

# Littlewood–Richardson coefficients and birational combinatorics

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BIRS Workshop 20w5164: Dynamical Algebraic Combinatorics  
28 October 2020



**This meeting is being recorded.**

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**slides:** [http:](http://www.cip.ifi.lmu.de/~grinberg/algebra/birs2020.pdf)

[//www.cip.ifi.lmu.de/~grinberg/algebra/birs2020.pdf](http://www.cip.ifi.lmu.de/~grinberg/algebra/birs2020.pdf)

**paper:** [arXiv:2008.06128](https://arxiv.org/abs/2008.06128) aka [http:](http://www.cip.ifi.lmu.de/~grinberg/algebra/lrhspr.pdf)

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- I will then state a “hidden symmetry” conjectured by Pelletier and Ressayre ([arXiv:2005.09877](https://arxiv.org/abs/2005.09877)) and outline how I proved it.
- The proof is a nice example of **birational combinatorics**: the use of birational transformations in elementary combinatorics (specifically, here, in finding and proving a bijection).

# CHAPTER 1

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## Littlewood–Richardson coefficients

References (among many):

- Richard Stanley, *Enumerative Combinatorics, vol. 2*, Chapter 7.
- Darij Grinberg, Victor Reiner, *Hopf Algebras in Combinatorics*, arXiv:1409.8356.
- Emmanuel Briand, Mercedes Rosas, *The 144 symmetries of the Littlewood-Richardson coefficients of  $SL_3$* , arXiv:2004.04995.
- Igor Pak, Ernesto Vallejo, *Combinatorics and geometry of Littlewood-Richardson cones*, arXiv:math/0407170.
- Emmanuel Briand, Rosa Orellana, Mercedes Rosas, *Rectangular symmetries for coefficients of symmetric functions*, arXiv:1410.8017.

- Fix a commutative ring  $\mathbf{k}$  with unity. We shall work over  $\mathbf{k}$ .
- Let  $\Lambda$  be the *ring of symmetric functions* over  $\mathbf{k}$ .  
This is the set of all symmetric bounded-degree power series in countably many variables  $x_1, x_2, x_3, \dots$  over  $\mathbf{k}$ .

## Schur functions, part 1: Young diagrams

- Let  $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$  be a *partition* (i.e., a weakly decreasing sequence of nonnegative integers such that  $\lambda_i = 0$  for all  $i \gg 0$ ).

We commonly omit trailing zeroes: e.g., the partition  $(4, 2, 2, 1, 0, 0, 0, 0, \dots)$  is identified with the tuple  $(4, 2, 2, 1)$ .

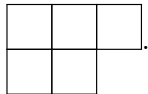
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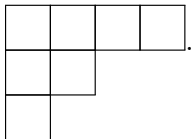
We commonly omit trailing zeroes: e.g., the partition  $(4, 2, 2, 1, 0, 0, 0, \dots)$  is identified with the tuple  $(4, 2, 2, 1)$ . The *Young diagram* of  $\lambda$  is like a matrix, but the rows have different lengths, and are left-aligned; the  $i$ -th row has  $\lambda_i$  cells.

### Examples:

- The Young diagram of  $(3, 2)$  has the form



- The Young diagram of  $(4, 2, 1)$  has the form



- A *semistandard tableau* of shape  $\lambda$  is the Young diagram of  $\lambda$ , filled with positive integers, such that
  - the entries in each **row** are **weakly** increasing;
  - the entries in each **column** are **strictly** increasing.

### Examples:

- A semistandard tableau of shape  $(3, 2)$  is

2	3	3
3	5	

- A semistandard tableau of shape  $(4, 2, 1)$  is

2	2	3	4
3	4		
5			

## Schur functions, part 3: definition of Schur functions

- Given a partition  $\lambda$ , we define the *Schur function*  $s_\lambda$  as the power series

$$s_\lambda = \sum_{\substack{T \text{ is a semistandard} \\ \text{tableau of shape } \lambda}} x_T, \quad \text{where } x_T = \prod_{p \text{ is a cell of } T} x_{T(p)}$$

(where  $T(p)$  denotes the entry of  $T$  in  $p$ ).

- Examples:**

- 

$$s_{(3,2)} = \sum_{\substack{a \leq b \leq c, d \leq e, \\ a < d, b < e}} x_a x_b x_c x_d x_e,$$

because the semistandard tableau

$$T = \begin{array}{|c|c|c|} \hline a & b & c \\ \hline d & e & \\ \hline \end{array}$$

contributes the addend  $x_T = x_a x_b x_c x_d x_e$ .

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- Examples:**

- For any  $n \geq 0$ , we have

$$s_{(n)} = \sum_{i_1 \leq i_2 \leq \dots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

since the semistandard tableaux of shape  $(n)$  are the fillings

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This symmetric function  $s_{(n)}$  is commonly called  $h_n$ .

- **Theorem:** The Schur function  $s_\lambda$  is a symmetric function (= an element of  $\Lambda$ ) for any partition  $\lambda$ .
- **Theorem:** The family  $(s_\lambda)_{\lambda \text{ a partition}}$  is a basis of the  $\mathbf{k}$ -module  $\Lambda$ .

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- **Theorem:** Fix  $n \geq 0$ . Let  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  be a partition with at most  $n$  nonzero entries. Then,

$$\begin{aligned}
 & s_\lambda(x_1, x_2, \dots, x_n) \\
 &= \underbrace{\det \left( \left( x_i^{\lambda_j + n - j} \right)_{1 \leq i, j \leq n} \right)}_{\text{this is called an } \textit{alternant}} \bigg/ \underbrace{\det \left( \left( x_i^{n-j} \right)_{1 \leq i, j \leq n} \right)}_{= \prod_{1 \leq i < j \leq n} (x_i - x_j)} \cdot \\
 & \hspace{15em} (= \text{the Vandermonde determinant})
 \end{aligned}$$

Here, for any  $f \in \Lambda$ , we let  $f(x_1, x_2, \dots, x_n)$  denote the result of substituting 0 for  $x_{n+1}, x_{n+2}, x_{n+3}, \dots$  in  $f$ ; this is a symmetric **polynomial** in  $x_1, x_2, \dots, x_n$ .

## Schur functions, part 4: classical properties

- **Theorem:** The Schur function  $s_\lambda$  is a symmetric function (= an element of  $\Lambda$ ) for any partition  $\lambda$ .
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- For proofs, see any text on symmetric functions (e.g., Stanley's EC2, or Grinberg-Reiner, or [Mark Wildon's notes](#)).

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- If  $\mu$  and  $\nu$  are two partitions, then  $s_\mu s_\nu$  belongs to  $\Lambda$  (since  $\Lambda$  is a ring)

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- **Example:**

$$\begin{aligned} s_{(2,1)} s_{(3,1)} &= s_{(3,2,1,1)} + s_{(3,2,2)} + s_{(3,3,1)} \\ &\quad + s_{(4,1,1,1)} + 2s_{(4,2,1)} + s_{(4,3)} \\ &\quad + s_{(5,1,1)} + s_{(5,2)} \end{aligned}$$

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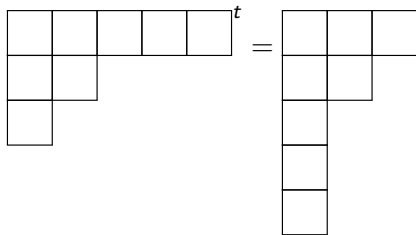
- **Theorem:** The coefficients  $c_{\mu,\nu}^\lambda$  are **nonnegative integers**. Various combinatorial interpretations (“*Littlewood–Richardson rules*”) for them are known.

- **Gradedness:**  $c_{\mu,\nu}^{\lambda} = 0$  unless  $|\lambda| = |\mu| + |\nu|$ , where  $|\kappa|$  denotes the *size* (i.e., the sum of the entries) of a partition  $\kappa$ . (This is because  $\Lambda$  is a graded ring and the  $s_{\lambda}$  are homogeneous.)

## Basic properties of Littlewood–Richardson coefficients

- **Gradedness:**  $c_{\mu,\nu}^{\lambda} = 0$  unless  $|\lambda| = |\mu| + |\nu|$ , where  $|\kappa|$  denotes the *size* (i.e., the sum of the entries) of a partition  $\kappa$ . (This is because  $\Lambda$  is a graded ring and the  $s_{\lambda}$  are homogeneous.)
- **Transposition symmetry:**  $c_{\mu,\nu}^{\lambda} = c_{\mu^t,\nu^t}^{\lambda^t}$ , where  $\kappa^t$  denotes the *transpose* of a partition  $\kappa$  (i.e., the partition whose Young diagram is obtained from that of  $\kappa$  by flipping across the main diagonal).

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- **Commutativity:**  $c_{\mu,\nu}^{\lambda} = c_{\nu,\mu}^{\lambda}$ . (Obvious from the definition.)

- Fix  $n \in \mathbb{N}$ . Let  $\text{Par}[n]$  be the set of all partitions having at most  $n$  nonzero entries.

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- If  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \text{Par}[n]$ , and if  $k \geq 0$  is such that all entries of  $\lambda$  are  $\leq k$ , then  $\lambda^{\vee k}$  shall denote the partition

$$(k - \lambda_n, k - \lambda_{n-1}, \dots, k - \lambda_1) \in \text{Par}[n].$$

This is called the  *$k$ -complement* of  $\lambda$ .

**Example:** If  $n = 5$ , then

$$\begin{aligned}(3, 1, 1)^{\vee 7} &= (3, 1, 1, 0, 0)^{\vee 7} = (7 - 0, 7 - 0, 7 - 1, 7 - 1, 7 - 3) \\ &= (7, 7, 6, 6, 4).\end{aligned}$$

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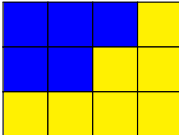
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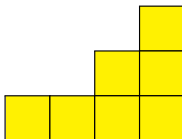
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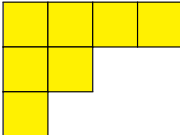


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(This can be proved by applying skew Schur functions to  $x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}$ , or by interpreting Schur functions as fundamental classes in the cohomology of the Grassmannian. See Exercise 2.9.15 in Grinberg-Reiner for the former proof.)

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- **Complementation symmetry II:** Let  $\lambda, \mu, \nu \in \text{Par}[n]$  and  $q, r \geq 0$  be such that all entries of  $\mu$  are  $\leq q$ , and all entries of  $\nu$  are  $\leq r$ . Then:

- If all entries of  $\lambda$  are  $\leq q + r$ , then  $c_{\mu, \nu}^{\lambda} = c_{\mu^{\vee q}, \nu^{\vee r}}^{\lambda^{\vee(q+r)}}$ .
- If not, then  $c_{\mu, \nu}^{\lambda} = 0$ .

(See, e.g., Exercise 2.9.16 in Grinberg-Reiner.)

- In [arXiv:2004.04995](#), Emmanuel Briand and Mercedes Rosas have used a computer (and prior work of Rassart, Knutson and Tao, which made the problem computable) to classify all such “symmetries” of Littlewood–Richardson coefficients  $c_{\mu,\nu}^{\lambda}$  with  $\lambda, \mu, \nu \in \text{Par}[n]$  for fixed  $n \in \{3, 4, \dots, 7\}$ .

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- For  $n \in \{4, 5, \dots, 7\}$ , they only found the complementation symmetries above, as well as the trivial translation symmetries (adding 1 to each entry of  $\lambda$  and  $\nu$  does not change  $c_{\mu,\nu}^\lambda$ ; nor does adding 1 to each entry of  $\lambda$  and  $\mu$ ).

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- For  $n = 3$ , they found an extra symmetry:

$$c_{(\mu_1, \mu_2), (\nu_1, \nu_2)}^{(\lambda_1, \lambda_2, \lambda_3)} = c_{(\mu_1 + \nu_1 - \lambda_2, \mu_2 + \nu_1 - \lambda_2), (\lambda_2, \nu_2)}^{(\lambda_1, \nu_1, \lambda_3)} \cdot$$

(Read the right hand side as 0 if the tuples are not partitions.)

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(Read the right hand side as 0 if the tuples are not partitions.)

**Question:** Is there a non-computer proof? What is the meaning of this identity?

# CHAPTER 2

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## The Pelletier–Ressayre symmetry

References (among many):

- Darij Grinberg, *The Pelletier–Ressayre hidden symmetry for Littlewood–Richardson coefficients*, arXiv:2008.06128.
- Maxime Pelletier, Nicolas Ressayre, *Some unexpected properties of Littlewood–Richardson coefficients*, arXiv:2005.09877.
- Robert Coquereaux, Jean-Bernard Zuber, *On sums of tensor and fusion multiplicities*, 2011.

- **Theorem (Coquereaux and Zuber, 2011):** Let  $n \geq 0$  and  $\mu, \nu \in \text{Par}[n]$ . Let  $k \geq 0$  be such that all entries of  $\mu$  are  $\leq k$ . Then,

$$\sum_{\lambda \in \text{Par}[n]} c_{\mu, \nu}^{\lambda} = \sum_{\lambda \in \text{Par}[n]} c_{\mu \vee k, \nu}^{\lambda}.$$

(See <https://mathoverflow.net/a/236220/> for a hint at a combinatorial proof.)

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- This can be interpreted in terms of Schur **polynomials**. For any  $\lambda \in \text{Par}[n]$ , the *Schur polynomial*

$$s_{\lambda}(x_1, x_2, \dots, x_n)$$

is the symmetric polynomial in  $x_1, x_2, \dots, x_n$  obtained by setting  $x_{n+1} = x_{n+2} = x_{n+3} = \dots = 0$  in  $s_{\lambda}$ .

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- The family  $(s_{\lambda}(x_1, x_2, \dots, x_n))_{\lambda \in \text{Par}[n]}$  is a basis of the  $\mathbf{k}$ -module of symmetric polynomials in  $x_1, x_2, \dots, x_n$ . We call it the *Schur basis*.

- The theorem of Coquereaux and Zuber says that the products

$$s_{\mu}(x_1, x_2, \dots, x_n) s_{\nu}(x_1, x_2, \dots, x_n)$$

and  $s_{\mu \vee k}(x_1, x_2, \dots, x_n) s_{\nu}(x_1, x_2, \dots, x_n)$

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**Question:** Does this hold for  $n \leq 4$ ? (Proved for  $n = 3$ .)

- **Conjecture (Pelletier and Ressayre, 2020):** It does hold when  $\mu$  is *near-rectangular* – i.e., when  $\mu = (a + b, a^{n-2})$  for some  $a, b \geq 0$ . Here,  $a^{n-2}$  means  $\underbrace{a, a, \dots, a}_{n-2 \text{ times}}$ .

In this case, for  $k = a + b$ , we have  $\mu^{\vee k} = (a + b, b^{n-2})$ .  
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## The Pelletier–Ressayre conjecture

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- In other words:

**Conjecture (Pelletier and Ressayre, 2020):** Let  $n \geq 0$  and  $\nu \in \text{Par}[n]$ . Let  $a, b \geq 0$ . Let  $\alpha = (a + b, a^{n-2})$  and  $\beta = (a + b, b^{n-2})$ . Then,

$$\left\{ c_{\alpha, \nu}^{\lambda} \mid \lambda \in \text{Par}[n] \right\}_{\text{multiset}} = \left\{ c_{\beta, \nu}^{\lambda} \mid \lambda \in \text{Par}[n] \right\}_{\text{multiset}}.$$

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- This means that there should be a bijection  $\varphi : \text{Par}[n] \rightarrow \text{Par}[n]$  such that

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- **Theorem (G., 2020):** This is true. Moreover, this bijection  $\varphi$  can more or less be defined explicitly in terms of maxima of sums of entries of  $\lambda$  and  $\nu$ .  
 (“More or less” means that we find a bijection  $\varphi : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ , not  $\varphi : \text{Par}[n] \rightarrow \text{Par}[n]$ , where we set  $c_{\alpha, \nu}^{\lambda} = c_{\beta, \nu}^{\varphi(\lambda)} = 0$  for all  $\lambda \in \mathbb{Z}^n \setminus \text{Par}[n]$ .)

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- The rest of this talk will sketch how this bijection  $\varphi$  was found.

- First, we notice that

$$\begin{aligned}\alpha &= (a + b, a^{n-2}) = (a + b, a^{n-2}, 0) && \text{(as } n\text{-tuple)} \\ &= (b, 0^{n-2}, -a) + a\end{aligned}$$

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- Formally: A *snake* will mean an  $n$ -tuple  $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n$  with  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . Thus,

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- Snakes index rational representations of  $\text{GL}(n)$ : See **John R. Stembridge, *Rational tableaux and the tensor algebra of  $\mathfrak{gl}_n$ , 1987.***

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- If  $\lambda \in \mathbb{Z}^n$  is any  $n$ -tuple, then
  - we let  $\lambda_i$  denote the  $i$ -th entry of  $\lambda$  (for any  $i$ );
  - we let  $\lambda + a := (\lambda_1 + a, \lambda_2 + a, \dots, \lambda_n + a)$ ;
  - we let  $\lambda - a := (\lambda_1 - a, \lambda_2 - a, \dots, \lambda_n - a)$ .

- We have defined a Schur polynomial  $s_\lambda(x_1, x_2, \dots, x_n) \in \mathbf{k}[x_1, x_2, \dots, x_n]$  for any  $\lambda \in \text{Par}[n]$ . We now denote it by  $\bar{s}_\lambda$ .

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- It is easy to see that 
$$\bar{s}_{\lambda+a} = (x_1 x_2 \cdots x_n)^a \bar{s}_\lambda \quad \text{for any } \lambda \in \text{Par}[n] \text{ and } a \geq 0.$$

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- This allows us to extend the definition of  $\bar{s}_\lambda$  from the case  $\lambda \in \text{Par}[n]$  to the more general case  $\lambda \in \{\text{snakes}\}$ :  
If  $\lambda$  is a snake, then we choose some  $a \geq 0$  such that  $\lambda + a \in \text{Par}[n]$ , and define

$$\bar{s}_\lambda = (x_1 x_2 \cdots x_n)^{-a} \bar{s}_{\lambda+a}.$$

This is a Laurent polynomial in  $\mathbf{k}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ .

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This is a Laurent polynomial in  $\mathbf{k}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ .

- Alternatively, we can define  $\bar{s}_\lambda$  explicitly by

$$\bar{s}_\lambda = \det \left( \left( x_i^{\lambda_j + n - j} \right)_{1 \leq i, j \leq n} \right) / \det \left( \left( x_i^{n-j} \right)_{1 \leq i, j \leq n} \right)$$

(same formula as before).

- For any  $k \geq 0$ , define the two Laurent polynomials

$$h_k^+ = h_k(x_1, x_2, \dots, x_n),$$

$$h_k^- = h_k(x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}).$$

(Recall:  $h_k = s_{(k)} = \sum_{i_1 \leq i_2 \leq \dots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$ .)

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- Proposition:** Let  $a, b \geq 0$ . Then,

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- Corollary:** Let  $a, b \geq 0$ . Let  $\alpha = (a + b, a^{n-2})$  and  $\beta = (a + b, b^{n-2})$ . Then,

$$\bar{s}_\alpha = (x_1 x_2 \cdots x_n)^a \cdot (h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+);$$

$$\bar{s}_\beta = (x_1 x_2 \cdots x_n)^b \cdot (h_b^- h_a^+ - h_{b-1}^- h_{a-1}^+).$$

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- Corollary:** Let  $a, b \geq 0$ . Let  $\alpha = (a + b, a^{n-2})$  and  $\beta = (a + b, b^{n-2})$ . Then,

$$\bar{s}_\alpha = (x_1 x_2 \cdots x_n)^a \cdot (h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+);$$

$$\bar{s}_\beta = (x_1 x_2 \cdots x_n)^b \cdot (h_b^- h_a^+ - h_{b-1}^- h_{a-1}^+).$$

- Thus, if we “know how to multiply by”  $h_k^-$  and  $h_k^+$ , then we “know how to multiply by”  $\bar{s}_\alpha$  and  $\bar{s}_\beta$ .

- **Theorem ( $h^+$ -Pieri rule for Laurent polynomials):**

Let  $\lambda \in \{\text{snakes}\}$ . Let  $k \in \mathbb{Z}$ . Then,

$$h_k^+ \cdot \bar{s}_\lambda = \sum_{\substack{\mu \in \{\text{snakes}\}; \\ |\mu| - |\lambda| = k; \\ \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \dots \geq \mu_n \geq \lambda_n}} \bar{s}_\mu.$$

Here:

- We let  $|\kappa|$  denote the **size** (i.e., the sum of the entries) of any  $n$ -tuple  $\kappa$ .
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- So we know how to multiply  $\bar{s}_\lambda$  by  $h_k^+$ . What about  $h_k^-$ ?

- Theorem ( $h^-$ -Pieri rule for Laurent polynomials):**

Let  $\lambda \in \{\text{snakes}\}$ . Let  $k \in \mathbb{Z}$ . Then,

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- This follows from the  $h^+$ -Pieri rule by substituting  $x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}$  for  $x_1, x_2, \dots, x_n$ , using the following fact:

**Proposition:** For any snake  $\lambda$ , we have

$$\bar{s}_{\lambda^\vee} = \bar{s}_\lambda(x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}).$$

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- So we now know how to multiply  $\bar{s}_\lambda$  by  $h_k^-$ .

- A consequence of the above:

**Corollary:** Let  $\mu$  be a snake. Let  $a, b \in \mathbb{Z}$ . Then,

$$h_a^- h_b^+ \bar{s}_\mu = \sum_{\gamma \text{ is a snake}} |R_{\mu,a,b}(\gamma)| \bar{s}_\gamma,$$

where  $R_{\mu,a,b}(\gamma)$  is the set of all snakes  $\nu$  satisfying

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- **Corollary:** Let  $\nu \in \text{Par}[n]$ . Let  $a, b \geq 0$ . Define the partition  $\alpha = (a + b, a^{n-2})$ . Then, every  $\lambda \in \mathbb{Z}^n$  satisfies

$$c_{\alpha,\nu}^\lambda = |R_{\nu,a,b}(\lambda - a)| - |R_{\nu,a-1,b-1}(\lambda - a)|.$$

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- Recall that we want a bijection  $\varphi : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$  such that

$$c_{\alpha,\mu}^\lambda = c_{\beta,\mu}^{\varphi(\lambda)} \quad \text{for each } \lambda \in \text{Par}[n].$$

## Closing in on the bijection, 1

- So we want a bijection  $\varphi : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$  such that

$$\begin{aligned} & |R_{\mu,a,b}(\lambda - a)| - |R_{\mu,a-1,b-1}(\lambda - a)| \\ &= |R_{\mu,b,a}(\varphi(\lambda) - b)| - |R_{\mu,b-1,a-1}(\varphi(\lambda) - b)| \end{aligned}$$

for all  $\lambda \in \mathbb{Z}^n$ .

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- So we want a bijection  $\mathbf{f} : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$  such that

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- In other words, if  $\mathbf{f}(\gamma) = \eta$ , then we want

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- Forget at first about the size conditions ( $|\mu| - |\nu| = a$ , etc.). Then the former snakes satisfy

$$\mu \rightarrow \nu \quad \text{and} \quad \gamma \rightarrow \nu$$

$$\iff (\mu_i \geq \nu_i \text{ for all } i \leq n) \wedge (\nu_i \geq \mu_{i+1} \text{ for all } i < n) \\ \wedge (\gamma_i \geq \nu_i \text{ for all } i \leq n) \wedge (\gamma_i \geq \gamma_{i+1} \text{ for all } i < n)$$

$$\iff (\min \{\mu_i, \gamma_i\} \geq \nu_i \text{ for all } i \leq n) \\ \wedge (\nu_i \geq \max \{\mu_{i+1}, \gamma_{i+1}\} \text{ for all } i < n)$$

$$\iff (\nu_i \in [\max \{\mu_{i+1}, \gamma_{i+1}\}, \min \{\mu_i, \gamma_i\}] \text{ for all } i < n) \\ \wedge (\min \{\mu_n, \gamma_n\} \geq \nu_n).$$

- Compare the condition

$$\nu_i \in [\max \{\mu_{i+1}, \gamma_{i+1}\}, \min \{\mu_i, \gamma_i\}] \text{ for all } i < n$$

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- It is thus reasonable to hope for

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- These conditions do not suffice to determine  $\mathbf{f}(\gamma) = \eta$  (nor probably to guarantee  $|R_{\mu,a,b}(\gamma)| = |R_{\mu,b,a}(\eta)|$ ), but let's see what they tell us.

- Let  $n = 3$ . We want  $\mathbf{f}(\gamma) = \eta$  to satisfy

$$\min \{ \mu_1, \gamma_1 \} - \max \{ \mu_2, \gamma_2 \} = \min \{ \mu_1, \eta_1 \} - \max \{ \mu_2, \eta_2 \};$$

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- This is a system of equations that only involves the operations  $+$ ,  $-$  and  $\min$ . (Recall:  $2a = a + a$ .)
- There is a trick for studying such systems: **detropicalization**.

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The same construction works for any totally ordered abelian group instead of  $\mathbb{Z}$ .

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- It is particularly useful if you just want **one** solution (rather than all of them). Often, solutions over  $\mathbb{Q}_+$  are unique, while those over the min tropical semifield are not.

- Recall our system

$$\min \{ \mu_1, \gamma_1 \} + \min \{ -\mu_2, -\gamma_2 \} = \min \{ \mu_1, \eta_1 \} + \min \{ -\mu_2, -\eta_2 \};$$

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$$(\gamma_1 + \gamma_2 + \gamma_3) + (\eta_1 + \eta_2 + \eta_3) = 2(\mu_1 + \mu_2 + \mu_3)$$

(where  $\eta_1, \eta_2, \eta_3$  are unknown).

- Recall our system

$$\min \{\mu_1, \gamma_1\} + \min \{-\mu_2, -\gamma_2\} = \min \{\mu_1, \eta_1\} + \min \{-\mu_2, -\eta_2\};$$

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- Detropicalization transforms this into

$$(\mu_1 + \gamma_1) \left( \frac{1}{\mu_2} + \frac{1}{\gamma_2} \right) = (\mu_1 + \eta_1) \left( \frac{1}{\mu_2} + \frac{1}{\eta_2} \right);$$

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- So we now need to solve the system

$$(\mu_1 + \gamma_1) \left( \frac{1}{\mu_2} + \frac{1}{\gamma_2} \right) = (\mu_1 + \eta_1) \left( \frac{1}{\mu_2} + \frac{1}{\eta_2} \right);$$

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- Let us rename  $\mu, \gamma, \eta$  as  $u, x, y$ . Then, this becomes

$$(u_1 + x_1) \left( \frac{1}{u_2} + \frac{1}{x_2} \right) = (u_1 + y_1) \left( \frac{1}{u_2} + \frac{1}{y_2} \right);$$

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- This is a system of polynomial equations, so we can give it to a computer. The answer is:

## Solving the detropicalized system ( $n = 3$ )

- *Solution 1:*

$$y_1 = \frac{u_1 (u_1 u_2 u_3 + x_1 u_2 u_3 + x_1 x_2 u_3 + x_1 x_2 x_3)}{u_1 x_2 u_3 - x_1 x_2 x_3},$$

$$y_2 = \frac{-u_1 u_2 u_3}{x_1 x_3},$$

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- Solution 1 is useless, since we want  $y_1, y_2, y_3 \in \mathbb{Q}_+$ .

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- But Solution 2 looks promising. Note in particular the (unexpected) **cyclic symmetry!**

## The map $f$ : definition

- Reverse-engineering Solution 2, we come up with the following

**Definition:** Let  $\mathbb{K}$  be a semifield, let  $n \geq 1$ , and let  $u \in \mathbb{K}^n$ .

We define a map  $f : \mathbb{K}^n \rightarrow \mathbb{K}^n$  as follows:

Let  $x \in \mathbb{K}^n$  be an  $n$ -tuple. For each  $j \in \mathbb{Z}$  and  $r \geq 0$ , define an element  $t_{r,j} \in \mathbb{K}$  by

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(Here and in the following, all indices are cyclic modulo  $n$ .)

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- Note that  $\mathbf{f}$  depends on  $u$  (whence I call it  $\mathbf{f}_u$  in the paper).

- **Theorem.** Let  $\mathbb{K}$  be a semifield,  $n \geq 1$  and  $u \in \mathbb{K}^n$ . Then:
  - (a) The map  $\mathbf{f}$  is an involution (i.e., we have  $\mathbf{f} \circ \mathbf{f} = \text{id}$ ).

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- In short:  $f(x)$  solves our system and more. (Note that the  $i = n$  case of part (c) is not part of our original system!)

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- The proof is heavily computational but not too hard (various auxiliary identities had to be discovered).

- Recall that we were looking for a bijection  $\mathbf{f} : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$  (independent on  $a$  and  $b$ ) such that

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- Shifting by  $a$  and  $b$  thus produces the bijection  $\varphi$  needed for the Pelletier–Ressayre conjecture. Explicitly:

- **Theorem (G., 2020):** Assume that  $n \geq 2$ . Let  $a, b \geq 0$ , and set  $\alpha = (a + b, a^{n-2})$  and  $\beta = (a + b, b^{n-2})$ .

Fix any partition  $\mu \in \text{Par}[n]$ .

Define a map  $\varphi : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$  as follows:

Let  $\omega \in \mathbb{Z}^n$ . Set  $\nu = \omega - a \in \mathbb{Z}^n$ . For each  $j \in \mathbb{Z}$ , set

$$\begin{aligned} \tau_j = \min \{ & (\nu_{j+1} + \nu_{j+2} + \cdots + \nu_{j+k}) \\ & + (\mu_{j+k+1} + \mu_{j+k+2} + \cdots + \mu_{j+n-1}) \\ & \mid k \in \{0, 1, \dots, n-1\} \}, \end{aligned}$$

where (unusually for partitions!) all indices are cyclic modulo  $n$ .

Define an  $n$ -tuple  $\eta = (\eta_1, \eta_2, \dots, \eta_n) \in \mathbb{Z}^n$  by setting

$$\eta_i = \mu_i + (\mu_{i-1} + \tau_{i-1}) - (\nu_{i+1} + \tau_{i+1}) \quad \text{for each } i.$$

Let  $\varphi(\omega)$  be the  $n$ -tuple  $\eta + b \in \mathbb{Z}^n$ . Thus, we have defined a map  $\varphi : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ .

- **Theorem (cont'd):** Then:
  - (a) The map  $\varphi$  is a bijection.
  - (b) We have

$$c_{\alpha,\mu}^{\omega} = c_{\beta,\mu}^{\varphi(\omega)} \quad \text{for each } \omega \in \mathbb{Z}^n.$$

Here, we are using the convention that every  $n$ -tuple  $\omega \in \mathbb{Z}^n$  that is not a partition satisfies  $c_{\alpha,\mu}^{\omega} = 0$  and  $c_{\beta,\mu}^{\omega} = 0$ .

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- **Question:** Can  $\varphi$  be written as a composition of “toggles” (i.e., “local” transformations, each affecting only one entry of the tuple)?

## Uniqueness questions, 1

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- No if  $\mathbb{K} = \mathbb{Z}_{\text{trop}}$ .

- **Question:** Given a semifield  $\mathbb{K}$  and  $n \geq 2$  and  $u \in \mathbb{K}^n$ . Assume that  $x \in \mathbb{K}^n$  and  $y \in \mathbb{K}^n$  satisfy

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- No if  $\mathbb{K} = \mathbb{Z}_{\text{trop}}$ .
- Thus, detropicalization has made the solution unique by removing the “extraneous” solutions.

- **Maxime Pelletier** and **Nicolas Ressayre** for the conjecture.
- **Tom Roby** for the invitation.
- **Tom Roby** and **Grigori Olshanski** for enlightening discussions.
- **you** for your patience.

## Why Littlewood–Richardson coefficients? 1

- Before we say more about Littlewood–Richardson coefficients, let us see where else they appear.

- For  $\mathbf{k} = \mathbb{Z}$ , the cohomology ring

$$H^*(\mathrm{Gr}(k, n))$$

of the complex Grassmannian  $\mathrm{Gr}(k, n)$  (of  $k$ -subspaces in  $\mathbb{C}^n$ ) is isomorphic to

$$\Lambda / (h_{n-k+1}, h_{n-k+2}, \dots, h_n, e_{k+1}, e_{k+2}, e_{k+3}, \dots)_{\text{ideal}}.$$

The cohomology classes corresponding to the Schur functions  $s_\lambda$  are the *Schubert classes* – the classes of the *Schubert varieties*. Roughly speaking, these subdivide  $\mathrm{Gr}(k, n)$  according to the positions of the pivots in the row-reduced echelon form.

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- For details, see:
  - Laurent Manivel, *Symmetric Functions, Schubert Polynomials and Degeneracy Loci*, AMS/SMF 1998.

## Why Littlewood–Richardson coefficients? 2

- Here is another interpretation of Littlewood–Richardson coefficients, also related to subspaces of a vector space.

## Why Littlewood–Richardson coefficients? 2

- Let  $V$  be a finite-dimensional vector space.
- The *Jordan type*  $J(A)$  of a nilpotent endomorphism  $A \in \text{End } V$  is the partition  $(\lambda_1, \lambda_2, \lambda_3, \dots)$  with  $\lambda_i$  being the size of the  $i$ -th largest Jordan block of  $A$ .

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- Pick a nilpotent endomorphism  $A \in \text{End } V$ , and let  $\lambda = J(A)$  be its Jordan type. Let  $\mu$  and  $\nu$  be two further partitions. When is there an  $A$ -invariant vector subspace  $W \subseteq V$  with

$$J(A) = \lambda, \quad J(A|_W) = \mu, \quad J(A/W) = \nu?$$

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- Moreover, the set of all such  $W$  is a subvariety of  $\text{Gr}(k, n)$ , and has  $c_{\mu, \nu}^{\lambda}$  irreducible components.
- For details, see:
  - *Marc van Leeuwen, Flag Varieties and Interpretations of Young Tableau Algorithms.*

## Why Littlewood–Richardson coefficients? 3

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- Their *characters* are the Schur functions  $s_\lambda$ .
- The Littlewood–Richardson coefficients tell how to decompose the tensor product of two such representations:

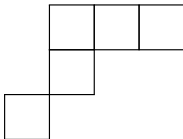
$$V_\mu \otimes V_\nu = \bigoplus_{\lambda} V_\lambda^{\oplus c_{\mu,\nu}^\lambda}.$$

- For details, see:
  - William Fulton, *Young Tableaux*, CUP 1997.

- In order to formulate the classic (or, at least, best known) Littlewood–Richardson rule, we need a
- **Definition:**
  - Two partitions  $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$  and  $\mu = (\mu_1, \mu_2, \mu_3, \dots)$  are said to satisfy  $\mu \subseteq \lambda$  if each  $i \geq 1$  satisfies  $\mu_i \leq \lambda_i$ .  
(Equivalently: if the Young diagram of  $\mu$  is contained in that of  $\lambda$ .)

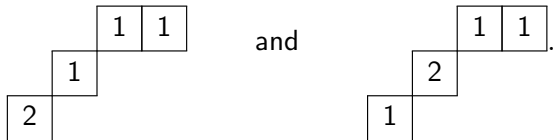
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  - If  $\lambda/\mu$  is a skew partition, then the *Young diagram* of  $\lambda/\mu$  is obtained from the Young diagram  $\lambda$  when all cells of the Young diagram of  $\mu$  are removed.
- **Example:** The Young diagram of  $(4, 2, 1) / (1, 1)$  is

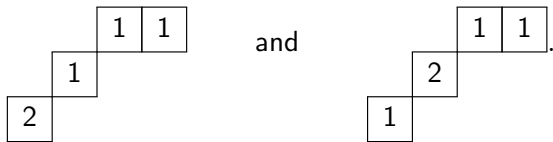


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  - If  $\lambda/\mu$  is a skew partition, then the *Young diagram* of  $\lambda/\mu$  is obtained from the Young diagram  $\lambda$  when all cells of the Young diagram of  $\mu$  are removed.
  - *Semistandard tableaux* of shape  $\lambda/\mu$  are defined just as ones of shape  $\lambda$ , except that we are now only filling the cells of  $\lambda/\mu$ .

- Littlewood–Richardson rule:** Let  $\lambda$ ,  $\mu$  and  $\nu$  be three partitions. Then,  $c_{\mu,\nu}^\lambda$  is the number of semistandard tableaux  $T$  of shape  $\lambda/\mu$  such that  $\text{cont } T = \nu$  and such that  $\text{cont}(T|_{\text{cols} \geq j})$  is a partition for each  $j$ . Here,
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- The shortest proof is due to Stembridge (using ideas by Gasharov); see [John R. Stembridge, \*A Concise Proof of the Littlewood-Richardson Rule\*, 2002](#), or Section 2.6 in Grinberg-Reiner.