{\in I_f^{(v)}} - v\overset{g}{\llcorner}\underbrace{\alpha^g\left(S\otimes(v\otimes v - f(v,v))\otimes U\right)}_{\in I_f^{(v)}}\\
&\in \underbrace{v\otimes I_f^{(v)}}_{\subseteq I_f^{(v)}} - \underbrace{v\overset{g}{\llcorner} I_f^{(v)}}_{\substack{\subseteq I_f^{(v)} \text{ (this was shown} \\ \text{in the proof of Theorem 27)}}} \subseteq I_f^{(v)} - I_f^{(v)} = I_f^{(v)}.
\end{aligned}$$

So, if we know that $\alpha^g(S\otimes(v\otimes v - f(v,v))\otimes U)\in I_f^{(v)}$, we can conclude that $\alpha^g(V\otimes(v\otimes v - f(v,v))\otimes U)\in I_f^{(v)}$. Since the tensor S has a smaller degree than the tensor V, this allows us to prove $\alpha^g(V\otimes(v\otimes v - f(v,v))\otimes U)\in I_f^{(v)}$ by induction over $|V|$. The details are left to the reader (who can find them in [0] anyway).

The next section will show that Theorem 30 can be strengthened:

Theorem 31. Let $w\in L$. Let $f : L\times L\to k$ and $g : L\times L\to k$ be two bilinear forms. Then, $\alpha^g(I_f) = I_{f+g}$.

7. $\alpha^f\circ\alpha^g = \alpha^{f+g}$

Until now, each of our results involved α^f only for one bilinear form f. Though we sometimes called it g instead of f, never did we consider the maps α^f for two *different* forms f together in one and the same theorem. Let us change this now:

Theorem 32. **(a)** Let $f : L\times L\to k$ and $g : L\times L\to k$ be two bilinear forms. Then, $\alpha^f\circ\alpha^g = \alpha^{f+g}$.

(b) The bilinear form $\mathbf{0} : L\times L\to k$ defined by ($\mathbf{0}(x,y) = 0$ for every $x\in L$ and $y\in L$) satisfies $\alpha^{\mathbf{0}} = \mathrm{id}$.

(c) Let $f : L\times L\to k$ be a bilinear form. Then, the map α^f is invertible, and its inverse is α^{-f}.

Proof of Theorem 32. **(a)** It is clearly enough to show that for every $p\in\mathbb{N}$, we have

$$\alpha^f\left(\alpha^g(U)\right) = \alpha^{f+g}(U) \tag{29}$$

for every $U\in L^{\otimes p}$ (because every $U\in\otimes L$ is a k-linear combination of elements of $L^{\otimes p}$ for various $p\in\mathbb{N}$, and since the equation (29) is k-linear).

In order to prove (29), we can proceed by induction over p. The base case $p=0$ is trivial, and in the induction step, we can WLOG assume that the tensor U is of the form $U = u\otimes\ddot{U}$ for some $u\in L$ and $\ddot{U}\in L^{\otimes(p-1)}$ (because every tensor in $L^{\otimes p}$ is a linear combination of such tensors, and the equation (29) is k-linear), and then prove that $\alpha^f\left(\alpha^g\left(u\otimes\ddot{U}\right)\right) = \alpha^{f+g}\left(u\otimes\ddot{U}\right)$ using the induction assumption (along with (21) and (26)). The details are explained in [0]. So much for **(a)**.

(b) is trivial, and **(c)** follows from **(a)** and **(b)**.

Theorem 31 readily follows from Theorems 30 and 32.

Now we are able to give a *proof of Theorem 1.* First a definition:

Definition 12. Let $f : L \times L \to k$ and $g : L \times L \to k$ be two bilinear forms. Theorem 25 yields $\alpha^g (I_f) \subseteq I_{f+g}$. Therefore, the k-module homomorphism $\alpha^g : \otimes L \to \otimes L$ induces a k-module homomorphism $(\otimes L) \diagup I_f \to (\otimes L) \diagup I_{f+g}$. We denote this homomorphism by $\overline{\alpha}_f^g$. Since $(\otimes L) \diagup I_f = \mathrm{Cl}(L, f)$ and $(\otimes L) \diagup I_{f+g} = \mathrm{Cl}(L, f+g)$, this homomorphism $\overline{\alpha}_f^g$ is a homomorphism $\overline{\alpha}_f^g : \mathrm{Cl}(L, f) \to \mathrm{Cl}(L, f+g)$.

Now consider two bilinear forms f and g. According to Theorem 32 **(c)** (applied to g instead of f), the map α^g is invertible, and its inverse is α^{-g}. Thus, $\alpha^g \circ \alpha^{-g} = \mathrm{id}$ and $\alpha^{-g} \circ \alpha^g = \mathrm{id}$. Now, the homomorphism $\overline{\alpha}_{f+g}^{-g}$ is a homomorphism from $\mathrm{Cl}(L, f+g)$ to $\mathrm{Cl}\left(L, \underbrace{(f+g)+(-g)}_{=f}\right) = \mathrm{Cl}(L, f)$, while the homomorphism $\overline{\alpha}_f^g$ is a homomorphism from $\mathrm{Cl}(L, f)$ to $\mathrm{Cl}(L, f+g)$. Therefore, $\alpha^g \circ \alpha^{-g} = \mathrm{id}$ becomes $\overline{\alpha}_f^g \circ \overline{\alpha}_{f+g}^{-g} = \mathrm{id}$, and for the same reason $\alpha^{-g} \circ \alpha^g = \mathrm{id}$ becomes $\overline{\alpha}_{f+g}^{-g} \circ \overline{\alpha}_f^g = \mathrm{id}$. Thus, the homomorphism $\overline{\alpha}_f^g$ has an inverse - namely, the homomorphism $\overline{\alpha}_{f+g}^{-g}$. Therefore, $\overline{\alpha}_f^g$ and $\overline{\alpha}_{f+g}^{-g}$ are isomorphisms. We have thus proven the following fact:

Theorem 33. Let $f : L \times L \to k$ and $g : L \times L \to k$ be two bilinear forms. Then, the k-modules $\mathrm{Cl}(L, f)$ and $\mathrm{Cl}(L, f+g)$ are isomorphic, and the maps $\overline{\alpha}_f^g : \mathrm{Cl}(L, f) \to \mathrm{Cl}(L, f+g)$ and $\overline{\alpha}_{f+g}^{-g} : \mathrm{Cl}(L, f+g) \to \mathrm{Cl}(L, f)$ are two mutually inverse isomorphisms between them.

In particular, this (when applied to $g = -f$) yields the following fact:

Theorem 34. Let $f : L \times L \to k$ be a bilinear form. Then, the k-modules $\mathrm{Cl}(L, f)$ and $\wedge L$ are isomorphic, and the maps $\overline{\alpha}_f^{-f} : \mathrm{Cl}(L, f) \to \wedge L$ and $\overline{\alpha}_{\mathbf{0}}^f : \wedge L \to \mathrm{Cl}(L, f)$ are two mutually inverse isomorphisms between them.

Clearly, Theorem 34 immediately yields Theorem 1. Theorem 3 is a simple consequence, as well:

Proof of Theorem 3. Let $\mathrm{proj}_f : \otimes L \to \mathrm{Cl}(L, f)$ denote the canonical projection of the k-algebra $\otimes L$ onto its factor algebra $(\otimes L) \diagup I_f = \mathrm{Cl}(L, f)$, and let $\mathrm{proj}_{\mathbf{0}} : \otimes L \to \wedge L$ denote the canonical projection of the k-algebra $\otimes L$ onto its factor algebra $(\otimes L) \diagup I_{\mathbf{0}} = \wedge L$. The isomorphism $\overline{\alpha}_{\mathbf{0}}^f$ is the map from $\wedge L$ to $\mathrm{Cl}(L, f)$ induced by the homomorphism $\alpha^f : \otimes L \to \otimes L$; in other words, $\overline{\alpha}_{\mathbf{0}}^f \circ \mathrm{proj}_{\mathbf{0}} = \mathrm{proj}_f \circ \alpha^f$.

We identify any vector $v \in L$ with the 1-tensor $\mathrm{inj}(v)$ in the tensor algebra $\otimes L$. In other words, we write $\mathrm{inj}(v) = v$ for every vector $v \in L$. This makes L a subspace of $\otimes L$. It is known that the map $\mathrm{proj}_{\mathbf{0}} \mid_L : L \to \wedge L$ (this is the canonical map from the k-module L to the exterior algebra of L) is injective. Also, the map $\overline{\alpha}_{\mathbf{0}}^f : \wedge L \to \mathrm{Cl}(L, f)$ is injective (since it is an isomorphism, according to Theorem 34). Thus, the composition $\overline{\alpha}_{\mathbf{0}}^f \circ (\mathrm{proj}_{\mathbf{0}} \mid_L)$ is also an injective map (because the two maps $\mathrm{proj}_{\mathbf{0}} \mid_L$ and $\overline{\alpha}_{\mathbf{0}}^f$ are injective). But every $v \in L$ satisfies

$$\left(\overline{\alpha}_{\mathbf{0}}^{f} \circ (\operatorname{proj}_{\mathbf{0}} |_L)\right)(v) = \overline{\alpha}_{\mathbf{0}}^{f}\left(\underbrace{\operatorname{proj}_{\mathbf{0}} |_L (v)}_{=\operatorname{proj}_{\mathbf{0}}(v)}\right) = \overline{\alpha}_{\mathbf{0}}^{f}(\operatorname{proj}_{\mathbf{0}}(v)) = \underbrace{\left(\overline{\alpha}_{\mathbf{0}}^{f} \circ \operatorname{proj}_{\mathbf{0}}\right)}_{=\operatorname{proj}_f \circ \alpha^f}(v)$$

$$= \left(\operatorname{proj}_f \circ \alpha^f\right)(v) = \operatorname{proj}_f\left(\underbrace{\alpha^f(v)}_{=v \text{ (by (20))}}\right) = \operatorname{proj}_f(v) = \varphi_f(v)$$

(since we identify any vector $v \in L$ with its image $\operatorname{inj}(v)$ in the tensor algebra $\otimes L$, and thus $\operatorname{proj}_f(v) = \operatorname{proj}_f(\operatorname{inj}(v)) = \underbrace{\left(\operatorname{proj}_f \circ \operatorname{inj}\right)}_{=\varphi_f}(v) = \varphi_f(v)$). In other words, $\overline{\alpha}_{\mathbf{0}}^{f} \circ (\operatorname{proj}_{\mathbf{0}} |_L) = \varphi_f$. Since the map $\overline{\alpha}_{\mathbf{0}}^{f} \circ (\operatorname{proj}_{\mathbf{0}} |_L)$ is injective, this yields that the map φ_f is injective, and Theorem 3 is proven.

8. A simple formula for α^f on special pure tensors

We record the following simple formula to compute α^f of certain kinds of pure tensors. It doesn't help us to compute α^f generally, but can be used to compute $\overline{\alpha}_{\mathbf{0}}^{f}$ and $\overline{\alpha}_{f}^{-f}$.

Theorem 35. Let $p \in \mathbb{N}$. Let $u_1, u_2, ..., u_p$ be p elements of L such that

$$\left(f(u_i, u_j) = 0 \text{ for every } i \in \{1, 2, ..., p\} \text{ and } j \in \{1, 2, ..., p\} \text{ satisfying } i < j\right). \tag{30}$$

Then,

$$\alpha^f(u_1 \otimes u_2 \otimes ... \otimes u_p) = u_1 \otimes u_2 \otimes ... \otimes u_p.$$

Before we prove this, a lemma about the right interior product:

Theorem 36. Let $p \in \mathbb{N}$. Let $u_1, u_2, ..., u_p$ be p elements of L, and let v be another element of L such that

$$\left(f(u_i, v) = 0 \text{ for every } i \in \{1, 2, ..., p\}\right). \tag{31}$$

Then,

$$(u_1 \otimes u_2 \otimes ... \otimes u_p) \overset{f}{\lrcorner} v = 0.$$

As usual, detailed proofs of these results can be found in [0]. But the reader should have no trouble deriving Theorem 36 from the definitions and Theorem 35 from Theorem 36 by induction.

9. The Clifford basis theorem

We now come closer to proving Theorem 2 - the Clifford basis theorem. First let us make Theorem 20 a bit more precise:

Theorem 37. Let $U \in L^{\otimes p}$ for some $p \in \mathbb{N}$. Then,

$$\alpha^f(U) - U \in \bigoplus_{\substack{i \in \{0,1,...,p-2\}; \\ i \equiv p \bmod 2}} L^{\otimes i}. \tag{32}$$

The proof of this is just an obvious refinement of the proof of Theorem 20 (look at the highest-degree terms).

Before we can finally prove Theorem 2, some preliminary work is needed. First, we define some notations:

In Definition 4, we defined the ascending product $\overrightarrow{\prod\limits_{i \in I}} a_i$ of a finite family $(a_i)_{i \in I}$ of elements of a ring A. However, this notation can turn out to be ambiguous if a_i are elements of two different rings with different multiplications. For instance, we consider every vector in L both as an element of the tensor algebra $\otimes L$ and as an element of the exterior algebra $\wedge L$. So, if a_i is a vector in L for each $i \in I$, then what exactly does the product $\overrightarrow{\prod\limits_{i \in I}} a_i$ mean: does it mean the ascending product of the vectors a_i seen as elements of $\otimes L$, or does it mean the ascending product of the vectors a_i seen as elements of $\wedge L$? In order to avoid this ambiguity, we shall rename the ascending product $\overrightarrow{\prod\limits_{i \in I}} a_i$ in the algebra $\otimes L$ as $\overrightarrow{\bigotimes\limits_{i \in I}} a_i$, and we shall rename the ascending product $\overrightarrow{\prod\limits_{i \in I}} a_i$ in the algebra $\wedge L$ as $\overrightarrow{\bigwedge\limits_{i \in I}} a_i$. In other words, we declare the following notation:

> **Definition 13.** **(a)** Let I be a finite subset of $\mathbb{Z}$. Let a_i be an element of $\otimes L$ for each $i \in I$. Then, we will denote by $\overrightarrow{\bigotimes\limits_{i \in I}} a_i$ the ascending product of the elements a_i of $\otimes L$ (this product is built using the multiplication in the ring $\otimes L$, i. e., using the tensor product multiplication).
>
> **(b)** Let I be a finite subset of $\mathbb{Z}$. Let a_i be an element of $\wedge L$ for each $i \in I$. Then, we will denote by $\overrightarrow{\bigwedge\limits_{i \in I}} a_i$ the ascending product of the elements a_i of $\wedge L$ (this product is built using the multiplication in the ring $\wedge L$, i. e., using the exterior product multiplication).

One more definition:

> **Definition 14.** If N is a set, and $\ell \in \mathbb{N}$, then we denote by $\mathcal{P}_\ell(N)$ the set of all ℓ-element subsets of the set N.

It is known that if $(e_1, e_2, ..., e_n)$ is a basis of the k-module L, then

$$\left(\overrightarrow{\bigwedge_{i \in I}} e_i \right)_{I \in \mathcal{P}_\ell(\{1,2,...,n\})} \text{ is a basis of the } k\text{-module } \wedge^\ell L. \tag{33}$$

Proof of Theorem 2. We want to prove that the family $\left(\overrightarrow{\prod\limits_{i \in I}} \varphi_f(e_i) \right)_{I \in \mathcal{P}(\{1,2,...,n\})}$ is a basis of the k-module $\mathrm{Cl}(L, f)$. In order to prove this, we must show that this family is linearly independent, and that it generates the k-module $\mathrm{Cl}(L,f)$. Let us first prove that it is linearly independent:

Proof of the linear independence of the family $\left(\overrightarrow{\prod\limits_{i\in I}}\varphi_f(e_i)\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$*:*

Let $(\lambda_I)_{I\in\mathcal{P}(\{1,2,...,n\})}$ be a family of elements of k such that

$$\sum_{I\in\mathcal{P}(\{1,2,...,n\})}\lambda_I\cdot\overrightarrow{\prod_{i\in I}}\varphi_f(e_i)=0. \tag{34}$$

Our goal is to prove that this family $(\lambda_I)_{I\in\mathcal{P}(\{1,2,...,n\})}$ satisfies $\lambda_I=0$ for all $I\in\mathcal{P}(\{1,2,...,n\})$. In order to do this, we assume the contrary. This means that we assume $\lambda_I\neq 0$ for some $I\in\mathcal{P}(\{1,2,...,n\})$. Let $\mathbf{i}\in\{0,1,...,n\}$ be the greatest element j of $\{0,1,...,n\}$ such that there is some j-element subset I of $\{1,2,...,n\}$ satisfying $\lambda_I\neq 0$. This means that (34) can be rewritten as

$$\sum_{\substack{I\in\mathcal{P}(\{1,2,...,n\});\\ |I|\leq\mathbf{i}}}\lambda_I\cdot\overrightarrow{\prod_{i\in I}}\varphi_f(e_i)=0 \tag{35}$$

(because all addends with $|I|>\mathbf{i}$ are zero), but on the other hand there exists some $\mathbf{i}$-element subset I_1 of $\{1,2,...,n\}$ satisfying $\lambda_{I_1}\neq 0$.

Let us now denote by $\wedge^{<\mathbf{i}}L$ the sub-k-module $\sum\limits_{\ell=0}^{\mathbf{i}-1}\wedge^\ell L$ of $\wedge L$. Clearly, $\left(\wedge^{<\mathbf{i}}L\right)\cap\wedge^{\mathbf{i}}L=0$. We are now going to show that

$$\sum_{\substack{I\in\mathcal{P}(\{1,2,...,n\});\\ |I|=\mathbf{i}}}\lambda_I\cdot\overrightarrow{\bigwedge_{i\in I}}e_i\in\wedge^{<\mathbf{i}}L. \tag{36}$$

Once this is proven, we will be able to conclude that the element $\sum\limits_{\substack{I\in\mathcal{P}(\{1,2,...,n\});\\ |I|=\mathbf{i}}}\lambda_I\cdot\overrightarrow{\bigwedge\limits_{i\in I}}e_i$ of $\wedge L$ is zero (since it lies in $\wedge^{<\mathbf{i}}L$ and in $\wedge^{\mathbf{i}}L$ at the same time, but $\left(\wedge^{<\mathbf{i}}L\right)\cap\wedge^{\mathbf{i}}L=0$), which will yield that $\lambda_I=0$ for every $I\in\mathcal{P}(\{1,2,...,n\})$ satisfying $|I|=\mathbf{i}$ (because $\left(\overrightarrow{\bigwedge\limits_{i\in I}}e_i\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$ is a basis of $\wedge L$ and therefore linearly independent), contradicting the assumption that there exists some $\mathbf{i}$-element subset I_1 of $\{1,2,...,n\}$ satisfying $\lambda_{I_1}\neq 0$. This contradiction will then complete the proof of the linear independence of the family $\left(\overrightarrow{\prod\limits_{i\in I}}\varphi_f(e_i)\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$.

It is thus enough to prove (36).

For this, consider the map $\overline{\alpha}_f^{-f}:\mathrm{Cl}(L,f)\to\wedge L$. We have defined this map $\overline{\alpha}_f^{-f}$ as the map from $(\otimes L)\diagup I_f=\mathrm{Cl}(L,f)$ to $(\otimes L)\diagup I_{\mathbf{0}}=\mathrm{Cl}(L,0)=\wedge L$ canonically induced by the map $\alpha^{-f}:\otimes L\to\otimes L$. In other words, if we denote by $\mathrm{proj}_f:\otimes L\to\mathrm{Cl}(L,f)$ the canonical projection of the k-algebra $\otimes L$ onto its factor algebra $(\otimes L)\diagup I_f=\mathrm{Cl}(L,f)$, and if we denote by $\mathrm{proj}_{\mathbf{0}}:\otimes L\to\wedge L$ the canonical projection of the k-algebra $\otimes L$ onto its factor algebra $(\otimes L)\diagup I_{\mathbf{0}}=\wedge L$, then we have $\overline{\alpha}_f^{-f}\circ\mathrm{proj}_f=\mathrm{proj}_{\mathbf{0}}\circ\alpha^{-f}$. Note that

$$\wedge^\ell L=\mathrm{proj}_{\mathbf{0}}\left(L^{\otimes\ell}\right)\qquad\text{for every }\ell\in\mathbb{N}. \tag{37}$$

Also, just as we denoted by $\wedge^{<\mathbf{i}}L$ the submodule $\sum\limits_{\ell=0}^{\mathbf{i}-1}\wedge^{\ell}L$ of $\wedge L$, let us denote by $L^{\otimes<\mathbf{i}}$ the submodule $\sum\limits_{\ell=0}^{\mathbf{i}-1}L^{\otimes\ell}$ of $\otimes L$. Then, of course, (37) yields $\wedge^{<\mathbf{i}}L=\operatorname{proj}_{\mathbf{0}}\left(L^{\otimes<\mathbf{i}}\right)$.

Clearly, for every subset I of $\{1,2,...,n\}$, we have

$$\overrightarrow{\prod_{i\in I}}\underbrace{\varphi_f\left(e_i\right)}_{=\operatorname{proj}_f(e_i)}=\overrightarrow{\prod_{i\in I}}\operatorname{proj}_f\left(e_i\right)=\operatorname{proj}_f\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)$$

(because $\overrightarrow{\prod\limits_{i\in I}}$ denotes an ascending product in the algebra $\operatorname{Cl}(L,f)$, whereas $\overrightarrow{\bigotimes\limits_{i\in I}}$ denotes an ascending product in the algebra $\otimes L$, and because taking products commutes with proj_f since proj_f is a k-algebra homomorphism). Therefore,

$$\begin{aligned}\overline{\alpha}_f^{-f}\left(\overrightarrow{\prod_{i\in I}}\varphi_f\left(e_i\right)\right)&=\overline{\alpha}_f^{-f}\left(\operatorname{proj}_f\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\right)=\underbrace{\left(\overline{\alpha}_f^{-f}\circ\operatorname{proj}_f\right)}_{=\operatorname{proj}_{\mathbf{0}}\circ\alpha^{-f}}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\\&=\left(\operatorname{proj}_{\mathbf{0}}\circ\alpha^{-f}\right)\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\\&=\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\right).\end{aligned}\qquad(38)$$

But (34) yields

$$\overline{\alpha}_f^{-f}\left(\sum_{I\in\mathcal{P}(\{1,2,\ldots,n\})}\lambda_I\cdot\overrightarrow{\prod_{i\in I}}\varphi_f\left(e_i\right)\right)=\overline{\alpha}_f^{-f}(0)=0.$$

This, in view of

$$\begin{aligned}&\overline{\alpha}_f^{-f}\left(\sum_{\substack{I\in\mathcal{P}(\{1,2,\ldots,n\});\\|I|\leq\mathbf{i}}}\lambda_I\cdot\overrightarrow{\prod_{i\in I}}\varphi_f\left(e_i\right)\right)\\&=\sum_{\substack{I\in\mathcal{P}(\{1,2,\ldots,n\});\\|I|\leq\mathbf{i}}}\lambda_I\cdot\overline{\alpha}_f^{-f}\left(\overrightarrow{\prod_{i\in I}}\varphi_f\left(e_i\right)\right)\qquad\left(\text{since }\overline{\alpha}_f^{-f}\text{ is }k\text{-linear}\right)\\&=\sum_{\substack{I\in\mathcal{P}(\{1,2,\ldots,n\});\\|I|\leq\mathbf{i}}}\lambda_I\cdot\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\right)\qquad(\text{by (38)}),\end{aligned}$$

becomes

$$\sum_{\substack{I\in\mathcal{P}(\{1,2,\ldots,n\});\\|I|\leq\mathbf{i}}}\lambda_I\cdot\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\right)=0.\qquad(39)$$

Now, every $I \in \mathcal{P}(\{1,2,...,n\})$ satisfying $|I| \leq \mathbf{i}$ satisfies $\overrightarrow{\bigotimes_{i\in I}} e_i \in L^{\otimes |I|}$ and thus

$$\begin{aligned}
&\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}} e_i\right) - \overrightarrow{\bigotimes_{i\in I}} e_i \\
&\in \bigoplus_{\substack{i\in\{0,1,...,|I|-2\};\\ i\equiv |I| \bmod 2}} L^{\otimes i} \\
&\qquad \left(\text{due to Theorem 37, applied to } \overrightarrow{\bigotimes_{i\in I}} e_i,\ |I| \text{ and } -f \text{ instead of } U,\, p \text{ and } f\right) \\
&\subseteq L^{\otimes < \mathbf{i}} \qquad (\text{since } |I| \leq \mathbf{i})
\end{aligned}$$

and therefore

$$\begin{aligned}
\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}} e_i\right)\right) - \operatorname{proj}_{\mathbf{0}}\left(\overrightarrow{\bigotimes_{i\in I}} e_i\right) &= \operatorname{proj}_{\mathbf{0}}\left(\underbrace{\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}} e_i\right) - \overrightarrow{\bigotimes_{i\in I}} e_i}_{\in L^{\otimes < \mathbf{i}}}\right) \\
&\in \operatorname{proj}_{\mathbf{0}}\left(L^{\otimes < \mathbf{i}}\right) = \wedge^{<\mathbf{i}} L.
\end{aligned}$$

In other words,

$$\begin{aligned}
&\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}} e_i\right)\right) \\
&\equiv \operatorname{proj}_{\mathbf{0}}\left(\overrightarrow{\bigotimes_{i\in I}} e_i\right) = \overrightarrow{\bigwedge_{i\in I}} \operatorname{proj}_{\mathbf{0}}(e_i) \\
&\qquad \left(\begin{array}{c}\text{since } \overrightarrow{\bigotimes_{i\in I}} \text{ denotes the ascending product in the algebra } \otimes L \text{, while } \overrightarrow{\bigwedge_{i\in I}} \\ \text{denotes the ascending product in the algebra } \wedge L \text{, and since the map} \\ \operatorname{proj}_{\mathbf{0}} \text{ commutes with taking products (because } \operatorname{proj}_{\mathbf{0}} \text{ is a } k\text{-algebra} \\ \text{homomorphism)}\end{array}\right) \\
&= \overrightarrow{\bigwedge_{i\in I}} e_i \bmod \wedge^{<\mathbf{i}} L \qquad \left(\begin{array}{c}\text{since } \operatorname{proj}_{\mathbf{0}}(e_i) = e_i \text{, because we identify any} \\ \text{vector } v \in L \text{ with its images in } \otimes L \text{ and in } \wedge L\end{array}\right).
\end{aligned}$$

Therefore, (39) yields

$$\begin{aligned}
0 &= \sum_{\substack{I\in\mathcal{P}(\{1,2,...,n\});\\ |I|\leq \mathbf{i}}} \lambda_I \cdot \underbrace{\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}} e_i\right)\right)}_{\equiv \overrightarrow{\bigwedge_{i\in I}} e_i \bmod \wedge^{<\mathbf{i}} L} \equiv \sum_{\substack{I\in\mathcal{P}(\{1,2,...,n\});\\ |I|\leq \mathbf{i}}} \lambda_I \cdot \overrightarrow{\bigwedge_{i\in I}} e_i \\
&\equiv \sum_{\substack{I\in\mathcal{P}(\{1,2,...,n\});\\ |I|= \mathbf{i}}} \lambda_I \cdot \overrightarrow{\bigwedge_{i\in I}} e_i \bmod \wedge^{<\mathbf{i}} L
\end{aligned}$$

(where, in the last step, we stripped the sum of all addends with $|I| < \mathbf{i}$, since these addends all lie in $\wedge^{<\mathbf{i}} L$). In other words, we have proven (36). As we know, this completes the proof that the family $\left(\overrightarrow{\prod\limits_{i\in I}}\varphi_f\left(e_i\right)\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$ is linearly independent.

Proof that the family $\left(\overrightarrow{\prod\limits_{i\in I}}\varphi_f\left(e_i\right)\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$ *generates the whole* k*-module* $\mathrm{Cl}\left(L,f\right)$*:*

Next we must prove that the family $\left(\overrightarrow{\prod\limits_{i\in I}}\varphi_f\left(e_i\right)\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$ generates the k-module $\mathrm{Cl}\left(L,f\right)$. We are not going to do this here, but instead refer the reader to [0] (the proof uses roughly the same ideas as the proof of linear independency, but is actually easier).

Altogether, everything necessary for the proof of Theorem 2 is done.

10. The antisymmetrizer formula

We have constructed the Chevalley map $\overline{\alpha}^f_{\mathbf{0}} : \wedge L \to \mathrm{Cl}\left(L,f\right)$ through a canonical, inductively defined map $\alpha^f : \otimes L \to \otimes L$. This, however, is not the most common definition of the Chevalley map. The purpose of this section is to prove a different formula for $\overline{\alpha}^f_{\mathbf{0}}$ (although the word "formula" is not to be taken too seriously here, since it gives a unique value for $\overline{\alpha}^f_{\mathbf{0}}$ only if k is a $\mathbb{Q}$-algebra), at least in the case when the form f is symmetric:

Theorem 38. Let $f : L\times L \to k$ be a symmetric bilinear form. Let $p \in \mathbb{N}$, and let u_1, u_2, ..., u_p be p vectors in L. Then,

$$p!\cdot\overline{\alpha}^f_{\mathbf{0}}\left(u_1\wedge u_2\wedge...\wedge u_p\right) = \sum_{\sigma\in S_p}\left(-1\right)^{\sigma}\varphi^f\left(u_{\sigma(1)}\right)\varphi^f\left(u_{\sigma(2)}\right)...\varphi^f\left(u_{\sigma(p)}\right).$$

Here and in the following, we denote by S_p the group of all permutations of the set $\{1,2,...,p\}$, and we denote by $\left(-1\right)^{\sigma}$ the sign of the permutation σ for every $\sigma\in S_p$.

Theorem 38 is often used as a definition of the map $\overline{\alpha}^f_{\mathbf{0}}$ in the case when k is a $\mathbb{Q}$-algebra (because in this case, we can divide by $p!$). However, it does not yield a unique value of $\overline{\alpha}^f_{\mathbf{0}}\left(u_1\wedge u_2\wedge...\wedge u_p\right)$ if the characteristic of k is too small, and therefore I believe my definition of $\overline{\alpha}^f_{\mathbf{0}}$ (through the map α^f introduced in Definition 10 above) to be a better one.

Theorem 38 is an equality in the Clifford algebra $\mathrm{Cl}\left(L,f\right)$. However, it can be "lifted" into $\otimes L$:

Theorem 39. Let $f : L\times L \to k$ be a symmetric bilinear form. Let $p \in \mathbb{N}$, and let u_1, u_2, ..., u_p be p vectors in L. Then,

$$\alpha^f\left(\sum_{\sigma\in S_p}\left(-1\right)^{\sigma}u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(p)}\right) = \sum_{\sigma\in S_p}\left(-1\right)^{\sigma}u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(p)}.$$

We will prove this... you guessed right, by induction. In the induction step we will use a lemma which is interesting for its own merit:

Theorem 40. Let $f : L \times L \to k$ be a symmetric bilinear form. Let $p \in \mathbb{N}$, and let u_1, u_2, ..., u_p be p vectors in L. Then,

$$\alpha^f \left(\sum_{\sigma \in S_p} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)} \right)$$
$$= \sum_{\sigma \in S_p} (-1)^\sigma \alpha^f \left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p-1)} \right) \otimes u_{\sigma(p)}.$$

This, in turn, will be concluded from the following result:

Theorem 41. Let $f : L \times L \to k$ be a symmetric bilinear form. Let $p \in \mathbb{N}$, and let u_1, u_2, ..., u_p be p vectors in L. Then,

$$\sum_{\sigma \in S_p} (-1)^\sigma \left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p-1)} \right) \overset{f}{\lrcorner} u_{\sigma(p)} = 0.$$

The proof of Theorem 41 (which, again, can be found in [0] in full detail) proceeds by expanding $\left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p-1)} \right) \overset{f}{\lrcorner} u_{\sigma(p)}$ using (9), and noticing that for every $i \in \{1, 2, ..., p-1\}$, the sum

$$\sum_{\sigma \in S_p} (-1)^\sigma f \left(u_{\sigma(i)}, u_{\sigma(p)} \right) \cdot u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes \widehat{u_{\sigma(i)}} \otimes ... \otimes u_{\sigma(p-1)}$$

is zero (because it breaks up into two sums consisting of exactly the same addends with opposite signs).

Theorem 41 has a "left" analogue:

Theorem 42. Let $f : L \times L \to k$ be a symmetric bilinear form. Let $p \in \mathbb{N}$, and let u_1, u_2, ..., u_p be p vectors in L. Then,

$$\sum_{\sigma \in S_p} (-1)^\sigma u_{\sigma(1)} \overset{f}{\llcorner} \left(u_{\sigma(2)} \otimes u_{\sigma(3)} \otimes ... \otimes u_{\sigma(p)} \right) = 0.$$

Theorem 40 easily follows from either Theorem 41 or Theorem 42. Now Theorem 39 follows from Theorem 40 by induction (details in [0]), and Theorem 38 from Theorem 39 by direct inspection.

11. Some more identities

Let us prove some more curious properties of $\overset{f}{\llcorner}$, $\overset{f}{\lrcorner}$ and α^f for a symmetric bilinear form f. The following theorems 43-45 bear a certain similarity to theorems 40-42 (and can actually be used to give an alternative proof of Theorem 39, although we are not going to elaborate on this proof).

Theorem 43. Let $f : L \times L \to k$ be a symmetric bilinear form. Let $p \in \mathbb{N}$, and let u_1, u_2, ..., u_p be p vectors in L. Then,

$$\sum_{i=1}^{p} (-1)^{i-1} \alpha^f \left((u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p) \otimes u_i \right)$$
$$= \sum_{i=1}^{p} (-1)^{i-1} \alpha^f \left(u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p \right) \otimes u_i.$$

Here, the hat over the vector u_i means that the vector u_i is being omitted from the tensor product; in other words, $u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p$ is just another way to write $\underbrace{u_1 \otimes u_2 \otimes ... \otimes u_{i-1}}_{\substack{\text{tensor product of the} \\ \text{first } i-1 \text{ vectors } u_\ell}} \otimes \underbrace{u_{i+1} \otimes u_{i+2} \otimes ... \otimes u_p}_{\substack{\text{tensor product of the} \\ \text{last } p-i \text{ vectors } u_\ell}}$.

This, in turn, will be concluded from the following result:

Theorem 44. Let $f : L \times L \to k$ be a symmetric bilinear form. Let $p \in \mathbb{N}$, and let u_1, u_2, ..., u_p be p vectors in L. Then,

$$\sum_{i=1}^{p} (-1)^{i-1} \left(u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p \right) \overset{f}{\lrcorner} u_i = 0.$$

(For the meaning of the term $u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p$, see Theorem 43.)

Theorem 44 is trivial from the definitions, and it yields Theorem 43 almost immediately. Theorem 44 has a "left" analogue:

Theorem 45. Let $f : L \times L \to k$ be a symmetric bilinear form. Let $p \in \mathbb{N}$, and let u_1, u_2, ..., u_p be p vectors in L. Then,

$$\sum_{i=1}^{p} (-1)^{i-1} u_i \overset{f}{\llcorner} \left(u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p \right) = 0.$$

(For the meaning of the term $u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p$, see Theorem 43.)

12. The invariant module of the α^f maps for all symmetric bilinear f

Let us consider a fixed commutative ring k, and a fixed k-module L. However, in this section, we are *not* going to fix a bilinear form f on L, but we will consider all bilinear forms f at once. Each bilinear form f gives rise to an endomorphism $\alpha^f : \otimes L \to \otimes L$, and we are going to study the module $\operatorname{Fix} \alpha^{\text{symm}}$ of all tensors in $\otimes L$ that are fixed under α^f for *all symmetric* bilinear forms f. [16]

[16] As for the space $\operatorname{Fix} \alpha$ of all tensors in $\otimes L$ that are fixed under α^f for *all* (not only symmetric) bilinear forms f, we are planning to study this space later.

Definition 15. Let k be a commutative ring, and L be a k-module. We denote by $\operatorname{Fix}\alpha^{\text{symm}}$ the subset

$$\left\{U \in \otimes L \mid \text{ every symmetric bilinear form } f : L \times L \to k \text{ satisfies } \alpha^f(U) = U\right\}$$
$$= \bigcap_{\substack{f:L\times L\to k \text{ is a} \\ \text{symmetric bilinear form}}} \operatorname{Ker}\left(\alpha^f - \operatorname{id}\right)$$

of $\otimes L$. Clearly, this subset $\operatorname{Fix}\alpha^{\text{symm}}$ is a sub-k-module of $\otimes L$.

It seems to be a nontrivial question to further characterize $\operatorname{Fix}\alpha^{\text{symm}}$. First we note that antisymmetrizers always lie in $\operatorname{Fix}\alpha^{\text{symm}}$:

Corollary 46. Let $p \in \mathbb{N}$, and let u_1, u_2, ..., u_p be p vectors in L. Then,

$$\sum_{\sigma\in S_p} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)} \in \operatorname{Fix}\alpha^{\text{symm}}.$$

This follows directly from Theorem 39.

However, elements of the form $\sum\limits_{\sigma\in S_p} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)}$ as in Corollary 46 are not the only inhabitants of $\operatorname{Fix}\alpha^{\text{symm}}$. There are more. I do not claim that I know all of them, but here is a result that construct at least a part:

Theorem 47. Let k be a commutative ring. Let L be a k-module.

(a) We have $k \subseteq \operatorname{Fix}\alpha^{\text{symm}}$ (where k is regarded as a sub-k-module of $\otimes L$ because $k = L^{\otimes 0} \subseteq L^{\otimes 0} \oplus L^{\otimes 1} \oplus L^{\otimes 2} \oplus ... = \otimes L$) and $L \subseteq \operatorname{Fix}\alpha^{\text{symm}}$ (where L is regarded as a sub-k-module of $\otimes L$ because $L = L^{\otimes 1} \subseteq L^{\otimes 0} \oplus L^{\otimes 1} \oplus L^{\otimes 2} \oplus ... = \otimes L$).

(b) Let $m \in \mathbb{N}$. Any two elements $u \in L$ and $V \in L^{\otimes m} \cap \operatorname{Fix}\alpha^{\text{symm}}$ satisfy $u \otimes V + (-1)^m V \otimes u \in \operatorname{Fix}\alpha^{\text{symm}}$.

The proof of Theorem 47 relies on the following result:

Lemma 48. Let $m \in \mathbb{N}$. Let k be a commutative ring. Let L be a k-module. Let $f : L \times L \to k$ be a symmetric bilinear form. Then, any $u \in L$ and $U \in L^{\otimes m}$ satisfy $\alpha^f(u \otimes U + (-1)^m U \otimes u) = u \otimes \alpha^f(U) + (-1)^m \alpha^f(U) \otimes u$ (where u is regarded as an element of $\otimes L$ because $u \in L = L^{\otimes 1} \subseteq L^{\otimes 0} \oplus L^{\otimes 1} \oplus L^{\otimes 2} \oplus ... = \otimes L$).

Lemma 49. Let $m \in \mathbb{N}$. Let k be a commutative ring. Let L be a k-module. Let $f : L \times L \to k$ be a bilinear form. Then, any $u \in L$ and $V \in L^{\otimes m}$ satisfy $u \overset{f}{\llcorner} V = (-1)^{m-1} V \overset{f^t}{\lrcorner} u$.

Lemma 50. Let $m \in \mathbb{N}$. Let k be a commutative ring. Let L be a k-module. Let $f : L \times L \to k$ be a bilinear form. Then, any $u \in L$ and $V \in \bigoplus\limits_{\substack{i\in\mathbb{N};\\ i\equiv m \bmod 2}} L^{\otimes i}$ satisfy $u \overset{f}{\llcorner} V = (-1)^{m-1} V \overset{f^t}{\lrcorner} u$.

The proofs all are easy: First, show Lemma 49 directly. Then, conclude Lemma 50, from which Lemma 48 easily follows (use Theorem 20). Then, Theorem 47 becomes obvious.

Theorem 47 yields an inductive way to construct elements of $\operatorname{Fix}\alpha^{\mathrm{symm}}$ beginning from elements of L. For example, for any two vectors $u \in L$ and $v \in L$, Theorem 47 shows that $u \otimes v - v \otimes u \in \operatorname{Fix}\alpha^{\mathrm{symm}}$ (not surprisingly). For any three vectors $u \in L$, $v \in L$ and $w \in L$, Theorem 47 shows that $u \otimes (v \otimes w - w \otimes v) + (v \otimes w - w \otimes v) \otimes u \in \operatorname{Fix}\alpha^{\mathrm{symm}}$. For any four vectors $u \in L$, $v \in L$, $w \in L$ and $x \in L$, Theorem 47 shows that

$$u \otimes (v \otimes (w \otimes x - x \otimes w) + (w \otimes x - x \otimes w) \otimes v) \\ - (v \otimes (w \otimes x - x \otimes w) + (w \otimes x - x \otimes w) \otimes v) \otimes u \tag{40}$$

lies in $\operatorname{Fix}\alpha^{\mathrm{symm}}$. And so on.

Do we get all elements of $\operatorname{Fix}\alpha^{\mathrm{symm}}$ this way? No. For example, for any four vectors $a \in L$, $b \in L$, $c \in L$ and $d \in L$, the tensor

$$a \otimes b \otimes (c \otimes d + d \otimes c) - (c \otimes d + d \otimes c) \otimes a \otimes b$$

lies in $\operatorname{Fix}\alpha^{\mathrm{symm}}$ (and is even fixed under α^f for *all* (not only symmetric) bilinear forms f). In general, this tensor cannot be written as a linear combination of elements of the form (40) for $u, v, w, x \in L$, even if the underlying ring k is a field of characteristic 0. (This was computed by Andrew Rupinski in [4].)

Another interesting question would be to generalize α^f to super-vector spaces, thus obtaining results about Weyl algebras rather than just Clifford algebras.

14. The α^f morphisms and direct sums

In this section we are going to deal with the behaviour of α^f morphisms when the k-module L is a direct sum of two smaller k-modules.

First a relative triviality on submodules:

Lemma 60. Let k be a commutative ring. Let L be a k-module. Let $f : L \times L \to k$ be a bilinear form. Let M be a k-submodule of L such that $f(L \times M) = 0$. Then:

(a) Every $U \in \otimes L$ and every $m \in M$ satisfy $U \overset{f}{\lrcorner} m = 0$.

(b) Every $U \in \otimes L$ and every $m \in M$ satisfy $\alpha^f (U \otimes m) = \alpha^f (U) \otimes m$.

(c) We have $\alpha^f ((\otimes L) \cdot M) = (\otimes L) \cdot M$.

Lemma 60 **(a)** is proven in a straightforward way (either by induction or with the help of Theorem 11), and the rest of Lemma 60 follows from that.

Now we can prove:

Theorem 61. Let k be a commutative ring. Let L be a k-module. Let $h : L \times L \to k$ be a bilinear form. Let M and N be two k-submodules of L such that $h(M \times M) = 0$ and $L = M \oplus N$. Then:

(a) For every bilinear form $g : L \times L \to k$, there exists a k-module isomorphism $\mathrm{Cl}(L, g) \to \mathrm{Cl}(L, h+g)$ which sends the k-submodule $\mathrm{Cl}(L, g) \cdot \varphi_g(M)$ of $\mathrm{Cl}(L, g)$ to the k-submodule $\mathrm{Cl}(L, h+g) \cdot \varphi_{h+g}(M)$ of $\mathrm{Cl}(L, h+g)$.

(b) There exists a k-module isomorphism $\wedge L \to \mathrm{Cl}(L, h)$ which sends the k-submodule $(\wedge L) \cdot \varphi_{\mathbf{0}}(M)$ of $\wedge L$ to the k-submodule $\mathrm{Cl}(L, h) \cdot \varphi_h(M)$ of $\mathrm{Cl}(L, h)$. Therefore,

$$(\mathrm{Cl}(L, h)) \diagup (\mathrm{Cl}(L, h) \cdot \varphi_h(M)) \cong (\wedge L) \diagup ((\wedge L) \cdot \varphi_{\mathbf{0}}(M)) \cong \wedge (L \diagup M) \cong \wedge N$$

as k-modules.

(c) Let proj_M be the projection from L on M with kernel N, and let proj_N be the projection from L on N with kernel M. (These two projections are well-defined because $L = M \oplus N$). Define a map $f : L \times L \to k$ by

$$(f(u, v) = h(\mathrm{proj}_M u, v) + h(\mathrm{proj}_N v, u) \qquad \text{for every } (u, v) \in L \times L). \tag{41}$$

Then, f is a bilinear form satisfying $f(L \times M) = 0$. Also,

$$f(v, v) = h(v, v) \qquad \text{for every } v \in L. \tag{42}$$

As a consequence, $I_f = I_h$ and $\mathrm{Cl}(L, f) = \mathrm{Cl}(L, h)$. Moreover, $I_{f+g} = I_{h+g}$, and $\mathrm{Cl}(L, f+g) = \mathrm{Cl}(L, h+g)$ for every bilinear form $g : L \times L \to k$. We also have $\alpha^f((\otimes L) \cdot M) = (\otimes L) \cdot M$. Finally, for every bilinear form $g : L \times L \to k$, the isomorphism $\overline{\alpha}_g^f : \mathrm{Cl}(L, g) \to \mathrm{Cl}(L, g+f)$ is a k-module isomorphism from $\mathrm{Cl}(L, g)$ to $\mathrm{Cl}(L, h+g)$ satisfying

$$\overline{\alpha}_g^f(\mathrm{Cl}(L, g) \cdot \varphi_g(M)) = \mathrm{Cl}(L, h+g) \cdot \varphi_{h+g}(M). \tag{43}$$

There is nothing about Theorem 61 that is not simple computation (provided one begins with proving Theorem 61 **(c)**, and then derives **(a)** and **(b)** from it), so we will not delve into the proof. It can be found in [0] in all its detail.

Note that Theorem 61 **(b)** was inspired by the results of the paper [6] by Calaque, Căldăraru and Tu. They considered, instead of a bilinear form h, a Lie bracket on L, and instead of $h(M \times M) = 0$ they required $[M, M] \subseteq M$. In this situation, analogues of Theorem 61 **(b)** for the universal enveloping algebra instead of the Clifford algebra were shown; however, these analogues are much harder and require some additional conditions.

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