A basis for a quotient of symmetric polynomials (draft)

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Abstract. Consider the ring $\mathcal{S}$ of symmetric polynomials in k variables over an arbitrary base ring $\mathbf{k}$. Fix k scalars $a_1, a_2, \ldots, a_k \in \mathbf{k}$.

Let I be the ideal of $\mathcal{S}$ generated by $h_{n-k+1} - a_1, h_{n-k+2} - a_2, \ldots, h_n - a_k$, where h_i is the i-th complete homogeneous symmetric polynomial.

The quotient ring $\mathcal{S}/I$ generalizes both the usual and the quantum cohomology of the Grassmannian.

We show that $\mathcal{S}/I$ has a $\mathbf{k}$-module basis consisting of (residue classes of) Schur polynomials fitting into an $(n-k) \times k$-rectangle; and that its multiplicative structure constants satisfy the same S_3-symmetry as those of the Grassmannian cohomology. We prove a Pieri rule and conjecture a positivity property generalizing that of Gromov-Witten invariants.

We also study the quotient of the whole polynomial ring (not just the symmetric polynomials) by the ideal generated by the same k polynomials as I.

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1. Introduction

This is a rough draft – all proofs are merely outlines. The results will also likely be shuffled around.

This work is devoted to a certain construction that generalizes both the regular and the quantum cohomology ring of the Grassmannian [Postni05]. This construction is purely algebraic – we do not conjecture any geometric meaning for it at this point – but shares some basic properties with quantum cohomology, such as an S_3-symmetry of its structure constants (generalizing the S_3-symmetry for Littlewood-Richardson coefficients and Gromov-Witten invariants). All our arguments are algebraic and combinatorial.

1.1. Acknowledgments

We thank Dongkwan Kim, Alex Postnikov, Victor Reiner, Mark Shimozono, Josh Swanson, Kaisa Taipale, and Anders Thorup for enlightening conversations. The SageMath computer algebra system [SageMath] has been used for experimentation leading up to some of the results below.

2. The basis theorems

2.1. Definitions and notations

Let $\mathbb{N}$ denote the set $\{0, 1, 2, \ldots\}$.

Let $\mathbf{k}$ be a commutative ring. Let $k \in \mathbb{N}$.

Let $\mathcal{P}$ denote the polynomial ring $\mathbf{k}[x_1, x_2, \ldots, x_k]$. This is a graded ring, where the grading is by total degree (so $\deg x_i = 1$ for each $i \in \{1, 2, \ldots, k\}$).

For each $\alpha \in \mathbb{N}^k$ and each $i \in \{1, 2, \ldots, k\}$, we denote the i-th entry of α by α_i (so that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$). For each $\alpha \in \mathbb{N}^k$, we define a monomial x^α by $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$.

Let $\mathcal{S}$ denote the ring of symmetric polynomials in $\mathcal{P}$; in other words, $\mathcal{S}$ is the ring of invariants of the symmetric group S_k acting on $\mathcal{P}$. The following fact is well-known (going back to Emil Artin) and is proven (e.g.) in [LLPT95, (DIFF.1.3)]:

Proposition 2.1. The $\mathcal{S}$-module $\mathcal{P}$ is free with basis $(x^\alpha)_{\alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$.

Now, fix an integer $n \geq k$. For each $i \in \{1, 2, \ldots, k\}$, let a_i be an element of $\mathcal{P}$ with degree $< n - k + i$. (This is clearly satisfied when $a_1, a_2, \ldots, a_k$ are constants in $\mathbf{k}$, but also in some other cases. Note that the a_i do not have to be homogeneous.)

For each $\alpha \in \mathbb{N}^k$, we let $|\alpha|$ denote the sum of the entries of the k-tuple α (that is, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$).

For each $m \in \mathbb{Z}$, we let h_m denote the m-th complete homogeneous symmetric polynomial; this is the element of $\mathcal{S}$ defined by

$$h_m = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq k} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} x^\alpha. \tag{1}$$

(Thus, $h_0 = 1$, and $h_m = 0$ when $m < 0$.)

Let J be the ideal of $\mathcal{P}$ generated by the k differences

$$h_{n-k+1} - a_1, h_{n-k+2} - a_2, \ldots, h_n - a_k. \tag{2}$$

If M is a $\mathbf{k}$-module and N is a submodule of M, then the projection of any $m \in M$ onto the quotient M/N (that is, the congruence class of m modulo N) will be denoted by $\overline{m}$.

2.2. The basis theorem for $\mathcal{P}/J$

We claim the following result:

Theorem 2.2. The $\mathbf{k}$-module $\mathcal{P}/J$ is free with basis $\left(\overline{x^\alpha}\right)_{\alpha\in\mathbb{N}^k;\ \alpha_i<n-k+i \text{ for each } i}$.

Example 2.3. Let $n = 5$ and $k = 2$. Then, $\mathcal{P} = \mathbf{k}[x_1, x_2]$, and J is the ideal of $\mathcal{P}$ generated by the 2 differences

$$h_4 - a_1 = \left(x_1^4 + x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + x_2^4\right) - a_1 \qquad \text{and}$$
$$h_5 - a_2 = \left(x_1^5 + x_1^4x_2 + x_1^3x_2^2 + x_1^2x_2^3 + x_1x_2^4 + x_2^5\right) - a_2.$$

Theorem 2.2 yields that the $\mathbf{k}$-module $\mathcal{P}/J$ is free with basis $\left(\overline{x^\alpha}\right)_{\alpha\in\mathbb{N}^2;\ a_i<3+i \text{ for each } i}$; this basis can also be rewritten as $\left(\overline{x_1^{\alpha_1}x_2^{\alpha_2}}\right)_{\alpha_1\in\{0,1,2,3\};\ \alpha_2\in\{0,1,2,3,4\}}$. As a consequence, any monomial in $\overline{x_1}$ and $\overline{x_2}$ can be written as a linear combination of elements of this basis. For example,

$$\overline{x_1^4} = a_1 - \overline{x_1^3x_2} - \overline{x_1^2x_2^2} - \overline{x_1x_2^3} - \overline{x_2^4} \qquad \text{and}$$
$$\overline{x_2^5} = a_2 - a_1\overline{x_1}.$$

These expressions will become more complicated for higher values of n and k.

Theorem 2.2 is related to the second part of [CoKrWa09, Proposition 2.9] (and our proof below can be viewed as an elaboration of the argument sketched in the last paragraph of [CoKrWa09, proof of Proposition 2.9]).

2.3. The basis theorem for $\mathcal{S}/I$

To state our next result, we need some more notations.

Definition 2.4. (a) We define the concept of *partitions* (of an integer) as in [GriRei18, Chapter 2].

(b) A *part* of a partition λ means a nonzero entry of λ.

(c) Let $P_{k,n}$ denote the set of all partitions that have at most k parts and have the property that each of their parts is $\leq n-k$. (Visually speaking, $P_{k,n}$ is the set of all partitions whose Young diagram fits into a $k \times (n-k)$-rectangle.)

(b) For any partition λ, we let s_λ denote the Schur polynomial (in $x_1, x_2, \ldots, x_k$) corresponding to the partition λ. This Schur polynomial is what is called $s_\lambda(x_1, x_2, \ldots, x_k)$ in [GriRei18, Chapter 2]. Note that

$$s_\lambda = 0 \qquad \text{if } \lambda \text{ has more than } k \text{ parts.} \tag{3}$$

Theorem 2.5. Assume that $a_1, a_2, \ldots, a_k$ belong to $\mathcal{S}$. Let I be the ideal of $\mathcal{S}$ generated by the k differences (2). Then, the $\mathbf{k}$-module $\mathcal{S}/I$ is free with basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$.

The $\mathbf{k}$-algebra $\mathcal{S}/I$ generalizes several constructions in the literature:

- If $\mathbf{k} = \mathbb{Z}$ and $a_1 = a_2 = \cdots = a_k = 0$, then $\mathcal{S}/I$ becomes the cohomology ring of the Grassmannian of k-dimensional subspaces in an n-dimensional space (see, e.g., [Fulton99, §9.4]); the elements of the basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ correspond to the Schubert classes.
- If $\mathbf{k} = \mathbb{Z}[q]$ and $a_1 = a_2 = \cdots = a_{k-1} = 0$ and $a_k = -(-1)^k q$, then $\mathcal{S}/I$ becomes isomorphic to the quantum cohomology ring of the same Grassmannian (see [Postni05]). Indeed, our ideal I becomes the J^q_{kn} of [Postni05, (6)] in this case, and Theorem 2.5 generalizes the fact that the quotient $(\Lambda_k \otimes \mathbb{Z}[q]) / J^q_{kn}$ in [Postni05, (6)] has basis $(s_\lambda)_{\lambda \in P_{kn}}$.

One goal of this paper is to provide a purely algebraic foundation for the study of the standard and quantum cohomology rings of the Grassmannian, without having to resort to geometry for proofs of the basic properties of these rings. In particular, Theorem 2.5 shows that the "abstract Schubert classes" $\overline{s_\lambda}$ (with $\lambda \in P_{k,n}$) form a basis of the $\mathbf{k}$-module $\mathcal{S}/I$, whereas Corollary 6.24 further below shows that the structure constants of the $\mathbf{k}$-algebra $\mathcal{S}/I$ with respect to this basis (we may call them "generalized Gromov-Witten invariants") satisfy an S_3-symmetry. These two properties are two of the facts for whose proofs [Postni05] relies on algebro-geometric literature; thus, our paper helps provide an alternative footing for [Postni05] using only combinatorics and algebra[1].

Remark 2.6. The $\mathbf{k}$-algebra $\mathcal{P}/J$ somewhat resembles the "splitting algebra" $\operatorname{Split}^d_A(p)$ from [LakTho12, §1.3]; further analogies between these concepts can be made as we study the former. For example, the basis we give in Theorem 2.2 is like the basis in [LakTho12, (1.5)]. It is not currently clear to us whether there is more than analogies.

---

[1] This, of course, presumes that one is willing to forget the cohomological definition of the ring $\operatorname{QH}^*(\operatorname{Gr}_{kn})$, and instead to define it algebraically as the quotient ring $(\Lambda_k \otimes \mathbb{Z}[q]) / J^q_{kn}$, using the notations of [Postni05].

3. A fundamental identity

Let us use the notations h_m and e_m for complete homogeneous symmetric polynomials and elementary symmetric polynomials in general. Thus, for any $m \in \mathbb{Z}$ and any p elements $y_1, y_2, \ldots, y_p$ of a commutative ring, we set

$$h_m\left(y_1, y_2, \ldots, y_p\right) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq p} y_{i_1} y_{i_2} \cdots y_{i_m} \qquad \text{and} \tag{4}$$

$$e_m\left(y_1, y_2, \ldots, y_p\right) = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq p} y_{i_1} y_{i_2} \cdots y_{i_m}. \tag{5}$$

(Thus, $h_0\left(y_1, y_2, \ldots, y_p\right) = 1$ and $e_0\left(y_1, y_2, \ldots, y_p\right) = 1$. Also, $e_m\left(y_1, y_2, \ldots, y_p\right) = 0$ for all $m > p$. Also, for any $m < 0$, we have $h_m\left(y_1, y_2, \ldots, y_p\right) = 0$ and $e_m\left(y_1, y_2, \ldots, y_p\right) = 0$. Finally, what we have previously called h_m without any arguments can now be rewritten as $h_m\left(x_1, x_2, \ldots, x_k\right)$. Similarly, we shall occasionally abbreviate $e_m\left(x_1, x_2, \ldots, x_k\right)$ as e_m.)

Lemma 3.1. Let $i \in \{1, 2, \ldots, k+1\}$ and $p \in \mathbb{N}$. Then,

$$h_p\left(x_i, x_{i+1}, \ldots, x_k\right) = \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) h_{p-t}\left(x_1, x_2, \ldots, x_k\right).$$

Notice that if $i = k+1$, then the term $h_p\left(x_i, x_{i+1}, \ldots, x_k\right)$ on the left hand side of Lemma 3.1 is understood to be h_p of an empty alphabet; this is 1 when $p = 0$ and 0 otherwise.

Proof of Lemma 3.1. This formula is actually a particular case of [Grinbe16a, detailed version, Theorem 3.15] (applied to $a = x_i \in \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ and $b = h_p = h_p\left(x_1, x_2, x_3, \ldots\right) \in \operatorname{QSym}$). But it can also be checked elementarily. For example, consider the ring $\mathcal{P}\left[\left[u\right]\right]$ of power series. In this ring, we have

$$\sum_{q \in \mathbb{N}} h_q\left(x_i, x_{i+1}, \ldots, x_k\right) u^q = \prod_{j=i}^{k} \frac{1}{1 - x_j u},$$

$$\sum_{q \in \mathbb{N}} (-1)^q e_q\left(x_1, x_2, \ldots, x_{i-1}\right) u^q = \prod_{j=1}^{i-1} \left(1 - x_j u\right),$$

$$\sum_{q \in \mathbb{N}} h_q\left(x_1, x_2, \ldots, x_k\right) u^q = \prod_{j=1}^{k} \frac{1}{1 - x_j u}.$$

Thus,

$$\begin{aligned}
&\sum_{q\in\mathbb{N}} h_q\left(x_i, x_{i+1}, \ldots, x_k\right) u^q \\
&= \prod_{j=i}^{k} \frac{1}{1-x_j u} = \left(\prod_{j=1}^{k} \frac{1}{1-x_j u}\right) \Big/ \left(\prod_{j=1}^{i-1} \frac{1}{1-x_j u}\right) \\
&= \underbrace{\left(\prod_{j=1}^{i-1} \left(1-x_j u\right)\right)}_{=\sum_{q\in\mathbb{N}} (-1)^q e_q(x_1,x_2,\ldots,x_{i-1}) u^q} \underbrace{\left(\prod_{j=1}^{k} \frac{1}{1-x_j u}\right)}_{=\sum_{q\in\mathbb{N}} h_q(x_1,x_2,\ldots,x_k) u^q} \\
&= \left(\sum_{q\in\mathbb{N}} (-1)^q e_q\left(x_1, x_2, \ldots, x_{i-1}\right) u^q\right) \left(\sum_{q\in\mathbb{N}} h_q\left(x_1, x_2, \ldots, x_k\right) u^q\right).
\end{aligned}$$

Comparing the coefficient before u^p in this equality of power series, we obtain

$$\begin{aligned}
h_p\left(x_i, x_{i+1}, \ldots, x_k\right) &= \sum_{t=0}^{p} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) h_{p-t}\left(x_1, x_2, \ldots, x_k\right) \\
&= \sum_{t=0}^{\infty} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) h_{p-t}\left(x_1, x_2, \ldots, x_k\right) \\
&\qquad \left(\text{since } h_{p-t}\left(x_1, x_2, \ldots, x_k\right) = 0 \text{ for all } t > p\right) \\
&= \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) h_{p-t}\left(x_1, x_2, \ldots, x_k\right) \\
&\qquad \left(\text{since } e_t\left(x_1, x_2, \ldots, x_{i-1}\right) = 0 \text{ for all } t > i-1\right).
\end{aligned}$$

This proves Lemma 3.1. □

Corollary 3.2. Let p be a positive integer. Then,

$$h_p = -\sum_{t=1}^{k} (-1)^t e_t h_{p-t}.$$

Proof of Corollary 3.2. Lemma 3.1 (applied to $i = k+1$) yields

$$\begin{aligned}
h_p\left(x_{k+1}, x_{k+2}, \ldots, x_k\right) &= \sum_{t=0}^{k} (-1)^t \underbrace{e_t\left(x_1, x_2, \ldots, x_k\right)}_{=e_t} \underbrace{h_{p-t}\left(x_1, x_2, \ldots, x_k\right)}_{=h_{p-t}} \\
&= \sum_{t=0}^{k} (-1)^t e_t h_{p-t}.
\end{aligned}$$

Comparing this with

$$h_p\left(x_{k+1}, x_{k+2}, \ldots, x_k\right) = h_p\left(\text{an empty list of variables}\right) = 0 \qquad \left(\text{since } p > 0\right),$$

we obtain

$$0 = \sum_{t=0}^{k} \left(-1\right)^t e_t h_{p-t} = \underbrace{\left(-1\right)^0}_{=1} \underbrace{e_0}_{=1} \underbrace{h_{p-0}}_{=h_p} + \sum_{t=1}^{k} \left(-1\right)^t e_t h_{p-t} = h_p + \sum_{t=1}^{k} \left(-1\right)^t e_t h_{p-t}.$$

Hence,

$$h_p = -\sum_{t=1}^{k} \left(-1\right)^t e_t h_{p-t}.$$

This proves Corollary 3.2. □

4. Proof of Theorem 2.2

We shall next prove Theorem 2.2 using Gröbner bases. For the concept of Gröbner bases over a commutative ring, see [Grinbe17, detailed version, §3].

We define a degree-lexicographic term order on the monomials in $\mathcal{P}$, where the variables are ordered by $x_1 > x_2 > \cdots > x_k$. Explicitly, this term order is the total order on the set of monomials in $x_1, x_2, \ldots, x_k$ defined as follows: Two monomials $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ and $x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k}$ satisfy $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} > x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k}$ if and only if

- **either** $\alpha_1 + \alpha_2 + \cdots + \alpha_k > \beta_1 + \beta_2 + \cdots + \beta_k$,
- **or** $\alpha_1 + \alpha_2 + \cdots + \alpha_k = \beta_1 + \beta_2 + \cdots + \beta_k$ and there exists some $i \in \{1, 2, \ldots, k\}$ such that $\alpha_i > \beta_i$ and $\left(\alpha_j = \beta_j \text{ for all } j < i\right)$.

This total order is a term order (in the sense of [Grinbe17, detailed version, Definition 3.5]). Fix this term order; thus it makes sense to speak of Gröbner bases of ideals.

Proposition 4.1. The family

$$\left(h_{n-k+i}\left(x_i, x_{i+1}, \ldots, x_k\right) - \sum_{t=0}^{i-1} \left(-1\right)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) a_{i-t}\right)_{i \in \{1,2,\ldots,k\}}$$

is a Gröbner basis of the ideal J. (Recall that we are using the notations from (4) and (5).)

Proposition 4.1 is somewhat similar to [Sturmf08, Theorem 1.2.7] (or, equivalently, [CoLiOs15, §7.1, Proposition 5]), but not the same.[2] Our proof of it relies on the following elementary fact:

> **Lemma 4.2.** Let A be a commutative ring. Let $b_1, b_2, \ldots, b_k \in A$ and $c_1, c_2, \ldots, c_k \in A$. Assume that
>
> $$b_i \in c_i + \sum_{t=1}^{i-1} c_{i-t} A \tag{6}$$
>
> for each $i \in \{1, 2, \ldots, k\}$. Then, $b_1 A + b_2 A + \cdots + b_k A = c_1 A + c_2 A + \cdots + c_k A$ (as ideals of A).

Proof of Lemma 4.2. We claim that

$$\sum_{p=1}^{j} b_p A = \sum_{p=1}^{j} c_p A \qquad \text{for each } j \in \{0, 1, \ldots, k\}. \tag{7}$$

[*Proof of (7):* We shall prove (7) by induction on j:

Induction base: For $j = 0$, both sides of the equality (7) are the zero ideal of A (since they are empty sums of ideals of A). Thus, (7) holds for $j = 0$. This completes the induction base.

Induction step: Let $i \in \{1, 2, \ldots, k\}$. Assume that (7) holds for $j = i - 1$. We must prove that (7) holds for $j = i$.

We have assumed that (7) holds for $j = i - 1$. In other words, we have $\sum_{p=1}^{i-1} b_p A = \sum_{p=1}^{i-1} c_p A$. But (6) yields $b_i \in c_i + \sum_{t=1}^{i-1} c_{i-t} A = c_i + \sum_{p=1}^{i-1} c_p A$ (here, we have substituted p for $i - t$ in the sum). Thus,

$$c_i \in b_i - \sum_{p=1}^{i-1} c_p A = b_i + \sum_{p=1}^{i-1} c_p A,$$

so that

$$c_i A \subseteq \left(b_i + \sum_{p=1}^{i-1} c_p A \right) A \subseteq b_i A + \sum_{p=1}^{i-1} c_p A.$$

But from $b_i \in c_i + \sum_{p=1}^{i-1} c_p A$, we obtain

$$b_i A \subseteq \left(c_i + \sum_{p=1}^{i-1} c_p A \right) A \subseteq c_i A + \sum_{p=1}^{i-1} c_p A = \sum_{p=1}^{i} c_p A.$$

---

[2] For example, our $a_1, a_2, \ldots, a_k$ are elements of $\mathbf{k}$ rather than indeterminates (although they **can** be indeterminates if $\mathbf{k}$ itself is a polynomial ring), and our term order is degree-lexicographic rather than lexicographic. Thus, it should not be surprising that the families are different.

Now,

$$\sum_{p=1}^{i} b_pA = \underbrace{\sum_{p=1}^{i-1} b_pA}_{\substack{=\sum_{p=1}^{i-1} c_pA \subseteq \sum_{p=1}^{i} c_pA \\ \text{(since } i-1\leq i\text{)}}} + \underbrace{b_iA}_{\subseteq \sum_{p=1}^{i} c_pA} \subseteq \sum_{p=1}^{i} c_pA + \sum_{p=1}^{i} c_pA = \sum_{p=1}^{i} c_pA.$$

Combining this inclusion with

$$\begin{aligned}\sum_{p=1}^{i} c_pA &= \sum_{p=1}^{i-1} c_pA + \underbrace{c_iA}_{\subseteq b_iA+\sum_{p=1}^{i-1} c_pA} \subseteq \sum_{p=1}^{i-1} c_pA + b_iA + \sum_{p=1}^{i-1} c_pA \\ &= \underbrace{\sum_{p=1}^{i-1} c_pA + \sum_{p=1}^{i-1} c_pA}_{=\sum_{p=1}^{i-1} c_pA = \sum_{p=1}^{i-1} b_pA} + b_iA = \sum_{p=1}^{i-1} b_pA + b_iA \\ &= \sum_{p=1}^{i-1} b_pA,\end{aligned}$$

we obtain $\sum_{p=1}^{i} b_pA = \sum_{p=1}^{i} c_pA$. In other words, (7) holds for $j = i$. This completes the induction step. Thus, (7) is proven by induction.]

Now, (7) (applied to $j = k$) yields

$$\sum_{p=1}^{k} b_pA = \sum_{p=1}^{k} c_pA.$$

Thus,

$$b_1A + b_2A + \cdots + b_kA = \sum_{p=1}^{k} b_pA = \sum_{p=1}^{k} c_pA = c_1A + c_2A + \cdots + c_kA.$$

This proves Lemma 4.2. □

Proof of Proposition 4.1 (sketched). For each $i \in \{1, 2, \ldots, k\}$, we define a polynomial $b_i \in \mathcal{P}$ by

$$b_i = h_{n-k+i}(x_i, x_{i+1}, \ldots, x_k) - \sum_{t=0}^{i-1} (-1)^t e_t(x_1, x_2, \ldots, x_{i-1}) a_{i-t}.$$

Then, we must prove that the family $(b_i)_{i\in\{1,2,\ldots,k\}}$ is a Gröbner basis of the ideal J. We shall first prove that this family generates J.

For each $i \in \{1,2,\ldots,k\}$, we define $c_i \in \mathcal{P}$ by $c_i = h_{n-k+i} - a_i$. Then, J is the ideal of $\mathcal{P}$ generated by the k elements $c_1, c_2, \ldots, c_k$ (by the definition of J). In other words,

$$J = c_1\mathcal{P} + c_2\mathcal{P} + \cdots + c_k\mathcal{P}. \tag{8}$$

For each $i \in \{1,2,\ldots,k\}$, we have

$$\begin{aligned}
b_i &= \underbrace{h_{n-k+i}\left(x_i, x_{i+1}, \ldots, x_k\right)}_{\substack{=\sum_{t=0}^{i-1}(-1)^t e_t(x_1,x_2,\ldots,x_{i-1}) h_{n-k+i-t}(x_1,x_2,\ldots,x_k) \\ \text{(by Lemma 3.1 (applied to } p=n-k+i\text{))}}} - \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) a_{i-t} \\
&= \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) h_{n-k+i-t}\left(x_1, x_2, \ldots, x_k\right) \\
&\qquad - \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) a_{i-t} \\
&= \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) \left(\underbrace{h_{n-k+i-t}\left(x_1, x_2, \ldots, x_k\right)}_{=h_{n-k+i-t}} - a_{i-t} \right) \\
&= \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) \underbrace{\left(h_{n-k+i-t} - a_{i-t}\right)}_{\substack{=c_{i-t} \\ \text{(by the definition of } c_{i-t}\text{)}}} \\
&= \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) c_{i-t} \\
&= \underbrace{(-1)^0}_{=1} \underbrace{e_0\left(x_1, x_2, \ldots, x_{0-1}\right)}_{=1} \underbrace{c_{i-0}}_{=c_i} + \sum_{t=1}^{i-1} \underbrace{(-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right)}_{\in\mathcal{P}} c_{i-t} \\
&\in c_i + \sum_{t=1}^{i-1} \underbrace{\mathcal{P} c_{i-t}}_{=c_{i-t}\mathcal{P}} = c_i + \sum_{t=1}^{i-1} c_{i-t}\mathcal{P}.
\end{aligned}$$

Hence, Lemma 4.2 (applied to $A = \mathcal{P}$) yields that $b_1\mathcal{P} + b_2\mathcal{P} + \cdots + b_k\mathcal{P} = c_1\mathcal{P} + c_2\mathcal{P} + \cdots + c_k\mathcal{P}$ (as ideals of $\mathcal{P}$). Comparing this with (8), we obtain $J = c_1\mathcal{P} + c_2\mathcal{P} + \cdots + c_k\mathcal{P}$. Thus, the family $(b_i)_{i\in\{1,2,\ldots,k\}}$ generates the ideal J. Furthermore, for each $i \in \{1,2,\ldots,k\}$, the i-th term

$$b_i = h_{n-k+i}\left(x_i, x_{i+1}, \ldots, x_k\right) - \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) a_{i-t}$$

of this family has leading term x_i^{n-k+i}. Thus, the leading terms of the k elements of this family are disjoint (in the sense that no two of these leading terms have any indeterminates in common). Thus, clearly, Buchberger's first criterion (see, e.g., [Grinbe17, detailed version, Proposition 3.9]) shows that it is a Gröbner basis. ☐

Proof of Theorem 2.2 (sketched). This follows using the Macaulay-Buchberger basis theorem (e.g., [Grinbe17, detailed version, Proposition 3.10]) from Proposition 4.1. (Indeed, if we let G be the Gröbner basis of J constructed in Proposition 4.1, then the monomials $\overline{x^\alpha}$ for all $\alpha \in \mathbb{N}^k$ satisfying ($\alpha_i < n - k + i$ for each i) are precisely the G-reduced monomials[3].) ☐

5. Proof of Theorem 2.5

Next, we shall prove Theorem 2.5.

Convention 5.1. For the rest of Section 5, **we assume that** $a_1, a_2, \ldots, a_k$ **belong to** $\mathcal{S}$**.**

Let I be the ideal of $\mathcal{S}$ generated by the k differences (2). Hence, these differences belong to I. Thus,

$$h_{n-k+j} \equiv a_j \operatorname{mod} I \qquad \text{for each } j \in \{1, 2, \ldots, k\}. \tag{9}$$

Renaming the index j as $i - n + k$ in this statement, we obtain

$$h_i \equiv a_{i-n+k} \operatorname{mod} I \qquad \text{for each } i \in \{n-k+1, n-k+2, \ldots, n\}. \tag{10}$$

Lemma 5.2. Let A be a commutative $\mathbf{k}$-algebra. Let B be a commutative A-algebra. Assume that the A-module B is spanned by the family $(b_u)_{u \in U} \in B^U$. Let $\mathcal{I}$ be an ideal of A. Let $(a_v)_{v \in V} \in A^V$ be a family of elements of A such that the $\mathbf{k}$-module $A/\mathcal{I}$ is spanned by the family $(\overline{a_v})_{v \in V} \in (A/\mathcal{I})^V$. Then, the $\mathbf{k}$-module $B/(\mathcal{I}B)$ is spanned by the family $\left(\overline{a_v b_u}\right)_{(u,v) \in U \times V} \in (B/(\mathcal{I}B))^{U \times V}$.

Proof of Lemma 5.2. Easy. Here is the proof under the assumption that the set U is finite[4]:

Let $x \in B/(\mathcal{I}B)$. Thus, $x = \overline{b}$ for some $b \in B$. Consider this b. Recall that the A-module B is spanned by the family $(b_u)_{u \in U}$. Hence, $b = \sum\limits_{u \in U} p_u b_u$ for some family $(p_u)_{u \in U} \in A^U$ of elements of A. Consider this family $(p_u)_{u \in U}$.

Recall that the $\mathbf{k}$-module $A/\mathcal{I}$ is spanned by the family $(\overline{a_v})_{v \in V} \in (A/\mathcal{I})^V$. Thus, for each $u \in U$, there exists a family $(q_{u,v})_{v \in V} \in \mathbf{k}^V$ of elements of $\mathbf{k}$ such

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[3] because the i-th entry of the Gröbner basis G has head term x_i^{n-k+i}

[4] The case when U is infinite needs only minor modifications. But we shall only use the case when U is finite.

that $\overline{p_u} = \sum_{v \in V} q_{u,v} \overline{a_v}$ (and such that all but finitely many $v \in V$ satisfy $q_{u,v} = 0$). Consider this family $(q_{u,v})_{v \in V}$.

Now, recall that $B / (\mathcal{I} B)$ is an $A / \mathcal{I}$-module (since B is an A-module, but each $i \in \mathcal{I}$ clearly acts as 0 on $B / (\mathcal{I} B)$). Now,

$$\begin{aligned} x = \overline{b} &= \overline{\sum_{u \in U} p_u b_u} \qquad \left(\text{since } b = \sum_{u \in U} p_u b_u\right) \\ &= \sum_{u \in U} \underbrace{\overline{p_u}}_{= \sum_{v \in V} q_{u,v} \overline{a_v}} \overline{b_u} = \underbrace{\sum_{u \in U} \sum_{v \in V}}_{= \sum_{(u,v) \in U \times V}} q_{u,v} \underbrace{\overline{a_v} \overline{b_u}}_{= \overline{a_v b_u}} = \sum_{(u,v) \in U \times V} q_{u,v} \overline{a_v b_u}. \end{aligned}$$

Thus, x belongs to the $\mathbf{k}$-submodule of $B / (\mathcal{I} B)$ spanned by the family $\left(\overline{a_v b_u}\right)_{(u,v) \in U \times V}$. Since we have proven this for all $x \in B / (\mathcal{I} B)$, we thus conclude that the $\mathbf{k}$-module $B / (\mathcal{I} B)$ is spanned by the family $\left(\overline{a_v b_u}\right)_{(u,v) \in U \times V} \in (B / (\mathcal{I} B))^{U \times V}$. This proves Lemma 5.2. $\square$

Lemma 5.3. Let M be a free $\mathbf{k}$-module with a finite basis $(b_s)_{s \in S}$. Let $(a_u)_{u \in U} \in M^U$ be a family that spans M. Assume that $|U| = |S|$. Then, $(a_u)_{u \in U}$ is a basis of the $\mathbf{k}$-module M. (In other words: A spanning family of M whose size equals the size of a basis must itself be a basis, as long as the sizes are finite.)

Proof of Lemma 5.3. Well-known (see, e.g., [GriRei18, Exercise 2.5.18 **(b)**]). $\square$

Lemma 5.4. Let i be an integer such that $i > n - k$. Then,

$$h_i \equiv (\text{some symmetric polynomial of degree } < i) \bmod I.$$

Proof of Lemma 5.4 (sketched). We shall prove Lemma 5.4 by strong induction on i. Thus, we assume (as the induction hypothesis) that

$$h_j \equiv (\text{some symmetric polynomial of degree } < j) \bmod I \tag{11}$$

for every $j \in \{n-k+1, n-k+2, \ldots, i-1\}$.

If $i \leq n$, then (10) yields $h_i \equiv a_{i-n+k} \bmod I$ (since $i \in \{n-k+1, n-k+2, \ldots, n\}$), which clearly proves Lemma 5.4. Thus, for the rest of this proof, we WLOG assume that $i > n$. Hence, each $t \in \{1, 2, \ldots, k\}$ satisfies $i - t \in \{n-k+1, n-k+2, \ldots, i-1\}$ (since $\underbrace{i}_{>n} - \underbrace{t}_{\leq k} > n - k$ and $i - \underbrace{t}_{\geq 1} \leq i - 1$) and therefore

$$h_{i-t} \equiv (\text{some symmetric polynomial of degree } < i - t) \bmod I \tag{12}$$

(by (11), applied to $j = i - t$).

But i is a positive integer (since $i > n \geq 0$). Hence, Corollary 3.2 (applied to $p = i$) yields

$$\begin{aligned} h_i &= -\sum_{t=1}^{k} (-1)^t e_t \underbrace{h_{i-t}}_{\substack{\equiv(\text{some symmetric polynomial of degree } <i-t) \bmod I \\ (\text{by (12)})}} \\ &\equiv -\sum_{t=1}^{k} (-1)^t e_t \cdot (\text{some symmetric polynomial of degree } < i-t) \\ &= (\text{some symmetric polynomial of degree } < i) \bmod I. \end{aligned}$$

This completes the induction step. Thus, Lemma 5.4 is proven. ☐

Definition 5.5. The *size* of a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ is defined as $\lambda_1 + \lambda_2 + \lambda_3 + \cdots$, and is denoted by $|\lambda|$.

Definition 5.6. Let P_k denote the set of all partitions with at most k parts. Thus, the elements of P_k are weakly decreasing k-tuples of nonnegative integers.

Proposition 5.7. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition in P_k. Then:

(a) We have

$$s_\lambda = \det\left((h_{\lambda_u - u + v})_{1 \leq u \leq k,\ 1 \leq v \leq k} \right).$$

(b) Let $p \in \{0, 1, \ldots, k\}$ be such that $\ell(\lambda) \leq p$. Then,

$$s_\lambda = \det\left((h_{\lambda_u - u + v})_{1 \leq u \leq p,\ 1 \leq v \leq p} \right).$$

Proof of Proposition 5.7. **(b)** Proposition 5.7 **(b)** is the well-known Jacobi-Trudi identity, and is proven in various places. (For instance, [GriRei18, (2.4.9)] states a similar formula for skew Schur functions; if we set $\mu = \varnothing$ in it and apply both sides to the variables $x_1, x_2, \ldots, x_k$, then we recover the claim of Proposition 5.7 **(b)**.)

(a) We have $\lambda \in P_k$ and thus $\ell(\lambda) \leq k$. Hence, Proposition 5.7 **(a)** is the particular case of Proposition 5.7 **(b)** for $p = k$. ☐

Lemma 5.8. Let $\lambda \in P_k$ be a partition such that $\lambda \notin P_{k,n}$. Then,

$$s_\lambda \equiv (\text{some symmetric polynomial of degree } < |\lambda|) \bmod I.$$

Proof of Lemma 5.8 (sketched). Write the partition λ as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. (This can be done, since $\lambda \in P_k$.) Note that $k > 0$ (since otherwise, $\lambda \in P_k$ would lead to $\lambda = \varnothing \in P_{k,n}$, which would contradict $\lambda \notin P_{k,n}$).

From $\lambda \in P_k$ and $\lambda \notin P_{k,n}$, we conclude that not all parts of the partition λ are $\leq n-k$. Thus, the first entry λ_1 of λ is $> n-k$ (since $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$). But $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Thus, Proposition 5.7 **(a)** yields

$$s_\lambda = \det\left(\left(h_{\lambda_u - u + v}\right)_{1\leq u\leq k,\ 1\leq v\leq k} \right) = \sum_{j=1}^{k} h_{\lambda_1 - 1 + j} \cdot C_j, \tag{13}$$

where C_j denotes the $(1,j)$-th cofactor of the $k \times k$-matrix $\left(h_{\lambda_u - u + v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}$ (here, we have expanded the determinant of the matrix $\left(h_{\lambda_u - u + v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}$ along its first row).

For each $j \in \{1, 2, \ldots, k\}$, the cofactor C_j is a symmetric polynomial of degree

$$\begin{aligned}
&\underbrace{\sum_{\substack{u\in\{1,2,\ldots,k\};\\ u\neq 1}} (\lambda_u - u)}_{= \sum_{u\in\{1,2,\ldots,k\}} (\lambda_u - u) - (\lambda_1 - 1)} + \underbrace{\sum_{\substack{v\in\{1,2,\ldots,k\};\\ u\neq j}} v}_{= \sum_{v\in\{1,2,\ldots,k\}} v - j} \\
&= \underbrace{\sum_{u\in\{1,2,\ldots,k\}} (\lambda_u - u)}_{= \sum_{u\in\{1,2,\ldots,k\}} \lambda_u - \sum_{u\in\{1,2,\ldots,k\}} u} - (\lambda_1 - 1) + \underbrace{\sum_{v\in\{1,2,\ldots,k\}} v}_{= \sum_{u\in\{1,2,\ldots,k\}} u} - j \\
&= \sum_{u\in\{1,2,\ldots,k\}} \lambda_u - \sum_{u\in\{1,2,\ldots,k\}} u - (\lambda_1 - 1) + \sum_{u\in\{1,2,\ldots,k\}} u - j \\
&= \underbrace{\sum_{u\in\{1,2,\ldots,k\}} \lambda_u}_{= |\lambda|} - (\lambda_1 - 1) - j = |\lambda| - (\lambda_1 - 1) - j = |\lambda| - (\lambda_1 - 1 + j).
\end{aligned}$$

Therefore, (13) becomes

$$\begin{aligned}
s_\lambda &= \sum_{j=1}^{k} \underbrace{h_{\lambda_1 - 1 + j}}_{\substack{\equiv (\text{some symmetric polynomial of degree } < \lambda_1 - 1 + j) \bmod I \\ (\text{by Lemma 5.4, since } \lambda_1 - 1 + j \geq \lambda_1 - 1 + 1 = \lambda_1 > n - k)}} \\
&\qquad \cdot \underbrace{C_j}_{= (\text{some symmetric polynomial of degree } |\lambda| - (\lambda_1 - 1 + j))} \\
&\equiv \sum_{j=1}^{k} (\text{some symmetric polynomial of degree } < \lambda_1 - 1 + j) \\
&\qquad \cdot (\text{some symmetric polynomial of degree } |\lambda| - (\lambda_1 - 1 + j)) \\
&= (\text{some symmetric polynomial of degree } < |\lambda|) \bmod I.
\end{aligned}$$

This proves Lemma 5.8. ☐

Recall Definition 5.6.

Lemma 5.9. For each $\mu \in P_k$, the element $\overline{s_\mu} \in \mathcal{S}/I$ belongs to the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$.

Proof of Lemma 5.9. Let M be the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$. We thus must prove that $\overline{s_\mu} \in M$ for each $\mu \in P_k$.

We shall prove this by strong induction on $|\mu|$. Thus, we fix some $N \in \mathbb{N}$, and we assume (as induction hypothesis) that

$$\overline{s_\lambda} \in M \qquad \text{for each } \lambda \in P_k \text{ satisfying } |\lambda| < N. \tag{14}$$

Now, let $\mu \in P_k$ be such that $|\mu| = N$. We then must show that $\overline{s_\mu} \in M$.

If $\mu \in P_{k,n}$, then this is obvious (since $\overline{s_\mu}$ then belongs to the family that spans M). Thus, for the rest of this proof, we WLOG assume that $\mu \notin P_{k,n}$. Hence, Lemma 5.8 (applied to $\lambda = \mu$) yields

$$s_\mu \equiv (\text{some symmetric polynomial of degree } < |\mu|) \bmod I.$$

In other words, there exists some symmetric polynomials $f \in \mathcal{S}$ of degree $< |\mu|$ such that $s_\mu \equiv f \bmod I$. Consider this f.

Now, recall that the family $(s_\lambda)_{\lambda \in P_k}$ is a graded basis of the graded $\mathbf{k}$-module $\mathcal{S}$. Thus, for each $d \in \mathbb{N}$, each homogeneous symmetric polynomial of degree d can be written as a $\mathbf{k}$-linear combination of Schur polynomials s_λ with $\lambda \in P_k$ satisfying $|\lambda| = d$. Applying this to all homogeneous components of f, we thus conclude that f can be written as a $\mathbf{k}$-linear combination of Schur polynomials s_λ with $\lambda \in P_k$ satisfying $|\lambda| < |\mu|$ (since f has degree $< |\mu|$). In other words, there exists a family $(c_\lambda)_{\lambda \in P_k;\ |\lambda| < |\mu|}$ of elements of $\mathbf{k}$ such that $f = \sum_{\substack{\lambda \in P_k; \\ |\lambda| < |\mu|}} c_\lambda s_\lambda$. Consider this family. From $f = \sum_{\substack{\lambda \in P_k; \\ |\lambda| < |\mu|}} c_\lambda s_\lambda$, we obtain

$$\overline{f} = \overline{\sum_{\substack{\lambda \in P_k; \\ |\lambda| < |\mu|}} c_\lambda s_\lambda} = \sum_{\substack{\lambda \in P_k; \\ |\lambda| < |\mu|}} c_\lambda \underbrace{\overline{s_\lambda}}_{\substack{\in M \\ \text{(by (14))} \\ \text{(since } |\lambda| < |\mu| = N)}} \in \sum_{\substack{\lambda \in P_k; \\ |\lambda| < |\mu|}} c_\lambda M \subseteq M.$$

But from $s_\mu \equiv f \bmod I$, we obtain $\overline{s_\mu} = \overline{f} \in M$. This completes our induction step. Thus, we have proven by strong induction that $\overline{s_\mu} \in M$ for each $\mu \in P_k$. This proves Lemma 5.9. $\square$

Proof of Theorem 2.5 (sketched). Proposition 2.1 yields that $(x^\alpha)_{\alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ is a spanning set of the $\mathcal{S}$-module $\mathcal{P}$.

Recall Definition 5.6. It is well-known that $(s_\lambda)_{\lambda \in P_k}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}$. Hence, $(\overline{s_\lambda})_{\lambda \in P_k}$ is a spanning set of the $\mathbf{k}$-module $\mathcal{S}/I$. Thus, $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ is also a spanning set of the $\mathbf{k}$-module $\mathcal{S}/I$ (because Lemma 5.9 shows that every element of the first spanning set belongs to the span of the second). It remains to prove that this spanning set is also a basis.

In order to do so, we consider the family $\left(\overline{s_\lambda x^\alpha}\right)_{\lambda \in P_{k,n};\ \alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ in the $\mathbf{k}$-module $\mathcal{P}/J$. This family spans $\mathcal{P}/J$ (by Lemma 5.2), because the family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ spans $\mathcal{S}/I$ whereas the family $(x^\alpha)_{\alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ spans $\mathcal{P}$ over $\mathcal{S}$ (and because $I\mathcal{P} = J$). Moreover, this family $\left(\overline{s_\lambda x^\alpha}\right)_{\lambda \in P_{k,n};\ \alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ has size

$$\underbrace{|P_{k,n}|}_{=\binom{n}{k}} \cdot \underbrace{\left|\left\{\alpha \in \mathbb{N}^k \mid \alpha_i < i \text{ for each } i\right\}\right|}_{=k!} = \binom{n}{k} \cdot k!$$
$$= n(n-1)\cdots(n-k+1),$$

which is exactly the size of the basis $\left(\overline{x^\alpha}\right)_{\alpha \in \mathbb{N}^k;\ \alpha_i < n-k+i \text{ for each } i}$ of the $\mathbf{k}$-module $\mathcal{P}/J$ (this is a basis by Theorem 2.2). Thus, this family $\left(\overline{s_\lambda x^\alpha}\right)_{\lambda \in P_{k,n};\ \alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ must be a basis of the $\mathbf{k}$-module $\mathcal{P}/J$ (by Lemma 5.3), and hence is $\mathbf{k}$-linearly independent. Thus, its subfamily $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ is also $\mathbf{k}$-linearly independent.

The canonical $\mathbf{k}$-linear map $\mathcal{S}/I \to \mathcal{P}/J$ (obtained as a quotient of the inclusion $\mathcal{S} \to \mathcal{P}$) is injective (because it sends the spanning set $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ of $\mathcal{S}/I$ to the $\mathbf{k}$-linearly independent family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ in $\mathcal{P}/J$). Hence, the $\mathbf{k}$-linear independency of the family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ in $\mathcal{P}/J$ yields the $\mathbf{k}$-linear independency of the family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ in $\mathcal{S}/I$. Thus, the family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ in $\mathcal{S}/I$ is a basis of $\mathcal{S}/I$ (since it is $\mathbf{k}$-linearly independent and spans $\mathcal{S}/I$). This proves Theorem 2.5. □

6. Symmetry of the multiplicative structure constants

Convention 6.1. For the rest of Section 6, **we assume that** $a_1, a_2, \ldots, a_k$ **belong to k.**

If $m \in \mathcal{S}$, then the notation $\overline{m}$ shall always mean the projection of $m \in \mathcal{S}$ onto the quotient $\mathcal{S}/I$ (and not the projection of $m \in \mathcal{P}$ onto the quotient $\mathcal{P}/J$).

Definition 6.2. **(a)** Let ω be the partition $(n-k, n-k, \ldots, n-k)$ with k entries equal to $n-k$. (This is the largest partition in $P_{k,n}$.)

(b) Let I be the ideal of $\mathcal{S}$ generated by the k differences (2). For each $\mu \in P_{k,n}$, let $\operatorname{coeff}_\mu : \mathcal{S}/I \to \mathbf{k}$ be the $\mathbf{k}$-linear map that sends $\overline{s_\mu}$ to 1 while sending all other $\overline{s_\lambda}$ (with $\lambda \in P_{k,n}$) to 0. (This is well-defined by Theorem 2.5. Actually, $\left(\operatorname{coeff}_\mu\right)_{\mu \in P_{k,n}}$ is the dual basis to the basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ of $\mathcal{S}/I$.)

(c) If λ is any partition and if p is a positive integer, then λ_p shall always denote the p-th entry of λ. Thus, $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ for every partition λ.

(d) For every partition $\nu = (\nu_1, \nu_2, \ldots, \nu_k) \in P_{k,n}$, we let $\nu^\vee$ denote the partition $(n-k-\nu_k, n-k-\nu_{k-1}, \ldots, n-k-\nu_1) \in P_{k,n}$. This partition $\nu^\vee$ is called the *complement* of ν.

We can now make a more substantial claim:

Theorem 6.3. Each $\nu \in P_{k,n}$ and $f \in \mathcal{S}/I$ satisfy $\operatorname{coeff}_\omega \left(\overline{s_\nu} f\right) = \operatorname{coeff}_{\nu^\vee}(f)$.

The proof of this theorem requires some preliminary work.

We first recall some basic notations from [GriRei18, Chapter 2]. If λ and μ are two partitions, then we say that $\mu \subseteq \lambda$ if and only if each positive integer p satisfies $\mu_p \leq \lambda_p$. A *skew partition* means a pair (λ, μ) of two partitions satisfying $\mu \subseteq \lambda$; such a pair is denoted by λ/μ. We refer to [GriRei18, §2.7] for the definition of a *vertical i-strip* (where $i \in \mathbb{N}$).

Let Λ be the ring of symmetric functions in infinitely many indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. If $\mathbf{f} \in \Lambda$ is a symmetric function, then $\mathbf{f}(x_1, x_2, \ldots, x_k)$ is a symmetric polynomial in $\mathcal{S}$; the map

$$\Lambda \to \mathcal{S}, \qquad \mathbf{f} \mapsto \mathbf{f}(x_1, x_2, \ldots, x_k)$$

is a surjective $\mathbf{k}$-algebra homomorphism. We shall use boldfaced notations for symmetric functions in Λ in order to distinguish them from symmetric polynomials in $\mathcal{S}$. In particular:

- For any $i \in \mathbb{Z}$, we let $\mathbf{h}_i$ be the i-th complete homogeneous symmetric function in Λ. (This is called h_i in [GriRei18, Definition 2.2.1].)
- For any $i \in \mathbb{Z}$, we let $\mathbf{e}_i$ be the i-th elementary symmetric function in Λ. (This is called e_i in [GriRei18, Definition 2.2.1].)
- For any partition λ, we let $\mathbf{e}_\lambda$ be the corresponding elementary symmetric function in Λ. (This is called e_λ in [GriRei18, Definition 2.2.1].)
- For any partition λ, we let $\mathbf{s}_\lambda$ be the corresponding Schur function in Λ. (This is called s_λ in [GriRei18, Definition 2.2.1].)
- For any partitions λ and μ, we let $\mathbf{s}_{\lambda/\mu}$ be the corresponding skew Schur function in Λ. (This is called $s_{\lambda/\mu}$ in [GriRei18, §2.3]. Note that $\mathbf{s}_{\lambda/\mu} = 0$ unless $\mu \subseteq \lambda$.)

Also, we shall use the *skewing operators* as defined (e.g.) in [GriRei18, §2.8]. We recall their main properties:

- For each $\mathbf{f} \in \Lambda$, the skewing operator $\mathbf{f}^\perp$ is a $\mathbf{k}$-linear map $\Lambda \to \Lambda$. It depends $\mathbf{k}$-linearly on $\mathbf{f}$ (that is, we have $(\alpha\mathbf{f} + \beta\mathbf{g})^\perp = \alpha\mathbf{f}^\perp + \beta\mathbf{g}^\perp$ for any $\alpha, \beta \in \mathbf{k}$ and $\mathbf{f}, \mathbf{g} \in \Lambda$).

- For any partitions λ and μ, we have
$$\left(\mathbf{s}_\mu\right)^\perp \left(\mathbf{s}_\lambda\right) = \mathbf{s}_{\lambda/\mu}. \tag{15}$$
(This is [GriRei18, (2.8.2)].)

- For any $\mathbf{f}, \mathbf{g} \in \Lambda$, we have
$$(\mathbf{fg})^\perp = \mathbf{g}^\perp \circ \mathbf{f}^\perp. \tag{16}$$
(This is [GriRei18, Proposition 2.8.2(ii)], applied to $A = \Lambda$.)

- We have $1^\perp = \mathrm{id}$.

For each partition λ, let λ^t denote the *conjugate partition* of λ; see [GriRei18, Definition 2.2.8] for its definition.

Recall the second Jacobi-Trudi identity ([GriRei18, (2.4.10)]):

Proposition 6.4. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ be two partitions. Then,
$$\mathbf{s}_{\lambda^t/\mu^t} = \det\left(\left(\mathbf{e}_{\lambda_i - \mu_j - i + j}\right)_{1\leq i\leq \ell,\ 1\leq j\leq \ell}\right).$$

Corollary 6.5. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition. Then,
$$\mathbf{s}_{\lambda^t} = \det\left(\left(\mathbf{e}_{\lambda_i - i + j}\right)_{1\leq i\leq \ell,\ 1\leq j\leq \ell}\right).$$

Proof of Corollary 6.5. This follows from Proposition 6.4, applied to $\mu = \varnothing$ (since $\varnothing^t = \varnothing$ and thus $\mathbf{s}_{\lambda^t/\varnothing^t} = \mathbf{s}_{\lambda^t/\varnothing} = \mathbf{s}_{\lambda^t}$). □

We also recall one of the Pieri rules ([GriRei18, (2.7.2)]):

Proposition 6.6. Let λ be a partition, and let $i \in \mathbb{N}$. Then,
$$\mathbf{s}_\lambda \mathbf{e}_i = \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu.$$

From this, we can easily derive the following:

Corollary 6.7. Let λ be a partition, and let $i \in \mathbb{N}$. Then,
$$\left(\mathbf{e}_i\right)^\perp \mathbf{s}_\lambda = \sum_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu.$$

Corollary 6.7 is also proven in [GriRei18, version with solutions (ancillary file), Lemma 12.83.3(b)].

The next proposition is the claim of [GriRei18, Exercise 2.9.1(b)]:

Proposition 6.8. Let λ be a partition. Let $m \in \mathbb{Z}$ be such that $m \geq \lambda_1$. Then,

$$\sum_{i\in\mathbb{N}} (-1)^i \mathbf{h}_{m+i} (\mathbf{e}_i)^{\perp} \mathbf{s}_\lambda = \mathbf{s}_{(m,\lambda_1,\lambda_2,\lambda_3,\ldots)}.$$

We shall use this to derive the following corollary:

Corollary 6.9. Let λ be a partition with at most k parts. Let $\overline{\lambda}$ be the partition $(\lambda_2, \lambda_3, \lambda_4, \ldots)$. Then,

$$\mathbf{s}_\lambda = \sum_{i=0}^{k-1} (-1)^i \mathbf{h}_{\lambda_1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu.$$

Proof of Corollary 6.9. The partition $\overline{\lambda}$ is obtained from λ by removing the first part. Hence, this partition $\overline{\lambda}$ has at most $k-1$ parts (since λ has at most k parts). Thus, if $i \in \mathbb{N}$ satisfies $i \geq k$, then

$$\text{there exists no partition } \mu \text{ such that } \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip.} \tag{17}$$

We have $\overline{\lambda} = (\lambda_2, \lambda_3, \lambda_4, \ldots)$, so that $(\lambda_2, \lambda_3, \lambda_4, \ldots) = \overline{\lambda} = \left(\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \ldots\right)$. Hence,

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots) = \left(\lambda_1, \overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \ldots\right).$$

Also, clearly, $\lambda_1 \geq \overline{\lambda}_1$ (since $\lambda_1 \geq \lambda_2 = \overline{\lambda}_1$). Hence, Proposition 6.8 (applied to $\overline{\lambda}$ and λ_1 instead of λ and m) yields

$$\sum_{i\in\mathbb{N}} (-1)^i \mathbf{h}_{\lambda_1+i} (\mathbf{e}_i)^{\perp} \mathbf{s}_{\overline{\lambda}} = \mathbf{s}_{\left(\lambda_1,\overline{\lambda}_1,\overline{\lambda}_2,\overline{\lambda}_3,\ldots\right)} = \mathbf{s}_\lambda$$

(since $\left(\lambda_1, \overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \ldots\right) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots) = \lambda$). Therefore,

$$\begin{aligned}
\mathbf{s}_\lambda &= \sum_{i\in\mathbb{N}} (-1)^i \, \mathbf{h}_{\lambda_1+i} \underbrace{\left(\mathbf{e}_i\right)^{\perp} \mathbf{s}_{\overline{\lambda}}}_{\substack{= \sum\limits_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu \\ \text{(by Corollary 6.7)}}} = \sum_{i\in\mathbb{N}} (-1)^i \, \mathbf{h}_{\lambda_1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu \\
&= \sum_{i=0}^{k-1} (-1)^i \, \mathbf{h}_{\lambda_1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu + \sum_{i\geq k} (-1)^i \, \mathbf{h}_{\lambda_1+i} \underbrace{\sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu}_{\substack{=0 \\ \text{(by (17))}}} \\
&= \sum_{i=0}^{k-1} (-1)^i \, \mathbf{h}_{\lambda_1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu .
\end{aligned}$$

This proves Corollary 6.9. ☐

Convention 6.10. We WLOG assume that $k > 0$ for the rest of Section 6 (since otherwise, Theorem 6.3 is trivial).

Next, we define a filtration on the $\mathbf{k}$-module $\mathcal{S}/I$:

Definition 6.11. For each $p \in \mathbb{Z}$, we let Q_p denote the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\lambda_k \leq p$.

Thus, $0 = Q_{-1} \subseteq Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \cdots$. Theorem 2.5 shows that the $\mathbf{k}$-module $\mathcal{S}/I$ is free with basis $\left(\overline{s_\lambda}\right)_{\lambda\in P_{k,n}}$; hence, $\mathcal{S}/I = Q_{n-k}$ (since each $\lambda \in P_{k,n}$ satisfies $\lambda_k \leq n-k$).

Note that $(Q_0, Q_1, Q_2, \ldots)$ is a filtration of the $\mathbf{k}$-module $\mathcal{S}/I$, but not (in general) of the $\mathbf{k}$-algebra $\mathcal{S}/I$.

Lemma 6.12. We have $\operatorname{coeff}_\omega \left(Q_{n-k-1}\right) = 0$.

Proof of Lemma 6.12. The map $\operatorname{coeff}_\omega$ is $\mathbf{k}$-linear; thus, it suffices to prove that $\operatorname{coeff}_\omega \left(\overline{s_\lambda}\right) = 0$ for each $\lambda \in P_{k,n}$ satisfying $\lambda_k \leq n-k-1$ (because the $\mathbf{k}$-module Q_{n-k-1} is spanned by the $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\lambda_k \leq n-k-1$). So let us fix some $\lambda \in P_{k,n}$ satisfying $\lambda_k \leq n-k-1$. We must then prove that $\operatorname{coeff}_\omega \left(\overline{s_\lambda}\right) = 0$.

We have $\lambda_k \leq n-k-1 < n-k = \omega_k$. Thus, $\lambda_k \neq \omega_k$, so that $\lambda \neq \omega$.

The definition of the map $\operatorname{coeff}_\omega$ yields $\operatorname{coeff}_\omega \left(\overline{s_\lambda}\right) = \begin{cases} 1, & \text{if } \lambda = \omega; \\ 0, & \text{if } \lambda \neq \omega \end{cases} = 0$ (since $\lambda \neq \omega$). This completes our proof of Lemma 6.12. ☐

Lemma 6.13. Let λ be a partition with at most k parts. Assume that $\lambda_1 = n-k+1$. Let $\overline{\lambda}$ be the partition $(\lambda_2, \lambda_3, \lambda_4, \ldots)$. Then,

$$\overline{s_\lambda} = \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}.$$

Proof of Lemma 6.13. Corollary 6.9 yields

$$\mathbf{s}_\lambda = \sum_{i=0}^{k-1} (-1)^i \mathbf{h}_{\lambda_1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu.$$

This is an identity in Λ. Evaluating both of its sides at the k variables $x_1, x_2, \ldots, x_k$, we obtain

$$\begin{aligned} s_\lambda &= \sum_{i=0}^{k-1} (-1)^i \underbrace{h_{\lambda_1+i}}_{\substack{=h_{n-k+1+i} \\ (\text{since } \lambda_1 = n-k+1)}} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} s_\mu \\ &= \sum_{i=0}^{k-1} (-1)^i \underbrace{h_{n-k+1+i}}_{\substack{\equiv a_{1+i} \bmod I \\ (\text{by (9)})}} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} s_\mu \\ &\equiv \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} s_\mu \bmod I. \end{aligned}$$

Projecting both sides of this equality from $\mathcal{S}$ to $\mathcal{S}/I$, we obtain

$$\overline{s_\lambda} = \overline{\sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} s_\mu} = \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}.$$

This proves Lemma 6.13. □

Lemma 6.14. Let λ be a partition with at most k parts. Assume that $\lambda_1 = n-k+1$. Then, $\overline{s_\lambda} \in Q_0$.

Proof of Lemma 6.14. We shall prove Lemma 6.14 by strong induction on $|\lambda|$. Thus, we fix some $N \in \mathbb{N}$, and we assume (as induction hypothesis) that Lemma 6.14 is already proven whenever $|\lambda| < N$. We now must prove Lemma 6.14 in the case when $|\lambda| = N$.

So let λ be as in Lemma 6.14, and assume that $|\lambda| = N$. Let $\overline{\lambda}$ be the partition $(\lambda_2, \lambda_3, \lambda_4, \ldots)$. Then, Lemma 6.13 yields

$$\overline{s_{\lambda}} = \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_{\mu}}. \tag{18}$$

But if μ is a partition such that $\overline{\lambda}/\mu$ is a vertical i-strip, then

$$\overline{s_{\mu}} \in \mathcal{Q}_0. \tag{19}$$

[*Proof of (19):* The partition λ has at most k parts; thus, the partition $\overline{\lambda}$ has at most $k-1$ parts.

Now, let μ be a partition such that $\overline{\lambda}/\mu$ is a vertical i-strip. Then, $\mu \subseteq \overline{\lambda}$, so that μ has at most $k-1$ parts (since $\overline{\lambda}$ has at most $k-1$ parts). Thus, $\mu_k = 0 \leq 0$. Also, μ has at most k parts (since μ has at most $k-1$ parts). If $\mu_1 \leq n-k$, then this yields that $\mu \in P_{k,n}$ and therefore $\overline{s_{\mu}} \in \mathcal{Q}_0$ (since $\mu \in P_{k,n}$ and $\mu_k \leq 0$). Thus, (19) is proven if $\mu_1 \leq n-k$. Hence, for the rest of this proof, we WLOG assume that we don't have $\mu_1 \leq n-k$. Hence, $\mu_1 > n-k$.

But $\mu \subseteq \overline{\lambda}$, so that $\mu_1 \leq \overline{\lambda}_1 = \lambda_2 \leq \lambda_1 = n-k+1$. Combining this with $\mu_1 > n-k$, we obtain $\mu_1 = n-k+1$. Also, $\mu \subseteq \overline{\lambda}$, so that

$$|\mu| \leq \left|\overline{\lambda}\right| = |\lambda| - \underbrace{\lambda_1}_{=n-k+1 \geq 1 > 0} < |\lambda| = N.$$

Hence, we can apply Lemma 6.14 to μ instead of λ (by the induction hypothesis). We thus obtain $\overline{s_{\mu}} \in \mathcal{Q}_0$. This completes the proof of (19).]

Now, (18) becomes

$$\overline{s_{\lambda}} = \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \underbrace{\overline{s_{\mu}}}_{\substack{\in \mathcal{Q}_0 \\ (\text{by (19)})}} \in \mathcal{Q}_0.$$

Thus, we have proven Lemma 6.14 for our λ. This completes the induction step; thus, Lemma 6.14 is proven. ☐

Lemma 6.15. Let $i \in \mathbb{N}$ and $\lambda \in P_{k,n}$. Then,

$$\overline{e_i s_{\lambda}} \equiv \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a vertical } i\text{-strip}}} \overline{s_{\mu}} \operatorname{mod} \mathcal{Q}_0.$$

Proof of Lemma 6.15. If μ is a partition such that μ/λ is a vertical i-strip and $\mu \notin P_{k,n}$, then

$$\overline{s_{\mu}} \equiv 0 \operatorname{mod} \mathcal{Q}_0. \tag{20}$$

[*Proof of (20):* Let μ be a partition such that μ / λ is a vertical i-strip and $\mu \notin P_{k,n}$. We must prove (20).

If the partition μ has more than k parts, then (20) easily follows[5]. Hence, for the rest of this proof, we WLOG assume that the partition μ has at most k parts.

Since μ / λ is a vertical strip, we have $\mu_1 \leq \lambda_1 + 1$. But $\lambda_1 \leq n - k$ (since $\lambda \in P_{k,n}$). If $\mu_1 = n - k + 1$, then (20) easily follows[6]. Hence, for the rest of this proof, we WLOG assume that $\mu_1 \neq n - k + 1$. Combining this with $\mu_1 \leq \underbrace{\lambda_1}_{\leq n-k} + 1 \leq n - k + 1$, we obtain $\mu_1 < n - k + 1$, so that $\mu_1 \leq n - k$. Hence, $\mu \in P_{k,n}$ (since μ has at most k parts). This contradicts $\mu \notin P_{k,n}$. Thus, $\overline{s_\mu} \equiv 0 \operatorname{mod} Q_0$ (because *ex falso quodlibet*). Hence, (20) is proven.]

Proposition 6.6 yields

$$\mathbf{s}_\lambda \mathbf{e}_i = \sum_{\substack{\mu \text{ is a partition;} \\ \mu / \lambda \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu .$$

This is an identity in Λ. Evaluating both of its sides at the k variables $x_1, x_2, \ldots, x_k$, we obtain

$$s_\lambda e_i = \sum_{\substack{\mu \text{ is a partition;} \\ \mu / \lambda \text{ is a vertical } i\text{-strip}}} s_\mu .$$

Projecting both sides of this equality from $\mathcal{S}$ to $\mathcal{S} / I$, we obtain

$$\begin{aligned}
\overline{s_\lambda e_i} &= \overline{\sum_{\substack{\mu \text{ is a partition;} \\ \mu / \lambda \text{ is a vertical } i\text{-strip}}} s_\mu} = \sum_{\substack{\mu \text{ is a partition;} \\ \mu / \lambda \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \\
&= \sum_{\substack{\mu \text{ is a partition;} \\ \mu / \lambda \text{ is a vertical } i\text{-strip;} \\ \mu \in P_{k,n}}} \overline{s_\mu} + \sum_{\substack{\mu \text{ is a partition;} \\ \mu / \lambda \text{ is a vertical } i\text{-strip;} \\ \mu \notin P_{k,n}}} \underbrace{\overline{s_\mu}}_{\substack{\equiv 0 \operatorname{mod} Q_0 \\ \text{(by (20))}}} \\
&\equiv \sum_{\substack{\mu \text{ is a partition;} \\ \mu / \lambda \text{ is a vertical } i\text{-strip;} \\ \mu \in P_{k,n}}} \overline{s_\mu} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu / \lambda \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \operatorname{mod} Q_0 .
\end{aligned}$$

Thus, $\overline{e_i s_\lambda} = \overline{s_\lambda e_i} \equiv \sum_{\substack{\mu \in P_{k,n}; \\ \mu / \lambda \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \operatorname{mod} Q_0$. This proves Lemma 6.15. ☐

Lemma 6.16. Let $i \in \mathbb{N}$ and $p \in \mathbb{Z}$. Then, $\overline{e_i} Q_p \subseteq Q_{p+1}$.

Proof of Lemma 6.16. Due to the definition of Q_p, it suffices that every $\lambda \in P_{k,n}$ satisfying $\lambda_k \leq p$ satisfies $\overline{e_i s_\lambda} \in Q_{p+1}$. So let us fix $\lambda \in P_{k,n}$ satisfying $\lambda_k \leq p$. We must prove that $\overline{e_i s_\lambda} \in Q_{p+1}$.

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[5] *Proof.* Assume that the partition μ has more than k parts. Thus, (3) (applied to μ instead of λ) yields $s_\mu = 0$. Thus, $\overline{s_\mu} = 0 \equiv 0 \operatorname{mod} Q_0$. Thus, (20) holds.

[6] *Proof.* Assume that $\mu_1 = n - k + 1$. Then, Lemma 6.14 (applied to μ instead of λ) yields $\overline{s_\mu} \in Q_0$. Hence, $\overline{s_\mu} \equiv 0 \operatorname{mod} Q_0$. Thus, (20) holds.

If $\mu \in P_{k,n}$ is such that μ / λ is a vertical i-strip, then

$$\overline{s_\mu} \equiv 0 \operatorname{mod} Q_{p+1}. \tag{21}$$

[*Proof of (21):* Let $\mu \in P_{k,n}$ be such that μ / λ is a vertical i-strip. We must prove (21).

Since μ / λ is a vertical strip, we have $\mu_k \leq \underbrace{\lambda_k}_{\leq p} + 1 \leq p + 1$. From $\mu \in P_{k,n}$ and $\mu_k \leq p + 1$, we obtain $\overline{s_\mu} \in Q_{p+1}$. In other words, $\overline{s_\mu} \equiv 0 \operatorname{mod} Q_{p+1}$. Thus, (21) is proven.]

Lemma 6.15 yields

$$\overline{e_i s_\lambda} \equiv \sum_{\substack{\mu \in P_{k,n}; \\ \mu / \lambda \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \operatorname{mod} Q_0.$$

Hence,

$$\overline{e_i s_\lambda} - \sum_{\substack{\mu \in P_{k,n}; \\ \mu / \lambda \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \in Q_0 \subseteq Q_{p+1}.$$

Thus,

$$\overline{e_i s_\lambda} \equiv \sum_{\substack{\mu \in P_{k,n}; \\ \mu / \lambda \text{ is a vertical } i\text{-strip}}} \underbrace{\overline{s_\mu}}_{\substack{\equiv 0 \operatorname{mod} Q_{p+1} \\ \text{(by (21))}}} \equiv 0 \operatorname{mod} Q_{p+1}.$$

In other words, $\overline{e_i s_\lambda} \in Q_{p+1}$. This completes our proof of Lemma 6.16. ☐

The next fact that we use from the theory of symmetric functions are some basic properties of the Littlewood-Richardson coefficients. For any partitions λ, μ, ν, we let $c^\lambda_{\mu,\nu}$ be the Littlewood-Richardson coefficient as defined in [GriRei18, Definition 2.5.8]. Then, we have the following fact (part of [GriRei18, Remark 2.5.9]):

Proposition 6.17. Let λ and μ be two partitions.

(a) We have

$$\mathbf{s}_{\lambda / \mu} = \sum_{\nu \text{ is a partition}} c^\lambda_{\mu,\nu} \mathbf{s}_\nu.$$

(b) If ν is a partition, then $c^\lambda_{\mu,\nu} = 0$ unless $\nu \subseteq \lambda$.

(c) If ν is a partition, then $c^\lambda_{\mu,\nu} = 0$ unless $|\mu| + |\nu| = |\lambda|$.

Next, let $\mathcal{Z}$ be the $\mathbf{k}$-submodule of Λ spanned by the $\mathbf{s}_\lambda$ with $\lambda \in P_{k,n}$. Then, $(\mathbf{s}_\lambda)_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{Z}$ (since $(\mathbf{s}_\lambda)_{\lambda \text{ is a partition}}$ is a basis of the $\mathbf{k}$-module Λ). We thus can define a $\mathbf{k}$-linear map $\delta : \mathcal{Z} \to \mathcal{S} / I$ by setting

$$\delta(\mathbf{s}_\lambda) = \overline{s_{\lambda^\vee}} \qquad \text{for every } \lambda \in P_{k,n}.$$

Notice that a partition λ satisfies $\lambda \in P_{k,n}$ if and only if $\lambda \subseteq \omega$.

Lemma 6.18. We have $\mathbf{f}^{\perp}(\mathcal{Z}) \subseteq \mathcal{Z}$ for each $\mathbf{f} \in \Lambda$.

Proof of Lemma 6.18. Since $\mathbf{f}^{\perp}$ depends $\mathbf{k}$-linearly on $\mathbf{f}$, it suffices to check that $(\mathbf{s}_{\mu})^{\perp}(\mathcal{Z}) \subseteq \mathcal{Z}$ for each partition μ. So let us fix a partition μ; we then must prove that $(\mathbf{s}_{\mu})^{\perp}(\mathcal{Z}) \subseteq \mathcal{Z}$.

Recall that $\mathcal{Z}$ is the $\mathbf{k}$-module spanned by the $\mathbf{s}_{\lambda}$ with $\lambda \in P_{k,n}$. Hence, in order to prove that $(\mathbf{s}_{\mu})^{\perp}(\mathcal{Z}) \subseteq \mathcal{Z}$, it suffices to check that $(\mathbf{s}_{\mu})^{\perp}(\mathbf{s}_{\lambda}) \in \mathcal{Z}$ for each $\lambda \in P_{k,n}$. So let us fix $\lambda \in P_{k,n}$; we must then prove that $(\mathbf{s}_{\mu})^{\perp}(\mathbf{s}_{\lambda}) \in \mathcal{Z}$.

From (15), we obtain

$$\begin{aligned}
(\mathbf{s}_{\mu})^{\perp}(\mathbf{s}_{\lambda}) = \mathbf{s}_{\lambda/\mu} &= \sum_{\nu \text{ is a partition}} c^{\lambda}_{\mu,\nu} \mathbf{s}_{\nu} \qquad (\text{by Proposition 6.17 } \textbf{(a)}) \\
&= \sum_{\substack{\nu \text{ is a partition;} \\ \nu \subseteq \lambda}} c^{\lambda}_{\mu,\nu} \mathbf{s}_{\nu} + \sum_{\substack{\nu \text{ is a partition;} \\ \text{not } \nu \subseteq \lambda}} \underbrace{c^{\lambda}_{\mu,\nu}}_{\substack{=0 \\ (\text{by Proposition 6.17 } \textbf{(b)})}} \mathbf{s}_{\nu} \\
&= \sum_{\substack{\nu \text{ is a partition;} \\ \nu \subseteq \lambda}} c^{\lambda}_{\mu,\nu} \underbrace{\mathbf{s}_{\nu}}_{\substack{\in \mathcal{Z} \\ (\text{because } \nu \subseteq \lambda \\ \text{and } \lambda \in P_{k,n} \text{ lead to} \\ \nu \in P_{k,n})}} \in \mathcal{Z}.
\end{aligned}$$

This completes our proof of Lemma 6.18. ☐

Lemma 6.19. Let $i \in \mathbb{Z}$ and $\mathbf{f} \in \mathcal{Z}$. Then,

$$\delta\left((\mathbf{e}_i)^{\perp} \mathbf{f}\right) \equiv \overline{e_i} \delta(\mathbf{f}) \bmod Q_0.$$

(Note that $\delta\left((\mathbf{e}_i)^{\perp} \mathbf{f}\right)$ is well-defined, since Lemma 6.18 yields $(\mathbf{e}_i)^{\perp} \mathbf{f} \in \mathcal{Z}$.)

Proof of Lemma 6.19. Both sides of the claim are $\mathbf{k}$-linear in $\mathbf{f}$. Hence, we can WLOG assume that $\mathbf{f} = \mathbf{s}_{\lambda}$ for some $\lambda \in P_{k,n}$ (since $(\mathbf{s}_{\lambda})_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{Z}$). Assume this, and consider this λ.

It is easy to see that if $\mu \in P_{k,n}$, then we have the following equivalence of statements:

$$(\lambda/\mu \text{ is a vertical } i\text{-strip}) \iff \left(\mu^{\vee}/\lambda^{\vee} \text{ is a vertical } i\text{-strip}\right). \qquad (22)$$

(Indeed, the skew Young diagram of $\mu^{\vee}/\lambda^{\vee}$ is obtained from the skew Young diagram of λ/μ by a rotation by 180°.)

We must prove that $\delta\left((\mathbf{e}_i)^{\perp} \mathbf{f}\right) \equiv \overline{e_i} \delta(\mathbf{f}) \bmod Q_0$. If $i < 0$, then this is obvious (because if $i < 0$, then both $\mathbf{e}_i$ and e_i equal 0, and therefore both sides of the congruence $\delta\left((\mathbf{e}_i)^{\perp} \mathbf{f}\right) \equiv \overline{e_i} \delta(\mathbf{f}) \bmod Q_0$ are equal to 0). Hence, for the rest of this proof, we WLOG assume that we don't have $i < 0$. Thus, $i \geq 0$, so that $i \in \mathbb{N}$.

From $\mathbf{f} = \mathbf{s}_\lambda$, we obtain

$$\left(\mathbf{e}_i\right)^\perp \mathbf{f} = \left(\mathbf{e}_i\right)^\perp \mathbf{s}_\lambda = \sum_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu \qquad \text{(by Corollary 6.7)}$$
$$= \sum_{\substack{\mu \in P_{k,n}; \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu$$

(because if μ is a partition such that λ/μ is a vertical i-strip, then $\mu \in P_{k,n}$ (since $\mu \subseteq \lambda$ and $\lambda \in P_{k,n}$)). Applying the map δ to both sides of this equality, we find

$$\delta\left(\left(\mathbf{e}_i\right)^\perp \mathbf{f}\right) = \delta\left(\sum_{\substack{\mu \in P_{k,n}; \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu\right) = \underbrace{\sum_{\substack{\mu \in P_{k,n}; \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}}}_{\substack{= \sum_{\substack{\mu \in P_{k,n}; \\ \mu^\vee/\lambda^\vee \text{ is a vertical } i\text{-strip}}} \\ \text{(by (22))}}} \underbrace{\delta\left(\mathbf{s}_\mu\right)}_{\substack{= \overline{s_{\mu^\vee}} \\ \text{(by the definition of } \delta\text{)}}}$$
$$= \sum_{\substack{\mu \in P_{k,n}; \\ \mu^\vee/\lambda^\vee \text{ is a vertical } i\text{-strip}}} \overline{s_{\mu^\vee}} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda^\vee \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \tag{23}$$

(here, we have substituted μ for $\mu^\vee$ in the sum, since the map $P_{k,n} \to P_{k,n}$, $\mu \mapsto \mu^\vee$ is a bijection).

On the other hand, from $\mathbf{f} = \mathbf{s}_\lambda$, we obtain $\delta\left(\mathbf{f}\right) = \delta\left(\mathbf{s}_\lambda\right) = \overline{s_{\lambda^\vee}}$ (by the definition of δ) and thus

$$\overline{e_i}\delta\left(\mathbf{f}\right) = \overline{e_i s_{\lambda^\vee}} \equiv \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda^\vee \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \bmod Q_0 \qquad \left(\begin{array}{c}\text{by Lemma 6.15, applied} \\ \text{to } \lambda^\vee \text{ instead of } \lambda\end{array}\right).$$

Comparing this with (23), we obtain $\delta\left(\left(\mathbf{e}_i\right)^\perp \mathbf{f}\right) \equiv \overline{e_i}\delta\left(\mathbf{f}\right) \bmod Q_0$. This proves Lemma 6.19. $\square$

Lemma 6.20. Let $p \in \mathbb{N}$. Let $i_1, i_2, \ldots, i_p \in \mathbb{Z}$ and $\mathbf{f} \in \mathcal{Z}$. Then,

$$\delta\left(\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_p}\right)^\perp \mathbf{f}\right) \equiv \overline{e_{i_1}e_{i_2}\cdots e_{i_p}}\delta\left(\mathbf{f}\right) \bmod Q_{p-1}.$$

Proof of Lemma 6.20. We proceed by induction on p.

The *induction base* (the case $p = 0$) is obvious (since $1^\perp = \mathrm{id}$ and thus $1^\perp \mathbf{f} = \mathbf{f}$).

Induction step: Let $q \in \mathbb{N}$. Assume (as the induction hypothesis) that Lemma 6.20 holds for $p = q$. We must now prove that Lemma 6.20 holds for $p = q + 1$. In other words, we must prove that every $i_1, i_2, \ldots, i_{q+1} \in \mathbb{Z}$ and $\mathbf{f} \in \mathcal{Z}$ satisfy

$$\delta\left(\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_{q+1}}\right)^{\perp}\mathbf{f}\right) \equiv \overline{e_{i_1}e_{i_2}\cdots e_{i_{q+1}}}\delta\left(\mathbf{f}\right) \operatorname{mod} Q_q. \tag{24}$$

So let $i_1, i_2, \ldots, i_{q+1} \in \mathbb{Z}$ and $\mathbf{f} \in \mathcal{Z}$. We must prove (24).

Lemma 6.16 (applied to i_{q+1} and $q-1$ instead of i and p) yields $\overline{e_{i_{q+1}}}Q_{q-1} \subseteq Q_q$.

The induction hypothesis yields

$$\delta\left(\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_q}\right)^{\perp}\mathbf{f}\right) \equiv \overline{e_{i_1}e_{i_2}\cdots e_{i_q}}\delta\left(\mathbf{f}\right) \operatorname{mod} Q_{q-1}.$$

Multiplying both sides of this congruence by $\overline{e_{i_{q+1}}}$, we obtain

$$\overline{e_{i_{q+1}}}\delta\left(\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_q}\right)^{\perp}\mathbf{f}\right) \equiv \overline{e_{i_{q+1}}e_{i_1}e_{i_2}\cdots e_{i_q}}\delta\left(\mathbf{f}\right) \operatorname{mod} Q_q \tag{25}$$

(since $\overline{e_{i_{q+1}}}Q_{q-1} \subseteq Q_q$).

Applying Lemma 6.18 to $\mathbf{f} = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_q}$, we obtain $\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_q}\right)^{\perp}(\mathcal{Z}) \subseteq \mathcal{Z}$. Hence, $\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_q}\right)^{\perp}\mathbf{f} \in \mathcal{Z}$ (since $\mathbf{f} \in \mathcal{Z}$).

But (16) (applied to $\mathbf{f} = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_q}$ and $\mathbf{g} = \mathbf{e}_{i_{q+1}}$) yields

$$\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_{q+1}}\right)^{\perp} = \left(\mathbf{e}_{i_{q+1}}\right)^{\perp} \circ \left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_q}\right)^{\perp}.$$

Hence,

$$\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_{q+1}}\right)^{\perp}\mathbf{f} = \left(\left(\mathbf{e}_{i_{q+1}}\right)^{\perp} \circ \left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_q}\right)^{\perp}\right)\mathbf{f} = \left(\mathbf{e}_{i_{q+1}}\right)^{\perp}\left(\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_q}\right)^{\perp}\mathbf{f}\right).$$

Applying the map δ to both sides of this equality, we find

$$\begin{aligned}\delta\left(\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_{q+1}}\right)^{\perp}\mathbf{f}\right) &= \delta\left(\left(\mathbf{e}_{i_{q+1}}\right)^{\perp}\left(\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_q}\right)^{\perp}\mathbf{f}\right)\right) \\ &\equiv \overline{e_{i_{q+1}}}\delta\left(\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_q}\right)^{\perp}\mathbf{f}\right) \operatorname{mod} Q_0\end{aligned}$$

(by Lemma 6.19, applied to i_{q+1} and $\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_q}\right)^{\perp}\mathbf{f}$ instead of i and $\mathbf{f}$). Since $Q_0 \subseteq Q_q$, this yields

$$\begin{aligned}\delta\left(\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_{q+1}}\right)^{\perp}\mathbf{f}\right) &\equiv \overline{e_{i_{q+1}}}\delta\left(\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_q}\right)^{\perp}\mathbf{f}\right) \\ &\equiv \overline{e_{i_{q+1}}e_{i_1}e_{i_2}\cdots e_{i_q}}\delta\left(\mathbf{f}\right) \qquad \text{(by (25))} \\ &= \overline{e_{i_1}e_{i_2}\cdots e_{i_{q+1}}}\delta\left(\mathbf{f}\right) \operatorname{mod} Q_q.\end{aligned}$$

Thus, (24) is proven. This completes the induction step. Thus, Lemma 6.20 is proven. ☐

Lemma 6.21. Let $\lambda \in P_{k,n}$ and $\mathbf{f} \in \mathcal{Z}$. Then,

$$\delta\left(\left(\mathbf{s}_\lambda\right)^\perp \mathbf{f}\right) \equiv \overline{s_\lambda}\delta\left(\mathbf{f}\right) \operatorname{mod} Q_{n-k-1}.$$

Proof of Lemma 6.21. Let $\ell = n-k$. From $\lambda \in P_{k,n}$, we have $\lambda_1 \leq n-k = \ell$.

Consider the conjugate partition λ^t of λ. Then, λ^t has exactly λ_1 parts. Thus, λ^t has $\leq \ell$ parts (since $\lambda_1 \leq \ell$). Therefore, $\lambda^t = \left(\left(\lambda^t\right)_1, \left(\lambda^t\right)_2, \ldots, \left(\lambda^t\right)_\ell\right)$. Hence, Corollary 6.5 (applied to λ^t instead of λ) yields

$$\mathbf{s}_{\left(\lambda^t\right)^t} = \det\left(\left(\mathbf{e}_{\left(\lambda^t\right)_i - i + j}\right)_{1\leq i\leq \ell,\ 1\leq j\leq \ell}\right).$$

In view of $\left(\lambda^t\right)^t = \lambda$, this rewrites as

$$\mathbf{s}_\lambda = \det\left(\left(\mathbf{e}_{\left(\lambda^t\right)_i - i + j}\right)_{1\leq i\leq \ell,\ 1\leq j\leq \ell}\right) = \sum_{\sigma\in S_\ell} \left(-1\right)^\sigma \prod_{i=1}^{\ell} \mathbf{e}_{\left(\lambda^t\right)_i - i + \sigma(i)} \tag{26}$$

(where S_ℓ denotes the symmetric group of the set $\{1, 2, \ldots, \ell\}$, and where $\left(-1\right)^\sigma$ denotes the sign of a permutation $\sigma \in S_\ell$). Hence,

$$\left(\mathbf{s}_\lambda\right)^\perp \mathbf{f} = \left(\sum_{\sigma\in S_\ell} \left(-1\right)^\sigma \prod_{i=1}^{\ell} \mathbf{e}_{\left(\lambda^t\right)_i - i + \sigma(i)}\right)^\perp \mathbf{f} = \sum_{\sigma\in S_\ell} \left(-1\right)^\sigma \left(\prod_{i=1}^{\ell} \mathbf{e}_{\left(\lambda^t\right)_i - i + \sigma(i)}\right)^\perp \mathbf{f}.$$

Applying the map δ to this equality, we obtain

$$\begin{aligned}
\delta\left(\left(\mathbf{s}_\lambda\right)^\perp \mathbf{f}\right) &= \delta\left(\sum_{\sigma\in S_\ell} \left(-1\right)^\sigma \left(\prod_{i=1}^{\ell} \mathbf{e}_{\left(\lambda^t\right)_i - i + \sigma(i)}\right)^\perp \mathbf{f}\right) \\
&= \sum_{\sigma\in S_\ell} \left(-1\right)^\sigma \underbrace{\delta\left(\left(\prod_{i=1}^{\ell} \mathbf{e}_{\left(\lambda^t\right)_i - i + \sigma(i)}\right)^\perp \mathbf{f}\right)}_{\substack{\equiv \overline{\prod_{i=1}^{\ell} e_{\left(\lambda^t\right)_i - i + \sigma(i)}}\delta(\mathbf{f}) \operatorname{mod} Q_{\ell-1} \\ \text{(by Lemma 6.20, applied} \\ \text{to } p=\ell \text{ and } i_j = \left(\lambda^t\right)_j - j + \sigma(j))}} \qquad \text{(since } \delta \text{ is } \mathbf{k}\text{-linear)} \\
&\equiv \sum_{\sigma\in S_\ell} \left(-1\right)^\sigma \overline{\prod_{i=1}^{\ell} e_{\left(\lambda^t\right)_i - i + \sigma(i)}}\delta\left(\mathbf{f}\right) \operatorname{mod} Q_{\ell-1}.
\end{aligned} \tag{27}$$

On the other hand, (26) is an identity in Λ. Evaluating both of its sides at the k variables $x_1, x_2, \ldots, x_k$, we obtain

$$s_\lambda = \sum_{\sigma \in S_\ell} (-1)^\sigma \prod_{i=1}^{\ell} e_{(\lambda^t)_i - i + \sigma(i)}.$$

Hence,

$$\overline{s_\lambda} \delta(\mathbf{f}) = \overline{\sum_{\sigma \in S_\ell} (-1)^\sigma \prod_{i=1}^{\ell} e_{(\lambda^t)_i - i + \sigma(i)}} \delta(\mathbf{f}) = \sum_{\sigma \in S_\ell} (-1)^\sigma \prod_{i=1}^{\ell} \overline{e_{(\lambda^t)_i - i + \sigma(i)}} \delta(\mathbf{f}).$$

Thus, (27) rewrites as $\delta\left((\mathbf{s}_\lambda)^\perp \mathbf{f}\right) \equiv \overline{s_\lambda} \delta(\mathbf{f}) \bmod Q_{\ell-1}$. In other words, $\delta\left((\mathbf{s}_\lambda)^\perp \mathbf{f}\right) \equiv \overline{s_\lambda} \delta(\mathbf{f}) \bmod Q_{n-k-1}$ (since $\ell = n-k$). This proves Lemma 6.21. $\square$

Lemma 6.22. Let $\lambda \in P_{k,n}$ and $\mu \in P_{k,n}$. Then,

$$\operatorname{coeff}_\omega\left(\overline{s_\lambda s_\mu}\right) = \begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee \end{cases}.$$

Proof of Lemma 6.22. From $\mu \in P_{k,n}$, we obtain $\mu^\vee \in P_{k,n}$. Hence, $\mathbf{s}_{\mu^\vee} \in \mathcal{Z}$ and

$$\begin{aligned} \delta\left(\mathbf{s}_{\mu^\vee}\right) &= \overline{s_{(\mu^\vee)^\vee}} && (\text{by the definition of } \delta) \\ &= \overline{s_\mu} && \left(\text{since } (\mu^\vee)^\vee = \mu\right). \end{aligned}$$

Also, Lemma 6.21 (applied to $\mathbf{f} = \mathbf{s}_{\mu^\vee}$) yields

$$\delta\left((\mathbf{s}_\lambda)^\perp \mathbf{s}_{\mu^\vee}\right) \equiv \overline{s_\lambda} \delta\left(\mathbf{s}_{\mu^\vee}\right) \bmod Q_{n-k-1}$$

(since $\mathbf{s}_{\mu^\vee} \in \mathcal{Z}$). In other words, $\delta\left((\mathbf{s}_\lambda)^\perp \mathbf{s}_{\mu^\vee}\right) - \overline{s_\lambda} \delta\left(\mathbf{s}_{\mu^\vee}\right) \in Q_{n-k-1}$. Hence,

$$\operatorname{coeff}_\omega\left(\delta\left((\mathbf{s}_\lambda)^\perp \mathbf{s}_{\mu^\vee}\right) - \overline{s_\lambda} \delta\left(\mathbf{s}_{\mu^\vee}\right)\right) \in \operatorname{coeff}_\omega\left(Q_{n-k-1}\right) = 0$$

(by Lemma 6.12). Thus,

$$\begin{aligned} \operatorname{coeff}_\omega\left(\delta\left((\mathbf{s}_\lambda)^\perp \mathbf{s}_{\mu^\vee}\right)\right) &= \operatorname{coeff}_\omega\left(\overline{s_\lambda} \underbrace{\delta\left(\mathbf{s}_{\mu^\vee}\right)}_{=\overline{s_\mu}}\right) = \operatorname{coeff}_\omega\left(\overline{s_\lambda}\, \overline{s_\mu}\right) \\ &= \operatorname{coeff}_\omega\left(\overline{s_\lambda s_\mu}\right). \end{aligned} \tag{28}$$

Applying (15) to λ and $\mu^\vee$ instead of μ and λ, we obtain $(\mathbf{s}_\lambda)^\perp \mathbf{s}_{\mu^\vee} = \mathbf{s}_{\mu^\vee/\lambda}$. Thus, (28) rewrites as

$$\operatorname{coeff}_\omega\left(\delta\left(\mathbf{s}_{\mu^\vee/\lambda}\right)\right) = \operatorname{coeff}_\omega\left(\overline{s_\lambda s_\mu}\right). \tag{29}$$

We are in one of the following three cases:

Case 1: We have $\lambda = \mu^\vee$.

Case 2: We have $\lambda \subseteq \mu^\vee$ but not $\lambda = \mu^\vee$.

Case 3: We don't have $\lambda \subseteq \mu^\vee$.

Let us first consider Case 1. In this case, we have $\lambda = \mu^\vee$. Thus, $\mathbf{s}_{\mu^\vee/\lambda} = \mathbf{s}_{\mu^\vee/\mu^\vee} = 1 = \mathbf{s}_\varnothing$ and thus

$$\begin{aligned} \delta\left(\mathbf{s}_{\mu^\vee/\lambda}\right) &= \delta\left(\mathbf{s}_\varnothing\right) = \overline{s_{\varnothing^\vee}} \qquad (\text{by the definition of } \delta) \\ &= \overline{s_\omega} \qquad \left(\text{since } \varnothing^\vee = \omega\right). \end{aligned}$$

Therefore, $\operatorname{coeff}_\omega\left(\delta\left(\mathbf{s}_{\mu^\vee/\lambda}\right)\right) = \operatorname{coeff}_\omega\left(\overline{s_\omega}\right) = 1$ (by the definition of $\operatorname{coeff}_\omega$). Comparing this with

$$\begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee \end{cases} = 1 \qquad \left(\text{since } \lambda = \mu^\vee\right),$$

we obtain $\operatorname{coeff}_\omega\left(\overline{s_\lambda s_\mu}\right) = \begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee \end{cases}$. Hence, Lemma 6.22 is proven in Case 1.

Let us next consider Case 2. In this case, we have $\lambda \subseteq \mu^\vee$ but not $\lambda = \mu^\vee$. Hence, $|\lambda| < |\mu^\vee|$ and $\lambda \neq \mu^\vee$.

Now, every partition ν satisfying $|\lambda| + |\nu| = |\mu^\vee|$ and $\nu \subseteq \mu^\vee$ must satisfy

$$\nu \in P_{k,n} \text{ and } \operatorname{coeff}_\omega\left(\delta\left(\mathbf{s}_\nu\right)\right) = 0. \tag{30}$$

[*Proof of (30):* Let ν be a partition satisfying $|\lambda| + |\nu| = |\mu^\vee|$ and $\nu \subseteq \mu^\vee$. We must prove (30).

First of all, from $\nu \subseteq \mu^\vee$ and $\mu^\vee \in P_{k,n}$, we obtain $\nu \in P_{k,n}$. It thus remains to show that $\operatorname{coeff}_\omega\left(\delta\left(\mathbf{s}_\nu\right)\right) = 0$.

The definition of δ yields $\delta\left(\mathbf{s}_\nu\right) = \overline{s_{\nu^\vee}}$ (since $\nu \in P_{k,n}$). But $|\lambda| + |\nu| = |\mu^\vee|$ yields $|\nu| = |\mu^\vee| - |\lambda| > 0$ (since $|\lambda| < |\mu^\vee|$).

But every partition $\kappa \in P_{k,n}$ satisfies $|\kappa^\vee| = \underbrace{k(n-k)}_{=|\omega|} - |\kappa| = |\omega| - |\kappa|$. Applying this to $\kappa = \nu$, we obtain

$$\left|\nu^\vee\right| = |\omega| - \underbrace{|\nu|}_{>0} < |\omega|.$$

Hence, $|\nu^\vee| \neq |\omega|$, so that $\nu^\vee \neq \omega$.

But the definition of $\operatorname{coeff}_\omega$ yields $\operatorname{coeff}_\omega\left(\overline{s_{\nu^\vee}}\right) = \begin{cases} 1, & \text{if } \nu^\vee = \omega; \\ 0, & \text{if } \nu^\vee \neq \omega \end{cases} = 0$ (since $\nu^\vee \neq \omega$). In view of $\delta\left(\mathbf{s}_\nu\right) = \overline{s_{\nu^\vee}}$, this rewrites as $\operatorname{coeff}_\omega\left(\delta\left(\mathbf{s}_\nu\right)\right) = 0$. This completes the proof of (30).]

Proposition 6.17 **(a)** (applied to $\mu^\vee$ and λ instead of λ and μ) yields

$$\begin{aligned}
\mathbf{s}_{\mu^\vee/\lambda} &= \sum_{\nu \text{ is a partition}} c^{\mu^\vee}_{\lambda,\nu} \mathbf{s}_\nu \\
&= \sum_{\substack{\nu \text{ is a partition;} \\ \nu \subseteq \mu^\vee}} c^{\mu^\vee}_{\lambda,\nu} \mathbf{s}_\nu + \sum_{\substack{\nu \text{ is a partition;} \\ \text{not } \nu \subseteq \mu^\vee}} \underbrace{c^{\mu^\vee}_{\lambda,\nu}}_{\substack{=0 \\ \text{(by Proposition 6.17 (b),} \\ \text{applied to } \mu^\vee \text{ and } \lambda \text{ instead of } \lambda \text{ and } \mu)}} \mathbf{s}_\nu \\
&= \sum_{\substack{\nu \text{ is a partition;} \\ \nu \subseteq \mu^\vee}} c^{\mu^\vee}_{\lambda,\nu} \mathbf{s}_\nu \\
&= \sum_{\substack{\nu \text{ is a partition;} \\ \nu \subseteq \mu^\vee; \\ |\lambda|+|\nu|=|\mu^\vee|}} c^{\mu^\vee}_{\lambda,\nu} \mathbf{s}_\nu + \sum_{\substack{\nu \text{ is a partition;} \\ \nu \subseteq \mu^\vee; \\ \text{not } |\lambda|+|\nu|=|\mu^\vee|}} \underbrace{c^{\mu^\vee}_{\lambda,\nu}}_{\substack{=0 \\ \text{(by Proposition 6.17 (c),} \\ \text{applied to } \mu^\vee \text{ and } \lambda \text{ instead of } \lambda \text{ and } \mu)}} \mathbf{s}_\nu \\
&= \sum_{\substack{\nu \text{ is a partition;} \\ \nu \subseteq \mu^\vee; \\ |\lambda|+|\nu|=|\mu^\vee|}} c^{\mu^\vee}_{\lambda,\nu} \mathbf{s}_\nu .
\end{aligned}$$

Applying the map δ to this equality, we find

$$\delta\left(\mathbf{s}_{\mu^\vee/\lambda}\right) = \delta\left(\sum_{\substack{\nu \text{ is a partition;} \\ \nu \subseteq \mu^\vee; \\ |\lambda|+|\nu|=|\mu^\vee|}} c^{\mu^\vee}_{\lambda,\nu} \mathbf{s}_\nu \right) = \sum_{\substack{\nu \text{ is a partition;} \\ \nu \subseteq \mu^\vee; \\ |\lambda|+|\nu|=|\mu^\vee|}} c^{\mu^\vee}_{\lambda,\nu} \delta\left(\mathbf{s}_\nu\right)$$

$$\left(\begin{array}{c} \text{since every partition } \nu \text{ satisfying } \nu \subseteq \mu^\vee \text{ and } |\lambda| + |\nu| = |\mu^\vee| \\ \text{must satisfy } \nu \in P_{k,n} \text{ (by (30)) and thus } \mathbf{s}_\nu \in \mathcal{Z} \end{array} \right).$$

Applying the map $\operatorname{coeff}_\omega$ to this equality, we find

$$\begin{aligned}
\operatorname{coeff}_\omega\left(\delta\left(\mathbf{s}_{\mu^\vee/\lambda}\right)\right) &= \operatorname{coeff}_\omega\left(\sum_{\substack{\nu \text{ is a partition;} \\ \nu \subseteq \mu^\vee; \\ |\lambda|+|\nu|=|\mu^\vee|}} c^{\mu^\vee}_{\lambda,\nu} \delta\left(\mathbf{s}_\nu\right) \right) \\
&= \sum_{\substack{\nu \text{ is a partition;} \\ \nu \subseteq \mu^\vee; \\ |\lambda|+|\nu|=|\mu^\vee|}} c^{\mu^\vee}_{\lambda,\nu} \underbrace{\operatorname{coeff}_\omega\left(\delta\left(\mathbf{s}_\nu\right)\right)}_{\substack{=0 \\ \text{(by (30))}}} = 0.
\end{aligned}$$

Comparing this with

$$\begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee \end{cases} = 0 \qquad \left(\text{since } \lambda \neq \mu^\vee\right),$$

we obtain $\operatorname{coeff}_\omega\left(\overline{s_\lambda s_\mu}\right) = \begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee \end{cases}$. Hence, Lemma 6.22 is proven in Case 2.

Let us finally consider Case 3. In this case, we don't have $\lambda \subseteq \mu^\vee$. Hence, we don't have $\lambda = \mu^\vee$ either. Thus, $\lambda \neq \mu^\vee$.

Also, $\mathbf{s}_{\mu^\vee/\lambda} = 0$ (since we don't have $\lambda \subseteq \mu^\vee$). Thus,

$$\operatorname{coeff}_\omega\left(\delta\left(\underbrace{\mathbf{s}_{\mu^\vee/\lambda}}_{=0}\right)\right) = \operatorname{coeff}_\omega\left(\delta\left(0\right)\right) = 0.$$

Comparing this with

$$\begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee \end{cases} = 0 \qquad \left(\text{since } \lambda \neq \mu^\vee\right),$$

we obtain $\operatorname{coeff}_\omega\left(\overline{s_\lambda s_\mu}\right) = \begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee \end{cases}$. Hence, Lemma 6.22 is proven in Case 3.

We have now proven Lemma 6.22 in all three Cases 1, 2 and 3. Thus, Lemma 6.22 always holds. ☐

Proof of Theorem 6.3. Write $f \in \mathcal{S}/I$ in the form $f = \sum_{\lambda \in P_{k,n}} \alpha_\lambda \overline{s_\lambda}$ with $\alpha_\lambda \in \mathbf{k}$. (This is possible, since $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$.) Then, the definition of $\operatorname{coeff}_{\nu^\vee}$ yields $\operatorname{coeff}_{\nu^\vee}\left(f\right) = \alpha_{\nu^\vee}$.

On the other hand,

$$\begin{aligned}
\operatorname{coeff}_\omega\left(\overline{s_\nu} \underbrace{f}_{= \sum_{\lambda \in P_{k,n}} \alpha_\lambda \overline{s_\lambda}}\right) &= \operatorname{coeff}_\omega\left(\overline{s_\nu} \sum_{\lambda \in P_{k,n}} \alpha_\lambda \overline{s_\lambda}\right) = \sum_{\lambda \in P_{k,n}} \alpha_\lambda \operatorname{coeff}_\omega\left(\underbrace{\overline{s_\nu s_\lambda}}_{=\overline{s_\lambda s_\nu}}\right) \\
&= \sum_{\lambda \in P_{k,n}} \alpha_\lambda \underbrace{\operatorname{coeff}_\omega\left(\overline{s_\lambda s_\nu}\right)}_{\substack{= \begin{cases} 1, & \text{if } \lambda = \nu^\vee; \\ 0, & \text{if } \lambda \neq \nu^\vee \end{cases} \\ \text{(by Lemma 6.22, applied to } \mu = \nu\text{)}}} \\
&= \sum_{\lambda \in P_{k,n}} \alpha_\lambda \begin{cases} 1, & \text{if } \lambda = \nu^\vee; \\ 0, & \text{if } \lambda \neq \nu^\vee \end{cases} = \alpha_{\nu^\vee}
\end{aligned}$$

(since $\nu^\vee \in P_{k,n}$). Comparing this with $\operatorname{coeff}_{\nu^\vee}(f) = \alpha_{\nu^\vee}$, we obtain $\operatorname{coeff}_\omega\left(\overline{s_\nu}f\right) = \operatorname{coeff}_{\nu^\vee}(f)$. This proves Theorem 6.3. ☐

Definition 6.23. For any three partitions $\alpha, \beta, \gamma \in P_{k,n}$, let $g_{\alpha,\beta,\gamma} = \operatorname{coeff}_{\gamma^\vee}\left(\overline{s_\alpha s_\beta}\right) \in \mathbf{k}$.

These scalars $g_{\alpha,\beta,\gamma}$ are thus the structure constants of the $\mathbf{k}$-algebra $\mathcal{S}/I$ in the basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ (although slightly reindexed). As a consequence of Theorem 6.3, we obtain the following S_3-property of these structure constants:

Corollary 6.24. We have

$$g_{\alpha,\beta,\gamma} = g_{\alpha,\gamma,\beta} = g_{\beta,\alpha,\gamma} = g_{\beta,\gamma,\alpha} = g_{\gamma,\alpha,\beta} = g_{\gamma,\beta,\alpha} = \operatorname{coeff}_\omega\left(\overline{s_\alpha s_\beta s_\gamma}\right)$$

for any $\alpha, \beta, \gamma \in P_{k,n}$.

Proof of Corollary 6.24. Let $\alpha, \beta, \gamma \in P_{k,n}$. It clearly suffices to prove $g_{\alpha,\beta,\gamma} = \operatorname{coeff}_\omega\left(\overline{s_\alpha s_\beta s_\gamma}\right)$, since the rest of the claim then follows by analogy.

Theorem 6.3 (applied to $\nu = \gamma$ and $f = \overline{s_\alpha s_\beta}$) yields

$$\operatorname{coeff}_\omega\left(\overline{s_\gamma}\,\overline{s_\alpha s_\beta}\right) = \operatorname{coeff}_{\gamma^\vee}\left(\overline{s_\alpha s_\beta}\right) = g_{\alpha,\beta,\gamma}$$

(by the definition of $g_{\alpha,\beta,\gamma}$). Thus, $g_{\alpha,\beta,\gamma} = \operatorname{coeff}_\omega\left(\underbrace{\overline{s_\gamma}\,\overline{s_\alpha s_\beta}}_{=\overline{s_\alpha s_\beta s_\gamma}}\right) = \operatorname{coeff}_\omega\left(\overline{s_\alpha s_\beta s_\gamma}\right)$. This completes our proof of Corollary 6.24. ☐

7. Complete homogeneous symmetric polynomials

In this section, we shall further explore the projections $\overline{h_i}$ of complete homogeneous symmetric polynomials h_i onto $\mathcal{S}/I$. This exploration will culminate in a second proof of Theorem 6.3.

Convention 7.1. Convention 6.1 remains in place for the whole Section 7.

We shall also use all the notations introduced in Section 6.

If $j \in \mathbb{N}$, then the expression "1^j" in a tuple stands for j consecutive entries equal to 1 (that is, $\underbrace{1, 1, \ldots, 1}_{j \text{ times}}$). Thus, $\left(m, 1^j\right) = \left(m, \underbrace{1, 1, \ldots, 1}_{j \text{ times}}\right)$ for any $m \in \mathbb{N}$ and $j \in \mathbb{N}$.

7.1. A reduction formula for h_{n+m}

The following result helps us reduce complete homogeneous symmetric polynomials h_{n+m} modulo the ideal I:

Proposition 7.2. Let m be a positive integer. Then,

$$h_{n+m} \equiv \sum_{j=0}^{k-1} (-1)^j a_{k-j} s_{\left(m,1^j\right)} \mod I.$$

We shall derive Proposition 7.2 from the following identity between symmetric functions in Λ:

Proposition 7.3. Let m be a positive integer. Then,

$$\mathbf{h}_{n+m} = \sum_{j=0}^{n} (-1)^j \mathbf{h}_{n-j} \mathbf{s}_{\left(m,1^j\right)}.$$

Proof of Proposition 7.3. Let $j \in \mathbb{N}$.

In [GriRei18, Exercise 2.9.14(b)], it is shown that

$$\sum_{i=0}^{b} (-1)^i \mathbf{h}_{a+i+1} \mathbf{e}_{b-i} = \mathbf{s}_{\left(a+1,1^b\right)} \qquad (31)$$

for all $a, b \in \mathbb{N}$. Applying this equality to $a = m-1$ and $b = j$, we obtain

$$\sum_{i=0}^{j} (-1)^i \mathbf{h}_{m+i} \mathbf{e}_{j-i} = \mathbf{s}_{\left(m,1^j\right)}. \qquad (32)$$

Now, forget that we fixed j. We thus have proven (32) for each $j \in \mathbb{N}$.

Also, for any $N \in \mathbb{N}$, we have

$$\sum_{\substack{(i,j)\in\mathbb{N}^2;\\ i+j=N}} (-1)^i \mathbf{e}_i \mathbf{h}_j = \delta_{0,N}$$

(where $\delta_{0,N}$ is a Kronecker delta). (This is [GriRei18, (2.4.4)], with n renamed as N.) Thus, for any $N \in \mathbb{N}$, we have

$$\begin{aligned}
\delta_{0,N} &= \sum_{\substack{(i,j)\in\mathbb{N}^2;\\ i+j=N}} (-1)^i \mathbf{e}_i \mathbf{h}_j = \sum_{\substack{(i,j)\in\mathbb{N}^2;\\ i+j=N}} (-1)^i \mathbf{h}_j \mathbf{e}_i \\
&= \sum_{i=0}^{N} (-1)^i \mathbf{h}_{N-i} \mathbf{e}_i \qquad \left(\begin{array}{c}\text{here, we have substituted } (i, N-i) \\ \text{for } (i,j) \text{ in the sum}\end{array}\right) \\
&= \sum_{j=0}^{N} (-1)^j \mathbf{h}_{N-j} \mathbf{e}_j \qquad \left(\begin{array}{c}\text{here, we have renamed the} \\ \text{summation index } i \text{ as } j\end{array}\right).
\end{aligned}$$

Thus, for any $N \in \mathbb{N}$, we have

$$\sum_{j=0}^{N} (-1)^j \mathbf{h}_{N-j} \mathbf{e}_j = \delta_{0,N}. \tag{33}$$

For each $i \in \{0, 1, \ldots, n\}$, we have

$$\begin{aligned}
&\sum_{j=i}^{n} (-1)^{j-i} \mathbf{h}_{n-j} \mathbf{e}_{j-i} \\
&= \sum_{j=0}^{n-i} (-1)^j \mathbf{h}_{n-i-j} \mathbf{e}_j \qquad \text{(here, we have substituted } j \text{ for } j-i \text{ in the sum)} \\
&= \delta_{0,n-i} \qquad \text{(by (33), applied to } n-i \text{ instead of } N\text{)} \\
&= \delta_{i,n}.
\end{aligned} \tag{34}$$

Now,

$$\begin{aligned}
&\sum_{j=0}^{n} (-1)^j \mathbf{h}_{n-j} \underbrace{\mathbf{s}_{(m,1^j)}}_{\substack{= \sum_{i=0}^{j} (-1)^i \mathbf{h}_{m+i} \mathbf{e}_{j-i} \\ \text{(by (32))}}} \\
&= \sum_{j=0}^{n} (-1)^j \mathbf{h}_{n-j} \sum_{i=0}^{j} (-1)^i \mathbf{h}_{m+i} \mathbf{e}_{j-i} = \underbrace{\sum_{j=0}^{n} \sum_{i=0}^{j}}_{= \sum_{i=0}^{n} \sum_{j=i}^{n}} (-1)^j \mathbf{h}_{n-j} (-1)^i \mathbf{h}_{m+i} \mathbf{e}_{j-i} \\
&= \sum_{i=0}^{n} \sum_{j=i}^{n} (-1)^j \mathbf{h}_{n-j} (-1)^i \mathbf{h}_{m+i} \mathbf{e}_{j-i} = \sum_{i=0}^{n} \mathbf{h}_{m+i} \sum_{j=i}^{n} (-1)^{j-i} \mathbf{h}_{n-j} \mathbf{e}_{j-i} \\
&= \sum_{i=0}^{n} \mathbf{h}_{m+i} \underbrace{\sum_{j=i}^{n} (-1)^{j-i} \mathbf{h}_{n-j} \mathbf{e}_{j-i}}_{\substack{= \delta_{i,n} \\ \text{(by (34))}}} = \sum_{i=0}^{n} \mathbf{h}_{m+i} \delta_{i,n} = \mathbf{h}_{m+n} = \mathbf{h}_{n+m}.
\end{aligned}$$

This proves Proposition 7.3. ☐

Proof of Proposition 7.2. For each integer $j \geq k$, we have

$$s_{(m,1^j)} = 0. \tag{35}$$

[*Proof of (35):* Let $j \geq k$ be an integer. Then, the partition $(m, 1^j)$ has $j+1$ parts; thus, this partition has more than k parts (since $j+1 > j \geq k$). Thus, (3) (applied to $\lambda = (m, 1^j)$) yields $s_{(m,1^j)} = 0$. This proves (35).]

Proposition 7.3 yields

$$\mathbf{h}_{n+m} = \sum_{j=0}^{n} (-1)^j \mathbf{h}_{n-j} \mathbf{s}_{(m,1^j)}.$$

This is an identity in Λ. Evaluating both of its sides at the k variables $x_1, x_2, \ldots, x_k$, we obtain

$$\begin{aligned} h_{n+m} &= \sum_{j=0}^{n} (-1)^j h_{n-j} s_{(m,1^j)} \\ &= \sum_{j=0}^{k-1} (-1)^j h_{n-j} s_{(m,1^j)} + \sum_{j=k}^{n} (-1)^j h_{n-j} \underbrace{s_{(m,1^j)}}_{\substack{=0 \\ \text{(by (35))}}} \\ &= \sum_{j=0}^{k-1} (-1)^j \underbrace{h_{n-j}}_{\substack{=h_{n-k+(k-j)} \equiv a_{k-j} \operatorname{mod} I \\ \text{(by (9), applied to } k-j \\ \text{instead of } j)}} s_{(m,1^j)} \equiv \sum_{j=0}^{k-1} (-1)^j a_{k-j} s_{(m,1^j)} \operatorname{mod} I. \end{aligned}$$

This proves Proposition 7.2. □

7.2. Lemmas on free modules

Next, we state a basic lemma from commutative algebra:

Lemma 7.4. Let $r \in \mathbb{N}$. Let X and Y be two free $\mathbf{k}$-modules of rank r. Then, every surjective $\mathbf{k}$-linear map from X to Y is a $\mathbf{k}$-module isomorphism.

Proof of Lemma 7.4. Let $f : X \to Y$ be a surjective $\mathbf{k}$-linear map from X to Y. We must prove that f is a $\mathbf{k}$-module isomorphism.

There is clearly a $\mathbf{k}$-module isomorphism $j : Y \to X$ (since X and Y are free $\mathbf{k}$-modules of the same rank). Consider this j. Then, the composition $j \circ f$ is surjective (since j and f are surjective), and thus is a surjective endomorphism of the finitely generated $\mathbf{k}$-module X. But [GriRei18, Exercise 2.5.18(a)] shows that any surjective endomorphism of a finitely generated $\mathbf{k}$-module is a $\mathbf{k}$-module isomorphism. Hence, we conclude that $j \circ f$ is a $\mathbf{k}$-module isomorphism. Thus, f is a $\mathbf{k}$-module isomorphism (since j is a $\mathbf{k}$-module isomorphism). This proves Lemma 7.4. □

Lemma 7.5. Let Z be a $\mathbf{k}$-module. Let U, X and Y be $\mathbf{k}$-submodules of Z such that $Z = X \oplus Y$ and $X \subseteq U$. Let $r \in \mathbb{N}$. Assume that the $\mathbf{k}$-module X has a basis with r elements, whereas the $\mathbf{k}$-module U can be spanned by r elements. Then, $X = U$.

Proof of Lemma 7.5. Let $\pi : Z \to X$ be the canonical projection from the direct sum $Z = X \oplus Y$ onto its addend X. Let $\iota : X \to U$ be the canonical injection. Then, the composition

$$X \xrightarrow{\iota} U \xrightarrow{\pi|_U} X$$

is just id_X (since $\pi\mid_X = \operatorname{id}_X$). Hence, the map $\pi\mid_U$ is surjective.

We assumed that the $\mathbf{k}$-module U can be spanned by r elements. Thus, there is a surjective $\mathbf{k}$-module homomorphism $u : \mathbf{k}^r \to U$. Consider this u.

Both $\mathbf{k}$-modules $\mathbf{k}^r$ and X are free of rank r (since X has a basis with r elements). The composition

$$\mathbf{k}^r \xrightarrow{u} U \xrightarrow{\pi|_U} X$$

is surjective (since both u and $\pi\mid_U$ are surjective), and thus is a $\mathbf{k}$-module isomorphism (by Lemma 7.4, applied to $\mathbf{k}^r$ and X instead of X and Y). Hence, it is injective. Thus, u is injective. Since u is also surjective, we thus conclude that u is bijective, and therefore a $\mathbf{k}$-module isomorphism. Since both u and the composition $\mathbf{k}^r \xrightarrow{u} U \xrightarrow{\pi|_U} X$ are $\mathbf{k}$-module isomorphisms, we now conclude that the map $\pi\mid_U$ is a $\mathbf{k}$-module isomorphism. Hence, it has an inverse. But this inverse must be ι (since the composition $X \xrightarrow{\iota} U \xrightarrow{\pi|_U} X$ is id_X). Thus, ι is a $\mathbf{k}$-module isomorphism, too. Thus, in particular, ι is surjective. Therefore, $U = \iota(X) = X$. This proves Lemma 7.5. □

7.3. The symmetric polynomials h_ν

Definition 7.6. Let $\ell \in \mathbb{N}$, and let $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathbb{Z}^\ell$ be any ℓ-tuple of integers. Then, we define the symmetric polynomial $h_\nu \in \mathcal{S}$ as follows:

$$h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_\ell}.$$

Note that the polynomial h_ν does not change if we permute the entries of the ℓ-tuple ν. If an ℓ-tuple ν of integers contains any negative entries, then $h_\nu = 0$ (since $h_i = 0$ for any $i < 0$). Also, if an ℓ-tuple ν of integers contains any entry $= 0$, then we can remove this entry without changing h_ν (since $h_0 = 1$).

7.4. The submodules L_p and H_p of $\mathcal{S}/I$

It is time to define two further filtrations of the $\mathbf{k}$-module $\mathcal{S}/I$ (in addition to the filtration $(Q_p)_{p\in\mathbb{Z}}$ from Definition 6.11):

Definition 7.7. **(a)** If λ is a partition, then $\ell(\lambda)$ shall denote the *length* of λ; this is defined as the number of positive entries of λ. Note that $\ell(\lambda) \leq k$ for each $\lambda \in P_{k,n}$.

(b) For each $p \in \mathbb{Z}$, we let L_p denote the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq p$.

(c) For each $p \in \mathbb{Z}$, we let H_p denote the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{h_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq p$.

The only partition λ satisfying $\ell(\lambda) \leq 0$ is the empty partition $\varnothing = ()$; it belongs to $P_{k,n}$ and satisfies $\overline{s_\lambda} = 1$. Hence, L_0 is the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by 1. Similarly, H_0 is the same $\mathbf{k}$-submodule.

Also, L_k is the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by all $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ (because each $\lambda \in P_{k,n}$ satisfies $\ell(\lambda) \leq k$). But the latter $\mathbf{k}$-submodule is $\mathcal{S}/I$ itself (by Theorem 2.5). Thus, we conclude that L_k is $\mathcal{S}/I$ itself. In other words,

$$L_k = \mathcal{S}/I.$$

Clearly, $L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$ and $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$. We shall soon see that the families $(L_p)_{p\in\mathbb{Z}}$ and $(H_p)_{p\in\mathbb{Z}}$ are identical (Proposition 7.11) and are filtrations of the $\mathbf{k}$-algebra $\mathcal{S}/I$ (Proposition 7.15). First let us show a basic fact:

Lemma 7.8. Let $p \in \mathbb{N}$ be such that $p \leq k$. Let $\nu = (\nu_1, \nu_2, \ldots, \nu_p) \in \mathbb{Z}^p$. Assume that $\nu_i \leq n$ for each $i \in \{1, 2, \ldots, p\}$. Then, $\overline{h_\nu} \in H_p$.

(The condition "$p \leq k$" can be removed from this lemma, but we aren't yet at the point where this is easy to see. We will show this in Proposition 7.14 below.)

Proof of Lemma 7.8. We WLOG assume that $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_p$ (since otherwise, we can just permute the entries of ν to achieve this). Let j be the number of $i \in \{1, 2, \ldots, p\}$ satisfying $\nu_i > n-k$. Then,

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_j > n-k \geq \nu_{j+1} \geq \nu_{j+2} \geq \cdots \geq \nu_p.$$

We WLOG assume that all of the $\nu_1, \nu_2, \ldots, \nu_p$ are nonnegative (since otherwise, we have $h_\nu = 0$ and thus $\overline{h_\nu} = 0 \in H_p$).

Now,

$$\overline{h_{\nu_i}} \in \mathbf{k} \qquad \text{for each } i \in \{1, 2, \ldots, j\}. \tag{36}$$

[*Proof of (36):* Let $i \in \{1, 2, \ldots, j\}$. Then, $\nu_i > n-k$ (since $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_j > n-k$), but also $\nu_i \leq n$ (by the assumptions of Lemma 7.8). Thus, $n-k < \nu_i \leq n$, so that $\nu_i \in \{n-k+1, n-k+2, \ldots, n\}$ and thus $\nu_i - (n-k) \in \{1, 2, \ldots, k\}$. Hence, (9) (applied to $\nu_i - (n-k)$ instead of j) yields $h_{\nu_i} \equiv a_{\nu_i-(n-k)} \mod I$. Hence, $\overline{h_{\nu_i}} = \overline{a_{\nu_i-(n-k)}} \in \mathbf{k}$. This proves (36).]

Furthermore, $(\nu_{j+1}, \nu_{j+2}, \ldots, \nu_p)$ is a partition (since $\nu_{j+1} \geq \nu_{j+2} \geq \cdots \geq \nu_p$ and since all of the $\nu_1, \nu_2, \ldots, \nu_p$ are nonnegative) with at most k entries (indeed, its number of entries is $\leq p-j \leq p \leq k$), and all of its entries are $\leq n-k$ (since $n-k \geq \nu_{j+1} \geq \nu_{j+2} \geq \cdots \geq \nu_p$). Hence, $(\nu_{j+1}, \nu_{j+2}, \ldots, \nu_p)$ belongs to $P_{k,n}$.

From $\left(\nu_{j+1}, \nu_{j+2}, \ldots, \nu_p\right) \in P_{k,n}$ and $\ell\left(\nu_{j+1}, \nu_{j+2}, \ldots, \nu_p\right) \leq p - j \leq p$, we obtain $\overline{h_{\left(\nu_{j+1}, \nu_{j+2}, \ldots, \nu_p\right)}} \in H_p$ (by the definition of H_p).

Now, the definition of h_ν yields $h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_p}$, so that

$$\overline{h_\nu} = \overline{h_{\nu_1} h_{\nu_2} \cdots h_{\nu_p}} = \overline{h_{\nu_1}}\, \overline{h_{\nu_2}} \cdots \overline{h_{\nu_p}} = \underbrace{\left(\overline{h_{\nu_1}}\, \overline{h_{\nu_2}} \cdots \overline{h_{\nu_j}}\right)}_{\substack{\in \mathbf{k} \\ \text{(by (36))}}} \underbrace{\left(\overline{h_{\nu_{j+1}}}\, \overline{h_{\nu_{j+2}}} \cdots \overline{h_{\nu_p}}\right)}_{\substack{= \overline{h_{\nu_{j+1}} h_{\nu_{j+2}} \cdots h_{\nu_p}} \\ = \overline{h_{\left(\nu_{j+1}, \nu_{j+2}, \ldots, \nu_p\right)}} \in H_p}}$$

$$\in \mathbf{k} H_p \subseteq H_p.$$

This proves Lemma 7.8. □

Lemma 7.9. Let $p \in \mathbb{Z}$. Then, the family $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ is a basis of the $\mathbf{k}$-module L_p.

Proof of Lemma 7.9. Theorem 2.5 yields that $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$. Hence, this family $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ is $\mathbf{k}$-linearly independent. Thus, its subfamily $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ is $\mathbf{k}$-linearly independent as well. Moreover, this subfamily $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ spans the $\mathbf{k}$-module L_p (by the definition of L_p). Hence, this subfamily $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ is a basis of the $\mathbf{k}$-module L_p. This proves Lemma 7.9. □

Lemma 7.10. Let $p \in \{0, 1, \ldots, k\}$. Then, $L_p = H_p$.

(This lemma holds more generally for all $p \in \mathbb{Z}$, as we shall see in Lemma 7.11 below.)

Proof of Lemma 7.10. Let $\lambda \in P_{k,n}$ be such that $\ell(\lambda) \leq p$. We shall show that $\overline{s_\lambda} \in H_p$.

Indeed, let S_p denote the group of permutations of $\{1, 2, \ldots, p\}$. For each $\sigma \in S_p$, let $(-1)^\sigma$ denote the sign of σ.

For each $\sigma \in S_p$, we have

$$\overline{\prod_{i=1}^{p} h_{\lambda_i - i + \sigma(i)}} \in H_p. \tag{37}$$

[*Proof of (37):* Let $\sigma \in S_p$. Then, each $i \in \{1, 2, \ldots, p\}$ satisfies

$$\underbrace{\lambda_i}_{\substack{\leq n-k \\ \text{(since } \lambda \in P_{k,n})}} - \underbrace{i}_{\geq 0} + \underbrace{\sigma(i)}_{\leq p \leq k} \leq n - k + 0 + k = n.$$

Thus, Lemma 7.8 (applied to $\left(\lambda_1 - 1 + \sigma(1), \lambda_2 - 2 + \sigma(2), \ldots, \lambda_p - p + \sigma(p)\right)$ and $\lambda_i - i + \sigma(i)$ instead of ν and ν_i) yields

$$\overline{h_{\left(\lambda_1 - 1 + \sigma(1), \lambda_2 - 2 + \sigma(2), \ldots, \lambda_p - p + \sigma(p)\right)}} \in H_p$$

(since $p \leq k$). In view of

$$h_{\left(\lambda_1-1+\sigma(1),\lambda_2-2+\sigma(2),\ldots,\lambda_p-p+\sigma(p)\right)} = \prod_{i=1}^{p} h_{\lambda_i-i+\sigma(i)},$$

this rewrites as $\overline{\prod_{i=1}^{p} h_{\lambda_i-i+\sigma(i)}} \in H_p$. Thus, (37) is proven.]

We have $\ell(\lambda) \leq p$ and thus $\lambda = \left(\lambda_1, \lambda_2, \ldots, \lambda_p\right)$. Hence, Proposition 5.7 **(b)** yields

$$s_\lambda = \det\left(\left(h_{\lambda_u-u+v}\right)_{1\leq u\leq p,\ 1\leq v\leq p}\right) = \sum_{\sigma\in S_p} (-1)^\sigma \prod_{i=1}^{p} h_{\lambda_i-i+\sigma(i)}$$

(by the definition of a determinant). Projecting both sides of this equality onto $\mathcal{S}/I$, we obtain

$$\overline{s_\lambda} = \overline{\sum_{\sigma\in S_p} (-1)^\sigma \prod_{i=1}^{p} h_{\lambda_i-i+\sigma(i)}} = \sum_{\sigma\in S_p} (-1)^\sigma \underbrace{\overline{\prod_{i=1}^{p} h_{\lambda_i-i+\sigma(i)}}}_{\substack{\in H_p \\ \text{(by (37))}}} \in H_p.$$

Now, forget that we fixed λ. We thus have proven that

$$\overline{s_\lambda} \in H_p \qquad \text{for each } \lambda \in P_{k,n} \text{ satisfying } \ell(\lambda) \leq p.$$

Therefore, $L_p \subseteq H_p$ (since L_p is the **k**-submodule of $\mathcal{S}/I$ spanned by the $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq p$).

Lemma 7.9 yields that the family $\left(\overline{s_\lambda}\right)_{\lambda\in P_{k,n};\ \ell(\lambda)\leq p}$ is a basis of the **k**-module L_p.

Now, let L'_p be the **k**-submodule of $\mathcal{S}/I$ spanned by the $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) > p$. Recall (from Theorem 2.5) that $\left(\overline{s_\lambda}\right)_{\lambda\in P_{k,n}}$ is a basis of the **k**-module $\mathcal{S}/I$. Hence, $\mathcal{S}/I = L_p \oplus L'_p$ (since each $\lambda \in P_{k,n}$ satisfies either $\ell(\lambda) \leq p$ or $\ell(\lambda) > p$ but not both). Let r be the number of all $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq p$. Then, the **k**-module H_p can be spanned by r elements (namely, by the $\overline{h_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq p$), whereas the **k**-module L_p has a basis with r elements (namely, the family $\left(\overline{s_\lambda}\right)_{\lambda\in P_{k,n};\ \ell(\lambda)\leq p}$). Thus, Lemma 7.5 (applied to $Z = \mathcal{S}/I$, $X = L_p$, $Y = L'_p$ and $U = H_p$) yields $L_p = H_p$. This proves Lemma 7.10. □

Proposition 7.11. Let $p \in \mathbb{Z}$. Then, $L_p = H_p$.

Proof of Proposition 7.11. If p is negative, then both L_p and H_p equal 0 (since there exists no $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq p$ in this case). Thus, if p is negative, then

$L_p = H_p$ is obviously true. Hence, for the rest of this proof, we WLOG assume that p is not negative. Thus, $p \in \mathbb{N}$.

If $p \in \{0, 1, \ldots, k\}$, then $L_p = H_p$ follows from Lemma 7.10. Hence, for the rest of this proof, we WLOG assume that $p \notin \{0, 1, \ldots, k\}$. Thus, $p > k$ (since $p \in \mathbb{N}$). Hence, $k < p$, so that $H_k \subseteq H_p$ (since $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$). But Lemma 7.10 (applied to k instead of p) yields $L_k = H_k$.

But recall that $L_k = \mathcal{S}/I$. Thus, $\mathcal{S}/I = L_k = H_k \subseteq H_p$. Thus, $H_p \supseteq \mathcal{S}/I \supseteq L_p$.

On the other hand, $k < p$ and thus $L_k \subseteq L_p$ (since $L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$). Hence, $L_p \supseteq L_k = \mathcal{S}/I \supseteq H_p$. Combining this with $H_p \supseteq L_p$, we obtain $L_p = H_p$. This proves Proposition 7.11. ☐

Corollary 7.12. Let $p \in \mathbb{Z}$. Then, the family $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ is a basis of the $\mathbf{k}$-module L_p.

Proof of Corollary 7.12. Lemma 7.9 yields that the family $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ is a basis of the $\mathbf{k}$-module L_p. On the other hand, the family $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ spans the $\mathbf{k}$-module H_p (by the definition of H_p). In other words, the family $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ spans the $\mathbf{k}$-module L_p (since Proposition 7.11 yields $L_p = H_p$). Since $|\{\lambda \in P_{k,n} \mid \ell(\lambda) \leq p\}| = |\{\lambda \in P_{k,n} \mid \ell(\lambda) \leq p\}|$, we can therefore apply Lemma 5.3 to L_p, $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ and $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ instead of M, $(b_s)_{s \in S}$ and $(a_u)_{u \in U}$. We thus conclude that $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ is a basis of the $\mathbf{k}$-module L_p. This proves Corollary 7.12. ☐

Theorem 7.13. The family $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$.

Proof of Theorem 7.13. Corollary 7.12 (applied to $p = k$) shows that the family $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq k}$ is a basis of the $\mathbf{k}$-module L_k. In view of $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq k} = \left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ (since each $\lambda \in P_{k,n}$ satisfies $\ell(\lambda) \leq k$) and $L_k = \mathcal{S}/I$, this rewrites as follows: The family $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$. This proves Theorem 7.13. ☐

Proposition 7.14. Let $p \in \mathbb{N}$. Let $\nu = (\nu_1, \nu_2, \ldots, \nu_p) \in \mathbb{Z}^p$. Assume that $\nu_i \leq n$ for each $i \in \{1, 2, \ldots, p\}$. Then, $\overline{h_\nu} \in H_p$.

Proof of Proposition 7.14. If $p \leq k$, then this follows from Lemma 7.8. Thus, for the rest of this proof, we WLOG assume that $p > k$. Hence, $k < p$, so that $H_k \subseteq H_p$ (since $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$). But Proposition 7.11 (applied to k instead of p) yields $H_k = L_k = \mathcal{S}/I$. Now, $\overline{h_\nu} \in \mathcal{S}/I = H_k \subseteq H_p$. This proves Proposition 7.14. ☐

We recall that the $\mathbf{k}$-submodules of a given $\mathbf{k}$-algebra A form a monoid under multiplication: The product XY of two $\mathbf{k}$-submodules X and Y of A is defined as the $\mathbf{k}$-linear span of all products xy with $x \in X$ and $y \in Y$. The neutral element of this monoid is $\mathbf{k} \cdot 1_A$. We shall specifically use this monoid in the case when $A = \mathcal{S}/I$.

Proposition 7.15. The family $\left(L_p\right)_{p\in\mathbb{N}}$ is a filtration of the $\mathbf{k}$-algebra $\mathcal{S}/I$; that is, we have

$$L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots, \qquad \bigcup_{p\in\mathbb{N}} L_p = \mathcal{S}/I,$$
$$1 \in L_0, \qquad \text{and}$$
$$L_a L_b \subseteq L_{a+b} \qquad \text{for every } a, b \in \mathbb{N}. \tag{38}$$

Proof of Proposition 7.15. We already know that $L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$. Also, $1 \in L_0$ (since L_0 is the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by 1). Also, $L_k = \mathcal{S}/I$, so that $\mathcal{S}/I = L_k \subseteq \bigcup_{p\in\mathbb{N}} L_p$. Combining this with $\bigcup_{p\in\mathbb{N}} L_p \subseteq \mathcal{S}/I$, we obtain $\bigcup_{p\in\mathbb{N}} L_p = \mathcal{S}/I$.

Hence, it remains to prove that $L_a L_b \subseteq L_{a+b}$ for every $a, b \in \mathbb{N}$. So let us fix $a, b \in \mathbb{N}$. We must prove that $L_a L_b \subseteq L_{a+b}$.

If $a + b \geq k$, then this is obvious (because if $a + b \geq k$, then $k \leq a + b$, hence $L_k \subseteq L_{a+b}$ (since $L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$), hence $L_a L_b \subseteq \mathcal{S}/I = L_k \subseteq L_{a+b}$). Hence, we WLOG assume that $a + b < k$.

We must prove that $L_a L_b \subseteq L_{a+b}$. It clearly suffices to show that $fg \in L_{a+b}$ for each $f \in L_a$ and $g \in L_b$. So let us fix $f \in L_a$ and $g \in L_b$; we must prove that $fg \in L_{a+b}$.

Proposition 7.11 yields that $L_a = H_a$. Thus, $f \in L_a = H_a$, so that f is a $\mathbf{k}$-linear combination of the $\overline{h_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq a$ (because H_a is the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by these $\overline{h_\lambda}$). Since the claim we are proving (that is, $fg \in L_{a+b}$) depends $\mathbf{k}$-linearly on f, we can thus WLOG assume that f is one of those $\overline{h_\lambda}$. In other words, we can WLOG assume that $f = \overline{h_\alpha}$ for some $\alpha \in P_{k,n}$ satisfying $\ell(\alpha) \leq a$. Assume this, and consider this α. For similar reasons, we WLOG assume that $g = \overline{h_\beta}$ for some $\beta \in P_{k,n}$ satisfying $\ell(\beta) \leq b$. Consider this β.

Note that each entry of α is $\leq n - k$ (since $\alpha \in P_{k,n}$), and therefore $\leq n$. Thus, we can consider α as an a-tuple of elements of $\{0, 1, \ldots, n\}$ (since $\ell(\alpha) \leq a$). Likewise, consider β as a b-tuple of elements of $\{0, 1, \ldots, n\}$.

Let γ be the concatenation of the a-tuple α with the b-tuple β. Thus, γ is an $(a + b)$-tuple of elements of $\{0, 1, \ldots, n\}$ (since α is an a-tuple of elements of $\{0, 1, \ldots, n\}$ and since β is a b-tuple of elements of $\{0, 1, \ldots, n\}$), and satisfies $h_\gamma = h_\alpha h_\beta$. (But γ is not necessarily a partition.) Moreover, $a + b \leq k$ (since $a + b < k$). Finally, write γ in the form $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{a+b})$; then, we have $\gamma_i \leq n$ for each $i \in \{1, 2, \ldots, a + b\}$ (because γ is an $(a + b)$-tuple of elements of

$\{0,1,\ldots,n\}$). Hence, Lemma 7.8 (applied to $p = a+b$, $\nu = \gamma$ and $\nu_i = \gamma_i$) yields $\overline{h_\gamma} \in H_{a+b}$. But Proposition 7.11 yields that $L_{a+b} = H_{a+b}$.

From $f = \overline{h_\alpha}$ and $g = \overline{h_\beta}$, we obtain $fg = \overline{h_\alpha}\,\overline{h_\beta} = \overline{h_\alpha h_\beta} = \overline{h_\gamma}$ (since $h_\alpha h_\beta = h_\gamma$). Thus, $fg = \overline{h_\gamma} \in H_{a+b} = L_{a+b}$ (since $L_{a+b} = H_{a+b}$). This completes our proof of Proposition 7.15. □

Corollary 7.16. We have $(L_1)^m \subseteq L_m$ for each $m \in \mathbb{N}$.

Proof of Corollary 7.16. This follows by induction on m, using the facts (which we proved in Proposition 7.15) that $1 \in L_0$ and that $L_a L_b \subseteq L_{a+b}$ for every $a, b \in \mathbb{N}$. □

7.5. A formula for hook-shaped Schur functions

Lemma 7.17. Let m be a positive integer. Let $j \in \mathbb{N}$. Then,

$$\mathbf{s}_{(m,1^j)} = \sum_{i=1}^{m} (-1)^{i-1} \mathbf{h}_{m-i} \mathbf{e}_{j+i}.$$

Proof of Lemma 7.17. For each $N \in \mathbb{N}$, we have

$$\sum_{p=0}^{N} (-1)^p \mathbf{h}_{N-p} \mathbf{e}_p = \delta_{0,N}. \tag{39}$$

(This is just the equality (33), with j renamed as p.)

From $m > 0$ and $j \geq 0$, we obtain $m + j > 0$, so that $\delta_{0,m+j} = 0$. The equality (39) (applied to $N = m + j$) becomes

$$\sum_{p=0}^{m+j} (-1)^p \mathbf{h}_{m+j-p} \mathbf{e}_p = \delta_{0,m+j} = 0.$$

Thus,

$$\begin{aligned}
0 &= \sum_{p=0}^{m+j} (-1)^p \mathbf{h}_{m+j-p}\mathbf{e}_p = \sum_{p=0}^{m+j} (-1)^p \mathbf{h}_{m+j-p}\mathbf{e}_p \\
&= \sum_{i=-m}^{j} (-1)^{j-i} \mathbf{h}_{m+i}\mathbf{e}_{j-i} \qquad \left(\begin{array}{c}\text{here, we have substituted } j-i \\ \text{for } p \text{ in the sum}\end{array}\right) \\
&= \underbrace{\sum_{i=-m}^{-1} (-1)^{j-i} \mathbf{h}_{m+i}\mathbf{e}_{j-i}}_{\substack{=\sum_{i=1}^{m} (-1)^{j+i}\mathbf{h}_{m-i}\mathbf{e}_{j+i} \\ \text{(here, we have substituted } -i \text{ for } i \\ \text{in the sum)}}} + \sum_{i=0}^{j} \underbrace{(-1)^{j-i}}_{=(-1)^j(-1)^i} \mathbf{h}_{m+i}\mathbf{e}_{j-i} \\
&= \sum_{i=1}^{m} (-1)^{j+i} \mathbf{h}_{m-i}\mathbf{e}_{j+i} + (-1)^j \underbrace{\sum_{i=0}^{j} (-1)^i \mathbf{h}_{m+i}\mathbf{e}_{j-i}}_{\substack{=\mathbf{s}_{\left(m,1^j\right)} \\ \text{(by (32))}}} \\
&= \sum_{i=1}^{m} (-1)^{j+i} \mathbf{h}_{m-i}\mathbf{e}_{j+i} + (-1)^j \mathbf{s}_{\left(m,1^j\right)}.
\end{aligned}$$

Solving this equality for $\mathbf{s}_{\left(m,1^j\right)}$, we obtain

$$\mathbf{s}_{\left(m,1^j\right)} = -\frac{1}{(-1)^j} \sum_{i=1}^{m} (-1)^{j+i} \mathbf{h}_{m-i}\mathbf{e}_{j+i} = \sum_{i=1}^{m} (-1)^{i-1} \mathbf{h}_{m-i}\mathbf{e}_{j+i}.$$

This proves Lemma 7.17. □

7.6. The submodules C and R_p of $\mathcal{S}/I$

Next, we introduce some more $\mathbf{k}$-submodules of $\mathcal{S}/I$:

Definition 7.18. **(a)** Let C be the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{e_i}$ with $i \in \mathbb{N}$.

(b) For each $p \in \mathbb{Z}$, we let R_p be the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{h_i}$ with $i \in \mathbb{N}$ satisfying $i \leq p$.

We recall that $e_i = 0$ for every $i > k$. Thus, $\overline{e_i} = 0$ for every $i > k$. Hence, the $\mathbf{k}$-module C is spanned by $\overline{e_0}, \overline{e_1}, \ldots, \overline{e_k}$ (because all the other among its designated generators $\overline{e_i}$ are 0). Also, the definition of C yields $\overline{e_0} \in C$, so that $1 = \overline{e_0} \in C$. Thus, each $i \in \mathbb{N}$ satisfies $C^i = \underbrace{1}_{\in C} C^i \subseteq CC^i = C^{i+1}$. In other words, $C^0 \subseteq C^1 \subseteq C^2 \subseteq \cdots$.

Note that $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$. Also:

Proposition 7.19. We have $R_{n-k} = L_1$.

Proof of Proposition 7.19. We WLOG assume that $k \neq 0$, because the case when $k = 0$ is trivial for its own reasons[7]. Thus, $k > 0$, and therefore the partition (i) belongs to $P_{k,n}$ for each $i \in \{0, 1, \ldots, n-k\}$.

Recall that L_1 was defined as the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq 1$. But the $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq 1$ are exactly the partitions of the form (i) for $i \in \{0, 1, \ldots, n-k\}$. Hence, L_1 is the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{s_{(i)}}$ with $i \in \{0, 1, \ldots, n-k\}$. Since we have $s_{(i)} = h_i$ for each $i \in \{0, 1, \ldots, n-k\}$, we can rewrite this as follows: L_1 is the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{h_i}$ with $i \in \{0, 1, \ldots, n-k\}$. In other words, L_1 is the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{h_i}$ with $i \in \mathbb{N}$ satisfying $i \leq n-k$. But this is precisely the definition of the $\mathbf{k}$-submodule R_{n-k}. Hence, $L_1 = R_{n-k}$. This proves Proposition 7.19. □

It is easy to see that $R_{n-k} = R_{n-k+1} = \cdots = R_n$, but the sequence $(R_0, R_1, R_2, \ldots)$ may and may not grow after its n-th term depending on the choice of $a_1, a_2, \ldots, a_k$. So the family $(R_p)_{p \in \mathbb{Z}}$ is a filtration of some $\mathbf{k}$-submodule of $\mathcal{S}/I$, but it isn't easy to say which specific $\mathbf{k}$-submodule it is.

Lemma 7.20. We have $R_p \subseteq C^p$ for each $p \in \mathbb{N}$.

Proof of Lemma 7.20. We have

$$\overline{e_i} \in C \qquad \text{for each } i \in \mathbb{N} \tag{40}$$

(by the definition of C).

Let $p \in \mathbb{N}$. Recall that R_p is the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{h_i}$ with $i \in \mathbb{N}$ satisfying $i \leq p$. Hence, in order to prove that $R_p \subseteq C^p$, it suffices to show that $\overline{h_i} \in C^p$ for each $i \in \mathbb{N}$ satisfying $i \leq p$.

We first claim that

$$\overline{h_i} \in C^i \qquad \text{for each } i \in \mathbb{N}. \tag{41}$$

[*Proof of (41):* We shall prove (41) by strong induction on i. So we fix $j \in \mathbb{N}$, and we assume (as induction hypothesis) that (41) holds for all $i < j$. We must now prove that (41) holds for $i = j$. In other words, we must prove that $\overline{h_j} \in C^j$.

If $j = 0$, then this is obvious (because in this case, we have $\overline{h_j} = \overline{h_0} = \overline{1} = 1 \in C^0$). Thus, we WLOG assume that $j \neq 0$. Hence, j is a positive integer. Thus, Corollary 3.2 (applied to j instead of p) yields

$$h_j = -\sum_{t=1}^{k} (-1)^t e_t h_{j-t}.$$

---

[7] *Proof.* Assume that $k = 0$. Then, $\mathcal{S} = \mathbf{k}$ and $I = 0$, whence $\mathcal{S}/I = \mathbf{k} \cdot 1$. Both $\mathbf{k}$-submodules R_{n-k} and L_1 contain 1 (since $1 = \overline{h_0}$ and since $1 = \overline{s_\varnothing}$); hence, both of these $\mathbf{k}$-submodules must be the whole $\mathcal{S}/I$ (since $\mathcal{S}/I = \mathbf{k} \cdot 1$) and therefore must be equal. So we have proven $R_{n-k} = L_1$. In other words, we have proven Proposition 7.19 under the assumption that $k = 0$.

Hence,

$$\begin{aligned}\overline{h_j} &= -\overline{\sum_{t=1}^{k} (-1)^t e_t h_{j-t}} = -\sum_{t=1}^{k} (-1)^t \underbrace{\overline{e_t}}_{\substack{\in C \\ \text{(by (40))}}} \underbrace{\overline{h_{j-t}}}_{\substack{\in C^{j-t} \\ \text{(by the induction} \\ \text{hypothesis, since } j-t<j)}} \\ &\in -\sum_{t=1}^{k} (-1)^t \underbrace{C C^{j-t}}_{\substack{= C^{j-t+1} \subseteq C^j \\ \text{(since } j-t+1 \leq j \text{ and} \\ C^0 \subseteq C^1 \subseteq C^2 \subseteq \cdots)}} \subseteq -\sum_{t=1}^{k} (-1)^t C^j \subseteq C^j.\end{aligned}$$

In other words, (41) holds for $i = j$. This completes the induction step. Thus, (41) is proven.]

Now, let us fix $i \in \mathbb{N}$ satisfying $i \leq p$. Then, $C^i \subseteq C^p$ (since $i \leq p$ and $C^0 \subseteq C^1 \subseteq C^2 \subseteq \cdots$). But (41) yields $\overline{h_i} \in C^i \subseteq C^p$.

Now, forget that we fixed i. We thus have shown that $\overline{h_i} \in C^p$ for each $i \in \mathbb{N}$ satisfying $i \leq p$. As we have said, this proves Lemma 7.20. □

Lemma 7.21. Let m be a positive integer. Let $j \in \mathbb{N}$. Then, $\overline{s_{(m,1^j)}} \in R_{m-1} C$.

Proof of Lemma 7.21. Lemma 7.17 yields

$$\mathbf{s}_{(m,1^j)} = \sum_{i=1}^{m} (-1)^{i-1} \mathbf{h}_{m-i} \mathbf{e}_{j+i}.$$

This is an equality in Λ. If we evaluate both of its sides at $x_1, x_2, \ldots, x_k$, then we obtain

$$s_{(m,1^j)} = \sum_{i=1}^{m} (-1)^{i-1} h_{m-i} e_{j+i}.$$

Thus,

$$\begin{aligned}\overline{s_{(m,1^j)}} &= \overline{\sum_{i=1}^{m} (-1)^{i-1} h_{m-i} e_{j+i}} = \sum_{i=1}^{m} (-1)^{i-1} \underbrace{\overline{h_{m-i}}}_{\substack{\in R_{m-1} \\ \text{(by the definition of } R_{m-1}, \\ \text{since } m-i \leq m-1)}} \underbrace{\overline{e_{j+i}}}_{\substack{\in C \\ \text{(by the definition of } C)}} \\ &\in \sum_{i=1}^{m} (-1)^{i-1} R_{m-1} C \subseteq R_{m-1} C.\end{aligned}$$

This proves Lemma 7.21. □

Corollary 7.22. Let m be a positive integer. Then, $\overline{h_{n+m}} \in R_{m-1} C$.

Proof of Corollary 7.22. Proposition 7.2 yields

$$h_{n+m} \equiv \sum_{j=0}^{k-1} (-1)^j a_{k-j} s_{\left(m,1^j\right)} \operatorname{mod} I.$$

Thus,

$$\begin{aligned}\overline{h_{n+m}} &= \overline{\sum_{j=0}^{k-1} (-1)^j a_{k-j} s_{\left(m,1^j\right)}} = \sum_{j=0}^{k-1} (-1)^j a_{k-j} \underbrace{\overline{s_{\left(m,1^j\right)}}}_{\substack{\in R_{m-1}C \\ \text{(by Lemma 7.21)}}} \\ &\in \sum_{j=0}^{k-1} (-1)^j a_{k-j} R_{m-1}C \subseteq R_{m-1}C.\end{aligned}$$

This proves Corollary 7.22. ☐

Lemma 7.23. Let $j \in \mathbb{N}$ be such that $j \leq n$.

(a) We have $\overline{h_j} \in L_1$.

(b) Assume that $n > k$ and $j \neq n-k$. Then, $\overline{h_j} \in R_{n-k-1}$.

Proof of Lemma 7.23. **(a)** We are in one of the following two cases:

Case 1: We have $j \leq n-k$.

Case 2: We have $j > n-k$.

Let us first consider Case 1. In this case, we have $j \leq n-k$. Recall that R_{n-k} was defined as the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{h_i}$ with $i \in \mathbb{N}$ satisfying $i \leq n-k$. Hence, $\overline{h_j} \in R_{n-k}$ (since $j \in \mathbb{N}$ and $j \leq n-k$). Thus, $\overline{h_j} \in R_{n-k} = L_1$ (by Proposition 7.19). Thus, Lemma 7.23 **(a)** is proven in Case 1.

Let us now consider Case 2. In this case, we have $j > n-k$. Hence, $n-k < j \leq n$, so that $j \in \{n-k+1, n-k+2, \ldots, n\}$ and therefore $j-(n-k) \in \{1, 2, \ldots, k\}$. Hence, (9) (applied to $j-(n-k)$ instead of j) yields $h_j \equiv a_{j-(n-k)} \operatorname{mod} I$. Hence, $\overline{h_j} = \overline{a_{j-(n-k)}} \in \mathbf{k}$.

But $0 \leq n-k$ and thus $\overline{h_0} \in R_{n-k}$ (by the definition of R_{n-k}). Hence, $1 = \overline{h_0} \in R_{n-k}$, so that $\mathbf{k} \subseteq R_{n-k}$ and thus $\overline{h_j} \in \mathbf{k} \subseteq R_{n-k} = L_1$ (by Proposition 7.19). Thus, Lemma 7.23 **(a)** is proven in Case 2.

We have now proven Lemma 7.23 **(a)** in each of the two Cases 1 and 2. Thus, Lemma 7.23 **(a)** is proven.

(b) We are in one of the following two cases:

Case 1: We have $j \leq n-k$.

Case 2: We have $j > n-k$.

Let us first consider Case 1. In this case, we have $j \leq n-k$. Thus, $j < n-k$ (since $j \neq n-k$), so that $j \leq n-k-1$. Thus, $n-k-1 \geq j \geq 0$, so that $n-k-1 \in \mathbb{N}$. Recall that R_{n-k-1} is defined as the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{h_i}$ with $i \in \mathbb{N}$ satisfying $i \leq n-k-1$. Hence, $\overline{h_j} \in R_{n-k-1}$ (since $j \in \mathbb{N}$ and $j \leq n-k-1$). Thus, Lemma 7.23 **(b)** is proven in Case 1.

Let us now consider Case 2. In this case, we have $j > n-k$. Hence, $n-k < j \leq n$, so that $j \in \{n-k+1, n-k+2, \ldots, n\}$ and therefore $j-(n-k) \in \{1,2,\ldots,k\}$. Hence, (9) (applied to $j-(n-k)$ instead of j) yields $h_j \equiv a_{j-(n-k)} \operatorname{mod} I$. Hence, $\overline{h_j} = \overline{a_{j-(n-k)}} \in \mathbf{k}$.

But $n-k > 0$ (since $n > k$), and thus $1 \leq n-k$, so that $0 \leq n-k-1$. Hence, $\overline{h_0} \in R_{n-k-1}$ (by the definition of R_{n-k-1}). Hence, $1 = \overline{h_0} \in R_{n-k-1}$, so that $\mathbf{k} \subseteq R_{n-k-1}$ and thus $\overline{h_j} \in \mathbf{k} \subseteq R_{n-k-1}$. Thus, Lemma 7.23 **(b)** is proven in Case 2.

We have now proven Lemma 7.23 **(b)** in each of the two Cases 1 and 2. Thus, Lemma 7.23 **(b)** is proven. □

7.7. Connection to the Q_p

Convention 7.24. We WLOG assume that $k > 0$ from now on.

Now, let us recall Definition 6.11.

Proposition 7.25. We have $L_{k-1} = Q_0$.

Proof of Proposition 7.25. Recall the following:

- We have defined L_{k-1} as the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq k-1$.
- We have defined Q_0 as the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\lambda_k \leq 0$.

Comparing these two definitions, we conclude that $L_{k-1} = Q_0$ (because for any $\lambda \in P_{k,n}$, the statement $(\ell(\lambda) \leq k-1)$ is equivalent to the statement $(\lambda_k \leq 0)$). This proves Proposition 7.25. □

Lemma 7.26. We have $(L_1)^{k-1} \subseteq Q_0$.

Proof of Lemma 7.26. Corollary 7.16 yields $(L_1)^{k-1} \subseteq L_{k-1} = Q_0$ (by Proposition 7.25). This proves Lemma 7.26. □

Lemma 7.27. Let $p \in \mathbb{Z}$. Then, $CQ_p \subseteq Q_{p+1}$.

Proof of Lemma 7.27. Lemma 6.16 shows that $\overline{e_i} Q_p \subseteq Q_{p+1}$ for each $i \in \mathbb{N}$. Thus, $CQ_p \subseteq Q_{p+1}$ (since the $\mathbf{k}$-module C is spanned by the $\overline{e_i}$ with $i \in \mathbb{N}$). This proves Lemma 7.27. □

Corollary 7.28. Let $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then, $C^q Q_p \subseteq Q_{p+q}$.

Proof of Corollary 7.28. This follows by induction on q, where the induction step uses Lemma 7.27. □

7.8. Criteria for $\operatorname{coeff}_\omega\left(\overline{h_\nu}\right) = 0$

We shall now show two sufficient criteria for when a p-tuple $\nu \in \mathbb{Z}^p$ satisfies $\operatorname{coeff}_\omega\left(\overline{h_\nu}\right) = 0$.

> **Theorem 7.29.** Let $p \in \mathbb{N}$ be such that $p \leq k$. Let $\nu = \left(\nu_1, \nu_2, \ldots, \nu_p\right) \in \mathbb{Z}^p$ be an p-tuple of integers. Let $q \in \{1, 2, \ldots, p\}$ be such that
>
> $$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_q > n \geq \nu_{q+1} \geq \nu_{q+2} \geq \cdots \geq \nu_p$$
>
> and $\nu_q \leq 2n - k - q$.
>
> Assume also that
>
> $$\nu_i \leq 2n - k + 1 \qquad \text{for each } i \in \{1, 2, \ldots, p\}. \tag{42}$$
>
> Then, $\operatorname{coeff}_\omega\left(\overline{h_\nu}\right) = 0$.

Proof of Theorem 7.29. From $\nu_q \leq 2n - k - q$, we obtain $2n - k - q \geq \nu_q > n$, so that $n - k - q > 0$. Thus, $n - k - q - 1 \in \mathbb{N}$.

If any of the entries $\nu_1, \nu_2, \ldots, \nu_p$ of ν is negative, then Theorem 7.29 holds for easy reasons[8]. Hence, we WLOG assume that none of the entries $\nu_1, \nu_2, \ldots, \nu_p$ of ν is negative. Thus, all of the entries $\nu_1, \nu_2, \ldots, \nu_p$ are nonnegative integers.

From $p \leq k$, we obtain $p - 1 \leq k - 1$ and thus $L_{p-1} \subseteq L_{k-1}$ (since $L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$). Thus,

$$L_{p-1} \subseteq L_{k-1} = Q_0 \tag{43}$$

(by Proposition 7.25).

From $n \geq \nu_{q+1} \geq \nu_{q+2} \geq \cdots \geq \nu_p$, we conclude that $\nu_j \leq n$ for each $j \in \{q+1, q+2, \ldots, p\}$. In other words, $\nu_{q+i} \leq n$ for each $i \in \{1, 2, \ldots, p-q\}$. Hence, Proposition 7.14 (applied to $p - q$, $\left(\nu_{q+1}, \nu_{q+2}, \ldots, \nu_p\right)$ and ν_{q+i} instead of p, ν and ν_i) yields $\overline{h_{\left(\nu_{q+1}, \nu_{q+2}, \ldots, \nu_p\right)}} \in H_{p-q}$. But Proposition 7.11 (applied to $p - q$ instead of p) yields $L_{p-q} = H_{p-q}$. Thus,

$$\overline{h_{\left(\nu_{q+1}, \nu_{q+2}, \ldots, \nu_p\right)}} \in H_{p-q} = L_{p-q}. \tag{44}$$

Next, we claim that

$$\overline{h_{\nu_i}} \in L_1 C \qquad \text{for each } i \in \{1, 2, \ldots, q-1\}. \tag{45}$$

---

[8] Indeed, in this case we have $\nu_i < 0$ for some $i \in \{1, 2, \ldots, p\}$, and therefore $h_{\nu_i} = 0$ for this i, and therefore

$$h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_p} = \left(h_{\nu_1} h_{\nu_2} \cdots h_{\nu_{i-1}}\right) \underbrace{h_{\nu_i}}_{=0} \left(h_{\nu_{i+1}} h_{\nu_{i+2}} \cdots h_{\nu_p}\right) = 0,$$

and therefore $\operatorname{coeff}_\omega\left(\overline{h_\nu}\right) = 0$, qed.

[*Proof of (45):* Let $i \in \{1, 2, \ldots, q-1\}$. Then, $\nu_i > n$ (since $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_q > n$), so that $\nu_i - n$ is a positive integer. Thus, Corollary 7.22 (applied to $m = \nu_i - n$) yields $\overline{h_{\nu_i}} \in R_{\nu_i - n - 1} C$.

But $\nu_i \leq 2n - k + 1$ (by (42)), so that $\nu_i - n - 1 \leq n - k$. Thus, $R_{\nu_i - n - 1} \subseteq R_{n-k}$ (since $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$). Thus, $R_{\nu_i - n - 1} \subseteq R_{n-k} = L_1$ (by Proposition 7.19). Hence, $\overline{h_{\nu_i}} \in \underbrace{R_{\nu_i - n - 1}}_{\subseteq L_1} C \subseteq L_1 C$. This proves (45).]

From (45), we obtain

$$\overline{h_{\nu_1} h_{\nu_2}} \cdots \overline{h_{\nu_{q-1}}} \in (L_1 C)^{q-1} = \underbrace{(L_1)^{q-1}}_{\substack{\subseteq L_{q-1} \\ \text{(by Corollary 7.16)}}} C^{q-1} \subseteq L_{q-1} C^{q-1}.$$

Also, $\nu_q > n$, so that $\nu_q - n$ is a positive integer. Thus, Corollary 7.22 (applied to $m = \nu_q - n$) yields $\overline{h_{\nu_q}} \in R_{\nu_q - n - 1} C$. But $\nu_q \leq 2n - k - q$ and thus $\nu_q - n - 1 \leq n - k - q - 1$. Hence, $R_{\nu_q - n - 1} \subseteq R_{n-k-q-1}$ (since $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$). Thus, $\overline{h_{\nu_q}} \in \underbrace{R_{\nu_q - n - 1}}_{\subseteq R_{n-k-q-1}} C \subseteq R_{n-k-q-1} C$.

Recall that $h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_p}$. Thus,

$$\begin{aligned}
\overline{h_\nu} &= \overline{h_{\nu_1} h_{\nu_2} \cdots h_{\nu_p}} = \overline{h_{\nu_1} h_{\nu_2}} \cdots \overline{h_{\nu_p}} \\
&= \underbrace{\left(\overline{h_{\nu_1} h_{\nu_2}} \cdots \overline{h_{\nu_{q-1}}} \right)}_{\in L_{q-1} C^{q-1}} \underbrace{\overline{h_{\nu_q}}}_{\in R_{n-k-q-1} C} \underbrace{\left(\overline{h_{\nu_{q+1}} h_{\nu_{q+2}}} \cdots \overline{h_{\nu_p}} \right)}_{\substack{= \overline{h_{\nu_{q+1}} h_{\nu_{q+2}} \cdots h_{\nu_p}} \\ = \overline{h_{\left(\nu_{q+1}, \nu_{q+2}, \ldots, \nu_p\right)}} \in L_{p-q} \\ \text{(by (44))}}} \\
&\in L_{q-1} C^{q-1} R_{n-k-q-1} C L_{p-q} = \underbrace{C^{q-1} C}_{= C^q} \underbrace{R_{n-k-q-1}}_{\substack{\subseteq C^{n-k-q-1} \\ \text{(by Lemma 7.20,} \\ \text{applied to } n-k-q-1 \text{ instead of } p)}} \underbrace{L_{q-1} L_{p-q}}_{\substack{\subseteq L_{(q-1)+(p-q)} \\ \text{(by (38))}}} \\
&\subseteq \underbrace{C^q C^{n-k-q-1}}_{= C^{q+(n-k-q-1)} = C^{n-k-1}} \underbrace{L_{(q-1)+(p-q)}}_{\substack{= L_{p-1} \subseteq Q_0 \\ \text{(by (43))}}} \subseteq C^{n-k-1} Q_0 \\
&\subseteq Q_{0+(n-k-1)}
\end{aligned}$$

(by Corollary 7.28, applied to $n-k-1$ and 0 instead of q and p). In other words, $\overline{h_\nu} \in Q_{n-k-1}$. Hence, $\operatorname{coeff}_\omega \left(\overline{h_\nu} \right) \in \operatorname{coeff}_\omega \left(Q_{n-k-1} \right) = 0$ (by Lemma 6.12), and thus $\operatorname{coeff}_\omega \left(\overline{h_\nu} \right) = 0$. This proves Theorem 7.29. □

Theorem 7.30. Assume that $n > k$. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{Z}^k$ be a k-tuple of integers such that $\gamma \neq \omega$.

Assume that

$$\gamma_i \leq 2n - k - i \qquad \text{for each } i \in \{1, 2, \ldots, k\} . \tag{46}$$

Then, $\operatorname{coeff}_\omega \left(\overline{h_\gamma} \right) = 0$.

Proof of Theorem 7.30. We have $k \neq 0$ [9]. Thus, $k > 0$; hence, γ_1 is well-defined.

If any of the entries $\gamma_1, \gamma_2, \ldots, \gamma_k$ of γ is negative, then Theorem 7.30 holds for easy reasons[10]. Hence, we WLOG assume that none of the entries $\gamma_1, \gamma_2, \ldots, \gamma_k$ of γ is negative. Thus, all of the entries $\gamma_1, \gamma_2, \ldots, \gamma_k$ are nonnegative integers. In other words, $(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{N}^k$.

Let $\nu = (\nu_1, \nu_2, \ldots, \nu_k) \in \mathbb{Z}^k$ be the weakly decreasing permutation of the k-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$. Thus, $h_{\nu_1} h_{\nu_2} \cdots h_{\nu_k} = h_{\gamma_1} h_{\gamma_2} \cdots h_{\gamma_k}$. Hence, $h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_k} = h_{\gamma_1} h_{\gamma_2} \cdots h_{\gamma_k} = h_\gamma$.

Recall that $(\nu_1, \nu_2, \ldots, \nu_k)$ is a permutation of $(\gamma_1, \gamma_2, \ldots, \gamma_k)$. In other words, there exists a permutation $\sigma \in S_k$ such that

$$\left(\nu_i = \gamma_{\sigma(i)} \text{ for each } i \in \{1, 2, \ldots, k\} \right) . \tag{47}$$

Consider this σ.

Recall that $(\nu_1, \nu_2, \ldots, \nu_k)$ is weakly decreasing. Thus, $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k$. Also, $(\nu_1, \nu_2, \ldots, \nu_k) \in \mathbb{N}^k$ (since $(\nu_1, \nu_2, \ldots, \nu_k)$ is a permutation of $(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{N}^k$).

For each $i \in \{1, 2, \ldots, k\}$, we have

$$\begin{aligned} \nu_i &= \gamma_{\sigma(i)} \qquad \text{(by (47))} \\ &\leq 2n - k - \sigma(i) \end{aligned} \tag{48}$$

(by (46), applied to $\sigma(i)$ instead of i).

We are in one of the following two cases:

Case 1: We have $\nu_1 \leq n$.

Case 2: We have $\nu_1 > n$.

---

[9] *Proof.* Assume the contrary. Thus, $k = 0$. Now, $\gamma \in \mathbb{Z}^k = \mathbb{Z}^0$ (since $k = 0$), whence $\gamma = ()$. But $k = 0$ also leads to $\omega = ()$, and thus $\gamma = () = \omega$. But this contradicts $\gamma \neq \omega$. This contradiction shows that our assumption was false. Qed.

[10] Indeed, in this case we have $\gamma_i < 0$ for some $i \in \{1, 2, \ldots, k\}$, and therefore $h_{\gamma_i} = 0$ for this i, and therefore

$$h_\gamma = h_{\gamma_1} h_{\gamma_2} \cdots h_{\gamma_k} = \left(h_{\gamma_1} h_{\gamma_2} \cdots h_{\gamma_{i-1}} \right) \underbrace{h_{\gamma_i}}_{=0} \left(h_{\gamma_{i+1}} h_{\gamma_{i+2}} \cdots h_{\gamma_k} \right) = 0,$$

and therefore $\operatorname{coeff}_\omega \left(\overline{h_\gamma} \right) = 0$, qed.

Let us first consider Case 1. In this case, we have $\nu_1 \leq n$. But recall that $\gamma \neq \omega$. Hence, there exists at least one $q \in \{1, 2, \ldots, k\}$ satisfying $\nu_q \neq n-k$ [11]. Consider such a q.

Next, we claim that

$$\overline{h_{\nu_i}} \in L_1 \qquad \text{for each } i \in \{1, 2, \ldots, k\}. \tag{49}$$

[*Proof of (49):* Let $i \in \{1, 2, \ldots, k\}$. We have $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k$, thus $\nu_i \leq \nu_1 \leq n$. Now, $\nu_i \leq n$ and $\nu_i \in \mathbb{N}$ (since $(\nu_1, \nu_2, \ldots, \nu_k) \in \mathbb{N}^k$). Hence, Lemma 7.23 **(a)** (applied to $j = \nu_i$) yields $\overline{h_{\nu_i}} \in L_1$. This proves (49).]

Also, $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k$, thus $\nu_q \leq \nu_1 \leq n$. Also, $n > k$ and $\nu_q \in \mathbb{N}$ (since $(\nu_1, \nu_2, \ldots, \nu_k) \in \mathbb{N}^k$) and $\nu_q \neq n-k$. Hence, Lemma 7.23 **(b)** (applied to $j = \nu_q$) yields $\overline{h_{\nu_q}} \in R_{n-k-1}$. From $n > k$, we obtain $n-k > 0$, so that $n-k \geq 1$, and thus $n-k-1 \in \mathbb{N}$.

Now, $h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_k} = \prod\limits_{i=1}^{k} h_{\nu_i}$, so that

$$\begin{aligned}
\overline{h_\nu} &= \overline{\prod_{i=1}^{k} h_{\nu_i}} = \prod_{i=1}^{k} \overline{h_{\nu_i}} = \left(\prod_{\substack{i\in\{1,2,\ldots,k\};\\ i\neq q}} \underbrace{\overline{h_{\nu_i}}}_{\substack{\in L_1\\ \text{(by (49))}}} \right) \underbrace{\overline{h_{\nu_q}}}_{\in R_{n-k-1}} \\
&\in \underbrace{\left(\prod_{\substack{i\in\{1,2,\ldots,k\};\\ i\neq q}} L_1 \right)}_{\substack{=(L_1)^{k-1}\subseteq L_{k-1}\\ \text{(by Corollary 7.16)}}} \underbrace{R_{n-k-1}}_{\substack{\subseteq C^{n-k-1}\\ \text{(by Lemma 7.20,}\\ \text{applied to } n-k-1 \text{ instead of } p)}} \subseteq \underbrace{L_{k-1}}_{\substack{=Q_0\\ \text{(by Proposition 7.25)}}} C^{n-k-1} \\
&= Q_0 C^{n-k-1} = C^{n-k-1} Q_0 \subseteq Q_{0+(n-k-1)}
\end{aligned}$$

(by Corollary 7.28, applied to $n-k-1$ and 0 instead of q and p). In other words, $\overline{h_\nu} \in Q_{n-k-1}$. In view of $h_\nu = h_\gamma$, this rewrites as $\overline{h_\gamma} \in Q_{n-k-1}$. Hence, $\operatorname{coeff}_\omega \left(\overline{h_\gamma} \right) \in \operatorname{coeff}_\omega \left(Q_{n-k-1} \right) = 0$ (by Lemma 6.12), and thus $\operatorname{coeff}_\omega \left(\overline{h_\gamma} \right) = 0$. Thus, Theorem 7.30 is proven in Case 1.

Let us now consider Case 2. In this case, we have $\nu_1 > n$. Hence, there exists at least one $r \in \{1, 2, \ldots, k\}$ such that $\nu_r > n$ (namely, $r = 1$). Let q be the **largest** such r. Thus, $\nu_q > n$, but each $r > q$ satisfies $\nu_r \leq n$. Hence,

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_q > n \geq \nu_{q+1} \geq \nu_{q+2} \geq \cdots \geq \nu_k$$

---

[11] *Proof.* Assume the contrary. Thus, $\nu_i = n-k$ for each $i \in \{1, 2, \ldots, k\}$. Now, let $j \in \{1, 2, \ldots, k\}$ be arbitrary. Then, $\nu_{\sigma^{-1}(j)} = n-k$ (since $\nu_i = n-k$ for each $i \in \{1, 2, \ldots, k\}$). But (47) (applied to $i = \sigma^{-1}(j)$) yields $\nu_{\sigma^{-1}(j)} = \gamma_{\sigma\left(\sigma^{-1}(j)\right)} = \gamma_j$. Hence, $\gamma_j = \nu_{\sigma^{-1}(j)} = n-k$. Now, forget that we fixed j. We thus have proven that $\gamma_j = n-k$ for each $j \in \{1, 2, \ldots, k\}$. Hence, $\gamma = (n-k, n-k, \ldots, n-k) = \omega$. This contradicts $\gamma \neq \omega$. This contradiction shows that our assumption was false, qed.

(since $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k$). Also, $\nu_q \leq 2n - k - q$ [12]. Furthermore, (48) shows that

$$\nu_i \leq 2n - k - \underbrace{\sigma(i)}_{\geq -1} \leq 2n - k + 1$$

for each $i \in \{1, 2, \ldots, k\}$. Hence, Theorem 7.29 (applied to $p = k$) yields $\operatorname{coeff}_\omega \left(\overline{h_\nu}\right) = 0$. In view of $h_\nu = h_\gamma$, this rewrites as $\operatorname{coeff}_\omega \left(\overline{h_\gamma}\right) = 0$. Thus, Theorem 7.30 is proven in Case 2.

We have now proven Theorem 7.30 in both Cases 1 and 2. Hence, Theorem 7.30 always holds. $\square$

7.9. A criterion for $\operatorname{coeff}_\omega \left(\overline{s_\lambda}\right) = 0$

Theorem 7.31. Let λ be a partition with at most k parts. Assume that $\lambda_1 \leq 2(n-k)$ and $\lambda \neq \omega$. Then, $\operatorname{coeff}_\omega \left(\overline{s_\lambda}\right) = 0$.

Proof of Theorem 7.31. We have $n > k$ [13].

We have $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ (since the partition λ has at most k parts). Proposition 5.7 **(a)** yields

$$s_\lambda = \det\left(\left(h_{\lambda_u - u + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right) = \sum_{\sigma \in S_k} (-1)^\sigma \prod_{i=1}^{k} h_{\lambda_i - i + \sigma(i)}$$

(by the definition of a determinant). Hence,

$$\overline{s_\lambda} = \overline{\sum_{\sigma \in S_k} (-1)^\sigma \prod_{i=1}^{k} h_{\lambda_i - i + \sigma(i)}} = \sum_{\sigma \in S_k} (-1)^\sigma \overline{\prod_{i=1}^{k} h_{\lambda_i - i + \sigma(i)}}. \tag{50}$$

---

[12] *Proof.* Assume the contrary. Thus, $\nu_q > 2n - k - q$.

The map σ is a permutation, and thus injective. Hence, $|\sigma(\{1, 2, \ldots, q\})| = |\{1, 2, \ldots, q\}| = q$. Thus, $\sigma(\{1, 2, \ldots, q\})$ cannot be a subset of $\{1, 2, \ldots, q-1\}$ (because this would lead to $|\sigma(\{1, 2, \ldots, q\})| \leq |\{1, 2, \ldots, q-1\}| = q - 1 < q$, which would contradict $|\sigma(\{1, 2, \ldots, q\})| = q$). In other words, not every $i \in \{1, 2, \ldots, q\}$ satisfies $\sigma(i) \in \{1, 2, \ldots, q-1\}$. In other words, there exists some $i \in \{1, 2, \ldots, q\}$ that satisfies $\sigma(i) \notin \{1, 2, \ldots, q-1\}$. Consider such an i.

From $i \in \{1, 2, \ldots, q\}$, we obtain $i \leq q$ and thus $\nu_i \geq \nu_q$ (since $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k$). From $\sigma(i) \notin \{1, 2, \ldots, q-1\}$, we obtain $\sigma(i) > q - 1$, so that $\sigma(i) \geq q$. Now, (48) yields $\nu_i \leq 2n - k - \underbrace{\sigma(i)}_{\geq q} \leq 2n - k - q$; but this contradicts $\nu_i \geq \nu_q > 2n - k - q$. This contradiction shows that our assumption was false, qed.

[13] *Proof.* Assume the contrary. Thus, $n \leq k$ and therefore $n = k$ (since $n \geq k$). Hence, $n - k = 0$. Thus, $\lambda_1 \leq 2\underbrace{(n-k)}_{=0} = 0$, so that $\lambda_1 = 0$ and thus $\lambda = \varnothing$ (since λ is a partition). But from $n - k = 0$, we also obtain $\omega = \varnothing$ (since $\omega = (n-k, n-k, \ldots, n-k)$). Thus, $\lambda = \varnothing = \omega$. But this contradicts $\lambda \neq \omega$. This contradiction shows that our assumption was wrong, qed.

Now, we claim that each $\sigma \in S_k$ satisfies

$$\operatorname{coeff}_\omega \left(\overline{\prod_{i=1}^{k} h_{\lambda_i - i + \sigma(i)}} \right) = 0. \tag{51}$$

[*Proof of (51):* Let $\sigma \in S_k$. Define a k-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{Z}^k$ of integers by

$$(\gamma_i = \lambda_i - i + \sigma(i) \qquad \text{for each } i \in \{1, 2, \ldots, k\}). \tag{52}$$

Then, $\gamma \neq \omega$ [14]. Moreover,

$$\gamma_i \leq 2n - k - i \qquad \text{for each } i \in \{1, 2, \ldots, k\}$$

[15]. Hence, Theorem 7.30 yields $\operatorname{coeff}_\omega \left(\overline{h_\gamma} \right) = 0$. In view of

$$h_\gamma = h_{\gamma_1} h_{\gamma_2} \cdots h_{\gamma_k} = \prod_{i=1}^{k} \underbrace{h_{\gamma_i}}_{\substack{= h_{\lambda_i - i + \sigma(i)} \\ \text{(by (52))}}} = \prod_{i=1}^{k} h_{\lambda_i - i + \sigma(i)},$$

this rewrites as $\operatorname{coeff}_\omega \left(\overline{\prod_{i=1}^{k} h_{\lambda_i - i + \sigma(i)}} \right) = 0$. Thus, (51) is proven.]

---

[14] *Proof.* Assume the contrary. Thus, $\gamma = \omega$.

Let $i \in \{1, 2, \ldots, k-1\}$. From $\gamma = \omega$, we obtain $\gamma_i = \omega_i = n - k$. Comparing this with (52), we find $\lambda_i - i + \sigma(i) = n - k$. The same argument (applied to $i+1$ instead of i) yields $\lambda_{i+1} - (i+1) + \sigma(i+1) = n - k$. But $\lambda_i \geq \lambda_{i+1}$ (since λ is a partition). Hence,

$$\underbrace{\lambda_i}_{\geq \lambda_{i+1}} - \underbrace{i}_{< i+1} + \sigma(i+1) > \lambda_{i+1} - (i+1) + \sigma(i+1) = n - k = \lambda_i - i + \sigma(i)$$

(since $\lambda_i - i + \sigma(i) = n - k$). If we subtract $\lambda_i - i$ from this inequality, we obtain $\sigma(i+1) > \sigma(i)$. In other words, $\sigma(i) < \sigma(i+1)$.

Now, forget that we fixed i. We thus have shown that each $i \in \{1, 2, \ldots, k-1\}$ satisfies $\sigma(i) < \sigma(i+1)$. In other words, we have $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$. Hence, σ is a strictly increasing map from $\{1, 2, \ldots, k\}$ to $\{1, 2, \ldots, k\}$. But the only such map is id. Thus, $\sigma = \operatorname{id}$. Hence, for each $i \in \{1, 2, \ldots, k\}$, we have

$$\begin{aligned} \gamma_i &= \lambda_i - i + \underbrace{\sigma}_{= \operatorname{id}}(i) \qquad \text{(by (52))} \\ &= \lambda_i - i + \operatorname{id}(i) = \lambda_i - i + i = \lambda_i. \end{aligned}$$

Thus, $\gamma = \lambda$. Comparing this with $\gamma = \omega$, we obtain $\lambda = \omega$. This contradicts $\lambda \neq \omega$. This contradiction shows that our assumption was wrong, qed.

[15] *Proof.* Let $i \in \{1, 2, \ldots, k\}$. Then, $\lambda_1 \geq \lambda_i$ (since λ is a partition), so that $\lambda_i \leq \lambda_1 \leq 2(n-k)$. Now, (52) yields

$$\gamma_i = \underbrace{\lambda_i}_{\leq 2(n-k)} - i + \underbrace{\sigma(i)}_{\leq k} \leq 2(n-k) - i + k = 2n - k - i,$$

qed.

From (50), we obtain

$$\begin{aligned}\operatorname{coeff}_\omega\left(\overline{s_\lambda}\right) &= \operatorname{coeff}_\omega\left(\sum_{\sigma\in S_k}(-1)^\sigma \overline{\prod_{i=1}^k h_{\lambda_i-i+\sigma(i)}}\right)\\ &= \sum_{\sigma\in S_k}(-1)^\sigma \underbrace{\operatorname{coeff}_\omega\left(\overline{\prod_{i=1}^k h_{\lambda_i-i+\sigma(i)}}\right)}_{\substack{=0\\ \text{(by (51))}}} = 0.\end{aligned}$$

This proves Theorem 7.31. □

8. Another proof of Theorem 6.3

We can use Theorem 7.31 to obtain a second proof of Theorem 6.3. To that end, we shall use a few more basic facts about Littlewood-Richardson coefficients. First we introduce a few notations (only for this section):

Convention 8.1. Convention 6.1 remains in place for the whole Section 8.

We shall also use all the notations introduced in Section 6.

8.1. Some basics on Littlewood-Richardson coefficients

Definition 8.2. Let $a \in \mathbb{N}$.

(a) We let Par_a denote the set of all partitions with size a. (That is, $\operatorname{Par}_a = \{\lambda \text{ is a partition} \mid |\lambda| = a\}$.)

(b) If λ and μ are two partitions with size a, then we write $\lambda \rhd \mu$ if and only if we have

$$\lambda_1+\lambda_2+\cdots+\lambda_i \geq \mu_1+\mu_2+\cdots+\mu_i \qquad \text{for each } i \in \{1,2,\ldots,a\}.$$

This defines a binary relation $\rhd$ on Par_a. This relation is the smaller-or-equal relation of a partial order on Par_a, which is called the *dominance order*.

Definition 8.3. Let μ and ν be two partitions. Then, we define two new partitions $\mu+\nu$ and $\mu \sqcup \nu$ as follows:

- The partition $\mu+\nu$ is defined as $(\mu_1+\nu_1, \mu_2+\nu_2, \mu_3+\nu_3, \ldots)$.
- The partition $\mu \sqcup \nu$ is defined as the result of sorting the list $\left(\mu_1,\mu_2,\ldots,\mu_{\ell(\mu)},\nu_1,\nu_2,\ldots,\nu_{\ell(\nu)}\right)$ in decreasing order.

We shall use the following fact:

Proposition 8.4. Let $a \in \mathbb{N}$ and $b \in \mathbb{N}$ be such that $a \leq b$. Let $\mu \in \operatorname{Par}_a$, $\nu \in \operatorname{Par}_{b-a}$ and $\lambda \in \operatorname{Par}_b$ be such that $c^{\lambda}_{\mu,\nu} \neq 0$. Then, $\mu + \nu \rhd \lambda \rhd \mu \sqcup \nu$.

Proposition 8.4 is precisely [GriRei18, Exercise 2.9.17(c)] (with k and n renamed as a and b).

Corollary 8.5. Let λ, μ and ν be three partitions such that $\lambda_1 > \mu_1 + \nu_1$. Then, $c^{\lambda}_{\mu,\nu} = 0$.

Proof of Corollary 8.5. Assume the contrary. Thus, $c^{\lambda}_{\mu,\nu} \neq 0$.

Let $a = |\mu|$; thus, $\mu \in \operatorname{Par}_a$. Let $b = |\lambda|$; thus, $\lambda \in \operatorname{Par}_b$.

Proposition 6.17 **(c)** shows that $c^{\lambda}_{\mu,\nu} = 0$ unless $|\mu| + |\nu| = |\lambda|$. Hence, $|\mu| + |\nu| = |\lambda|$ (since $c^{\lambda}_{\mu,\nu} \neq 0$). Thus, $|\nu| = \underbrace{|\lambda|}_{=b} - \underbrace{|\mu|}_{=a} = b - a$. Hence, $b - a = |\nu| \geq 0$, so that $a \leq b$. Also, from $|\nu| = b - a$, we obtain $\nu \in \operatorname{Par}_{b-a}$. Thus, Proposition 8.4 yields $\mu + \nu \rhd \lambda \rhd \mu \sqcup \nu$.

But $b = |\lambda| \geq \lambda_1 > \mu_1 + \nu_1 \geq 0$, so that $1 \in \{1, 2, \ldots, b\}$.

Now, from $\mu + \nu \rhd \lambda$, we conclude that

$$(\mu + \nu)_1 + (\mu + \nu)_2 + \cdots + (\mu + \nu)_i \geq \lambda_1 + \lambda_2 + \cdots + \lambda_i \qquad \text{for each } i \in \{1, 2, \ldots, b\}$$

(by the definition of the relation $\rhd$, since $\mu + \nu$ and λ are two partitions of size b). Applying this to $i = 1$, we obtain $(\mu + \nu)_1 \geq \lambda_1$ (since $1 \in \{1, 2, \ldots, b\}$). But the definition of $\mu + \nu$ yields $(\mu + \nu)_1 = \mu_1 + \nu_1 < \lambda_1$ (since $\lambda_1 > \mu_1 + \nu_1$). This contradicts $(\mu + \nu)_1 \geq \lambda_1$. This contradiction shows that our assumption was false. Hence, Corollary 8.5 is proven. ☐

Next, we recall the Littlewood-Richardson rule itself:

Proposition 8.6. Let λ and μ be two partitions. Then,

$$\mathbf{s}_\lambda \mathbf{s}_\mu = \sum_{\rho \text{ is a partition}} c^{\rho}_{\lambda,\mu} \mathbf{s}_\rho .$$

Proposition 8.6 is precisely [GriRei18, (2.5.6)] (with λ, μ and ν renamed as ρ, λ and μ).

Corollary 8.7. Let $\lambda \in P_{k,n}$ and $\mu \in P_{k,n}$. Then,

$$s_\lambda s_\mu = \sum_{\substack{\rho \text{ is a partition with at most } k \text{ parts;} \\ \rho_1 \leq 2(n-k)}} c^{\rho}_{\lambda,\mu} s_\rho .$$

Proof of Corollary 8.7. If ρ is a partition satisfying $\rho_1 > 2(n-k)$, then

$$c^{\rho}_{\lambda,\mu} = 0. \tag{53}$$

[*Proof of (53):* Let ρ be a partition satisfying $\rho_1 > 2(n-k)$.

We have $\lambda \in P_{k,n}$; thus, each part of λ is $\leq n-k$. Thus, $\lambda_1 \leq n-k$. Similarly, $\mu_1 \leq n-k$. Hence, $\underbrace{\lambda_1}_{\leq n-k} + \underbrace{\mu_1}_{\leq n-k} \leq 2(n-k) < \rho_1$. In other words, $\rho_1 > \lambda_1 + \mu_1$. Hence, Corollary 8.5 (applied to ρ, λ and μ instead of λ, μ and ν) yields $c^{\rho}_{\lambda,\mu} = 0$. This proves (53).]

Proposition 8.6 yields

$$\mathbf{s}_\lambda \mathbf{s}_\mu = \sum_{\rho \text{ is a partition}} c^{\rho}_{\lambda,\mu} \mathbf{s}_\rho .$$

This is an equality in Λ. Evaluating both of its sides at the k indeterminates $x_1, x_2, \ldots, x_k$, we find

$$\begin{aligned}
s_\lambda s_\mu &= \sum_{\rho \text{ is a partition}} c^{\rho}_{\lambda,\mu} s_\rho \\
&= \sum_{\substack{\rho \text{ is a partition;} \\ \rho_1 \leq 2(n-k)}} c^{\rho}_{\lambda,\mu} s_\rho + \sum_{\substack{\rho \text{ is a partition;} \\ \rho_1 > 2(n-k)}} \underbrace{c^{\rho}_{\lambda,\mu}}_{\substack{=0 \\ \text{(by (53))}}} s_\rho \\
&\qquad \left(\begin{array}{c} \text{since each partition } \rho \text{ satisfies either } \rho_1 \leq 2(n-k) \\ \text{or } \rho_1 > 2(n-k) \text{ (but not both)} \end{array} \right) \\
&= \sum_{\substack{\rho \text{ is a partition;} \\ \rho_1 \leq 2(n-k)}} c^{\rho}_{\lambda,\mu} s_\rho \\
&= \sum_{\substack{\rho \text{ is a partition with at most } k \text{ parts;} \\ \rho_1 \leq 2(n-k)}} c^{\rho}_{\lambda,\mu} s_\rho + \sum_{\substack{\rho \text{ is a partition with more than } k \text{ parts;} \\ \rho_1 \leq 2(n-k)}} c^{\rho}_{\lambda,\mu} \underbrace{s_\rho}_{\substack{=0 \\ \text{(by (3))}}} \\
&= \sum_{\substack{\rho \text{ is a partition with at most } k \text{ parts;} \\ \rho_1 \leq 2(n-k)}} c^{\rho}_{\lambda,\mu} s_\rho .
\end{aligned}$$

This proves Corollary 8.7. ☐

Next, let us recall another known fact on skew Schur functions:

Proposition 8.8. Let λ be any partition. Then, $\mathbf{s}_{\omega/\lambda^\vee} = \mathbf{s}_\lambda$.

Proof of Proposition 8.8. From [GriRei18, Exercise 2.9.15(a)] (applied to $n-k$ and $\varnothing$ instead of m and μ), we obtain $\mathbf{s}_{\lambda/\varnothing} = \mathbf{s}_{\varnothing^\vee/\lambda^\vee}$. In view of $\varnothing^\vee = \omega$, this rewrites as $\mathbf{s}_{\lambda/\varnothing} = \mathbf{s}_{\omega/\lambda^\vee}$. Thus, $\mathbf{s}_{\omega/\lambda^\vee} = \mathbf{s}_{\lambda/\varnothing} = \mathbf{s}_\lambda$. This proves Proposition 8.8. ☐

Corollary 8.9. Let λ and μ be two partitions. Then,

$$c_{\lambda,\mu}^{\omega} = \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \mu = \lambda^{\vee}; \\ 0, & \text{else} \end{cases}.$$

Proof of Corollary 8.9. Proposition 6.17 **(a)** (applied to ω and λ instead of λ and μ) shows that

$$\mathbf{s}_{\omega/\lambda} = \sum_{\nu \text{ is a partition}} c_{\lambda,\nu}^{\omega} \mathbf{s}_{\nu}. \tag{54}$$

On the other hand, it is easy to see that

$$\mathbf{s}_{\omega/\lambda} = \sum_{\nu \text{ is a partition}} \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^{\vee}; \\ 0, & \text{else} \end{cases} \mathbf{s}_{\nu}. \tag{55}$$

[*Proof of (55):* We are in one of the following two cases:

Case 1: We have $\lambda \in P_{k,n}$.

Case 2: We have $\lambda \notin P_{k,n}$.

Let us first consider Case 1. In this case, we have $\lambda \in P_{k,n}$. Thus, $\lambda^{\vee}$ is well-defined, and we have $\left(\lambda^{\vee}\right)^{\vee} = \lambda$. Hence, Proposition 8.8 (applied to $\lambda^{\vee}$ instead of λ) yields

$$\mathbf{s}_{\omega/\left(\lambda^{\vee}\right)^{\vee}} = \mathbf{s}_{\lambda^{\vee}} = \sum_{\nu \text{ is a partition}} \underbrace{\begin{cases} 1, & \text{if } \nu = \lambda^{\vee}; \\ 0, & \text{else} \end{cases}}_{\substack{= \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^{\vee}; \\ 0, & \text{else} \end{cases} \\ \text{(since } \lambda \in P_{k,n} \text{ holds)}}} \mathbf{s}_{\nu}$$

$$= \sum_{\nu \text{ is a partition}} \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^{\vee}; \\ 0, & \text{else} \end{cases} \mathbf{s}_{\nu}.$$

In view of $\left(\lambda^{\vee}\right)^{\vee} = \lambda$, this rewrites as

$$\mathbf{s}_{\omega/\lambda} = \sum_{\nu \text{ is a partition}} \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^{\vee}; \\ 0, & \text{else} \end{cases} \mathbf{s}_{\nu}.$$

Thus, (55) is proven in Case 1.

Now, let us consider Case 2. In this case, we have $\lambda \notin P_{k,n}$. Hence, $\lambda \not\subseteq \omega$ (since $\lambda \subseteq \omega$ holds if and only if $\lambda \in P_{k,n}$). Thus, $\mathbf{s}_{\omega/\lambda} = 0$. Comparing this with

$$\sum_{\nu \text{ is a partition}} \underbrace{\begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^{\vee}; \\ 0, & \text{else} \end{cases}}_{\substack{=0 \\ \text{(since } \lambda \notin P_{k,n}\text{)}}} \mathbf{s}_{\nu} = 0,$$

we obtain

$$\mathbf{s}_{\omega/\lambda} = \sum_{\nu \text{ is a partition}} \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^\vee; \\ 0, & \text{else} \end{cases} \mathbf{s}_\nu .$$

Thus, (55) is proven in Case 2.

We have now proven (55) in each of the two Cases 1 and 2. Thus, (55) always holds.]

Now, comparing (55) with (54), we obtain

$$\sum_{\nu \text{ is a partition}} c^{\omega}_{\lambda,\nu} \mathbf{s}_\nu = \sum_{\nu \text{ is a partition}} \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^\vee; \\ 0, & \text{else} \end{cases} \mathbf{s}_\nu .$$

Since the family $(\mathbf{s}_\nu)_{\nu \text{ is a partition}}$ is a basis of the $\mathbf{k}$-module Λ, we can compare the coefficients of $\mathbf{s}_\mu$ on both sides of this equality. We thus obtain

$$c^{\omega}_{\lambda,\mu} = \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \mu = \lambda^\vee; \\ 0, & \text{else} \end{cases} .$$

This proves Corollary 8.9. □

8.2. Another proof of Theorem 6.3

We are now ready to prove Theorem 6.3 again. More precisely, we shall prove Lemma 6.22 (as we know that Theorem 6.3 quickly follows from Lemma 6.22).

Second proof of Lemma 6.22. If $k = 0$, then Lemma 6.22 holds[16]. Hence, for the rest of this proof, we WLOG assume that $k \neq 0$. Thus, $k > 0$. Hence, $\omega_1 = n - k \leq 2(n-k)$. Thus, ω is a partition ρ with at most k parts that satisfies $\rho_1 \leq 2(n-k)$ (since $\omega_1 \leq 2(n-k)$).

Corollary 8.7 yields

$$s_\lambda s_\mu = \sum_{\substack{\rho \text{ is a partition with at most } k \text{ parts;} \\ \rho_1 \leq 2(n-k)}} c^{\rho}_{\lambda,\mu} s_\rho .$$

Hence,

$$\overline{s_\lambda s_\mu} = \overline{\sum_{\substack{\rho \text{ is a partition with at most } k \text{ parts;} \\ \rho_1 \leq 2(n-k)}} c^{\rho}_{\lambda,\mu} s_\rho} = \sum_{\substack{\rho \text{ is a partition with at most } k \text{ parts;} \\ \rho_1 \leq 2(n-k)}} c^{\rho}_{\lambda,\mu} \overline{s_\rho} .$$

---

[16] *Proof.* Assume that $k = 0$. Then, $P_{k,n} = \{\varnothing\}$, so that $\lambda \in P_{k,n} = \{\varnothing\}$ and thus $\lambda = \varnothing$. Similarly, $\mu = \varnothing$. Therefore, $\lambda = \mu^\vee$ holds. Also, $\omega = \varnothing$. Moreover, from $\lambda = \varnothing$, we obtain $s_\lambda = s_\varnothing = 1$; similarly, $s_\mu = 1$. Thus, $\underbrace{s_\lambda}_{=1} \underbrace{s_\mu}_{=1} = 1 = s_\varnothing = s_\omega$ (since $\varnothing = \omega$). Hence, $\operatorname{coeff}_\omega(\overline{s_\lambda s_\mu}) = \operatorname{coeff}_\omega(\overline{s_\omega}) = 1$. Comparing this with $\begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee \end{cases} = 1$ (since $\lambda = \mu^\vee$ holds), we obtain $\operatorname{coeff}_\omega(\overline{s_\lambda s_\mu}) = \begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee \end{cases}$. Thus, Lemma 6.22 holds. Qed.

Thus,

$$\begin{aligned}
\operatorname{coeff}_{\omega}\left(\overline{s_\lambda s_\mu}\right) &= \operatorname{coeff}_{\omega}\left(\sum_{\substack{\rho \text{ is a partition with at most } k \text{ parts;}\\ \rho_1 \leq 2(n-k)}} c^{\rho}_{\lambda,\mu}\overline{s_\rho}\right)\\
&= \sum_{\substack{\rho \text{ is a partition with at most } k \text{ parts;}\\ \rho_1 \leq 2(n-k)}} c^{\rho}_{\lambda,\mu}\operatorname{coeff}_{\omega}\left(\overline{s_\rho}\right)\\
&= c^{\omega}_{\lambda,\mu}\operatorname{coeff}_{\omega}\left(\overline{s_\omega}\right) + \sum_{\substack{\rho \text{ is a partition with at most } k \text{ parts;}\\ \rho_1 \leq 2(n-k);\\ \rho\neq\omega}} c^{\rho}_{\lambda,\mu}\underbrace{\operatorname{coeff}_{\omega}\left(\overline{s_\rho}\right)}_{\substack{=0\\ \text{(by Theorem 7.31,}\\ \text{applied to } \rho \text{ instead of } \lambda)}}\\
&\qquad\left(\begin{array}{c}\text{here, we have split off the addend for } \rho=\omega\\ \text{from the sum, since } \omega \text{ is a partition } \rho \text{ with}\\ \text{at most } k \text{ parts that satisfies } \rho_1 \leq 2(n-k)\end{array}\right)\\
&= c^{\omega}_{\lambda,\mu}\underbrace{\operatorname{coeff}_{\omega}\left(\overline{s_\omega}\right)}_{\substack{=1\\ \text{(by the definition of } \operatorname{coeff}_{\omega})}}\\
&= c^{\omega}_{\lambda,\mu} = \begin{cases}1, & \text{if } \lambda\in P_{k,n} \text{ and } \mu=\lambda^{\vee};\\ 0, & \text{else}\end{cases} \qquad \text{(by Corollary 8.9)}\\
&= \begin{cases}1, & \text{if } \mu=\lambda^{\vee};\\ 0, & \text{if } \mu\neq\lambda^{\vee}\end{cases} \qquad \text{(since } \lambda\in P_{k,n} \text{ holds)}\\
&= \begin{cases}1, & \text{if } \lambda=\mu^{\vee};\\ 0, & \text{if } \lambda\neq\mu^{\vee}\end{cases}
\end{aligned}$$

(since $\mu = \lambda^{\vee}$ holds if and only if $\lambda = \mu^{\vee}$). Thus, Lemma 6.22 is proven again. □

9. Pieri rules for multiplying by $\overline{h_j}$

Convention 9.1. Convention 6.1 remains in place for the whole Section 9. We shall also use all the notations introduced in Section 6.

In this section, we shall explore formulas for expanding products of the form $\overline{s_\lambda h_j}$ in the basis $\left(\overline{s_\mu}\right)_{\mu\in P_{k,n}}$. We begin with the simplest case – that of $j = 1$:

9.1. Multiplying by $\overline{h_1}$

Proposition 9.2. Let $\lambda \in P_{k,n}$. Assume that $k > 0$.

(a) If $\lambda_1 < n - k$, then

$$\overline{s_\lambda h_1} = \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a single box}}} \overline{s_\mu}.$$

(b) Let $\overline{\lambda}$ be the partition $(\lambda_2, \lambda_3, \lambda_4, \ldots)$. If $\lambda_1 = n - k$, then

$$\overline{s_\lambda h_1} = \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a single box}}} \overline{s_\mu} + \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \in P_{k,n};\\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}.$$

Proof of Proposition 9.2. We have $\mathbf{h}_1 = \mathbf{e}_1$, thus

$$\mathbf{s}_\lambda \mathbf{h}_1 = \mathbf{s}_\lambda \mathbf{e}_1 = \sum_{\substack{\mu \text{ is a partition};\\ \mu/\lambda \text{ is a vertical 1-strip}}} \mathbf{s}_\mu$$

(by Proposition 6.6, applied to $i = 1$). Evaluating both sides of this identity at the k variables $x_1, x_2, \ldots, x_k$, we find

$$s_\lambda h_1 = \sum_{\substack{\mu \text{ is a partition};\\ \mu/\lambda \text{ is a vertical 1-strip}}} s_\mu = \sum_{\substack{\mu \text{ is a partition};\\ \mu/\lambda \text{ is a single box}}} s_\mu$$

(because a skew diagram μ/λ is a vertical 1-strip if and only if it is a single box). This becomes

$$\begin{aligned} s_\lambda h_1 &= \sum_{\substack{\mu \text{ is a partition};\\ \mu/\lambda \text{ is a single box}}} s_\mu \\ &= \sum_{\substack{\mu \text{ is a partition};\\ \mu/\lambda \text{ is a single box};\\ \mu \text{ has at most } k \text{ parts}}} s_\mu + \sum_{\substack{\mu \text{ is a partition};\\ \mu/\lambda \text{ is a single box};\\ \mu \text{ has more than } k \text{ parts}}} \underbrace{s_\mu}_{\substack{=0\\ \text{(by (3)}\\ \text{(applied to } \mu \text{ instead of } \lambda))}} \\ &= \sum_{\substack{\mu \text{ is a partition};\\ \mu/\lambda \text{ is a single box};\\ \mu \text{ has at most } k \text{ parts}}} s_\mu. \end{aligned} \tag{56}$$

(a) Assume that $\lambda_1 < n - k$. Then, each partition μ satisfying

$$(\mu/\lambda \text{ is a single box}) \wedge (\mu \text{ has at most } k \text{ parts}) \tag{57}$$

must satisfy

$$\mu \in P_{k,n}. \tag{58}$$

[*Proof of (58):* Let μ be a partition satisfying (57). We must prove that $\mu \in P_{k,n}$. We have $\mu_1 \leq \lambda_1 + 1$ (since μ/λ is a single box) and thus $\mu_1 \leq \lambda_1 + 1 \leq n - k$ (since $\lambda_1 < n - k$). Hence, each part of μ is $\leq n - k$ (since μ is a partition). Thus, $\mu \in P_{k,n}$ (since μ has at most k parts). This proves (58).]

Now, (56) becomes

$$s_\lambda h_1 = \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a single box;} \\ \mu \text{ has at most } k \text{ parts}}} s_\mu = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}} s_\mu$$

(because (58) yields the equality $\sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a single box;} \\ \mu \text{ has at most } k \text{ parts}}} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}}$ of summation signs). Projecting both sides of this equality onto $\mathcal{S}/I$, we obtain

$$\overline{s_\lambda h_1} = \overline{\sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}} s_\mu} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}} \overline{s_\mu}.$$

This proves Proposition 9.2 **(a)**.

(b) Assume that $\lambda_1 = n - k$. Let ν be the partition $(\lambda_1 + 1, \lambda_2, \lambda_3, \ldots)$. Then, ν/λ is a single box, which lies in the first row. The definition of ν yields $\nu_1 = \lambda_1 + 1 = n - k + 1$ (since $\lambda_1 = n - k$) and thus $\nu_1 > n - k$; hence, not all parts of ν are $\leq n - k$. Thus, $\nu \notin P_{k,n}$.

Clearly, $\overline{\lambda} \in P_{k,n}$. Hence, if $i \in \mathbb{N}$, and if μ is any partition such that $\overline{\lambda}/\mu$ is a vertical i-strip, then $\mu \in P_{k,n}$ (since $\mu \subseteq \overline{\lambda}$). Thus, for each $i \in \mathbb{N}$, we have the following equality of summation signs:

$$\sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} = \sum_{\substack{\mu \in P_{k,n}; \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} . \tag{59}$$

The partition ν has at most k parts (since λ has at most k parts, and since $k > 0$). The definition of ν yields $\nu_1 = \lambda_1 + 1 = n - k + 1$ (since $\lambda_1 = n - k$) and $(\nu_2, \nu_3, \nu_4, \ldots) = (\lambda_2, \lambda_3, \lambda_4, \ldots) = \overline{\lambda}$. Hence, Lemma 6.13 (applied to ν and ν_i instead of λ and λ_i) yields

$$\overline{s_\nu} = \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} = \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \in P_{k,n}; \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \tag{60}$$

(by (59)).

Each partition μ satisfying

$$(\mu/\lambda \text{ is a single box}) \wedge (\mu \text{ has at most } k \text{ parts}) \wedge (\mu \neq \nu) \tag{61}$$

must satisfy

$$\mu \in P_{k,n}. \tag{62}$$

[*Proof of (62):* Let μ be a partition satisfying (61). We must prove that $\mu \in P_{k,n}$.

We know that μ / λ is a single box. If we had $\mu_1 > \lambda_1$, then this box would lie in the first row, which would yield that $\mu = \nu$ (because ν is the partition obtained from λ by adding a box in the first row); but this would contradict $\mu \neq \nu$. Hence, we cannot have $\mu_1 > \lambda_1$. Thus, we have $\mu_1 \leq \lambda_1 = n-k$. Hence, each part of μ is $\leq n-k$ (since μ is a partition). Thus, $\mu \in P_{k,n}$ (since μ has at most k parts). This proves (62).]

Conversely, each $\mu \in P_{k,n}$ satisfies $\mu \neq \nu$ (because $\nu \notin P_{k,n}$) and has at most k parts. Combining this with (62), we obtain the following equality of summation signs:

$$\sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a single box;} \\ \mu \text{ has at most } k \text{ parts;} \\ \mu \neq \nu}} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}} . \tag{63}$$

Now, (56) becomes

$$\begin{aligned} s_\lambda h_1 &= \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a single box;} \\ \mu \text{ has at most } k \text{ parts}}} s_\mu = s_\nu + \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a single box;} \\ \mu \text{ has at most } k \text{ parts;} \\ \mu \neq \nu}} s_\mu \\ &\qquad \left(\begin{array}{c} \text{here, we have split off the addend for } \mu = \nu \text{ from the sum} \\ \text{(since } \nu/\lambda \text{ is a single box, and since } \nu \text{ has at most } k \text{ parts)} \end{array} \right) \\ &= s_\nu + \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}} s_\mu \qquad (\text{by (63)}) . \end{aligned}$$

Projecting both sides of this equality onto $\mathcal{S}/I$, we obtain

$$\begin{aligned} \overline{s_\lambda h_1} &= \overline{s_\nu + \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}} s_\mu} = \overline{s_\nu} + \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}} \overline{s_\mu} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}} \overline{s_\mu} + \overline{s_\nu} \\ &= \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}} \overline{s_\mu} + \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \in P_{k,n}; \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \end{aligned}$$

(by (60)). This proves Proposition 9.2 **(b)**. □

9.2. Multiplying by $\overline{h_{n-k}}$

On the other end of the spectrum is the case of $j = n-k$; this case also turns out to have a simple answer:

Proposition 9.3. Let $\lambda \in P_{k,n}$. Assume that $k > 0$.

(a) We have

$$\overline{s_\lambda h_{n-k}} = \overline{s_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)}} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\substack{\mu \in P_{k,n};\\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}.$$

(b) If $\lambda_k > 0$, then

$$\overline{s_\lambda h_{n-k}} = - \sum_{i=1}^{k} (-1)^i a_i \sum_{\substack{\mu \in P_{k,n};\\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}.$$

Proof of Proposition 9.3. We have $\lambda \in P_{k,n}$, thus $\lambda_1 \leq n-k$. Hence, $n-k \geq \lambda_1$. Thus, $(n-k, \lambda_1, \lambda_2, \lambda_3, \ldots)$ is a partition.

(a) We have

$$(\mathbf{e}_i)^\perp \mathbf{s}_\lambda = 0 \qquad \text{for every integer } i > k. \tag{64}$$

[*Proof of (64):* Let $i > k$ be an integer. The partition λ has at most k parts (since $\lambda \in P_{k,n}$). In other words, the Young diagram of λ contains at most k rows. Hence, this diagram contains no vertical i-strip (since a vertical i-strip would involve more than k rows (beucase $i > k$)). Thus, there exists no partition μ such that λ/μ is a vertical i-strip. Hence, $\sum_{\substack{\mu \text{ is a partition};\\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu = (\text{empty sum}) = 0$. But Corollary 6.7 yields $(\mathbf{e}_i)^\perp \mathbf{s}_\lambda = \sum_{\substack{\mu \text{ is a partition};\\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu = 0$. This proves (64).]

Recall that $\mathbf{e}_0 = 1$ and thus $(\mathbf{e}_0)^\perp = 1^\perp = \mathrm{id}$. Hence, $(\mathbf{e}_0)^\perp \mathbf{s}_\lambda = \mathrm{id}\, \mathbf{s}_\lambda = \mathbf{s}_\lambda$.

But $n-k \geq \lambda_1$. Hence, Proposition 6.8 (applied to $m = n-k$) yields

$$\sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{n-k+i} (\mathbf{e}_i)^\perp \mathbf{s}_\lambda = \mathbf{s}_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)}.$$

Hence,

$$\begin{aligned}
\mathbf{s}_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)} &= \sum_{i\in\mathbb{N}} (-1)^i \mathbf{h}_{n-k+i} (\mathbf{e}_i)^\perp \mathbf{s}_\lambda \\
&= \sum_{i=0}^{k} (-1)^i \mathbf{h}_{n-k+i} (\mathbf{e}_i)^\perp \mathbf{s}_\lambda + \sum_{i=k+1}^{\infty} (-1)^i \mathbf{h}_{n-k+i} \underbrace{(\mathbf{e}_i)^\perp \mathbf{s}_\lambda}_{\substack{=0 \\ \text{(by (64))}}} \\
&= \sum_{i=0}^{k} (-1)^i \mathbf{h}_{n-k+i} (\mathbf{e}_i)^\perp \mathbf{s}_\lambda \\
&= \underbrace{(-1)^0}_{=1} \underbrace{\mathbf{h}_{n-k+0}}_{=\mathbf{h}_{n-k}} \underbrace{(\mathbf{e}_0)^\perp \mathbf{s}_\lambda}_{=\mathbf{s}_\lambda} + \sum_{i=1}^{k} (-1)^i \mathbf{h}_{n-k+i} \underbrace{(\mathbf{e}_i)^\perp \mathbf{s}_\lambda}_{\substack{= \sum\limits_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu \\ \text{(by Corollary 6.7)}}} \\
&= \underbrace{\mathbf{h}_{n-k}\mathbf{s}_\lambda}_{=\mathbf{s}_\lambda \mathbf{h}_{n-k}} + \sum_{i=1}^{k} (-1)^i \mathbf{h}_{n-k+i} \sum_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu \\
&= \mathbf{s}_\lambda \mathbf{h}_{n-k} + \sum_{i=1}^{k} (-1)^i \mathbf{h}_{n-k+i} \sum_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu,
\end{aligned}$$

so that

$$\mathbf{s}_\lambda \mathbf{h}_{n-k} = \mathbf{s}_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)} - \sum_{i=1}^{k} (-1)^i \mathbf{h}_{n-k+i} \sum_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu .$$

This is an equality in Λ. If we evaluate both of its sides at $x_1, x_2, \ldots, x_k$, then we obtain

$$\begin{aligned}
s_\lambda h_{n-k} &= s_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)} - \sum_{i=1}^{k} (-1)^i \underbrace{h_{n-k+i}}_{\substack{\equiv a_i \operatorname{mod} I \\ \text{(by (9))}}} \underbrace{\sum_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}}}_{\substack{= \sum\limits_{\substack{\mu\in P_{k,n}; \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \\ \text{(because if } \mu \text{ is a partition such} \\ \text{that } \lambda/\mu \text{ is a vertical } i\text{-strip, then } \mu\in P_{k,n} \\ \text{(since } \mu\subseteq\lambda \text{ and } \lambda\in P_{k,n}))}} s_\mu \\
&\equiv s_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\substack{\mu\in P_{k,n}; \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} s_\mu \operatorname{mod} I.
\end{aligned}$$

In other words,

$$\begin{aligned}\overline{s_\lambda h_{n-k}} &= \overline{s_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\substack{\mu \in P_{k,n};\\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} s_\mu} \\ &= \overline{s_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)}} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\substack{\mu \in P_{k,n};\\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}.\end{aligned}$$

This proves Proposition 9.3 **(a)**.

(b) Assume that $\lambda_k > 0$. Hence, the partition $(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)$ has more than k parts (since its $(k+1)$-st entry is $\lambda_k > 0$). Thus, (3) (applied to $(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)$ instead of λ) yields $s_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)} = 0$. Hence, $\overline{s_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)}} = \overline{0} = 0$. Now, Proposition 9.3 **(a)** yields

$$\begin{aligned}\overline{s_\lambda h_{n-k}} &= \underbrace{\overline{s_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)}}}_{=0} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\substack{\mu \in P_{k,n};\\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \\ &= -\sum_{i=1}^{k} (-1)^i a_i \sum_{\substack{\mu \in P_{k,n};\\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}.\end{aligned}$$

This proves Proposition 9.3 **(b)**. □

9.3. Multiplying by $\overline{h_j}$

At last, let us give an explicit expansion for $\overline{s_\lambda h_j}$ in the basis $\left(\overline{s_\mu}\right)_{\mu \in P_{k,n}}$ that holds for all $j \in \{0,1,\ldots,n-k\}$. Before we state it, we need a notation:

Definition 9.4. Let $\mathbf{f} \in \Lambda$ be any symmetric function. Then, $\overline{\mathbf{f}} \in \mathcal{S}/I$ is defined to be $\overline{f}$, where $f \in \mathcal{S}$ is the result of evaluating the symmetric function $\mathbf{f} \in \Lambda$ at the k variables $x_1, x_2, \ldots, x_k$. Thus, for every partition λ, we have $\overline{\mathbf{s}_\lambda} = \overline{s_\lambda}$. Likewise, for any $m \in \mathbb{N}$, we have $\overline{\mathbf{h}_m} = \overline{h_m}$ and $\overline{\mathbf{e}_m} = \overline{e_m}$.

Theorem 9.5. Let $\lambda \in P_{k,n}$. Let $j \in \{0,1,\ldots,n-k\}$. Then,

$$\overline{s_\lambda h_j} = \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \overline{\left(\mathbf{s}_{(n-k-j+1,1^{i-1})}\right)^{\perp} \mathbf{s}_\lambda}.$$

Example 9.6. If $n = 7$ and $k = 3$, then

$$\begin{aligned}&\overline{s_{(4,3,2)}h_2}\\&= \overline{s_{(4,4,3)}} + a_1\left(\overline{s_{(4,2)}} + \overline{s_{(3,2,1)}} + \overline{s_{(3,3)}}\right) - a_2\left(\overline{s_{(4,1)}} + \overline{s_{(2,2,1)}} + \overline{s_{(3,1,1)}} + 2\overline{s_{(3,2)}}\right)\\&\qquad + a_3\left(\overline{s_{(2,2)}} + \overline{s_{(2,1,1)}} + \overline{s_{(3,1)}}\right).\end{aligned}$$

It is not hard to reveal Propositions 9.2 and 9.3 as particular cases of Theorem 9.5 (by setting $j = 1$ or $j = n - k$, respectively). Likewise, one can see that Theorem 9.5 generalizes [BeCiFu99, (22)]. Indeed, [BeCiFu99, (22)] says that if $a_1 = a_2 = \cdots = a_{k-1} = 0$, then every $\lambda \in P_{k,n}$ and $j \in \{0, 1, \ldots, n-k\}$ satisfy

$$\overline{s_\lambda h_j} = \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} - (-1)^k a_k \sum_\nu \overline{s_\nu},$$

where the second sum runs over all $\nu \in P_{k,n}$ satisfying

$$\begin{aligned}&(\lambda_i - 1 \geq \nu_i \text{ for all } i \in \{1, 2, \ldots, k\}) \qquad \text{and}\\&(\nu_i \geq \lambda_{i+1} - 1 \text{ for all } i \in \{1, 2, \ldots, k-1\}) \qquad \text{and}\\&|\nu| = |\lambda| + j - n.\end{aligned}$$

Note, however, that the sums in Theorem 9.5 contain multiplicities (see the "$2\overline{s_{(3,2)}}$" in Example 9.6), unlike those in [BeCiFu99, (22)].

We shall prove Theorem 9.5 by deriving it from an identity between genuine symmetric functions (in Λ, not in $\mathcal{S}$ or $\mathcal{S}/I$):

Theorem 9.7. Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \ldots, n-k\}$. Then,

$$\mathbf{s}_\lambda \mathbf{h}_j = \sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 \leq n-k;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu - \sum_{i=1}^{k} (-1)^i \mathbf{h}_{n-k+i} \left(\mathbf{s}_{(n-k-j+1, 1^{i-1})}\right)^\perp \mathbf{s}_\lambda.$$

Before we prove this theorem, we need several auxiliary results. First, we recall one of the Pieri rules ([GriRei18, (2.7.1)]):

Proposition 9.8. Let λ be a partition, and let $i \in \mathbb{N}$. Then,

$$\mathbf{s}_\lambda \mathbf{h}_i = \sum_{\substack{\mu \text{ is a partition;}\\ \mu/\lambda \text{ is a horizontal } i\text{-strip}}} \mathbf{s}_\mu.$$

From this, we can easily derive the following:

Corollary 9.9. Let λ be a partition, and let $i \in \mathbb{N}$. Then,

$$(\mathbf{h}_i)^{\perp} \mathbf{s}_\lambda = \sum_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a horizontal } i\text{-strip}}} \mathbf{s}_\mu .$$

Corollary 9.9 is also proven in [GriRei18, version with solutions (ancillary file), Lemma 12.83.3(a)].

Next, let us show some further lemmas:

Lemma 9.10. Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \ldots, n-k\}$. Let g be a positive integer. Then,

$$\sum_{\substack{\mu \text{ is a partition;} \\ \mu_1 = n-k+g; \\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu = \sum_{w \geq 1} (-1)^{w-g} \mathbf{h}_{n-k+w} \left(\mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g}\right)^{\perp} \mathbf{s}_\lambda .$$

Proof of Lemma 9.10 (sketched). First, we observe that $\lambda_1 \leq n-k$ (since $\lambda \in P_{k,n}$). Now, every partition μ satisfying $\mu_1 = n-k+g$ must automatically satisfy $\mu_1 \geq \lambda_1$ (because $\mu_1 = n-k+\underbrace{g}_{\geq 0} \geq n-k \geq \lambda_1$).

Let A be the set of all partitions μ such that $\mu_1 = n-k+g$ and such that μ/λ is a horizontal j-strip. Let B be the set of all partitions ν such that λ/ν is a horizontal $(n-k+g-j)$-strip. Then,[17]

$$\begin{aligned} A &= \{\mu \text{ is a partition} \mid \mu_1 = n-k+g \text{ and } |\mu| - |\lambda| = j \\ &\qquad \text{and } \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \mu_3 \geq \lambda_3 \geq \cdots\} \\ &= \{\mu \text{ is a partition} \mid \mu_1 = n-k+g \text{ and } |\mu| - |\lambda| = j \\ &\qquad \text{and } \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \mu_3 \geq \lambda_3 \geq \mu_4 \geq \cdots\} \end{aligned}$$

(since every partition μ satisfying $\mu_1 = n-k+g$ must automatically satisfy $\mu_1 \geq \lambda_1$) and

$$\begin{aligned} B = \{\nu \text{ is a partition} \mid\ & |\lambda| - |\nu| = n-k+g-j \\ & \text{and } \lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \lambda_3 \geq \nu_3 \geq \cdots\} . \end{aligned}$$

Hence, it is easy to check that the map

$$\begin{aligned} & B \to A, \\ & \nu \mapsto (n-k+g, \nu_1, \nu_2, \nu_3, \ldots) \end{aligned}$$

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[17] We are using Definition 6.2 **(c)** here.

is well-defined (because every $\nu \in B$ satisfies $\lambda_1 \geq \nu_1$ and thus $n-k+\underbrace{g}_{\geq 0} \geq n-k \geq \lambda_1 \geq \nu_1$) and is a bijection (its inverse map just sends each $\mu \in A$ to $(\mu_2, \mu_3, \mu_4, \ldots) \in B$). Thus, we can substitute $(n-k+g, \nu_1, \nu_2, \nu_3, \ldots)$ for μ in the sum $\sum\limits_{\mu \in A} \mathbf{s}_\mu$. We thus obtain

$$\sum_{\mu \in A} \mathbf{s}_\mu = \sum_{\nu \in B} \mathbf{s}_{(n-k+g, \nu_1, \nu_2, \nu_3, \ldots)}. \qquad (65)$$

But each $\nu \in B$ satisfies $n-k+\underbrace{g}_{\geq 0} \geq n-k \geq \lambda_1 \geq \nu_1$ and thus

$$\sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{n-k+g+i} (\mathbf{e}_i)^\perp \mathbf{s}_\nu = \mathbf{s}_{(n-k+g, \nu_1, \nu_2, \nu_3, \ldots)} \qquad (66)$$

(by Proposition 6.8, applied to ν and $n-k+g$ instead of λ and m). Hence, (65) becomes

$$\begin{aligned} \sum_{\mu \in A} \mathbf{s}_\mu &= \sum_{\nu \in B} \underbrace{\mathbf{s}_{(n-k+g, \nu_1, \nu_2, \nu_3, \ldots)}}_{\substack{= \sum\limits_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{n-k+g+i} (\mathbf{e}_i)^\perp \mathbf{s}_\nu \\ \text{(by (66))}}} = \sum_{\nu \in B} \sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{n-k+g+i} (\mathbf{e}_i)^\perp \mathbf{s}_\nu \\ &= \sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{n-k+g+i} (\mathbf{e}_i)^\perp \left(\sum_{\nu \in B} \mathbf{s}_\nu \right). \end{aligned} \qquad (67)$$

But Corollary 9.9 (applied to $i = n-k+g-j$) yields[18]

$$\begin{aligned} \left(\mathbf{h}_{n-k+g-j}\right)^\perp \mathbf{s}_\lambda &= \underbrace{\sum_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a horizontal } (n-k+g-j)\text{-strip}}}}_{\substack{= \sum\limits_{\mu \in B} \\ \text{(by the definition of } B\text{)}}} \mathbf{s}_\mu = \sum_{\mu \in B} \mathbf{s}_\mu \\ &= \sum_{\nu \in B} \mathbf{s}_\nu. \end{aligned} \qquad (68)$$

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[18] More precisely: This follows from Corollary 9.9 (applied to $i = n-k+g-j$) when $n-k+g-j \in \mathbb{N}$. But otherwise, it is obvious for trivial reasons ($0 = 0$).

Hence, (67) becomes

$$\begin{aligned}
\sum_{\mu\in A}\mathbf{s}_\mu &= \sum_{i\in\mathbb{N}}(-1)^i\,\mathbf{h}_{n-k+g+i}\,(\mathbf{e}_i)^\perp\left(\underbrace{\sum_{\nu\in B}\mathbf{s}_\nu}_{\substack{=\left(\mathbf{h}_{n-k+g-j}\right)^\perp\mathbf{s}_\lambda\\ \text{(by (68))}}}\right)\\
&= \sum_{i\in\mathbb{N}}(-1)^i\,\mathbf{h}_{n-k+g+i}\,\underbrace{(\mathbf{e}_i)^\perp\left(\left(\mathbf{h}_{n-k+g-j}\right)^\perp\mathbf{s}_\lambda\right)}_{=\left((\mathbf{e}_i)^\perp\circ\left(\mathbf{h}_{n-k+g-j}\right)^\perp\right)\mathbf{s}_\lambda}\\
&= \sum_{i\in\mathbb{N}}(-1)^i\,\mathbf{h}_{n-k+g+i}\qquad\underbrace{\left((\mathbf{e}_i)^\perp\circ\left(\mathbf{h}_{n-k+g-j}\right)^\perp\right)}_{\substack{=\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_i\right)^\perp\\ \text{(since (16)}\\ \text{yields }\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_i\right)^\perp=(\mathbf{e}_i)^\perp\circ\left(\mathbf{h}_{n-k+g-j}\right)^\perp)}}\qquad\mathbf{s}_\lambda\\
&= \sum_{i\in\mathbb{N}}(-1)^i\,\mathbf{h}_{n-k+g+i}\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_i\right)^\perp\mathbf{s}_\lambda\\
&= \sum_{w\geq g}(-1)^{w-g}\,\mathbf{h}_{n-k+w}\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}\right)^\perp\mathbf{s}_\lambda\\
&\qquad\qquad\left(\text{here, we have substituted } w-g \text{ for } i \text{ in the sum}\right).
\end{aligned}$$

Comparing this with

$$\begin{aligned}
&\sum_{w\geq 1}(-1)^{w-g}\,\mathbf{h}_{n-k+w}\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}\right)^\perp\mathbf{s}_\lambda\\
&= \sum_{w=1}^{g-1}(-1)^{w-g}\,\mathbf{h}_{n-k+w}\left(\mathbf{h}_{n-k+g-j}\qquad\underbrace{\mathbf{e}_{w-g}}_{\substack{=0\\ \text{(since } w-g<0\\ \text{(since } w\leq g-1<g))}}\right)^\perp\mathbf{s}_\lambda\\
&\qquad+\sum_{w\geq g}(-1)^{w-g}\,\mathbf{h}_{n-k+w}\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}\right)^\perp\mathbf{s}_\lambda\\
&\qquad\left(\text{since } g \text{ is a positive integer}\right)\\
&= \underbrace{\sum_{w=1}^{g-1}(-1)^{w-g}\,\mathbf{h}_{n-k+w}\left(\mathbf{h}_{n-k+g-j}0\right)^\perp\mathbf{s}_\lambda}_{=0}+\sum_{w\geq g}(-1)^{w-g}\,\mathbf{h}_{n-k+w}\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}\right)^\perp\mathbf{s}_\lambda\\
&= \sum_{w\geq g}(-1)^{w-g}\,\mathbf{h}_{n-k+w}\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}\right)^\perp\mathbf{s}_\lambda,
\end{aligned}$$

we obtain

$$\sum_{\mu\in A}\mathbf{s}_\mu=\sum_{w\geq 1}(-1)^{w-g}\,\mathbf{h}_{n-k+w}\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}\right)^{\perp}\mathbf{s}_\lambda. \tag{69}$$

In view of

$$\sum_{\mu\in A}=\sum_{\substack{\mu\text{ is a partition;}\\ \mu_1=n-k+g;\\ \mu/\lambda\text{ is a horizontal } j\text{-strip}}}\qquad\text{(by the definition of } A\text{)},$$

this rewrites as

$$\sum_{\substack{\mu\text{ is a partition;}\\ \mu_1=n-k+g;\\ \mu/\lambda\text{ is a horizontal } j\text{-strip}}}\mathbf{s}_\mu=\sum_{w\geq 1}(-1)^{w-g}\,\mathbf{h}_{n-k+w}\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}\right)^{\perp}\mathbf{s}_\lambda.$$

This proves Lemma 9.10. □

Our next lemma will be a slight generalization of Lemma 7.17; but first we extend our definition of $\mathbf{s}_{\left(m,1^j\right)}$:

Convention 9.11. Let $m\in\mathbb{N}$, and let j be a negative integer. Then, we shall understand the (otherwise undefined) expression $\mathbf{s}_{\left(m,1^j\right)}$ to mean $0\in\Lambda$.

We can now generalize Lemma 7.17 as follows:

Lemma 9.12. Let m be a positive integer. Let $j\in\mathbb{Z}$ be such that $m+j>0$. Then,

$$\mathbf{s}_{\left(m,1^j\right)}=\sum_{i=1}^{m}(-1)^{i-1}\,\mathbf{h}_{m-i}\mathbf{e}_{j+i}.$$

Proof of Lemma 9.12. If $j\in\mathbb{N}$, then this follows directly from Lemma 7.17. Hence, for the rest of this proof, we WLOG assume that $j\notin\mathbb{N}$. Hence, $j<0$. Now, the proof of Lemma 9.12 is the same as our above proof of Lemma 7.17, with two changes:

- The inequality $m+j>0$ no longer follows from $m>0$ and $j\geq 0$, but rather comes straight from the assumptions.
- The equality $\sum_{i=0}^{j}(-1)^i\,\mathbf{h}_{m+i}\mathbf{e}_{j-i}=\mathbf{s}_{\left(m,1^j\right)}$ no longer follows from (32), but rather comes from comparing $\sum_{i=0}^{j}(-1)^i\,\mathbf{h}_{m+i}\mathbf{e}_{j-i}=(\text{empty sum})=0$ with $\mathbf{s}_{\left(m,1^j\right)}=0$.

Thus, Lemma 9.12 is proven. □

Lemma 9.13. Let $j \in \{0, 1, \ldots, n-k\}$, and let w be a positive integer. Then,

$$\sum_{g=1}^{j} (-1)^{w-g} \mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g} = (-1)^{w-j} \mathbf{s}_{\left(n-k+1, 1^{w-j-1}\right)} - (-1)^{w} \mathbf{s}_{\left(n-k-j+1, 1^{w-1}\right)}.$$

Proof of Lemma 9.13. From $j \in \{0, 1, \ldots, n-k\}$, we obtain $0 \leq j \leq n-k \leq n-k+1$.

We have $\underbrace{n}_{\geq k} - k + 1 \geq k - k + 1 = 1$; thus, $n-k+1$ is a positive integer. Also, $(n-k+1) + \left(w - \underbrace{j}_{\leq n-k} - 1\right) \geq (n-k+1) + (w - (n-k) - 1) = w > 0$ (since w is a positive integer). Hence, Lemma 9.12 (applied to $n-k+1$ and $w-j-1$ instead of m and j) yields

$$\begin{aligned} \mathbf{s}_{\left(n-k+1, 1^{w-j-1}\right)} &= \sum_{i=1}^{n-k+1} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i} \\ &= \sum_{i=1}^{j} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i} + \sum_{i=j+1}^{n-k+1} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i} \end{aligned} \tag{70}$$

(since $0 \leq j \leq n-k+1$). Also, $n - k - \underbrace{j}_{\leq n-k} + 1 \geq n - k - (n-k) + 1 = 1$; thus, $n-k-j+1$ is a positive integer. Also, $\left(n - k - \underbrace{j}_{\leq n-k} + 1\right) + (w-1) \geq (n - k - (n-k) + 1) + (w-1) = w > 0$. Hence, Lemma 7.17 (applied to $n-k-j+1$ and $w-1$ instead of m and j) yields

$$\begin{aligned} \mathbf{s}_{\left(n-k-j+1, 1^{w-1}\right)} &= \sum_{i=1}^{n-k-j+1} (-1)^{i-1} \mathbf{h}_{n-k-j+1-i} \mathbf{e}_{w-1+i} \\ &= \sum_{i=j+1}^{n-k+1} \underbrace{(-1)^{i-j-1}}_{=(-1)^{j}(-1)^{i-1}} \underbrace{\mathbf{h}_{n-k-j+1-(i-j)}}_{=\mathbf{h}_{n-k+1-i}} \underbrace{\mathbf{e}_{w-1+i-j}}_{=\mathbf{e}_{w-j-1+i}} \\ &\qquad \left(\text{here, we have substituted } i-j \text{ for } i \text{ in the sum}\right) \\ &= (-1)^{j} \sum_{i=j+1}^{n-k+1} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i}. \end{aligned}$$

Multiplying this equality by $(-1)^j$, we find

$$(-1)^j \mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)} = \sum_{i=j+1}^{n-k+1} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i}.$$

Subtracting this equality from (70), we obtain

$$\begin{aligned}
&\mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)} - (-1)^j \mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)} \\
&= \left(\sum_{i=1}^{j} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i} + \sum_{i=j+1}^{n-k+1} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i} \right) \\
&\qquad - \sum_{i=j+1}^{n-k+1} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i} \\
&= \sum_{i=1}^{j} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i}. \qquad (71)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{g=1}^{j} (-1)^{w-g} \mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g} \\
&= \sum_{i=1}^{j} \underbrace{(-1)^{w-(j+1-i)}}_{=(-1)^{w-j}(-1)^{i-1}} \underbrace{\mathbf{h}_{n-k+(j+1-i)-j}}_{=\mathbf{h}_{n-k+1-i}} \underbrace{\mathbf{e}_{w-(j+1-i)}}_{=\mathbf{e}_{w-j-1+i}} \\
&\qquad \left(\text{here, we have substituted } j+1-i \text{ for } g \text{ in the sum}\right) \\
&= (-1)^{w-j} \underbrace{\sum_{i=1}^{j} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i}}_{\substack{=\mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)} - (-1)^j \mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)} \\ \text{(by (71))}}} \\
&= (-1)^{w-j} \left(\mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)} - (-1)^j \mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)} \right) \\
&= (-1)^{w-j} \mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)} - \underbrace{(-1)^{w-j} (-1)^j}_{=(-1)^w} \mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)} \\
&= (-1)^{w-j} \mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)} - (-1)^w \mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}.
\end{aligned}$$

This proves Lemma 9.13. □

Proof of Theorem 9.7. We have $j \in \{0, 1, \ldots, n-k\}$, thus $0 \leq j \leq n-k$. Also, we have $\lambda \in P_{k,n}$; thus, the partition λ has at most k parts and satisfies $\lambda_1 \leq n-k$.

Let g be an integer such that $g \geq j+1$. If μ is a partition such that μ/λ is a horizontal j-strip, then $\mu_1 \leq \underbrace{\lambda_1}_{\leq n-k} + j \leq n-k+\underbrace{j}_{<j+1\leq g} < n-k+g$ and thus $\mu_1 \neq n-k+g$. Thus, there exists no partition μ such that $\mu_1 = n-k+g$ and such that μ/λ is a horizontal j-strip. Hence,

$$\sum_{\substack{\mu \text{ is a partition;} \\ \mu_1 = n-k+g; \\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu = (\text{empty sum}) = 0. \tag{72}$$

Now, forget that we fixed g. We thus have proven the equality (72) for every integer g satisfying $g \geq j+1$.

On the other hand, let $g \in \{1,2,\ldots,j\}$. Thus, $g \leq j \leq n-k$. If w is an integer satisfying $w \geq n+1$, then $\underbrace{w}_{\geq n+1} - \underbrace{g}_{\leq n-k} \geq (n+1)-(n-k) = k+1 > k$, and thus the partition (1^{w-g}) does **not** satisfy $(1^{w-g}) \subseteq \lambda$ (because the partition λ has at most k parts, whereas the partition (1^{w-g}) has $w-g > k$ parts), and therefore we have

$$\begin{aligned}\Bigg(\underbrace{\mathbf{e}_{w-g}}_{=\mathbf{s}_{(1^{w-g})}}\Bigg)^{\perp}(\mathbf{s}_\lambda) &= \left(\mathbf{s}_{(1^{w-g})}\right)^{\perp}(\mathbf{s}_\lambda) = \mathbf{s}_{\lambda/(1^{w-g})} \qquad (\text{by (15)}) \\ &= 0 \qquad \left(\text{since we don't have } \left(1^{w-g}\right) \subseteq \lambda\right). \end{aligned} \tag{73}$$

Hence, if w is an integer satisfying $w \geq n+1$, then

$$\begin{aligned}\Bigg(\underbrace{\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}}_{=\mathbf{e}_{w-g}\mathbf{h}_{n-k+g-j}}\Bigg)^{\perp}\mathbf{s}_\lambda &= \underbrace{\left(\mathbf{e}_{w-g}\mathbf{h}_{n-k+g-j}\right)^{\perp}}_{\substack{=\left(\mathbf{h}_{n-k+g-j}\right)^{\perp}\circ\left(\mathbf{e}_{w-g}\right)^{\perp} \\ (\text{by (16)})}}\mathbf{s}_\lambda = \left(\left(\mathbf{h}_{n-k+g-j}\right)^{\perp}\circ\left(\mathbf{e}_{w-g}\right)^{\perp}\right)(\mathbf{s}_\lambda) \\ &= \left(\mathbf{h}_{n-k+g-j}\right)^{\perp}\underbrace{\left(\left(\mathbf{e}_{w-g}\right)^{\perp}(\mathbf{s}_\lambda)\right)}_{\substack{=0 \\ (\text{by (73)})}} = 0. \end{aligned} \tag{74}$$

Now,

$$\begin{aligned}
&\sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 = n-k+g;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu \\
&= \sum_{w \geq 1} (-1)^{w-g} \mathbf{h}_{n-k+w} \left(\mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g}\right)^\perp \mathbf{s}_\lambda \\
&\qquad \left(\text{by Lemma 9.10}\right) \\
&= \sum_{w=1}^{n} (-1)^{w-g} \mathbf{h}_{n-k+w} \left(\mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g}\right)^\perp \mathbf{s}_\lambda \\
&\qquad + \sum_{w \geq n+1} (-1)^{w-g} \mathbf{h}_{n-k+w} \underbrace{\left(\mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g}\right)^\perp \mathbf{s}_\lambda}_{\substack{=0\\ \text{(by (74))}}} \\
&= \sum_{w=1}^{n} (-1)^{w-g} \mathbf{h}_{n-k+w} \left(\mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g}\right)^\perp \mathbf{s}_\lambda. \qquad (75)
\end{aligned}$$

Now, forget that we fixed g. We thus have proven the equality (75) for each $g \in \{1, 2, \ldots, j\}$.

Proposition 9.8 (applied to $i = j$) yields

$$\mathbf{s}_\lambda \mathbf{h}_j = \sum_{\substack{\mu \text{ is a partition;}\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu = \sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 \leq n-k;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu + \sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 > n-k;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu$$

(since each partition μ satisfies either $\mu_1 \leq n-k$ or $\mu_1 > n-k$). Hence,

$$\begin{aligned}
&\mathbf{s}_\lambda \mathbf{h}_j - \sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 \leq n-k;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu \\
&= \sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 > n-k;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu = \sum_{g \geq 1} \sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 = n-k+g;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu \\
&\qquad \left(\begin{array}{c} \text{because the partitions } \mu \text{ satisfying } \mu_1 > n-k \text{ are precisely} \\ \text{the partitions } \mu \text{ satisfying } \mu_1 = n-k+g \text{ for some } g \geq 1, \\ \text{and moreover the } g \text{ is uniquely determined by the partition} \end{array} \right) \\
&= \sum_{g=1}^{j} \underbrace{\sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 = n-k+g;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu}_{\substack{= \sum_{w=1}^{n} (-1)^{w-g} \mathbf{h}_{n-k+w} \left(\mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g}\right)^{\perp} \mathbf{s}_\lambda \\ \text{(by (75))}}} + \sum_{g=j+1}^{\infty} \underbrace{\sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 = n-k+g;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu}_{\substack{=0 \\ \text{(by (72))}}} \\
&= \underbrace{\sum_{g=1}^{j} \sum_{w=1}^{n}}_{= \sum_{w=1}^{n} \sum_{g=1}^{j}} (-1)^{w-g} \mathbf{h}_{n-k+w} \left(\mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g}\right)^{\perp} \mathbf{s}_\lambda \\
&= \sum_{w=1}^{n} \sum_{g=1}^{j} (-1)^{w-g} \mathbf{h}_{n-k+w} \left(\mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g}\right)^{\perp} \mathbf{s}_\lambda \\
&= \sum_{w=1}^{n} \mathbf{h}_{n-k+w} \left(\underbrace{\sum_{g=1}^{j} (-1)^{w-g} \mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g}}_{\substack{= (-1)^{w-j} \mathbf{s}_{\left(n-k+1, 1^{w-j-1}\right)} - (-1)^{w} \mathbf{s}_{\left(n-k-j+1, 1^{w-1}\right)} \\ \text{(by Lemma 9.13)}}} \right)^{\perp} \mathbf{s}_\lambda \\
&= \sum_{w=1}^{n} \mathbf{h}_{n-k+w} \left((-1)^{w-j} \mathbf{s}_{\left(n-k+1, 1^{w-j-1}\right)} - (-1)^{w} \mathbf{s}_{\left(n-k-j+1, 1^{w-1}\right)} \right)^{\perp} \mathbf{s}_\lambda \\
&= \sum_{w=1}^{n} \mathbf{h}_{n-k+w} (-1)^{w-j} \left(\mathbf{s}_{\left(n-k+1, 1^{w-j-1}\right)} \right)^{\perp} \mathbf{s}_\lambda \\
&\qquad - \sum_{w=1}^{n} \mathbf{h}_{n-k+w} (-1)^{w} \left(\mathbf{s}_{\left(n-k-j+1, 1^{w-1}\right)} \right)^{\perp} \mathbf{s}_\lambda . \qquad (76)
\end{aligned}$$

Next, we claim that

$$\left(\mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)}\right)^{\perp}\mathbf{s}_{\lambda}=0\qquad\text{for each } w\in\{1,2,\ldots,n\}. \tag{77}$$

[*Proof of (77):* Let $w\in\{1,2,\ldots,n\}$. If $w-j-1$ is a negative integer, then $\mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)}=0$ (by Convention 9.11), and thus (77) holds in this case. Hence, for the rest of this proof of (77), we WLOG assume that $w-j-1$ is not a negative integer. Thus, $w-j-1\in\mathbb{N}$. Now, the partition $\left(n-k+1,1^{w-j-1}\right)$ has a bigger first entry than the partition λ (since its first entry is $n-k+1>n-k\geq\lambda_1$). Thus, we do not have $\left(n-k+1,1^{w-j-1}\right)\subseteq\lambda$. Hence, $\mathbf{s}_{\lambda/\left(n-k+1,1^{w-j-1}\right)}=0$. But (15) yields $\left(\mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)}\right)^{\perp}\mathbf{s}_{\lambda}=\mathbf{s}_{\lambda/\left(n-k+1,1^{w-j-1}\right)}=0$. This proves (77).]

Next, we claim that

$$\left(\mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}\right)^{\perp}\mathbf{s}_{\lambda}=0\qquad\text{for each } w\in\{k+1,k+2,\ldots,n\}. \tag{78}$$

[*Proof of (78):* Let $w\in\{k+1,k+2,\ldots,n\}$. Then, $w\geq k+1$. Now, the number of parts of the partition $\left(n-k-j+1,1^{w-1}\right)$ is $1+(w-1)=w\geq k+1>k$, which is bigger than the number of parts of λ (since λ has at most k parts). Hence, we don't have $\left(n-k-j+1,1^{w-1}\right)\subseteq\lambda$. Thus, $\mathbf{s}_{\lambda/\left(n-k-j+1,1^{w-1}\right)}=0$. But (15) yields $\left(\mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}\right)^{\perp}\mathbf{s}_{\lambda}=\mathbf{s}_{\lambda/\left(n-k-j+1,1^{w-1}\right)}=0$. This proves (78).]

Now, (76) becomes

$$\begin{aligned}
&\mathbf{s}_\lambda \mathbf{h}_j - \sum_{\substack{\mu \text{ is a partition;} \\ \mu_1 \leq n-k; \\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu \\
&= \sum_{w=1}^{n} \mathbf{h}_{n-k+w} (-1)^{w-j} \underbrace{\left(\mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)}\right)^{\perp} \mathbf{s}_\lambda}_{\substack{=0 \\ \text{(by (77))}}} \\
&\qquad - \sum_{w=1}^{n} \mathbf{h}_{n-k+w} (-1)^{w} \left(\mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}\right)^{\perp} \mathbf{s}_\lambda \\
&= - \sum_{w=1}^{n} \mathbf{h}_{n-k+w} (-1)^{w} \left(\mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}\right)^{\perp} \mathbf{s}_\lambda \\
&= - \left(\sum_{w=1}^{k} \mathbf{h}_{n-k+w} (-1)^{w} \left(\mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}\right)^{\perp} \mathbf{s}_\lambda \right. \\
&\qquad \qquad \left. + \sum_{w=k+1}^{n} \mathbf{h}_{n-k+w} (-1)^{w} \underbrace{\left(\mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}\right)^{\perp} \mathbf{s}_\lambda}_{\substack{=0 \\ \text{(by (78))}}} \right) \\
&\qquad (\text{since } 0 \leq k \leq n) \\
&= - \sum_{w=1}^{k} \mathbf{h}_{n-k+w} (-1)^{w} \left(\mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}\right)^{\perp} \mathbf{s}_\lambda \\
&= - \sum_{w=1}^{k} (-1)^{w} \mathbf{h}_{n-k+w} \left(\mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}\right)^{\perp} \mathbf{s}_\lambda \\
&= - \sum_{i=1}^{k} (-1)^{i} \mathbf{h}_{n-k+i} \left(\mathbf{s}_{\left(n-k-j+1,1^{i-1}\right)}\right)^{\perp} \mathbf{s}_\lambda
\end{aligned}$$

(here, we have renamed the summation index w as i). Hence,

$$\mathbf{s}_\lambda \mathbf{h}_j = \sum_{\substack{\mu \text{ is a partition;} \\ \mu_1 \leq n-k; \\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu - \sum_{i=1}^{k} (-1)^{i} \mathbf{h}_{n-k+i} \left(\mathbf{s}_{\left(n-k-j+1,1^{i-1}\right)}\right)^{\perp} \mathbf{s}_\lambda .$$

This proves Theorem 9.7. □

Proof of Theorem 9.5. Theorem 9.7 yields

$$\mathbf{s}_\lambda \mathbf{h}_j = \sum_{\substack{\mu \text{ is a partition;} \\ \mu_1 \leq n-k; \\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu - \sum_{i=1}^{k} (-1)^{i} \mathbf{h}_{n-k+i} \left(\mathbf{s}_{\left(n-k-j+1,1^{i-1}\right)}\right)^{\perp} \mathbf{s}_\lambda .$$

Both sides of this equality are symmetric functions in Λ. If we evaluate them at $x_1, x_2, \ldots, x_k$ and project the resulting symmetric polynomials onto $\mathcal{S}/I$, then we obtain

$$\begin{aligned}
\overline{s_\lambda h_j} &= \sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 \le n-k;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i \underbrace{\overline{h_{n-k+i}}}_{\substack{=a_i\\ \text{(since (9)}\\ \text{yields } h_{n-k+i}\equiv a_i \bmod I)}} \overline{\left(\mathbf{s}_{\left(n-k-j+1,1^{i-1}\right)}\right)^{\perp} \mathbf{s}_\lambda} \\
&= \sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 \le n-k;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \overline{\left(\mathbf{s}_{\left(n-k-j+1,1^{i-1}\right)}\right)^{\perp} \mathbf{s}_\lambda}. \qquad (79)
\end{aligned}$$

But every partition μ has either at most k parts or more than k parts. Hence,

$$\begin{aligned}
&\sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 \le n-k;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} \\
&= \sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 \le n-k;\\ \mu \text{ has at most } k \text{ parts;}\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} + \sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 \le n-k;\\ \mu \text{ has more than } k \text{ parts;}\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \underbrace{\overline{s_\mu}}_{\substack{=0\\ \text{(because (3) (applied to } \mu\\ \text{instead of } \lambda\text{) yields } s_\mu = 0)}} \\
&= \underbrace{\sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 \le n-k;\\ \mu \text{ has at most } k \text{ parts;}\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}}}_{\substack{= \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}}\\ \text{(because the partitions } \mu \text{ such that } \mu_1 \le n-k\\ \text{and such that } \mu \text{ has at most } k \text{ parts}\\ \text{are precisely the partitions } \mu \in P_{k,n})}} \overline{s_\mu} = \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu}.
\end{aligned}$$

Hence, (79) becomes

$$\begin{aligned}
\overline{s_\lambda h_j} &= \underbrace{\sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 \le n-k;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu}}_{= \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu}} - \sum_{i=1}^{k} (-1)^i a_i \overline{\left(\mathbf{s}_{\left(n-k-j+1,1^{i-1}\right)}\right)^{\perp} \mathbf{s}_\lambda} \\
&= \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \overline{\left(\mathbf{s}_{\left(n-k-j+1,1^{i-1}\right)}\right)^{\perp} \mathbf{s}_\lambda}.
\end{aligned}$$

This proves Theorem 9.5. □

Let us again use the notation $c^{\gamma}_{\alpha,\beta}$ for a Littlewood–Richardson coefficient (defined as in [GriRei18, Definition 2.5.8], for example). Then, we can restate Theorem 9.5 as follows:

Theorem 9.14. Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \ldots, n-k\}$. Then,

$$\overline{s_\lambda h_j} = \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\nu \subseteq \lambda} c^{\lambda}_{\left(n-k-j+1,1^{i-1}\right),\nu} \overline{s_\nu},$$

where the last sum ranges over all partitions ν satisfying $\nu \subseteq \lambda$.

Proof of Theorem 9.14. Let μ be a partition. Then, (15) yields

$$\begin{aligned}
\left(\mathbf{s}_\mu\right)^{\perp} \mathbf{s}_\lambda = \mathbf{s}_{\lambda/\mu} &= \sum_{\nu \text{ is a partition}} c^{\lambda}_{\mu,\nu} \mathbf{s}_\nu \\
&= \sum_{\substack{\nu \text{ is a partition};\\ \nu \subseteq \lambda}} c^{\lambda}_{\mu,\nu} \mathbf{s}_\nu + \sum_{\substack{\nu \text{ is a partition};\\ \text{we don't have } \nu \subseteq \lambda}} \underbrace{c^{\lambda}_{\mu,\nu}}_{\substack{=0\\ \text{(by Proposition 6.17 } \textbf{(b)}\text{)}}} \mathbf{s}_\nu \\
&= \underbrace{\sum_{\substack{\nu \text{ is a partition};\\ \nu \subseteq \lambda}}}_{= \sum_{\nu \subseteq \lambda}} c^{\lambda}_{\mu,\nu} \mathbf{s}_\nu = \sum_{\nu \subseteq \lambda} c^{\lambda}_{\mu,\nu} \mathbf{s}_\nu. \qquad (80)
\end{aligned}$$

Both sides of this equality are symmetric functions in Λ. If we evaluate them at $x_1, x_2, \ldots, x_k$ and project the resulting symmetric polynomials onto $\mathcal{S}/I$, then we obtain

$$\overline{\left(\mathbf{s}_\mu\right)^{\perp} \mathbf{s}_\lambda} = \sum_{\nu \subseteq \lambda} c^{\lambda}_{\mu,\nu} \overline{s_\nu}. \qquad (81)$$

Now, forget that we fixed μ. We thus have proven (81) for each partition μ. Theorem 9.5 yields

$$\begin{aligned}
\overline{s_\lambda h_j} &= \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \underbrace{\overline{\left(\mathbf{s}_{\left(n-k-j+1,1^{i-1}\right)}\right)^{\perp} \mathbf{s}_\lambda}}_{\substack{= \sum_{\nu \subseteq \lambda} c^{\lambda}_{\left(n-k-j+1,1^{i-1}\right),\nu} \overline{s_\nu} \\ \text{(by (81), applied to } \mu = \left(n-k-j+1,1^{i-1}\right)\text{)}}} \\
&= \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\nu \subseteq \lambda} c^{\lambda}_{\left(n-k-j+1,1^{i-1}\right),\nu} \overline{s_\nu}.
\end{aligned}$$

This proves Theorem 9.14. □

9.4. Triangularity between s-basis and h-basis

An application of the Pieri rule is a triangularity property for the change-of-basis matrix between the bases $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ and $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ of $\mathcal{S}/I$. We refer to [GriRei18, Definition 11.1.16(c)] for the concepts that we shall be using. The triangularity is defined with respect to a certain partial order on the set $P_{k,n}$:

Definition 9.15. We define a binary relation $\geq^*$ on the set $P_{k,n}$ as follows: For two partitions $\lambda \in P_{k,n}$ and $\mu \in P_{k,n}$, we set $\lambda \geq^* \mu$ if and only if

- **either** $|\lambda| > |\mu|$
- **or** $|\lambda| = |\mu|$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i$ for all $i \geq 1$ (that is, μ dominates λ).

It is clear that this relation $\geq^*$ is the greater-or-equal relation of a partial order on $P_{k,n}$. This order will be called the *size-then-antidominance order*.

Theorem 2.5 yields that the family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$.

Theorem 9.16. The family $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ expands unitriangularly in the family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$.

TODO: Proof of Theorem 9.16.

TODO: Theorem 7.13 yields that the family $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$. This also follows from [GriRei18, Corollary 11.1.19(e)] and Theorem 9.16.

9.5. Other bases

TODO: Explain why $(\overline{m_\lambda})_{\lambda \in P_{k,n}}$ (where m_λ is a monomial symmetric polynomial) is also a basis of the $\mathbf{k}$-module $\mathcal{S}/I$.

TODO: Mention that $(\overline{p_\lambda})_{\lambda \in P_{k,n}}$ (where p_λ are power sums) is not a basis when $n = 4$ and $k = 2$, even if $\mathbf{k} = \mathbb{Q}$.

9.6. Positivity?

Let us recall some background about the quantum cohomology ring $\operatorname{QH}^*(\operatorname{Gr}_{kn})$ discussed in [Postni05]. The structure constants of the $\mathbb{Z}[q]$-algebra $\operatorname{QH}^*(\operatorname{Gr}_{kn})$ are polynomials in the indeterminate q, whose coefficients are the famous Gromov-Witten invariants $C^d_{\lambda\mu\nu}$. These Gromov-Witten invariants $C^d_{\lambda\mu\nu}$ are nonnegative integers (as follows from their geometric interpretation, but also from the "Quantum Littlewood-Richardson Rule" [BKPT16, Theorem 2]). This appears to generalize to the general case of $\mathcal{S}/I$:

Conjecture 9.17. Let $b_i = (-1)^{n-k-1} a_i$ for each $i \in \{1, 2, \ldots, k\}$. Let λ, μ and ν be three partitions in $P_{k,n}$. Then, $(-1)^{|\lambda|+|\mu|-|\nu|} \operatorname{coeff}_\nu \left(\overline{s_\lambda s_\mu}\right)$ is a polynomial in $b_1, b_2, \ldots, b_k$ with nonnegative integer coefficients. (See Definition 6.2 **(b)** for the meaning of coeff_ν.)

We have verified this conjecture for all $n \leq 8$ using SageMath.

10. The "rim hook algorithm"

We shall next take aim at a recursive formula for "straightening" a Schur polynomial – i.e., representing an $\overline{s_\mu}$, where μ is a partition that does not belong to $P_{k,n}$, as a $\mathbf{k}$-linear combination of "smaller" $\overline{s_\lambda}$'s. However, before we can state this formula, we will have to introduce several new notations.

10.1. Schur polynomials for non-partitions

Recall Definition 5.6. Thus, the elements of P_k are weakly decreasing k-tuples in $\mathbb{N}^k$. For each $\lambda \in P_k$, a Schur polynomial $s_\lambda \in \mathcal{S}$ is defined. Let us extend this definition by defining s_λ for each $\lambda \in \mathbb{Z}^k$:

Definition 10.1. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{Z}^k$. Then, we define a symmetric polynomial $s_\lambda \in \mathcal{S}$ by

$$s_\lambda = \det\left(\left(h_{\lambda_u - u + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right). \qquad (82)$$

This new definition does not clash with the previous use of the notation s_λ, because when $\lambda \in P_k$, both definitions yield the same result (because of Proposition 5.7 **(a)**).

This definition is similar to the definition of $\overline{s}_{(\alpha_1, \alpha_2, \ldots, \alpha_n)}$ in [GriRei18, Exercise 2.9.1 (c)], but we are working with symmetric polynomials rather than symmetric functions here.

Definition 10.1 does not really open the gates to a new world of symmetric polynomials; indeed, each s_α (with $\alpha \in \mathbb{Z}^k$) defined in Definition 10.1 is either 0 or can be rewritten in the form $\pm s_\lambda$ for some $\lambda \in P_k$. Here is a more precise statement of this:

Proposition 10.2. Let $\alpha \in \mathbb{Z}^k$. Define a k-tuple $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ by

$$\left(\beta_i = \alpha_i + k - i \qquad \text{for each } i \in \{1, 2, \ldots, k\}\right).$$

(a) If β has at least one negative entry, then $s_\alpha = 0$.

(b) If β has two equal entries, then $s_\alpha = 0$.

(c) Assume that β has no negative entries and no two equal entries. Let $\sigma \in S_k$ be the permutation such that $\beta_{\sigma(1)} > \beta_{\sigma(2)} > \cdots > \beta_{\sigma(k)}$. (Such a permutation σ exists and is unique, since β has no two equal entries.) Define a k-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{Z}^k$ by

$$\left(\lambda_i = \beta_{\sigma(i)} - k + i \qquad \text{for each } i \in \{1, 2, \ldots, k\}\right).$$

Then, $\lambda \in P_k$ and $s_\alpha = (-1)^\sigma s_\lambda$.

Proof of Proposition 10.2. For each $u \in \{1, 2, \ldots, k\}$, we have $\beta_u = \alpha_u + k - u$ (by the definition of β_u) and thus

$$\underbrace{\beta_u}_{=\alpha_u + k - u} - k = (\alpha_u + k - u) - k = \alpha_u - u. \tag{83}$$

The definition of s_α yields

$$\begin{aligned} s_\alpha &= \det\left(\left(\underbrace{h_{\alpha_u - u + v}}_{\substack{=h_{\beta_u - k + v} \\ \text{(since (83) yields } \alpha_u - u = \beta_u - k)}}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right) \\ &= \det\left(\left(h_{\beta_u - k + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right). \end{aligned} \tag{84}$$

(b) Assume that β has two equal entries. In other words, there are two distinct elements i and j of $\{1, 2, \ldots, k\}$ such that $\beta_i = \beta_j$. Consider these i and j. The i-th and j-th rows of the matrix $\left(h_{\beta_u - k + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}$ are equal (since $\beta_i = \beta_j$). Hence, this matrix has two equal rows. Thus, its determinant is 0. In other words, $\det\left(\left(h_{\beta_u - k + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right) = 0$. Now, (84) becomes $s_\alpha = \det\left(\left(h_{\beta_u - k + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right) = 0$. This proves Proposition 10.2 **(b)**.

(a) Assume that β has at least one negative entry. In other words, there exists some $i \in \{1, 2, \ldots, k\}$ such that $\beta_i < 0$. Consider this i. For each $v \in \{1, 2, \ldots, k\}$, we have $\beta_i - k + \underbrace{v}_{\leq k} \leq \beta_i - k + k = \beta_i < 0$ and thus $h_{\beta_i - k + v} = 0$. Hence, all entries of the i-th row of the matrix $\left(h_{\beta_u - k + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}$ are 0. Hence, this matrix has a zero row. Thus, its determinant is 0. In other words, $\det\left(\left(h_{\beta_u - k + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right) = 0$. Now, (84) becomes $s_\alpha = \det\left(\left(h_{\beta_u - k + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right) = 0$. This proves Proposition 10.2 **(a)**.

(c) It is well-known that if we permute the rows of a $k \times k$-matrix using a permutation τ, then the determinant of the matrix gets multiplied by $(-1)^\tau$. In other words, every $k \times k$-matrix $(b_{u,v})_{1 \leq u \leq k,\ 1 \leq v \leq k}$ and every $\tau \in S_k$ satisfy

$\det\left(\left(b_{\tau(u),v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right) = (-1)^{\tau}\det\left(\left(b_{u,v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right)$. Applying this to $\left(b_{u,v}\right)_{1\leq u\leq k,\ 1\leq v\leq k} = \left(h_{\beta_u-k+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}$ and $\tau=\sigma$, we obtain

$$\det\left(\left(h_{\beta_{\sigma(u)}-k+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right) = (-1)^{\sigma}\det\left(\left(h_{\beta_u-k+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right).$$

Multiplying both sides of this equality by $(-1)^{\sigma}$, we find

$$\begin{aligned}(-1)^{\sigma}\det\left(\left(h_{\beta_{\sigma(u)}-k+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right) &= \underbrace{(-1)^{\sigma}(-1)^{\sigma}}_{=\left((-1)^{\sigma}\right)^2=1}\det\left(\left(h_{\beta_u-k+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right)\\ &= \det\left(\left(h_{\beta_u-k+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right). \qquad (85)\end{aligned}$$

For each $u\in\{1,2,\ldots,k\}$, we have $\lambda_u = \beta_{\sigma(u)}-k+u$ (by the definition of λ_u) and thus

$$\lambda_u - u = \beta_{\sigma(u)} - k. \qquad (86)$$

Now, (84) becomes

$$\begin{aligned}s_{\alpha} &= \det\left(\left(h_{\beta_u-k+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right)\\ &= (-1)^{\sigma}\det\left(\left(\underbrace{h_{\beta_{\sigma(u)}-k+v}}_{\substack{=h_{\lambda_u-u+v}\\ (\text{since (86) yields }\beta_{\sigma(u)}-k=\lambda_u-u)}}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right) \qquad (\text{by (85)})\\ &= (-1)^{\sigma}\underbrace{\det\left(\left(h_{\lambda_u-u+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right)}_{\substack{=s_{\lambda}\\ (\text{by (82)})}} = (-1)^{\sigma}s_{\lambda}.\end{aligned}$$

It remains to prove that $\lambda\in P_k$.

Let $i\in\{1,2,\ldots,k-1\}$. Then, $\beta_{\sigma(i)} > \beta_{\sigma(i+1)}$ (since $\beta_{\sigma(1)} > \beta_{\sigma(2)} > \cdots > \beta_{\sigma(k)}$) and thus $\beta_{\sigma(i)} \geq \beta_{\sigma(i+1)}+1$ (since $\beta_{\sigma(i)}$ and $\beta_{\sigma(i+1)}$ are integers). The definition of λ_{i+1} yields $\lambda_{i+1} = \beta_{\sigma(i+1)}-k+(i+1)$. The definition of λ_i yields

$$\lambda_i = \underbrace{\beta_{\sigma(i)}}_{\geq\beta_{\sigma(i+1)}+1} - k + i \geq \beta_{\sigma(i+1)}+1-k+i = \beta_{\sigma(i+1)}-k+(i+1) = \lambda_{i+1}.$$

Now, forget that we fixed i. We thus have proven that $\lambda_i\geq\lambda_{i+1}$ for each $i\in\{1,2,\ldots,k-1\}$. In other words, $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_k$.

Let $i\in\{1,2,\ldots,k\}$. Then, $1\leq i\leq k$ and thus $k\geq 1$, so that λ_k is well-defined. Furthermore, from $i\leq k$, we obtain $\lambda_i\geq\lambda_k$ (since $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_k$). But the definition of λ_k yields $\lambda_k = \beta_{\sigma(k)} - k + k = \beta_{\sigma(k)} \geq 0$ (since all entries of β are nonnegative (since β has no negative entries)). Thus, $\lambda_i \geq \lambda_k \geq 0$.

Now, forget that we fixed i. We thus have proven that $\lambda_i \geq 0$ for each $i \in \{1, 2, \ldots, k\}$. In other words, $\lambda_1, \lambda_2, \ldots, \lambda_k$ are nonnegative integers (since they are clearly integers). Hence, $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{N}^k$. Combining this with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, we obtain $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in P_k$. Hence, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in P_k$. This completes the proof of Proposition 10.2 **(c)**. □

Let us next recall the bialternant formula for Schur polynomials. We need a few definitions first:

Definition 10.3. **(a)** Let ρ denote the k-tuple $(k-1, k-2, \ldots, 0) \in \mathbb{N}^k$.

(b) We regard $\mathbb{Z}^k$ as a $\mathbb{Z}$-module in the obvious way: Addition is define entrywise (i.e., we set $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_k + \beta_k)$ for any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{Z}^k$ and any $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathbb{Z}^k$). This also defines subtraction on $\mathbb{Z}^k$ (which, too, works entrywise). We let $\mathbf{0}$ denote the k-tuple $\left(\underbrace{0, 0, \ldots, 0}_{k \text{ entries}}\right) \in \mathbb{N}^k \subseteq \mathbb{Z}^k$; this is the zero vector of $\mathbb{Z}^k$.

Definition 10.4. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$. Then, we define the *alternant* $a_\alpha \in \mathcal{P}$ by

$$a_\alpha = \det\left(\left(x_i^{\alpha_j}\right)_{1 \leq i \leq k,\ 1 \leq j \leq k}\right).$$

The two definitions we have just made match the notations in [GriRei18, §2.6], except that we are using k instead of n for the number of indeterminates.

Note that the element a_ρ of $\mathcal{P}$ is the Vandermonde determinant $\det\left(\left(x_i^{k-j}\right)_{1 \leq i \leq k,\ 1 \leq j \leq k}\right) = \prod_{1 \leq i < j \leq k} (x_i - x_j)$; it is a regular element of $\mathcal{P}$ (that is, a non-zero-divisor).

We recall the *bialternant formula* for Schur polynomials ([GriRei18, Corollary 2.6.6]):

Proposition 10.5. For any $\lambda \in P_k$, we have $s_\lambda = a_{\lambda+\rho} / a_\rho$ in $\mathcal{P}$.

Let us extend this fact to arbitrary $\lambda \in \mathbb{Z}^k$ satisfying $\lambda + \rho \in \mathbb{N}^k$ (and rename λ as α):

Proposition 10.6. Let $\alpha \in \mathbb{Z}^k$ be such that $\alpha + \rho \in \mathbb{N}^k$. Then, $s_\alpha = a_{\alpha+\rho} / a_\rho$ in $\mathcal{P}$.

Proof of Proposition 10.6. We have $\rho = (k-1, k-2, \ldots, 0)$. Thus,

$$\rho_i = k - i \qquad \text{for each } i \in \{1, 2, \ldots, k\}. \tag{87}$$

Define a k-tuple $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ as in Proposition 10.2. Thus, for each $i \in \{1, 2, \ldots, k\}$, we have

$$\beta_i = \alpha_i + \underbrace{k - i}_{\substack{=\rho_i \\ \text{(by (87))}}} = \alpha_i + \rho_i = (\alpha + \rho)_i .$$

In other words, $\beta = \alpha + \rho$. Hence, $\beta = \alpha + \rho \in \mathbb{N}^k$. Thus, the k-tuple β has no negative entries.

Moreover, from $\alpha + \rho = \beta$, we obtain

$$\begin{aligned} a_{\alpha+\rho} = a_\beta &= \det\left(\left(x_i^{\beta_j} \right)_{1 \le i \le k,\ 1 \le j \le k} \right) && \left(\text{by the definition of } a_\beta\right) \\ &= \det\left(\left(x_u^{\beta_v} \right)_{1 \le u \le k,\ 1 \le v \le k} \right) && (88) \end{aligned}$$

(here, we have renamed the indices i and j as u and v). Now, we are in one of the following two cases:

Case 1: The k-tuple β has two equal entries.

Case 2: The k-tuple β has no two equal entries.

Let us first consider Case 1. In this case, the k-tuple β has two equal entries. In other words, there are two distinct elements i and j of $\{1, 2, \ldots, k\}$ such that $\beta_i = \beta_j$. Consider these i and j. The i-th and j-th columns of the matrix $\left(x_u^{\beta_v} \right)_{1 \le u \le k,\ 1 \le v \le k}$ are equal (since $\beta_i = \beta_j$). Hence, this matrix has two equal columns. Thus, its determinant is 0. In other words, $\det\left(\left(x_u^{\beta_v} \right)_{1 \le u \le k,\ 1 \le v \le k} \right) = 0$. Now, (88) becomes $a_{\alpha+\rho} = \det\left(\left(x_u^{\beta_v} \right)_{1 \le u \le k,\ 1 \le v \le k} \right) = 0$. Hence, $a_{\alpha+\rho} / a_\rho = 0 / a_\rho = 0$. Comparing this with $s_\alpha = 0$ (which follows from Proposition 10.2 **(b)**), we obtain $s_\alpha = a_{\alpha+\rho} / a_\rho$. Thus, Proposition 10.6 is proven in Case 1.

Let us next consider Case 2. In this case, the k-tuple β has no two equal entries. Thus, there is a unique permutation $\sigma \in S_k$ that sorts this k-tuple into strictly decreasing order. In other words, there is a unique permutation $\sigma \in S_k$ such that $\beta_{\sigma(1)} > \beta_{\sigma(2)} > \cdots > \beta_{\sigma(k)}$. Consider this σ. Define a k-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{Z}^k$ by

$$\left(\lambda_i = \beta_{\sigma(i)} - k + i \qquad \text{for each } i \in \{1, 2, \ldots, k\} \right).$$

Then, Proposition 10.2 **(c)** yields $\lambda \in P_k$ and $s_\alpha = (-1)^\sigma s_\lambda$.

It is well-known that if we permute the columns of a $k \times k$-matrix using a permutation τ, then the determinant of the matrix gets multiplied by $(-1)^\tau$. In other words, every $k \times k$-matrix $(b_{u,v})_{1 \le u \le k,\ 1 \le v \le k}$ and every $\tau \in S_k$ satisfy $\det\left(\left(b_{u,\tau(v)} \right)_{1 \le u \le k,\ 1 \le v \le k} \right) = (-1)^\tau \det\left((b_{u,v})_{1 \le u \le k,\ 1 \le v \le k} \right)$. Applying this to

$(b_{u,v})_{1\leq u\leq k,\ 1\leq v\leq k} = \left(x_u^{\beta_v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}$ and $\tau = \sigma$, we obtain

$$\det\left(\left(x_u^{\beta_{\sigma(v)}}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right) = (-1)^{\sigma}\det\left(\left(x_u^{\beta_v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right). \tag{89}$$

But each $v \in \{1, 2, \ldots, k\}$ satisfies

$$\begin{aligned}(\lambda+\rho)_v &= \underbrace{\lambda_v}_{\substack{=\beta_{\sigma(v)}-k+v \\ \text{(by the definition of } \lambda_v)}} + \underbrace{\rho_v}_{\substack{=k-v \\ \text{(by (87))}}} = \left(\beta_{\sigma(v)} - k + v\right) + (k-v) \\ &= \beta_{\sigma(v)}.\end{aligned} \tag{90}$$

Now, the definition of $a_{\lambda+\rho}$ yields

$$\begin{aligned}a_{\lambda+\rho} &= \det\left(\left(x_i^{(\lambda+\rho)_j}\right)_{1\leq i\leq k,\ 1\leq j\leq k}\right) = \det\left(\left(\underbrace{x_u^{(\lambda+\rho)_v}}_{\substack{=x_u^{\beta_{\sigma(v)}} \\ \text{(by (90))}}}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right) \\ &\qquad \text{(here, we have renamed the indices } i \text{ and } j \text{ as } u \text{ and } v) \\ &= \det\left(\left(x_u^{\beta_{\sigma(v)}}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right) \\ &= (-1)^{\sigma}\underbrace{\det\left(\left(x_u^{\beta_v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right)}_{\substack{=a_{\alpha+\rho} \\ \text{(by (88))}}} \qquad \text{(by (89))} \\ &= (-1)^{\sigma} a_{\alpha+\rho}.\end{aligned}$$

But $\lambda \in P_k$. Hence, Proposition 10.5 yields

$$s_\lambda = \underbrace{a_{\lambda+\rho}}_{=(-1)^{\sigma}a_{\alpha+\rho}} / a_\rho = (-1)^{\sigma} a_{\alpha+\rho}/a_\rho.$$

Hence,

$$s_\alpha = (-1)^{\sigma}\underbrace{s_\lambda}_{=(-1)^{\sigma}a_{\alpha+\rho}/a_\rho} = \underbrace{(-1)^{\sigma}(-1)^{\sigma}}_{=\left((-1)^{\sigma}\right)^2=1} a_{\alpha+\rho}/a_\rho = a_{\alpha+\rho}/a_\rho.$$

Thus, Proposition 10.6 is proven in Case 2.

We have now proven Proposition 10.6 in both Cases 1 and 2. Thus, Proposition 10.6 is proven. □

10.2. The uncancelled Pieri rule

Having defined s_λ for all $\lambda \in \mathbb{Z}^k$ (rather than merely for partitions), we can state a nonstandard version of the Pieri rule for products of the form $s_\lambda h_i$, which will turn out rather useful:

Theorem 10.7. Let $\lambda \in \mathbb{Z}^k$ be such that $\lambda + \rho \in \mathbb{N}^k$. Let $m \in \mathbb{N}$. Then,

$$s_\lambda h_m = \sum_{\substack{\nu \in \mathbb{N}^k; \\ |\nu| = m}} s_{\lambda+\nu}.$$

Example 10.8. For this example, let $k = 3$ and $\lambda = (-2,2,1)$. Then, $\lambda + \rho = (-2,2,1) + (2,1,0) = (0,3,1)$. It is easy to see (using Proposition 10.2 **(c)**) that $s_\lambda = s_{(1)}$.

Furthermore, set $m = 2$. Then, the $\nu \in \mathbb{N}^k$ satisfying $|\nu| = m$ are the six 3-tuples

$$(2,0,0), \qquad (0,2,0), \qquad (0,0,2), \qquad (1,1,0), \qquad (1,0,1), \qquad (0,1,1).$$

Hence, Theorem 10.7 yields

$$\begin{aligned} s_{(-2,2,1)} h_2 &= \sum_{\substack{\nu \in \mathbb{N}^k; \\ |\nu| = m}} s_{(-2,2,1)+\nu} \\ &= \underbrace{s_{(-2,2,1)+(2,0,0)}}_{\substack{= s_{(0,2,1)} = -s_{(1,1,1)} \\ \text{(by Proposition 10.2 }\textbf{(c)}\text{)}}} + \underbrace{s_{(-2,2,1)+(0,2,0)}}_{\substack{= s_{(-2,4,1)} = s_{(3)} \\ \text{(by Proposition 10.2 }\textbf{(c)}\text{)}}} + \underbrace{s_{(-2,2,1)+(0,0,2)}}_{\substack{= s_{(-2,2,3)} = 0 \\ \text{(by Proposition 10.2 }\textbf{(b)}\text{)}}} \\ &\qquad + \underbrace{s_{(-2,2,1)+(1,1,0)}}_{\substack{= s_{(-1,3,1)} = 0 \\ \text{(by Proposition 10.2 }\textbf{(b)}\text{)}}} + \underbrace{s_{(-2,2,1)+(1,0,1)}}_{\substack{= s_{(-1,2,2)} = s_{(1,1,1)} \\ \text{(by Proposition 10.2 }\textbf{(c)}\text{)}}} + \underbrace{s_{(-2,2,1)+(0,1,1)}}_{\substack{= s_{(-2,3,2)} = s_{(2,1)} \\ \text{(by Proposition 10.2 }\textbf{(c)}\text{)}}} \\ &= -s_{(1,1,1)} + s_{(3)} + 0 + 0 + s_{(1,1,1)} + s_{(2,1)} = s_{(2,1)} + s_{(3)}. \end{aligned}$$

In view of $s_{(-2,2,1)} = s_{(1)}$, this rewrites as $s_{(1)} h_2 = s_{(2,1)} + s_{(3)}$, which is exactly what the usual Pieri rule would yield. Note that the expression we obtained from Theorem 10.7 involves both vanishing addends (here, $s_{(-2,2,1)+(0,0,2)}$ and $s_{(-2,2,1)+(1,1,0)}$) and mutually cancelling addends (here, $s_{(-2,2,1)+(2,0,0)}$ and $s_{(-2,2,1)+(1,0,1)}$); this is why I call it the "uncancelled Pieri rule".

We note that the idea of such an "uncancelled Pieri rule" as our Theorem 10.7 is not new (similar things appeared in [LakTho07, §2] and [Tamvak13]), but we have not seen it stated in this exact form anywhere in the literature. Thus, let us give a proof:

Proof of Theorem 10.7. Define $\beta \in \mathbb{N}^k$ by $\beta = \lambda + \rho$. (This is well-defined, since $\lambda + \rho \in \mathbb{N}^k$.)

From (1), we obtain

$$h_m = \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} \underbrace{x^\alpha}_{\substack{= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} \\ = \prod_{i=1}^k x_i^{\alpha_i}}} = \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} \prod_{i=1}^k x_i^{\alpha_i}. \tag{91}$$

For each permutation $\sigma \in S_k$, we have

$$\begin{aligned} h_m &= h_m \left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)} \right) \qquad \text{(since the polynomial } h_m \text{ is symmetric)} \\ &= \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} \prod_{i=1}^k x_{\sigma(i)}^{\alpha_i} \end{aligned} \tag{92}$$

(here, we have substituted $x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}$ for $x_1, x_2, \ldots, x_k$ in the equality (91)).

But Proposition 10.6 (applied to $\alpha = \lambda$) yields $s_\lambda = a_{\lambda+\rho} / a_\rho$ in $\mathcal{P}$. Thus,

$$\begin{aligned} a_\rho s_\lambda &= a_{\lambda+\rho} = a_\beta \qquad (\text{since } \lambda + \rho = \beta) \\ &= \det \left(\left(x_i^{\beta_j} \right)_{1 \le i \le k,\ 1 \le j \le k} \right) \qquad \left(\text{by the definition of } a_\beta \right) \\ &= \det \left(\left(x_j^{\beta_i} \right)_{1 \le i \le k,\ 1 \le j \le k} \right) \\ &\qquad \left(\begin{array}{c} \text{since the determinant of a matrix equals} \\ \text{the determinant of its transpose} \end{array} \right) \\ &= \sum_{\sigma \in S_k} (-1)^\sigma \prod_{i=1}^k x_{\sigma(i)}^{\beta_i} \qquad \left(\text{by the definition of a determinant} \right). \end{aligned}$$

Multiplying both sides of this equality with h_m, we find

$$\begin{aligned}
a_\rho s_\lambda h_m &= \left(\sum_{\sigma \in S_k} (-1)^\sigma \prod_{i=1}^k x_{\sigma(i)}^{\beta_i} \right) h_m = \sum_{\sigma \in S_k} (-1)^\sigma \left(\prod_{i=1}^k x_{\sigma(i)}^{\beta_i} \right) \underbrace{h_m}_{\substack{= \sum\limits_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} \prod\limits_{i=1}^k x_{\sigma(i)}^{\alpha_i} \\ \text{(by (92))}}} \\
&= \sum_{\sigma \in S_k} (-1)^\sigma \left(\prod_{i=1}^k x_{\sigma(i)}^{\beta_i} \right) \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} \prod_{i=1}^k x_{\sigma(i)}^{\alpha_i} \\
&= \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} \sum_{\sigma \in S_k} (-1)^\sigma \underbrace{\left(\prod_{i=1}^k x_{\sigma(i)}^{\beta_i} \right) \prod_{i=1}^k x_{\sigma(i)}^{\alpha_i}}_{= \prod\limits_{i=1}^k \left(x_{\sigma(i)}^{\beta_i} x_{\sigma(i)}^{\alpha_i} \right)} \\
&= \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} \sum_{\sigma \in S_k} (-1)^\sigma \prod_{i=1}^k \underbrace{\left(x_{\sigma(i)}^{\beta_i} x_{\sigma(i)}^{\alpha_i} \right)}_{\substack{= x_{\sigma(i)}^{\beta_i + \alpha_i} = x_{\sigma(i)}^{(\beta+\alpha)_i} \\ \text{(since } \beta_i + \alpha_i = (\beta+\alpha)_i \text{)}}} \\
&= \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} \sum_{\sigma \in S_k} (-1)^\sigma \prod_{i=1}^k x_{\sigma(i)}^{(\beta+\alpha)_i} \\
&= \sum_{\substack{\nu \in \mathbb{N}^k; \\ |\nu| = m}} \sum_{\sigma \in S_k} (-1)^\sigma \prod_{i=1}^k x_{\sigma(i)}^{(\beta+\nu)_i} \qquad (93)
\end{aligned}$$

(here, we have renamed the summation index α as ν).

On the other hand, let $\nu \in \mathbb{N}^k$. Then, $(\lambda + \nu) + \rho = \underbrace{(\lambda + \rho)}_{\in \mathbb{N}^k} + \underbrace{\nu}_{\in \mathbb{N}^k} \in \mathbb{N}^k$. Thus, Proposition 10.6 (applied to $\alpha = \lambda + \nu$) yields $s_{\lambda+\nu} = a_{(\lambda+\nu)+\rho} / a_\rho$ in $\mathcal{P}$.

Thus,

$$\begin{aligned}
a_\rho s_{\lambda+\nu} &= a_{(\lambda+\nu)+\rho} = a_{\beta+\nu} \qquad \left(\text{since } (\lambda+\nu)+\rho = \underbrace{\lambda+\rho}_{=\beta}+\nu = \beta+\nu\right) \\
&= \det\left(\left(x_i^{(\beta+\nu)_j}\right)_{1\leq i\leq k,\ 1\leq j\leq k}\right) \qquad \left(\text{by the definition of } a_{\beta+\nu}\right) \\
&= \det\left(\left(x_j^{(\beta+\nu)_i}\right)_{1\leq i\leq k,\ 1\leq j\leq k}\right) \\
&\qquad \left(\begin{array}{c}\text{since the determinant of a matrix equals} \\ \text{the determinant of its transpose}\end{array}\right) \\
&= \sum_{\sigma\in S_k} (-1)^\sigma \prod_{i=1}^{k} x_{\sigma(i)}^{(\beta+\nu)_i} \qquad (94)
\end{aligned}$$

(by the definition of a determinant).

Now, forget that we fixed ν. We thus have proven (94) for each $\nu \in \mathbb{N}^k$.

Now, (93) becomes

$$a_\rho s_\lambda h_m = \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=m}} \underbrace{\sum_{\sigma\in S_k} (-1)^\sigma \prod_{i=1}^{k} x_{\sigma(i)}^{(\beta+\nu)_i}}_{\substack{=a_\rho s_{\lambda+\nu} \\ \text{(by (94))}}} = \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=m}} a_\rho s_{\lambda+\nu} = a_\rho \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=m}} s_{\lambda+\nu}.$$

We can cancel a_ρ from this equality (since a_ρ is a regular element of $\mathcal{P}$), and thus obtain

$$s_\lambda h_m = \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=m}} s_{\lambda+\nu}.$$

This proves Theorem 10.7. □

10.3. The “rim hook algorithm”

For the rest of this section, we assume that $k > 0$.

We need one more weird definition:

Definition 10.9. Let V be the set of all k-tuples $(-n, \tau_2, \tau_3, \ldots, \tau_k) \in \mathbb{Z}^k$ satisfying

$$(\tau_i \in \{0,1\} \qquad \text{for each } i \in \{2,3,\ldots,k\}). \qquad (95)$$

Example 10.10. If $n = 6$ and $k = 3$, then

$$V = \{(-6,0,0), (-6,0,1), (-6,1,0), (-6,1,1)\}. \tag{96}$$

Proposition 10.11. Let $\tau \in V$. Then, $-|\tau| \in \{n-k+1, n-k+2, \ldots, n\}$.

Proof of Proposition 10.11. We have $\tau \in V$. Thus, τ has the form $\tau = (-n, \tau_2, \tau_3, \ldots, \tau_k) \in \mathbb{Z}^k$ for some $\tau_2, \tau_3, \ldots, \tau_k$ satisfying (95) (by the definition of V). Consider these $\tau_2, \tau_3, \ldots, \tau_k$. We have $\tau = (-n, \tau_2, \tau_3, \ldots, \tau_k)$ and thus

$$|\tau| = (-n) + \underbrace{\tau_2 + \tau_3 + \cdots + \tau_k}_{= \sum_{i=2}^{k} \tau_i} = (-n) + \sum_{i=2}^{k} \tau_i$$

and thus

$$\begin{aligned} -|\tau| &= -\left((-n) + \sum_{i=2}^{k} \tau_i\right) = n - \sum_{i=2}^{k} \underbrace{\tau_i}_{\substack{\leq 1 \\ \text{(since (95)} \\ \text{yields } \tau_i \in \{0,1\})}} \geq n - \underbrace{\sum_{i=2}^{k} 1}_{=k-1} \\ &= n - (k-1) = n - k + 1. \end{aligned}$$

Combining this with

$$-|\tau| = n - \sum_{i=2}^{k} \underbrace{\tau_i}_{\substack{\geq 0 \\ \text{(since (95)} \\ \text{yields } \tau_i \in \{0,1\})}} \leq n - \underbrace{\sum_{i=2}^{k} 0}_{=0} = n,$$

we obtain $n - k + 1 \leq -|\tau| \leq n$. Thus, $-|\tau| \in \{n-k+1, n-k+2, \ldots, n\}$ (since $-|\tau|$ is an integer). This proves Proposition 10.11. ☐

We are now ready to state the main theorem of this section: a generalization of the "rim hook algorithm" from [BeCiFu99, §2, Main Lemma]:

Theorem 10.12. Assume that $a_1, a_2, \ldots, a_k$ belong to $\mathbf{k}$.

Let $\mu \in P_k$ be such that $\mu_1 > n - k$. Then,

$$\overline{s_\mu} = \sum_{j=1}^{k} (-1)^{k-j} a_j \sum_{\substack{\tau \in V; \\ -|\tau| = n-k+j}} \overline{s_{\mu+\tau}}.$$

Example 10.13. For this example, set $n = 6$ and $k = 3$ and $\mu = (5, 4, 2)$. Then, Theorem 10.12 yields

$$\begin{aligned}
&\overline{s_{(5,4,2)}} \\
&= \sum_{j=1}^{k} (-1)^{k-j} a_j \sum_{\substack{\tau \in V; \\ -|\tau| = n-k+j}} \overline{s_{(5,4,2)+\tau}} \\
&= (-1)^{3-1} a_1 \overline{s_{(5,4,1)+(-6,1,1)}} + (-1)^{3-2} a_2 \left(\overline{s_{(5,4,1)+(-6,0,1)}} + \overline{s_{(5,4,1)+(-6,1,0)}} \right) \\
&\qquad + (-1)^{3-3} a_3 \overline{s_{(5,4,1)+(-6,0,0)}} \qquad \text{(by (96))} \\
&= a_1 \underbrace{\overline{s_{(5,4,1)+(-6,1,1)}}}_{\substack{= \overline{s_{(-1,5,2)}} \\ = \overline{s_{(4,1,1)}} \\ \text{(by Proposition 10.2 } \textbf{(c)}\text{)}}} - a_2 \left(\underbrace{\overline{s_{(5,4,1)+(-6,0,1)}}}_{\substack{= \overline{s_{(-1,4,2)}} \\ = \overline{s_{(3,1,1)}} \\ \text{(by Proposition 10.2 } \textbf{(c)}\text{)}}} + \underbrace{\overline{s_{(5,4,1)+(-6,1,0)}}}_{\substack{= \overline{s_{(-1,5,1)}} \\ = 0 \\ \text{(by Proposition 10.2 } \textbf{(b)}\text{)}}} \right) \\
&\qquad + a_3 \underbrace{\overline{s_{(5,4,1)+(-6,0,0)}}}_{\substack{= \overline{s_{(-1,4,1)}} \\ = 0 \\ \text{(by Proposition 10.2 } \textbf{(b)}\text{)}}} \\
&= a_1 \overline{s_{(4,1,1)}} - a_2 \overline{s_{(3,1,1)}}.
\end{aligned}$$

Note that this is **not** yet an expansion of $\overline{s_\mu}$ in the basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$. Indeed, we still have a term $\overline{s_{(4,1,1)}}$ on the right hand side which has $(4, 1, 1) \notin P_{k,n}$. But this term can, in turn, be rewritten using Theorem 10.12, and so on until we end up with an expansion of $\overline{s_\mu}$ in the basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$, namely

$$\overline{s_{(5,4,2)}} = -a_2 \overline{s_{(3,1,1)}} + a_1^2 \overline{s_{(1,1)}} - a_1 a_2 \overline{s_{(1)}} + a_1 a_3 \overline{s_{()}}.$$

As we saw in this example, when we apply Theorem 10.12, some of the $\overline{s_{\mu+\tau}}$ addends on the right hand side may be 0 (by Proposition 10.2 **(b)**). Once these addends are removed, the remaining addends can be rewritten in the form $\pm \overline{s_\lambda}$ for some $\lambda \in P_k$ satisfying $|\lambda| < |\mu|$ (using Proposition 10.2 **(c)**). The resulting sum is multiplicity-free – in the sense that no $\overline{s_\lambda}$ occurs more than once in it. (This is not difficult to check, but would take us too far afield.) However, this sum is (in general) not an expansion of $\overline{s_\mu}$ in the basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ yet, because it often contains terms $\overline{s_\lambda}$ with $\lambda \notin P_{k,n}$. If we keep applying Theorem 10.12 multiple times until we reach an expansion of $\overline{s_\mu}$ in the basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$, then this latter expansion may contain multiplicities: For example, for $n = 6$ and $k = 3$, we have

$$\overline{s_{(4,4,3)}} = -a_2\overline{s_{(3,3)}} + a_3\overline{s_{(3,2)}} + a_1^2\overline{s_{(3)}} - 2a_1a_2\overline{s_{(2)}} + a_2^2\overline{s_{(1)}}.$$

We owe the reader an explanation of why we call Theorem 10.12 a "rim hook algorithm". It owes this name to the fact that it generalizes the "rim hook algorithm" for quantum cohomology [BeCiFu99, §2, Main Lemma] (which can be obtained from it with some work by setting $a_i = 0$ for all $i < k$). Nevertheless, it does not visibly involve any rim hooks itself. I am, in fact, unaware of a way to restate it in the language of Young diagrams; the operation $\mu \mapsto \mu + \tau$ for $\tau \in V$ resembles both the removal of an n-rim hook (since it lowers the first entry by n) and the addition of a vertical strip (since it increases each of the remaining entries by 0 or 1), but it cannot be directly stated as one of these operations followed by the other.

We shall prove Theorem 10.12 by deriving it from an identity in $\mathcal{S}$:

Theorem 10.14. Let $\mu \in P_k$ be such that $\mu_1 > n - k$. Then,

$$s_\mu = \sum_{j=1}^{k} (-1)^{k-j} h_{n-k+j} \sum_{\substack{\tau \in V; \\ -|\tau| = n-k+j}} s_{\mu+\tau}.$$

Our proof of this identity, in turn, will rely on the following combinatorial lemmas:

Lemma 10.15. Let $j \in \{2, 3, \ldots, k\}$. Let Δ be the vector $(0, 0, \ldots, 0, 1, 0, 0, \ldots, 0) \in \mathbb{Z}^k$, where 1 is the j-th entry.

(a) If $\tau \in V$ satisfies $\tau_j = 0$, then $\tau + \Delta \in V$ and $(\tau + \Delta)_j = 1$.

(b) If $\tau \in V$ satisfies $\tau_j = 1$, then $\tau - \Delta \in V$ and $(\tau - \Delta)_j = 0$.

(c) If $\nu \in \mathbb{N}^k$ satisfies $\nu_j \neq 0$, then $\nu - \Delta \in \mathbb{N}^k$.

(d) If $\nu \in \mathbb{N}^k$, then $\nu + \Delta \in \mathbb{N}^k$.

Proof of Lemma 10.15. We have $j \in \{2, 3, \ldots, k\}$, thus $j \neq 1$.

We have $\Delta = (0, 0, \ldots, 0, 1, 0, 0, \ldots, 0) \in \mathbb{N}^k$. Thus, $\Delta_j = 1$ and

$$(\Delta_i = 0 \qquad \text{for each } i \in \{1, 2, \ldots, k\} \text{ satisfying } i \neq j). \tag{97}$$

Applying (97) to $i = 1$, we obtain $\Delta_1 = 0$ (since $1 \neq j$).

(a) Let $\tau \in V$ be such that $\tau_j = 0$.

We have $\tau \in V$. According to the definition of V, this means that τ is a k-tuple $(-n, \tau_2, \tau_3, \ldots, \tau_k) \in \mathbb{Z}^k$ satisfying (95). In other words, $\tau \in \mathbb{Z}^k$ and $\tau_1 = -n$ and

$$(\tau_i \in \{0, 1\} \qquad \text{for each } i \in \{2, 3, \ldots, k\}). \tag{98}$$

Define a k-tuple $\sigma \in \mathbb{Z}^k$ by $\sigma = \tau + \Delta$. Thus, $\sigma_1 = (\tau + \Delta)_1 = \tau_1 + \underbrace{\Delta_1}_{=0} = \tau_1 = -n$.

Furthermore, from $\sigma = \tau + \Delta$, we obtain $\sigma_j = (\tau + \Delta)_j = \underbrace{\tau_j}_{=0} + \underbrace{\Delta_j}_{=1} = 1 \in \{0, 1\}$.

Next, we have $\sigma_i \in \{0, 1\}$ for each $i \in \{2, 3, \ldots, k\}$ [19]. Altogether, we thus have shown that $\sigma \in \mathbb{Z}^k$ and $\sigma_1 = -n$ and

$$(\sigma_i \in \{0, 1\} \qquad \text{for each } i \in \{2, 3, \ldots, k\}) . \tag{99}$$

In other words, σ is a k-tuple $(-n, \sigma_2, \sigma_3, \ldots, \sigma_k) \in \mathbb{Z}^k$ satisfying (99). In other words, $\sigma \in V$ (by the definition of V). Thus, $\tau + \Delta = \sigma \in V$. So we have proven that $\tau + \Delta \in V$ and $(\tau + \Delta)_j = 1$. Thus, Lemma 10.15 **(a)** is proven.

(b) The proof of Lemma 10.15 **(b)** is analogous to the above proof of Lemma 10.15 **(a)**, and is left to the reader.

(c) Let $\nu \in \mathbb{N}^k$ be such that $\nu_j \neq 0$. We must prove that $\nu - \Delta \in \mathbb{N}^k$.

We have $\nu_j \in \mathbb{N}$ (since $\nu \in \mathbb{N}^k$). Hence, from $\nu_j \neq 0$, we conclude that $\nu_j \geq 1$. Thus, $\nu_j - 1 \in \mathbb{N}$. Also, the entries $\nu_1, \nu_2, \ldots, \nu_{j-1}, \nu_{j+1}, \nu_{j+2}, \ldots, \nu_k$ of ν belong to $\mathbb{N}$ (since $\nu \in \mathbb{N}^k$).

Recall that Δ is the vector $(0, 0, \ldots, 0, 1, 0, 0, \ldots, 0) \in \mathbb{Z}^k$, where 1 is the j-th entry. Hence,

$$\begin{aligned} \nu - \Delta &= \nu - (0, 0, \ldots, 0, 1, 0, 0, \ldots, 0) \\ &= \left(\nu_1, \nu_2, \ldots, \nu_{j-1}, \nu_j - 1, \nu_{j+1}, \nu_{j+2}, \ldots, \nu_k\right) \in \mathbb{N}^k \end{aligned}$$

(since $\nu_j - 1 \in \mathbb{N}$ and since the entries $\nu_1, \nu_2, \ldots, \nu_{j-1}, \nu_{j+1}, \nu_{j+2}, \ldots, \nu_k$ of ν belong to $\mathbb{N}$). This proves Lemma 10.15 **(c)**.

(d) Let $\nu \in \mathbb{N}^k$. Also, $\Delta \in \mathbb{N}^k$. Thus, $\underbrace{\nu}_{\in \mathbb{N}^k} + \underbrace{\Delta}_{\in \mathbb{N}^k} \in \mathbb{N}^k$. This proves Lemma 10.15 **(d)**. □

Lemma 10.16. Let $\gamma \in \mathbb{Z}^k$. Then,

$$\sum_{\tau \in V} \sum_{\substack{\nu \in \mathbb{N}^k; \\ |\nu| = -|\tau|; \\ \nu + \tau = \gamma}} (-1)^{n + |\tau|} = \begin{cases} 1, & \text{if } \gamma = \mathbf{0}; \\ 0, & \text{if } \gamma \neq \mathbf{0} \end{cases} .$$

(Recall that $\mathbf{0}$ denotes the vector $\left(\underbrace{0, 0, \ldots, 0}_{k \text{ zeroes}}\right) \in \mathbb{Z}^k$.)

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[19] *Proof.* Let $i \in \{2, 3, \ldots, k\}$. We must prove $\sigma_i \in \{0, 1\}$.

If $i = j$, then this follows from $\sigma_j \in \{0, 1\}$. Hence, for the rest of this proof, we WLOG assume that $i \neq j$. Thus, (97) yields $\Delta_i = 0$. Now, from $\sigma = \tau + \Delta$, we obtain $\sigma_i = (\tau + \Delta)_i = \tau_i + \underbrace{\Delta_i}_{=0} = \tau_i \in \{0, 1\}$ (by (98)). Qed.

Proof of Lemma 10.16. Let Q be the set of all pairs $(\tau,\nu) \in V \times \mathbb{N}^k$ satisfying $|\nu| = -|\tau|$ and $\nu + \tau = \gamma$. We have the following equality of summation signs:

$$\sum_{\tau \in V} \sum_{\substack{\nu \in \mathbb{N}^k; \\ |\nu| = -|\tau|; \\ \nu+\tau=\gamma}} = \sum_{\substack{(\tau,\nu) \in V \times \mathbb{N}^k; \\ |\nu| = -|\tau|; \\ \nu+\tau=\gamma}} = \sum_{(\tau,\nu) \in Q} \tag{100}$$

(since Q is the set of all pairs $(\tau,\nu) \in V \times \mathbb{N}^k$ satisfying $|\nu| = -|\tau|$ and $\nu + \tau = \gamma$).

We are in one of the following three cases:

Case 1: We have $(\gamma_2, \gamma_3, \ldots, \gamma_k) \neq (0,0,\ldots,0)$.

Case 2: We have $(\gamma_2, \gamma_3, \ldots, \gamma_k) = (0,0,\ldots,0)$ and $\gamma_1 \neq 0$.

Case 3: We have $(\gamma_2, \gamma_3, \ldots, \gamma_k) = (0,0,\ldots,0)$ and $\gamma_1 = 0$.

Let us first consider Case 1. In this case, we have $(\gamma_2, \gamma_3, \ldots, \gamma_k) \neq (0,0,\ldots,0)$. In other words, there exists a $j \in \{2,3,\ldots,k\}$ such that $\gamma_j \neq 0$. Consider such a j. Clearly, $\gamma \neq \mathbf{0}$ (since $\gamma_j \neq 0$). Hence, $\begin{cases} 1, & \text{if } \gamma = \mathbf{0}; \\ 0, & \text{if } \gamma \neq \mathbf{0} \end{cases} = 0$.

Let Δ be the vector $(0,0,\ldots,0,1,0,0,\ldots,0) \in \mathbb{Z}^k$, where 1 is the j-th entry. Clearly, $\Delta \in \mathbb{N}^k$ and $|\Delta| = 1$.

Let Q_0 be the set of all $(\tau,\nu) \in Q$ satisfying $\tau_j = 0$. (Recall that τ_j denotes the j-th entry of the k-tuple $\tau \in V \subseteq \mathbb{Z}^k$.) Let Q_1 be the set of all $(\tau,\nu) \in Q$ satisfying $\tau_j = 1$. Each $(\tau,\nu) \in Q$ satisfies $(\tau,\nu) \in V \times \mathbb{N}^k$ (by the definition of Q) and thus $\tau \in V$ and thus $\tau_j \in \{0,1\}$ (by (95), applied to $i = j$). In other words, each $(\tau,\nu) \in Q$ satisfies either $\tau_j = 0$ or $\tau_j = 1$ (but not both at the same time). In other words, each $(\tau,\nu) \in Q$ belongs to either Q_0 or Q_1 (but not both at the same time).

For each $(\tau,\nu) \in Q_0$, we have $(\tau + \Delta, \nu - \Delta) \in Q_1$ [20]. Thus, the map

$$Q_0 \to Q_1, \qquad (\tau,\nu) \mapsto (\tau+\Delta, \nu-\Delta) \tag{101}$$

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[20] *Proof.* Let $(\tau,\nu) \in Q_0$. According to the definition of Q_0, this means that $(\tau,\nu) \in Q$ and $\tau_j = 0$.

We have $(\tau,\nu) \in Q$. According to the definition of Q, this means that $(\tau,\nu) \in V \times \mathbb{N}^k$ and $|\nu| = -|\tau|$ and $\nu + \tau = \gamma$.

From $(\tau,\nu) \in V \times \mathbb{N}^k$, we obtain $\tau \in V$ and $\nu \in \mathbb{N}^k$.

From $\nu + \tau = \gamma$, we obtain $(\nu+\tau)_j = \gamma_j$. Hence, $\gamma_j = (\nu+\tau)_j = \nu_j + \underbrace{\tau_j}_{=0} = \nu_j$. Thus, $\nu_j = \gamma_j \neq 0$. Thus, Lemma 10.15 **(c)** yields $\nu - \Delta \in \mathbb{N}^k$. Also, Lemma 10.15 **(a)** yields that $\tau + \Delta \in V$ and $(\tau+\Delta)_j = 1$. Also, any two k-tuples $\alpha \in \mathbb{N}^k$ and $\beta \in \mathbb{N}^k$ satisfy $|\alpha+\beta| = |\alpha| + |\beta|$ and $|\alpha - \beta| = |\alpha| - |\beta|$. Thus, $|\tau + \Delta| = |\tau| + |\Delta|$ and $|\nu - \Delta| = \underbrace{|\nu|}_{=-|\tau|} - |\Delta| = -|\tau| - |\Delta| = -\underbrace{(|\tau| + |\Delta|)}_{=|\tau+\Delta|} = -|\tau+\Delta|$. Also, $(\nu-\Delta) + (\tau+\Delta) = \nu + \tau = \gamma$.

From $\tau + \Delta \in V$ and $\nu - \Delta \in \mathbb{N}^k$ and $|\nu - \Delta| = -|\tau+\Delta|$ and $(\nu-\Delta)+(\tau+\Delta) = \gamma$, we obtain $(\tau+\Delta, \nu-\Delta) \in Q$ (by the definition of Q). Combining this with $(\tau+\Delta)_j = 1$, we obtain $(\tau+\Delta, \nu-\Delta) \in Q_1$ (by the definition of Q_1), qed.

is well-defined.

For each $(\tau, \nu) \in Q_1$, we have $(\tau - \Delta, \nu + \Delta) \in Q_0$ [21]. Thus, the map

$$Q_1 \to Q_0, \qquad (\tau, \nu) \mapsto (\tau - \Delta, \nu + \Delta) \tag{102}$$

is well-defined.

The two maps (101) and (102) are mutually inverse (this is clear from their definitions), and thus are bijections. Hence, in particular, the map (101) is a bijection.

Also, each $\tau \in \mathbb{Z}^k$ satisfies

$$\begin{aligned} |\tau + \Delta| &= |\tau| + \underbrace{|\Delta|}_{=1} \qquad \left(\text{since } |\alpha + \beta| = |\alpha| + |\beta| \text{ for all } \alpha \in \mathbb{Z}^k \text{ and } \beta \in \mathbb{Z}^k\right) \\ &= |\tau| + 1 \end{aligned}$$

and thus

$$(-1)^{n+|\tau+\Delta|} = (-1)^{n+|\tau|+1} = -(-1)^{n+|\tau|}. \tag{103}$$

Now, recall that Q_0 and Q_1 are two subsets of Q such that each $(\tau, \nu) \in Q$ belongs to either Q_0 or Q_1 (but not both at the same time). In other words, Q_0 and Q_1 are two disjoint subsets of Q whose union is the whole set Q. Hence, we

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[21] *Proof.* Let $(\tau, \nu) \in Q_1$. According to the definition of Q_1, this means that $(\tau, \nu) \in Q$ and $\tau_j = 1$.

We have $(\tau, \nu) \in Q$. According to the definition of Q, this means that $(\tau, \nu) \in V \times \mathbb{N}^k$ and $|\nu| = -|\tau|$ and $\nu + \tau = \gamma$.

From $(\tau, \nu) \in V \times \mathbb{N}^k$, we obtain $\tau \in V$ and $\nu \in \mathbb{N}^k$.

Lemma 10.15 **(d)** yields $\nu + \Delta \in \mathbb{N}^k$. Also, Lemma 10.15 **(b)** yields that $\tau - \Delta \in V$ and $(\tau - \Delta)_j = 0$. Also, any two k-tuples $\alpha \in \mathbb{N}^k$ and $\beta \in \mathbb{N}^k$ satisfy $|\alpha + \beta| = |\alpha| + |\beta|$ and $|\alpha - \beta| = |\alpha| - |\beta|$. Thus, $|\tau - \Delta| = |\tau| - |\Delta|$ and $|\nu + \Delta| = \underbrace{|\nu|}_{=-|\tau|} + |\Delta| = -|\tau| + |\Delta| = -\underbrace{(|\tau| - |\Delta|)}_{=|\tau - \Delta|} = -|\tau - \Delta|$. Also, $(\nu + \Delta) + (\tau - \Delta) = \nu + \tau = \gamma$.

From $\tau - \Delta \in V$ and $\nu + \Delta \in \mathbb{N}^k$ and $|\nu + \Delta| = -|\tau - \Delta|$ and $(\nu + \Delta) + (\tau - \Delta) = \gamma$, we obtain $(\tau - \Delta, \nu + \Delta) \in Q$ (by the definition of Q). Combining this with $(\tau - \Delta)_j = 0$, we obtain $(\tau - \Delta, \nu + \Delta) \in Q_0$ (by the definition of Q_0), qed.

can split the sum $\sum\limits_{(\tau,\nu)\in Q} (-1)^{n+|\tau|}$ as follows:

$$\begin{aligned}
\sum_{(\tau,\nu)\in Q} (-1)^{n+|\tau|} &= \sum_{(\tau,\nu)\in Q_0} (-1)^{n+|\tau|} + \underbrace{\sum_{(\tau,\nu)\in Q_1} (-1)^{n+|\tau|}}_{\substack{=\sum\limits_{(\tau,\nu)\in Q_0} (-1)^{n+|\tau+\Delta|} \\ \text{(here, we have substituted } (\tau+\Delta,\nu-\Delta) \text{ for } (\tau,\nu) \\ \text{in the sum, since the map (101) is a bijection)}}} \\
&= \sum_{(\tau,\nu)\in Q_0} (-1)^{n+|\tau|} + \sum_{(\tau,\nu)\in Q_0} \underbrace{(-1)^{n+|\tau+\Delta|}}_{\substack{=-(-1)^{n+|\tau|} \\ \text{(by (103))}}} \\
&= \sum_{(\tau,\nu)\in Q_0} (-1)^{n+|\tau|} + \sum_{(\tau,\nu)\in Q_0} \left(-(-1)^{n+|\tau|}\right) \\
&= \sum_{(\tau,\nu)\in Q_0} (-1)^{n+|\tau|} - \sum_{(\tau,\nu)\in Q_0} (-1)^{n+|\tau|} = 0.
\end{aligned}$$

Now, (100) yields

$$\sum_{\tau\in V} \sum_{\substack{\nu\in\mathbb{N}^k; \\ |\nu|=-|\tau|; \\ \nu+\tau=\gamma}} (-1)^{n+|\tau|} = \sum_{(\tau,\nu)\in Q} (-1)^{n+|\tau|} = 0 = \begin{cases} 1, & \text{if } \gamma = \mathbf{0}; \\ 0, & \text{if } \gamma \neq \mathbf{0} \end{cases}$$

(since $\begin{cases} 1, & \text{if } \gamma = \mathbf{0}; \\ 0, & \text{if } \gamma \neq \mathbf{0} \end{cases} = 0$). Thus, Lemma 10.16 is proven in Case 1.

Let us now consider Case 2. In this case, we have $(\gamma_2, \gamma_3, \ldots, \gamma_k) = (0, 0, \ldots, 0)$ and $\gamma_1 \neq 0$. From $\gamma_1 \neq 0$, we obtain $\gamma \neq \mathbf{0}$ and thus $\begin{cases} 1, & \text{if } \gamma = \mathbf{0}; \\ 0, & \text{if } \gamma \neq \mathbf{0} \end{cases} = 0$.

Now, $Q = \varnothing$ [22]. But (100) yields

$$\sum_{\tau \in V} \sum_{\substack{\nu \in \mathbb{N}^k; \\ |\nu| = -|\tau|; \\ \nu + \tau = \gamma}} (-1)^{n+|\tau|} = \sum_{(\tau,\nu) \in Q} (-1)^{n+|\tau|} = (\text{empty sum}) \qquad (\text{since } Q = \varnothing)$$

$$= 0 = \begin{cases} 1, & \text{if } \gamma = \mathbf{0}; \\ 0, & \text{if } \gamma \neq \mathbf{0} \end{cases}$$

(since $\begin{cases} 1, & \text{if } \gamma = \mathbf{0}; \\ 0, & \text{if } \gamma \neq \mathbf{0} \end{cases} = 0$). Thus, Lemma 10.16 is proven in Case 2.

Let us finally consider Case 3. In this case, we have $(\gamma_2, \gamma_3, \ldots, \gamma_k) = (0, 0, \ldots, 0)$

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[22] *Proof.* Let $(\tau, \nu) \in Q$. We shall derive a contradiction.

Indeed, we have $(\tau, \nu) \in Q$. According to the definition of Q, this means that $(\tau, \nu) \in V \times \mathbb{N}^k$ and $|\nu| = -|\tau|$ and $\nu + \tau = \gamma$.

From $(\tau, \nu) \in V \times \mathbb{N}^k$, we obtain $\tau \in V$ and $\nu \in \mathbb{N}^k$.

We have $\tau \in V$. According to the definition of V, this means that τ is a k-tuple $(-n, \tau_2, \tau_3, \ldots, \tau_k) \in \mathbb{Z}^k$ satisfying (95). In other words, $\tau \in \mathbb{Z}^k$ and $\tau_1 = -n$ and the condition (95) holds.

Now, fix $j \in \{2, 3, \ldots, k\}$. Then, $\tau_j \in \{0, 1\}$ (by (95), applied to $i = j$). Hence, $\tau_j \geq 0$. Also, $\nu_j \in \mathbb{N}$ (since $\nu \in \mathbb{N}^k$), so that $\nu_j \geq 0$. But $(\gamma_2, \gamma_3, \ldots, \gamma_k) = (0, 0, \ldots, 0)$, and thus $\gamma_j = 0$ (since $j \in \{2, 3, \ldots, k\}$). But $\gamma = \nu + \tau$, and thus $\gamma_j = (\nu + \tau)_j = \nu_j + \tau_j$. Hence, $\nu_j + \tau_j = \gamma_j = 0$, so that $\nu_j = -\underbrace{\tau_j}_{\geq 0} \leq 0$. Combining this with $\nu_j \geq 0$, we obtain $\nu_j = 0$. Hence, $\nu_j = -\tau_j$ rewrites as $0 = -\tau_j$, so that $\tau_j = 0$.

Now, forget that we fixed j. Thus, we have shown that each $j \in \{2, 3, \ldots, k\}$ satisfies

$$\nu_j = 0 \tag{104}$$

and

$$\tau_j = 0. \tag{105}$$

Now,

$$|\tau| = \tau_1 + \tau_2 + \cdots + \tau_k = \sum_{j=1}^{k} \tau_j = \tau_1 + \sum_{j=2}^{k} \underbrace{\tau_j}_{\substack{=0 \\ (\text{by } (105))}} = \tau_1 = -n,$$

so that $-|\tau| = n$. Furthermore,

$$|\nu| = \nu_1 + \nu_2 + \cdots + \nu_k = \sum_{j=1}^{k} \nu_j = \nu_1 + \sum_{j=2}^{k} \underbrace{\nu_j}_{\substack{=0 \\ (\text{by } (104))}} = \nu_1,$$

so that $\nu_1 = |\nu| = -|\tau| = n$.

Now, from $\gamma = \nu + \tau$, we obtain $\gamma_1 = (\nu + \tau)_1 = \underbrace{\nu_1}_{=n} + \underbrace{\tau_1}_{=-n} = n + (-n) = 0$. This contradicts $\gamma_1 \neq 0$.

Now, forget that we fixed (τ, ν). We thus have found a contradiction for each $(\tau, \nu) \in Q$. Thus, there exists no $(\tau, \nu) \in Q$. In other words, $Q = \varnothing$.

and $\gamma_1 = 0$. Combining these two equalities, we obtain $\gamma_i = 0$ for all $i \in \{1, 2, \ldots, k\}$. In other words, $\gamma = \mathbf{0}$. Hence, $\begin{cases} 1, & \text{if } \gamma = \mathbf{0}; \\ 0, & \text{if } \gamma \neq \mathbf{0} \end{cases} = 1$.

Now, define two k-tuples $\tau_0 \in \mathbb{Z}^k$ and $\nu_0 \in \mathbb{Z}^k$ by

$$\tau_0 = (-n, 0, 0, \ldots, 0) \qquad \text{and} \qquad \nu_0 = (n, 0, 0, \ldots, 0).$$

Clearly, $\tau_0 \in V$ (by the definition of V) and $\nu_0 \in \mathbb{N}^k$ and $|\tau_0| = -n$ and $|\nu_0| = n$ and $\nu_0 + \tau_0 = \mathbf{0}$.

From $\tau_0 \in V$ and $\nu_0 \in \mathbb{N}^k$, we obtain $(\tau_0, \nu_0) \in V \times \mathbb{N}^k$. Also, $|\nu_0| = -|\tau_0|$ (since $\underbrace{|\nu_0|}_{=n} + \underbrace{|\tau_0|}_{=-n} = n + (-n) = 0$) and $\nu_0 + \tau_0 = \mathbf{0} = \gamma$. Thus, we have shown that $(\tau_0, \nu_0) \in V \times \mathbb{N}^k$ and $|\nu_0| = -|\tau_0|$ and $\nu_0 + \tau_0 = \gamma$. In other words, $(\tau_0, \nu_0) \in Q$ (by the definition of Q). In other words, $\{(\tau_0, \nu_0)\} \subseteq Q$.

On the other hand, $Q \subseteq \{(\tau_0, \nu_0)\}$ [23]. Combining this with $\{(\tau_0, \nu_0)\} \subseteq Q$, we obtain $Q = \{(\tau_0, \nu_0)\}$.

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[23] *Proof.* Let $(\tau, \nu) \in Q$. We shall prove that $(\tau, \nu) = (\tau_0, \nu_0)$.

Most of the following argument is copypasted from the previous footnote.

We have $(\tau, \nu) \in Q$. According to the definition of Q, this means that $(\tau, \nu) \in V \times \mathbb{N}^k$ and $|\nu| = -|\tau|$ and $\nu + \tau = \gamma$.

From $(\tau, \nu) \in V \times \mathbb{N}^k$, we obtain $\tau \in V$ and $\nu \in \mathbb{N}^k$.

We have $\tau \in V$. According to the definition of V, this means that τ is a k-tuple $(-n, \tau_2, \tau_3, \ldots, \tau_k) \in \mathbb{Z}^k$ satisfying (95). In other words, $\tau \in \mathbb{Z}^k$ and $\tau_1 = -n$ and the condition (95) holds.

Now, fix $j \in \{2, 3, \ldots, k\}$. Then, $\tau_j \in \{0, 1\}$ (by (95), applied to $i = j$). Hence, $\tau_j \geq 0$. Also, $\nu_j \in \mathbb{N}$ (since $\nu \in \mathbb{N}^k$), so that $\nu_j \geq 0$. But $(\gamma_2, \gamma_3, \ldots, \gamma_k) = (0, 0, \ldots, 0)$, and thus $\gamma_j = 0$ (since $j \in \{2, 3, \ldots, k\}$). But $\gamma = \nu + \tau$, and thus $\gamma_j = (\nu + \tau)_j = \nu_j + \tau_j$. Hence, $\nu_j + \tau_j = \gamma_j = 0$, so that $\nu_j = -\underbrace{\tau_j}_{\geq 0} \leq 0$. Combining this with $\nu_j \geq 0$, we obtain $\nu_j = 0$. Hence, $\nu_j = -\tau_j$ rewrites as $0 = -\tau_j$, so that $\tau_j = 0$.

Now, forget that we fixed j. Thus, we have shown that each $j \in \{2, 3, \ldots, k\}$ satisfies $\tau_j = 0$. In other words, $(\tau_2, \tau_3, \ldots, \tau_k) = (0, 0, \ldots, 0)$. Combining this with $\tau_1 = -n$, we obtain $\tau = (-n, 0, 0, \ldots, 0) = \tau_0$.

From $\nu + \tau = \gamma$, we obtain $\nu = \gamma - \underbrace{\tau}_{=\tau_0} = \gamma - \tau_0 = \nu_0$ (since $\nu_0 + \tau_0 = \gamma$). Combining this with $\tau = \tau_0$, we obtain $(\tau, \nu) = (\tau_0, \nu_0) \in \{(\tau_0, \nu_0)\}$.

Now, forget that we fixed (τ, ν). We thus have proven that $(\tau, \nu) \in \{(\tau_0, \nu_0)\}$ for each $(\tau, \nu) \in Q$. In other words, $Q \subseteq \{(\tau_0, \nu_0)\}$.

But (100) yields

$$\begin{aligned}
\sum_{\tau\in V}\ \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=-|\tau|;\\ \nu+\tau=\gamma}} (-1)^{n+|\tau|} &= \sum_{(\tau,\nu)\in Q} (-1)^{n+|\tau|} = (-1)^{n+|\tau_0|} \qquad (\text{since } Q=\{(\tau_0,\nu_0)\}) \\
&= (-1)^0 \qquad \left(\text{since } n+\underbrace{|\tau_0|}_{=-n} = n+(-n)=0\right) \\
&= 1 = \begin{cases} 1, & \text{if } \gamma=\mathbf{0}; \\ 0, & \text{if } \gamma\neq\mathbf{0} \end{cases}
\end{aligned}$$

(since $\begin{cases} 1, & \text{if } \gamma=\mathbf{0}; \\ 0, & \text{if } \gamma\neq\mathbf{0} \end{cases} = 1$). Thus, Lemma 10.16 is proven in Case 3.

We have now proven Lemma 10.16 in each of the three Cases 1, 2 and 3. Hence, Lemma 10.16 always holds. □

Proof of Theorem 10.14. Each $\tau\in V$ satisfies $-|\tau|\in\{n-k+1,n-k+2,\ldots,n\}$ (by Proposition 10.11). Thus, we have the following equality of summation signs:

$$\sum_{\tau\in V} = \sum_{i=n-k+1}^{n}\ \sum_{\substack{\tau\in V;\\ -|\tau|=i}} = \sum_{j=1}^{k}\ \sum_{\substack{\tau\in V;\\ -|\tau|=n-k+j}} \tag{106}$$

(here, we have substituted $n-k+j$ for i in the outer sum). Now,

$$\begin{aligned}
&\sum_{j=1}^{k} (-1)^{k-j} h_{n-k+j} \sum_{\substack{\tau\in V;\\ -|\tau|=n-k+j}} s_{\mu+\tau} \\
&= \underbrace{\sum_{j=1}^{k}\ \sum_{\substack{\tau\in V;\\ -|\tau|=n-k+j}}}_{\substack{=\sum_{\tau\in V}\\ (\text{by (106)})}} \underbrace{(-1)^{k-j}}_{\substack{=(-1)^{n+|\tau|}\\ (\text{since } k-j=n+|\tau|\\ (\text{because } -|\tau|=n-k+j))}} \underbrace{h_{n-k+j}}_{\substack{=h_{-|\tau|}\\ (\text{since } n-k+j=-|\tau|\\ (\text{because } -|\tau|=n-k+j))}} s_{\mu+\tau} \\
&= \sum_{\tau\in V} (-1)^{n+|\tau|} h_{-|\tau|} s_{\mu+\tau}.
\end{aligned} \tag{107}$$

But each $\tau\in V$ satisfies

$$h_{-|\tau|} s_{\mu+\tau} = \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=-|\tau|}} s_{\mu+(\nu+\tau)}. \tag{108}$$

[*Proof of (108):* Let $\tau\in V$. According to the definition of V, this means that τ is a k-tuple $(-n,\tau_2,\tau_3,\ldots,\tau_k)\in\mathbb{Z}^k$ satisfying (95). In other words, $\tau\in\mathbb{Z}^k$ and $\tau_1=-n$ and the relation (95) holds.

Proposition 10.11 yields $-|\tau| \in \{n-k+1, n-k+2, \ldots, n\} \subseteq \mathbb{N}$.
Also, $\mu \in P_k \subseteq \mathbb{N}^k$; hence,

$$\mu_i \geq 0 \qquad \text{for each } i \in \{1, 2, \ldots, k\}. \tag{109}$$

Also, $\rho_1 = k-1$ (by the definition of ρ) and $\rho \in \mathbb{N}^k$ (likewise). Now,

$$(\mu+\tau+\rho)_1 = \underbrace{\mu_1}_{>n-k} + \underbrace{\tau_1}_{=-n} + \underbrace{\rho_1}_{=k-1} > (n-k) + (-n) + (k-1) = -1.$$

Thus, $(\mu+\tau+\rho)_1 \geq 0$ (since $(\mu+\tau+\rho)_1$ is an integer). In other words, $(\mu+\tau+\rho)_1 \in \mathbb{N}$. Furthermore, for each $i \in \{2, 3, \ldots, k\}$, we have

$$(\mu+\tau+\rho)_i = \underbrace{\mu_i}_{\substack{\in\mathbb{N} \\ (\text{since } \mu\in\mathbb{N}^k)}} + \underbrace{\tau_i}_{\substack{\in\mathbb{N} \\ (\text{since (95)} \\ \text{yields } \tau_i\in\{0,1\}\subseteq\mathbb{N})}} + \underbrace{\rho_i}_{\substack{\in\mathbb{N} \\ (\text{since } \rho\in\mathbb{N}^k)}} \in \mathbb{N}.$$

This also holds for $i=1$ (since $(\mu+\tau+\rho)_1 \in \mathbb{N}$). Thus, we have $(\mu+\tau+\rho)_i \in \mathbb{N}$ for each $i \in \{1, 2, \ldots, k\}$. In other words, $\mu+\tau+\rho \in \mathbb{N}^k$. Hence, Theorem 10.7 (applied to $\lambda = \mu+\tau$ and $m = -|\tau|$) yields

$$s_{\mu+\tau} h_{-|\tau|} = \sum_{\substack{\nu\in\mathbb{N}^k; \\ |\nu|=-|\tau|}} \underbrace{s_{\mu+\tau+\nu}}_{=s_{\mu+(\nu+\tau)}} = \sum_{\substack{\nu\in\mathbb{N}^k; \\ |\nu|=-|\tau|}} s_{\mu+(\nu+\tau)}.$$

Thus,

$$h_{-|\tau|} s_{\mu+\tau} = s_{\mu+\tau} h_{-|\tau|} = \sum_{\substack{\nu\in\mathbb{N}^k; \\ |\nu|=-|\tau|}} s_{\mu+(\nu+\tau)}.$$

This proves (108).]

Now, (107) becomes

$$\begin{aligned}
&\sum_{j=1}^{k} (-1)^{k-j} h_{n-k+j} \sum_{\substack{\tau\in V;\\ -|\tau|=n-k+j}} s_{\mu+\tau} \\
&= \sum_{\tau\in V} (-1)^{n+|\tau|} \underbrace{h_{-|\tau|} s_{\mu+\tau}}_{\substack{=\sum\limits_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=-|\tau|}} s_{\mu+(\nu+\tau)} \\ \text{(by (108))}}} \\
&= \sum_{\tau\in V} (-1)^{n+|\tau|} \underbrace{\sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=-|\tau|}}}_{\substack{=\sum\limits_{\gamma\in\mathbb{Z}^k} \sum\limits_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=-|\tau|;\\ \nu+\tau=\gamma}}}} s_{\mu+(\nu+\tau)} = \sum_{\tau\in V} (-1)^{n+|\tau|} \sum_{\gamma\in\mathbb{Z}^k} \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=-|\tau|;\\ \nu+\tau=\gamma}} \underbrace{s_{\mu+(\nu+\tau)}}_{\substack{=s_{\mu+\gamma} \\ \text{(since } \nu+\tau=\gamma)}} \\
&= \sum_{\tau\in V} (-1)^{n+|\tau|} \sum_{\gamma\in\mathbb{Z}^k} \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=-|\tau|;\\ \nu+\tau=\gamma}} s_{\mu+\gamma} = \sum_{\gamma\in\mathbb{Z}^k} \underbrace{\left(\sum_{\tau\in V} \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=-|\tau|;\\ \nu+\tau=\gamma}} (-1)^{n+|\tau|} \right)}_{\substack{=\begin{cases} 1, & \text{if } \gamma=\mathbf{0}; \\ 0, & \text{if } \gamma\neq\mathbf{0} \end{cases} \\ \text{(by Lemma 10.16)}}} s_{\mu+\gamma} \\
&= \sum_{\gamma\in\mathbb{Z}^k} \begin{cases} 1, & \text{if } \gamma=\mathbf{0}; \\ 0, & \text{if } \gamma\neq\mathbf{0} \end{cases} s_{\mu+\gamma} = \underbrace{\begin{cases} 1, & \text{if } \mathbf{0}=\mathbf{0}; \\ 0, & \text{if } \mathbf{0}\neq\mathbf{0} \end{cases}}_{\substack{=1 \\ \text{(since } \mathbf{0}=\mathbf{0})}} s_{\mu+\mathbf{0}} + \sum_{\substack{\gamma\in\mathbb{Z}^k;\\ \gamma\neq\mathbf{0}}} \underbrace{\begin{cases} 1, & \text{if } \gamma=\mathbf{0}; \\ 0, & \text{if } \gamma\neq\mathbf{0} \end{cases}}_{\substack{=0 \\ \text{(since } \gamma\neq\mathbf{0})}} s_{\mu+\gamma} \\
&\qquad \left(\text{here, we have split off the addend for } \gamma=\mathbf{0} \text{ from the sum}\right) \\
&= s_{\mu+\mathbf{0}} = s_\mu .
\end{aligned}$$

This proves Theorem 10.14. □

Proof of Theorem 10.12. Theorem 10.14 yields

$$s_\mu = \sum_{j=1}^{k} (-1)^{k-j} \underbrace{h_{n-k+j}}_{\substack{\equiv a_j \operatorname{mod} I \\ \text{(by (9))}}} \sum_{\substack{\tau\in V;\\ -|\tau|=n-k+j}} s_{\mu+\tau} \equiv \sum_{j=1}^{k} (-1)^{k-j} a_j \sum_{\substack{\tau\in V;\\ -|\tau|=n-k+j}} s_{\mu+\tau} \operatorname{mod} I.$$

Thus, in $\mathcal{S}/I$, we have

$$\overline{s_\mu} = \overline{\sum_{j=1}^{k} (-1)^{k-j} a_j \sum_{\substack{\tau \in V; \\ -|\tau|=n-k+j}} s_{\mu+\tau}} = \sum_{j=1}^{k} (-1)^{k-j} a_j \sum_{\substack{\tau \in V; \\ -|\tau|=n-k+j}} \overline{s_{\mu+\tau}}.$$

This proves Theorem 10.12. □

11. Deforming symmetric functions

We have so far studied a quotient $\mathcal{S}/I$ of the ring $\mathcal{S}$ of symmetric polynomials in k variables $x_1, x_2, \ldots, x_k$. But $\mathcal{S}$ itself is a quotient of a larger ring – the ring Λ of symmetric functions in infinitely many variables. More precisely,

$$\mathcal{S} \cong \Lambda / \langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \rangle$$

(where $\langle \cdot \rangle$ means the ideal generated by whatever is inside the brackets). Hence,

$$\mathcal{S}/I \cong \Lambda / \left(\langle \mathbf{h}_{n-k+1} - a_1, \mathbf{h}_{n-k+2} - a_2, \ldots, \mathbf{h}_n - a_k \rangle + \langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \rangle \right).$$

This suggests a further generalization: What if we replace $\mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots$ by $\mathbf{e}_{k+1} - b_1, \mathbf{e}_{k+2} - b_2, \mathbf{e}_{k+3} - b_3, \ldots$ for some constants $b_1, b_2, b_3, \ldots$? In the next corollary, we take a quick look at this generalization, at least in the case when $a_1, a_2, \ldots, a_k \in \mathbf{k}$:

Theorem 11.1. Assume that $a_1, a_2, \ldots, a_k$ as well as $b_1, b_2, b_3, \ldots$ are elements of $\mathbf{k}$. Let Λ be the ring of symmetric functions in infinitely many indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. (See [GriRei18, Chapter 2] for more about this ring Λ.) Let $\mathbf{e}_m$ and $\mathbf{h}_m$ be the elementary symmetric functions and the complete homogeneous symmetric functions in Λ. For each partition λ, let $\mathbf{s}_\lambda$ be the Schur function in Λ corresponding to λ.

Let K be the ideal

$$\langle \mathbf{h}_{n-k+1} - a_1, \mathbf{h}_{n-k+2} - a_2, \ldots, \mathbf{h}_n - a_k \rangle + \langle \mathbf{e}_{k+1} - b_1, \mathbf{e}_{k+2} - b_2, \mathbf{e}_{k+3} - b_3, \ldots \rangle$$

of Λ (where $\langle \cdot \rangle$ means the ideal generated by whatever is inside the brackets). Then, Λ/K is a free $\mathbf{k}$-module with basis $\left(\overline{\mathbf{s}_\lambda} \right)_{\lambda \in P_{k,n}}$.

Proof of Theorem 11.1. Again, the Jacobi-Trudi identities ([GriRei18, (2.4.9) and (2.4.10)]) easily yield that the family $\left(\overline{\mathbf{s}_\lambda} \right)_{\lambda \in P_{k,n}}$ spans the $\mathbf{k}$-module Λ/K. We need to prove that it is a basis of Λ/K.

Let us first recall that $\Lambda / \langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \rangle \cong \mathcal{S}$. More precisely, there is a canonical surjective $\mathbf{k}$-algebra homomorphism $\Lambda \to \mathcal{S}$ which is given by substituting 0 for each of the variables $x_{k+1}, x_{k+2}, x_{k+3}, \ldots$; the kernel of this homomorphism is precisely the ideal $\langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \rangle$ of Λ. This homomorphism sends each $\mathbf{h}_m \in \Lambda$ to the polynomial $h_m \in \mathcal{S}$ defined in (1).

It is well-known that the commutative $\mathbf{k}$-algebra Λ is freely generated by its elements $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots$. Hence, we can define an endomorphism φ of the $\mathbf{k}$-algebra Λ by letting

$$\begin{aligned} &\varphi\left(\mathbf{e}_i\right) = \mathbf{e}_i && \text{for each } i \in \{1, 2, \ldots, k\}; \\ &\varphi\left(\mathbf{e}_i\right) = \mathbf{e}_i + b_{i-k} && \text{for each } i \in \{k+1, k+2, k+3, \ldots\}. \end{aligned}$$

Consider this φ. Then, clearly, φ is an automorphism of Λ (indeed, the inverse of φ can easily be constructed: it sends $\mathbf{e}_i$ to $\mathbf{e}_i$ for each $i \in \{1, 2, \ldots, k\}$, and sends $\mathbf{e}_i$ to $\mathbf{e}_i - b_{i-k}$ for each $i \in \{k+1, k+2, k+3, \ldots\}$). Thus,

$$\varphi\left(\left\langle \mathbf{e}_{k+1} - b_1, \mathbf{e}_{k+2} - b_2, \mathbf{e}_{k+3} - b_3, \ldots \right\rangle\right) = \left\langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \right\rangle. \tag{110}$$

For each $i \in \{1, 2, \ldots, k\}$, define $\mathbf{c}_i \in \Lambda$ by $\mathbf{c}_i = \mathbf{h}_{n-k+i} - \varphi\left(\mathbf{h}_{n-k+i}\right) + a_i$. Then, $\varphi\left(\mathbf{h}_{n-k+i} - a_i\right) = \mathbf{h}_{n-k+i} - \mathbf{c}_i$ for each $i \in \{1, 2, \ldots, k\}$. Thus,

$$\varphi\left(\left\langle \mathbf{h}_{n-k+1} - a_1, \mathbf{h}_{n-k+2} - a_2, \ldots, \mathbf{h}_n - a_k \right\rangle\right) = \left\langle \mathbf{h}_{n-k+1} - \mathbf{c}_1, \mathbf{h}_{n-k+2} - \mathbf{c}_2, \ldots, \mathbf{h}_n - \mathbf{c}_k \right\rangle.$$

Adding this together with (110), we obtain

$$\varphi\left(K\right) = \left\langle \mathbf{h}_{n-k+1} - \mathbf{c}_1, \mathbf{h}_{n-k+2} - \mathbf{c}_2, \ldots, \mathbf{h}_n - \mathbf{c}_k \right\rangle + \left\langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \right\rangle. \tag{111}$$

The definition of φ shows that the endomorphism φ respects the filtration of Λ; moreover, it shows that the endomorphism of the associated graded ring $\operatorname{gr}\Lambda$ induced by φ is the identity endomorphism. In other words, if $\mathbf{f} \in \Lambda$ is a symmetric function of degree $\leq g$, then $\mathbf{f} - \varphi\left(\mathbf{f}\right)$ is a symmetric function of degree $< g$. Hence, for each $i \in \{1, 2, \ldots, k\}$, the symmetric function $\mathbf{h}_{n-k+i} - \varphi\left(\mathbf{h}_{n-k+i}\right)$ is a symmetric function of degree $< n - k + i$. Hence, for each $i \in \{1, 2, \ldots, k\}$, the symmetric function $\mathbf{c}_i \in \Lambda$ is a symmetric function of degree $< n - k + i$ (since $a_i \in \mathbf{k}$).

But φ is an automorphism of Λ. Thus,

$$\begin{aligned} \Lambda / K &\cong \Lambda / \varphi\left(K\right) \\ &\cong \Lambda / \left(\left\langle \mathbf{h}_{n-k+1} - \mathbf{c}_1, \mathbf{h}_{n-k+2} - \mathbf{c}_2, \ldots, \mathbf{h}_n - \mathbf{c}_k \right\rangle + \left\langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \right\rangle\right) \\ &\qquad \left(\text{by (111)}\right) \\ &\cong \underbrace{\left(\Lambda / \left\langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \right\rangle\right)}_{\cong \mathcal{S}} / \left\langle \overline{\mathbf{h}_{n-k+1} - \mathbf{c}_1}, \overline{\mathbf{h}_{n-k+2} - \mathbf{c}_2}, \ldots, \overline{\mathbf{h}_n - \mathbf{c}_k} \right\rangle \\ &\qquad \left(\begin{array}{c} \text{where the projections } \overline{\mathbf{h}_{n-k+1} - \mathbf{c}_1}, \overline{\mathbf{h}_{n-k+2} - \mathbf{c}_2}, \ldots, \overline{\mathbf{h}_n - \mathbf{c}_k} \text{ are} \\ \text{projections onto } \Lambda / \left\langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \right\rangle \end{array}\right) \\ &\cong \mathcal{S} / \left\langle h_{n-k+1} - \overline{\mathbf{c}_1}, h_{n-k+2} - \overline{\mathbf{c}_2}, \ldots, h_n - \overline{\mathbf{c}_k} \right\rangle. \end{aligned} \tag{112}$$

But recall that for each $i \in \{1, 2, \ldots, k\}$, the symmetric function $\mathbf{c}_i \in \Lambda$ is a symmetric function of degree $< n - k + i$. Hence, for each $i \in \{1, 2, \ldots, k\}$, the symmetric polynomial $\overline{\mathbf{c}_i} \in \mathcal{S}$ is a symmetric polynomial of degree $< n -$

$k+i$. Thus, Theorem 2.5 (applied to $\overline{\mathbf{c}_i}$ instead of a_i) yields that the **k**-module $\mathcal{S}/\left\langle h_{n-k+1}-\overline{\mathbf{c}_1}, h_{n-k+2}-\overline{\mathbf{c}_2}, \ldots, h_n-\overline{\mathbf{c}_k}\right\rangle$ is free with basis $\left(\overline{s_\lambda}\right)_{\lambda\in P_{k,n}}$. Hence, this **k**-module is free and has a basis of size $\left|P_{k,n}\right|$. Thus, (112) shows that the **k**-module Λ/K also is free and has a basis of size $\left|P_{k,n}\right|$.

Now, recall that the family $\left(\overline{\mathbf{s}_\lambda}\right)_{\lambda\in P_{k,n}}$ spans the **k**-module Λ/K. Hence, this family must be a basis of Λ/K (since it has the same size as a basis of Λ/K). This proves Theorem 11.1. $\square$

TODO: Explain why Theorem 11.1 remains true if we replace the requirements $a_i\in\mathbf{k}$ and $b_i\in\mathbf{k}$ by the weaker requirements $\deg\left(a_i\right)<n-k+i$ and $\deg\left(b_i\right)<k+i$.

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