A basis for a quotient of symmetric polynomials (draft)

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June 29, 2019

Abstract. Consider the ring $S$ of symmetric polynomials in $k$ variables over an arbitrary base ring $k$. Fix $k$ scalars $a_1, a_2, \ldots, a_k \in k$.

Let $I$ be the ideal of $S$ generated by $h_{n-k+1} - a_1, h_{n-k+2} - a_2, \ldots, h_n - a_k$, where $h_i$ is the $i$-th complete homogeneous symmetric polynomial.

The quotient ring $S/I$ generalizes both the usual and the quantum cohomology of the Grassmannian.

We show that $S/I$ has a $k$-module basis consisting of (residue classes of) Schur polynomials fitting into an $(n-k) \times k$-rectangle; and that its multiplicative structure constants satisfy the same $S_3$-symmetry as those of the Grassmannian cohomology. We prove a Pieri rule and conjecture a positivity property generalizing that of Gromov-Witten invariants.

We also study the quotient of the whole polynomial ring (not just the symmetric polynomials) by the ideal generated by the same $k$ polynomials as $I$.

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1. Introduction

This is a rough draft – all proofs are merely outlines. The results will also likely be shuffled around.
This work is devoted to a certain construction that generalizes both the regular and the quantum cohomology ring of the Grassmannian [Postni05]. This construction is purely algebraic – we do not conjecture any geometric meaning for it at this point – but shares some basic properties with quantum cohomology, such as an $S_3$-symmetry of its structure constants (generalizing the $S_3$-symmetry for Littlewood-Richardson coefficients and Gromov-Witten invariants). All our arguments are algebraic and combinatorial.

1.1. Acknowledgments

We thank Dongkwan Kim, Alex Postnikov, Victor Reiner, Mark Shimozono, Josh Swanson, Kaisa Taipale, and Anders Thorup for enlightening conversations. The SageMath computer algebra system [SageMath] has been used for experimentation leading up to some of the results below.

2. The basis theorems

2.1. Definitions and notations

Let $\mathbb{N}$ denote the set $\{0,1,2,\ldots\}$.

Let $k$ be a commutative ring. Let $k \in \mathbb{N}$.

Let $P$ denote the polynomial ring $k[x_1,x_2,\ldots,x_k]$. This is a graded ring, where the grading is by total degree (so $\deg x_i = 1$ for each $i \in \{1,2,\ldots,k\}$).

For each $\alpha \in \mathbb{Z}^k$ and each $i \in \{1,2,\ldots,k\}$, we denote the $i$-th entry of $\alpha$ by $a_i$ (so that $\alpha = (a_1,a_2,\ldots,a_k)$). For each $\alpha \in \mathbb{N}^k$, we define a monomial $x^\alpha$ by $x^\alpha = x_1^{a_1}x_2^{a_2}\cdots x_k^{a_k}$.

Let $S$ denote the ring of symmetric polynomials in $P$; in other words, $S$ is the ring of invariants of the symmetric group $S_k$ acting on $P$. (The action here is the one you would expect: A permutation $\sigma \in S_k$ sends a monomial $x_{i_1}x_{i_2}\cdots x_{i_m}$ to $x_{\sigma(i_1)}x_{\sigma(i_2)}\cdots x_{\sigma(i_m)}$.)

The following fact is well-known (going back to Emil Artin) and is proven (e.g.) in [LLPT95, (DIFF.1.3)]:

**Proposition 2.1.** The $S$-module $P$ is free with basis $(x^\alpha)_{\alpha \in \mathbb{N}^k}; a_i < i$ for each $i$.

Now, fix an integer $n \geq k$. For each $i \in \{1,2,\ldots,k\}$, let $a_i$ be an element of $P$ with degree $< n - k + i$. (This is clearly satisfied when $a_1, a_2, \ldots, a_k$ are constants in $k$, but also in some other cases. Note that the $a_i$ do not have to be homogeneous.)

For each $\alpha \in \mathbb{Z}^k$, we let $|\alpha|$ denote the sum of the entries of the $k$-tuple $\alpha$ (that is, $|\alpha| = a_1 + a_2 + \cdots + a_k$).
For each \( m \in \mathbb{Z} \), we let \( h_m \) denote the \( m \)-th complete homogeneous symmetric polynomial; this is the element of \( S \) defined by
\[
h_m = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq k} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{a \in \mathbb{N}^k; \ |a| = m} x^a.
\] (1)

(Thus, \( h_0 = 1 \), and \( h_m = 0 \) when \( m < 0 \).)

Let \( J \) be the ideal of \( \mathcal{P} \) generated by the \( k \) differences
\[
h_{n-k+1} - a_1, h_{n-k+2} - a_2, \ldots, h_n - a_k.
\] (2)

If \( M \) is a \( k \)-module and \( N \) is a submodule of \( M \), then the projection of any \( m \in M \) onto the quotient \( M/N \) (that is, the congruence class of \( m \) modulo \( N \)) will be denoted by \( \overline{m} \).

2.2. The basis theorem for \( \mathcal{P}/J \)

We claim the following result:

**Theorem 2.2.** The \( k \)-module \( \mathcal{P}/J \) is free with basis \( (\overline{x^a})_{a \in \mathbb{N}^k; \ a_i < n-k+i \text{ for each } i} \).

**Example 2.3.** Let \( n = 5 \) and \( k = 2 \). Then, \( \mathcal{P} = k[x_1, x_2] \), and \( J \) is the ideal of \( \mathcal{P} \) generated by the 2 differences
\[
h_4 - a_1 = \left( x^4_1 + x^3_1 x_2 + x^2_1 x^2_2 + x_1 x^3_2 + x^4_2 \right) - a_1 \quad \text{and} \quad h_5 - a_2 = \left( x^5_1 + x^4_1 x_2 + x^3_1 x^2_2 + x^2_1 x^3_2 + x_1 x^4_2 + x^5_2 \right) - a_2.
\]

Theorem 2.2 yields that the \( k \)-module \( \mathcal{P}/J \) is free with basis \( (\overline{x^a})_{a \in \mathbb{N}^2; \ a_i < 3+i \text{ for each } i} \); this basis can also be rewritten as \( (\overline{x^a_1 x^a_2})_{a_1 \in \{0, 1, 2, 3\}; \ a_2 \in \{0, 1, 2, 3, 4\}} \). As a consequence, any \( \overline{x_1^{b_1} x_2^{b_2}} \in \mathcal{P}/J \) can be written as a linear combination of elements of this basis. For example,
\[
\overline{x^3_1} = a_1 - \overline{x^2_1 x_2} - \overline{x^2_1 x^2_2} - \overline{x_1 x^3_2} - \overline{x^3_1 x^3_2} \quad \text{and} \quad \overline{x^5_2} = a_2 - a_1 \overline{x_1}.
\]

These expressions will become more complicated for higher values of \( n \) and \( k \).

Theorem 2.2 is related to the second part of [CoKrWa09, Proposition 2.9] (and our proof below can be viewed as an elaboration of the argument sketched in the last paragraph of [CoKrWa09, proof of Proposition 2.9]).
2.3. The basis theorem for $S/I$

To state our next result, we need some more notations.

**Definition 2.4.** (a) We define the concept of *partitions* (of an integer) as in [GriRei18, Chapter 2]. Thus, a partition is a weakly decreasing infinite sequence $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ of nonnegative integers such that all but finitely many $i$ satisfy $\lambda_i = 0$. We identify each partition $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ with the finite list $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ whenever $p \in \mathbb{N}$ has the property that $(\lambda_i = 0$ for all $i > p)$.

For example, the partition \( \left( 3, 1, 1, 0, 0, \ldots \right) \) is identified with $(3, 1, 1, 0)$ and with $(3, 1, 1)$.

(b) A part of a partition $\lambda$ means a nonzero entry of $\lambda$.

(c) Let $P_{k,n}$ denote the set of all partitions that have at most $k$ parts and have the property that each of their parts is $\leq n - k$. (Visually speaking, $P_{k,n}$ is the set of all partitions whose Young diagram fits into a $k \times (n - k)$-rectangle.)

(d) We let $\emptyset$ denote the empty partition $()$.

**Example 2.5.** If $n = 4$ and $k = 2$, then

$$P_{k,n} = P_{2,4} = \{ \emptyset, (1), (2), (1,1), (2,1), (2,2) \}.$$ 

If $n = 5$ and $k = 2$, then

$$P_{k,n} = P_{2,5} = \{ \emptyset, (1), (2), (3), (1,1), (2,1), (3,1), (2,2), (3,2), (3,3) \}.$$ 

**Definition 2.6.** For any partition $\lambda$, we let $s_\lambda$ denote the Schur polynomial in $x_1, x_2, \ldots, x_k$ corresponding to the partition $\lambda$. This Schur polynomial is what is called $s_\lambda (x_1, x_2, \ldots, x_k)$ in [GriRei18, Chapter 2]. Note that

$$s_\lambda = 0 \quad \text{if } \lambda \text{ has more than } k \text{ parts.} \quad (3)$$

If $\lambda$ is any partition, then the Schur polynomial $s_\lambda = s_\lambda (x_1, x_2, \ldots, x_k)$ is symmetric and thus belongs to $S$.

**Theorem 2.7.** Assume that $a_1, a_2, \ldots, a_k$ belong to $S$. Let $I$ be the ideal of $S$ generated by the $k$ differences \( \nabla \). Then, the $k$-module $S/I$ is free with basis $(s_\lambda)_{\lambda \in P_{k,n}}$.

The $k$-algebra $S/I$ generalizes several constructions in the literature:

- If $k = \mathbb{Z}$ and $a_1 = a_2 = \cdots = a_k = 0$, then $S/I$ becomes the cohomology ring of the Grassmannian of $k$-dimensional subspaces in an $n$-dimensional space (see, e.g., [Fulton99, §9.4]); the elements of the basis $(s_\lambda)_{\lambda \in P_{k,n}}$ correspond to the Schubert classes.
• If $k = \mathbb{Z} [q]$ and $a_1 = a_2 = \cdots = a_{k-1} = 0$ and $a_k = (-1)^k q$, then $S / I$ becomes isomorphic to the quantum cohomology ring of the same Grassmannian (see [Postni05]). Indeed, our ideal $I$ becomes the $J^q_{kn}$ of [Postni05] (6) in this case, and Theorem 2.7 generalizes the fact that the quotient $(\Lambda_k \otimes \mathbb{Z} [q]) / J^q_{kn}$ in [Postni05, (6)] has basis $(s^\lambda)_{\lambda \in P_{kn}}$.

One goal of this paper is to provide a purely algebraic foundation for the study of the standard and quantum cohomology rings of the Grassmannian, without having to resort to geometry for proofs of the basic properties of these rings. In particular, Theorem 2.7 shows that the “abstract Schubert classes” $s^\lambda$ (with $\lambda \in P_{kn}$) form a basis of the $k$-module $S / I$, whereas Corollary 6.24 further below shows that the structure constants of the $k$-algebra $S / I$ with respect to this basis (we may call them “generalized Gromov-Witten invariants”) satisfy an $S_3$-symmetry. These two properties are two of the facts for whose proofs [Postni05] relies on algebro-geometric literature; thus, our paper helps provide an alternative footing for [Postni05] using only combinatorics and algebra.

Remark 2.8. The $k$-algebra $\mathcal{P} / J$ somewhat resembles the “splitting algebra” $\text{Split}^q_{d} (p)$ from [LakTho12, §1.3]; further analogies between these concepts can be made as we study the former. For example, the basis we give in Theorem 2.2 is like the basis in [LakTho12, (1.5)]. It is not currently clear to us whether there is more than analogies.

3. A fundamental identity

Let us use the notations $h_m$ and $e_m$ for complete homogeneous symmetric polynomials and elementary symmetric polynomials in general. Thus, for any $m \in \mathbb{Z}$ and any $p$ elements $y_1, y_2, \ldots, y_p$ of a commutative ring, we set

$$h_m (y_1, y_2, \ldots, y_p) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq p} y_{i_1} y_{i_2} \cdots y_{i_m}$$

and

$$e_m (y_1, y_2, \ldots, y_p) = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq p} y_{i_1} y_{i_2} \cdots y_{i_m}. \tag{5}$$

(Thus, $h_0 (y_1, y_2, \ldots, y_p) = 1$ and $e_0 (y_1, y_2, \ldots, y_p) = 1$. Also, $e_m (y_1, y_2, \ldots, y_p) = 0$ for all $m > p$. Also, for any $m < 0$, we have $h_m (y_1, y_2, \ldots, y_p) = 0$ and $e_m (y_1, y_2, \ldots, y_p) = 0$. Finally, what we have previously called $h_m$ without any arguments can now be rewritten as $h_m (x_1, x_2, \ldots, x_k)$. Similarly, we shall occasionally abbreviate $e_m (x_1, x_2, \ldots, x_k)$ as $e_m$.)

1This, of course, presumes that one is willing to forget the cohomological definition of the ring $\text{QH}^* (\text{Gr}_{kn})$, and instead to define it algebraically as the quotient ring $(\Lambda_k \otimes \mathbb{Z} [q]) / J^q_{kn}$, using the notations of [Postni05].
**Lemma 3.1.** Let \( i \in \{1, 2, \ldots, k + 1 \} \) and \( p \in \mathbb{N} \). Then,
\[
h_p(x_i, x_{i+1}, \ldots, x_k) = \sum_{t=0}^{i-1} (-1)^t e_t(x_1, x_2, \ldots, x_{i-1}) h_{p-t}(x_1, x_2, \ldots, x_k).
\]

Notice that if \( i = k + 1 \), then the term \( h_p(x_i, x_{i+1}, \ldots, x_k) \) on the left hand side of Lemma 3.1 is understood to be \( h_p \) of an empty list of vectors; this is 1 when \( p = 0 \) and 0 otherwise.

Lemma 3.1 is actually a particular case of [GrinRei16a, detailed version, Theorem 3.15] (applied to \( a = x_i \in k[[x_1, x_2, x_3, \ldots]] \) and \( b = h_p(x_1, x_2, x_3, \ldots) \in \text{QSym} \)). However, we shall give a more elementary proof of it here. This proof relies on the following two basic identities:

**Lemma 3.2.** Let \( A \) be a commutative ring. Let \( y_1, y_2, \ldots, y_p \) be some elements of \( A \). Consider the ring \( A[[u]] \) of formal power series in one indeterminate \( u \) over \( A \). Then, in this ring, we have
\[
\sum_{q \in \mathbb{N}} h_q(y_1, y_2, \ldots, y_p) u^q = \prod_{j=1}^{p} \frac{1}{1 - y_j u} \tag{6}
\]
and
\[
\sum_{q \in \mathbb{N}} (-1)^q e_q(y_1, y_2, \ldots, y_p) u^q = \prod_{j=1}^{p} (1 - y_j u). \tag{7}
\]

**Proof of Lemma 3.2** The identity (6) can be obtained from the identities [GriRei18 (2.4.1)] by substituting \( y_1, y_2, \ldots, y_p, 0, 0, 0, \ldots \) for the indeterminates \( x_1, x_2, x_3, \ldots \). The identity (7) can be obtained from the identities [GriRei18 (2.4.2)] by substituting \( y_1, y_2, \ldots, y_p, 0, 0, 0, \ldots \) for the indeterminates \( x_1, x_2, x_3, \ldots \) and substituting \( -u \) for \( u \). Thus, Lemma 3.2 is proven. \( \square \)

**Proof of Lemma 3.1** Consider the ring \( \mathcal{P}[[u]] \) of formal power series in one indeterminate \( u \) over \( \mathcal{P} \). Applying (6) to \( \mathcal{P} \) and \( (x_i, x_{i+1}, \ldots, x_k) \) instead of \( A \) and \( (y_1, y_2, \ldots, y_p) \), we obtain
\[
\sum_{q \in \mathbb{N}} h_q(x_i, x_{i+1}, \ldots, x_k) u^q = \prod_{j=1}^{k-i+1} \frac{1}{1 - x_{i+j-1} u} = \prod_{j=1}^{k} \frac{1}{1 - x_j u}
\]
(here, we have substituted \( j \) for \( i + j - 1 \) in the product). Applying (7) to \( \mathcal{P} \) and \( (x_1, x_2, \ldots, x_{i-1}) \) instead of \( A \) and \( (y_1, y_2, \ldots, y_p) \), we obtain
\[
\sum_{q \in \mathbb{N}} (-1)^q e_q(x_1, x_2, \ldots, x_{i-1}) u^q = \prod_{j=1}^{i-1} (1 - x_j u). \tag{8}
\]
Applying (6) to $P$ and $(x_1, x_2, \ldots, x_k)$ instead of $A$ and $(y_1, y_2, \ldots, y_p)$, we obtain

$$
\sum_{q \in \mathbb{N}} h_q (x_1, x_2, \ldots, x_k) \, u^q = \prod_{j=1}^{k} \frac{1}{1 - x_j u}.
$$

Thus,

$$
\sum_{q \in \mathbb{N}} h_q (x_i, x_{i+1}, \ldots, x_k) \, u^q
= \prod_{j=i}^{k} \frac{1}{1 - x_j u} = \left( \prod_{j=1}^{k} \frac{1}{1 - x_j u} \right) / \left( \prod_{j=1}^{i-1} \frac{1}{1 - x_j u} \right)
= \left( \prod_{j=1}^{i-1} (1 - x_j u) \right) \left( \prod_{j=1}^{k} \frac{1}{1 - x_j u} \right)
= \left( \sum_{q \in \mathbb{N}} (-1)^q e_q (x_1, x_2, \ldots, x_{i-1}) u^q \right) \left( \sum_{q \in \mathbb{N}} h_q (x_1, x_2, \ldots, x_k) u^q \right) \quad \text{(by (8))}
= \left( \sum_{q \in \mathbb{N}} (-1)^q e_q (x_1, x_2, \ldots, x_{i-1}) u^q \right) \left( \sum_{q \in \mathbb{N}} h_q (x_1, x_2, \ldots, x_k) u^q \right).
$$

Comparing the coefficient before $u^p$ in this equality of power series, we obtain

$$
h_p (x_i, x_{i+1}, \ldots, x_k) = \sum_{t=0}^{p} (-1)^t e_t (x_1, x_2, \ldots, x_{i-1}) h_{p-t} (x_1, x_2, \ldots, x_k)
= \sum_{t=0}^{\infty} (-1)^t e_t (x_1, x_2, \ldots, x_{i-1}) h_{p-t} (x_1, x_2, \ldots, x_k)
\quad \text{(since $h_{p-t} (x_1, x_2, \ldots, x_k) = 0$ for all $t > p$)}
= \sum_{t=0}^{i-1} (-1)^t e_t (x_1, x_2, \ldots, x_{i-1}) h_{p-t} (x_1, x_2, \ldots, x_k)
\quad \text{(since $e_t (x_1, x_2, \ldots, x_{i-1}) = 0$ for all $t > i - 1$)}.
$$

This proves Lemma 3.1.

**Corollary 3.3.** Let $p$ be a positive integer. Then,

$$
h_p = - \sum_{t=1}^{k} (-1)^t e_t h_{p-t}.
$$
Proof of Corollary 3.3: Lemma 3.1 (applied to $i = k + 1$) yields

$$h_p(x_{k+1}, x_{k+2}, \ldots, x_k) = \sum_{t=0}^{k} (-1)^t e_t(x_1, x_2, \ldots, x_k) h_{p-t}(x_1, x_2, \ldots, x_k) = \sum_{t=0}^{k} (-1)^t e_th_{p-t}.$$  

Comparing this with

$$h_p(x_{k+1}, x_{k+2}, \ldots, x_k) = h_p \text{ (an empty list of variables)} = 0 \quad \text{(since } p > 0),$$

we obtain

$$0 = \sum_{t=0}^{k} (-1)^t e_th_{p-t} = (-1)^0 e_0 h_{p-0} + \sum_{t=1}^{k} (-1)^t e_th_{p-t} = h_p + \sum_{t=1}^{k} (-1)^t e_th_{p-t}.$$  

Hence,

$$h_p = - \sum_{t=1}^{k} (-1)^t e_th_{p-t}.$$  

This proves Corollary 3.3. \Box

4. Proof of Theorem 2.2

We shall next prove Theorem 2.2 using Gröbner bases. For the concept of Gröbner bases over a commutative ring, see [Grinbe17, detailed version, §3].

We define a degree-lexicographic term order on the monomials in $P$, where the variables are ordered by $x_1 > x_2 > \cdots > x_k$. Explicitly, this term order is the total order on the set of monomials in $x_1, x_2, \ldots, x_k$ defined as follows: Two monomials $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_k^{\alpha_k}$ and $x_1^{\beta_1}x_2^{\beta_2}\cdots x_k^{\beta_k}$ satisfy $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_k^{\alpha_k} > x_1^{\beta_1}x_2^{\beta_2}\cdots x_k^{\beta_k}$ if and only if

- either $\alpha_1 + \alpha_2 + \cdots + \alpha_k > \beta_1 + \beta_2 + \cdots + \beta_k$,
- or $\alpha_1 + \alpha_2 + \cdots + \alpha_k = \beta_1 + \beta_2 + \cdots + \beta_k$ and there exists some $i \in \{1, 2, \ldots, k\}$ such that $\alpha_i > \beta_i$ and ($\alpha_j = \beta_j$ for all $j < i$).

This total order is a term order (in the sense of [Grinbe17, detailed version, Definition 3.5]). Fix this term order; thus it makes sense to speak of Gröbner bases of ideals.
Proposition 4.1. The family

\[
\left( h_{n-k+i} (x_i, x_{i+1}, \ldots, x_k) - \sum_{t=0}^{i-1} (-1)^t e_t (x_1, x_2, \ldots, x_{i-1}) a_{i-t} \right)_{i \in \{1, 2, \ldots, k\}}
\]

is a Gröbner basis of the ideal \( J \). (Recall that we are using the notations from (4) and (5).)

Proposition 4.1 is somewhat similar to [Sturmf08, Theorem 1.2.7] (or, equivalently, [CoLiOs15, §7.1, Proposition 5]), but not the same. Our proof of it relies on the following elementary fact:

Lemma 4.2. Let \( A \) be a commutative ring. Let \( b_1, b_2, \ldots, b_k \in A \) and \( c_1, c_2, \ldots, c_k \in A \). Assume that

\[
b_i \in c_i + \sum_{t=1}^{i-1} c_{i-t} A
\]

for each \( i \in \{1, 2, \ldots, k\} \). Then, \( b_1 A + b_2 A + \cdots + b_k A = c_1 A + c_2 A + \cdots + c_k A \) (as ideals of \( A \)).

Proof of Lemma 4.2. We claim that

\[
\sum_{p=1}^j b_p A = \sum_{p=1}^j c_p A \quad \text{for each } j \in \{0, 1, \ldots, k\}.
\]

[Proof of (11): We shall prove (11) by induction on \( j \):

Induction base: For \( j = 0 \), both sides of the equality (11) are the zero ideal of \( A \) (since they are empty sums of ideals of \( A \)). Thus, (11) holds for \( j = 0 \). This completes the induction base.

Induction step: Let \( i \in \{1, 2, \ldots, k\} \). Assume that (11) holds for \( j = i - 1 \). We must prove that (11) holds for \( j = i \).

We have assumed that (11) holds for \( j = i - 1 \). In other words, we have

\[
\sum_{p=1}^{i-1} b_p A = \sum_{p=1}^{i-1} c_p A.
\]

But (10) yields

\[
b_i \in c_i + \sum_{t=1}^{i-1} c_{i-t} A = c_i + \sum_{p=1}^{i-1} c_p A \quad \text{(here, we have substituted } p \text{ for } i-t \text{ in the sum)}.
\]

Thus,

\[
c_i \in b_i - \sum_{p=1}^{i-1} c_p A = b_i + \sum_{p=1}^{i-1} c_p A,
\]

\footnote{For example, our \( a_1, a_2, \ldots, a_t \) are elements of \( k \) rather than indeterminates (although they can be indeterminates if \( k \) itself is a polynomial ring), and our term order is degree-lexicographic rather than lexicographic. Thus, it should not be surprising that the families are different.}
so that
\[ c_i A \subseteq \left( b_i + \sum_{p=1}^{i-1} c_p A \right) A \subseteq b_i A + \sum_{p=1}^{i-1} c_p A. \]

But from \( b_i \in c_i + \sum_{p=1}^{i-1} c_p A \), we obtain
\[ b_i A \subseteq \left( c_i + \sum_{p=1}^{i-1} c_p A \right) A \subseteq c_i A + \sum_{p=1}^{i-1} c_p A = \sum_{p=1}^{i} c_p A. \]

Now,
\[ \sum_{p=1}^{i} b_p A = \sum_{p=1}^{i-1} b_p A + b_i A \subseteq \sum_{p=1}^{i} c_p A + \sum_{p=1}^{i} c_p A = \sum_{p=1}^{i} c_p A. \]

Combining this inclusion with
\[ \sum_{p=1}^{i} c_p A = \sum_{p=1}^{i-1} c_p A + c_i A \subseteq \sum_{p=1}^{i-1} c_p A + b_i A + \sum_{p=1}^{i-1} c_p A \]
\[ = \sum_{p=1}^{i-1} c_p A + \sum_{p=1}^{i-1} c_p A + b_i A = \sum_{p=1}^{i} b_p A + b_i A = \sum_{p=1}^{i} b_p A, \]
\[ = \sum_{p=1}^{i} c_p A = \sum_{p=1}^{i} b_p A \]
we obtain \( \sum_{p=1}^{i} b_p A = \sum_{p=1}^{i} c_p A \). In other words, (11) holds for \( j = i \). This completes the induction step. Thus, (11) is proven by induction.

Now, (11) (applied to \( j = k \)) yields
\[ \sum_{p=1}^{k} b_p A = \sum_{p=1}^{k} c_p A. \]

Thus,
\[ b_1 A + b_2 A + \cdots + b_k A = \sum_{p=1}^{k} b_p A = \sum_{p=1}^{k} c_p A = c_1 A + c_2 A + \cdots + c_k A. \]

This proves Lemma 4.2. \( \square \)
Proof of Proposition 4.1 (sketched). For each \( i \in \{1, 2, \ldots, k\} \), we define a polynomial \( b_i \in \mathcal{P} \) by

\[
b_i = h_{n-k+i}(x_i, x_{i+1}, \ldots, x_k) - \sum_{t=0}^{i-1} (-1)^t e_t(x_1, x_2, \ldots, x_{i-1}) a_{i-t}.
\]

Then, we must prove that the family \( \{b_i\}_{i \in \{1, 2, \ldots, k\}} \) is a Gröbner basis of the ideal \( J \). We shall first prove that this family generates \( J \).

For each \( i \in \{1, 2, \ldots, k\} \), we define \( c_i \in \mathcal{P} \) by \( c_i = h_{n-k+i} - a_i \). Then, \( J \) is the ideal of \( \mathcal{P} \) generated by the \( k \) elements \( c_1, c_2, \ldots, c_k \) (by the definition of \( J \)). In other words,

\[
J = c_1\mathcal{P} + c_2\mathcal{P} + \cdots + c_k\mathcal{P}.
\]

For each \( i \in \{1, 2, \ldots, k\} \), we have

\[
b_i = \left( h_{n-k+i}(x_i, x_{i+1}, \ldots, x_k) \right) - \sum_{t=0}^{i-1} (-1)^t e_t(x_1, x_2, \ldots, x_{i-1}) a_{i-t}
\]

(by Lemma 3.1 (applied to \( p = n-k+i \))

\[
= \sum_{t=0}^{i-1} (-1)^t e_t(x_1, x_2, \ldots, x_{i-1}) h_{n-k+i-t}(x_1, x_2, \ldots, x_k)
\]

\[
- \sum_{t=0}^{i-1} (-1)^t e_t(x_1, x_2, \ldots, x_{i-1}) a_{i-t}
\]

\[
= \sum_{t=0}^{i-1} (-1)^t e_t(x_1, x_2, \ldots, x_{i-1}) \left( h_{n-k+i-t}(x_1, x_2, \ldots, x_k) - a_{i-t} \right)
\]

(by the definition of \( c_{i-t} \))

\[
= \sum_{t=0}^{i-1} (-1)^t e_t(x_1, x_2, \ldots, x_{i-1}) \left( h_{n-k+i-t} - a_{i-t} \right)
\]

\[
= \sum_{t=0}^{i-1} (-1)^t e_t(x_1, x_2, \ldots, x_{i-1}) c_{i-t}
\]

\[
= \sum_{t=0}^{i-1} (-1)^t e_t(x_1, x_2, \ldots, x_{i-1}) c_{i-t}
\]

\[
= \left( \sum_{t=0}^{i-1} e_t(x_1, x_2, \ldots, x_{0-1}) c_{i-0} \right) + \sum_{t=1}^{i-1} (-1)^t e_t(x_1, x_2, \ldots, x_{i-1}) c_{i-t}
\]

\[
\in c_i + \sum_{t=1}^{i-1} \mathcal{P} c_{i-t} = c_i + \sum_{t=1}^{i-1} c_{i-t} \mathcal{P}.
\]

Hence, Lemma 4.2 (applied to \( A = \mathcal{P} \)) yields that \( b_1\mathcal{P} + b_2\mathcal{P} + \cdots + b_k\mathcal{P} = c_1\mathcal{P} + c_2\mathcal{P} + \cdots + c_k\mathcal{P} \) (as ideals of \( \mathcal{P} \)). Comparing this with (12), we obtain
\( J = c_1 \mathcal{P} + c_2 \mathcal{P} + \cdots + c_k \mathcal{P} \). Thus, the family \( (b_i)_{i \in \{1,2,\ldots,k\}} \) generates the ideal \( J \). Furthermore, for each \( i \in \{1,2,\ldots,k\} \), the \( i \)-th term

\[
b_i = h_{n-k+i} (x_i, x_{i+1}, \ldots, x_k) - \sum_{t=0}^{i-1} (-1)^t e_t (x_1, x_2, \ldots, x_{i-1}) a_{i-t}
\]

of this family has leading term \( x_i^{n-k+i} \) (because the polynomial

\[
\sum_{t=0}^{i-1} (-1)^t e_t (x_1, x_2, \ldots, x_{i-1}) a_{i-t}
\]

has degree \( < n-k+i \) \(^3\)). Thus, the leading terms of the \( k \) elements of this family are disjoint (in the sense that no two of these leading terms have any indeterminates in common). Thus, clearly, Buchberger’s first criterion (see, e.g., [Grinbe17, detailed version, Proposition 3.9]) shows that it is a Gröbner basis.

\[\square\]

**Proof of Theorem 2.7 (sketched).** This follows using the Macaulay-Buchberger basis theorem (e.g., [Grinbe17, detailed version, Proposition 3.10]) from Proposition 4.1. \( \) (Indeed, if we let \( G \) be the Gröbner basis of \( J \) constructed in Proposition 4.1, then the monomials \( \alpha^\mathcal{P} \) for all \( \alpha \in \mathbb{N}^k \) satisfying \( \alpha_i < n-k+i \) for each \( i \)) are precisely the \( G \)-reduced monomials \(^4\)).

\[\square\]

**5. Proof of Theorem 2.7**

Next, we shall prove Theorem 2.7

**Convention 5.1.** For the rest of Section 5 we assume that \( a_1, a_2, \ldots, a_k \) belong to \( \mathcal{S} \).

Thus, \( a_1, a_2, \ldots, a_k \) are symmetric polynomials. Moreover, recall that for each \( i \in \{1,2,\ldots,k\} \), the polynomial \( a_i \) has degree \( < n-k+i \). In other words, for each \( i \in \{1,2,\ldots,k\} \), we have

\[
\deg(a_i) < n-k+i. \tag{13}
\]

\(^3\)Proof. It clearly suffices to show that for each \( t \in \{0,1,\ldots,i-1\} \), the polynomial \( e_t (x_1, x_2, \ldots, x_{i-1}) a_{i-t} \) has degree \( < n-k+i \).

So let us do this. Let \( t \in \{0,1,\ldots,i-1\} \). Then, the polynomial \( a_{i-t} \) has degree \( < n-k+(i-t) \) (by the definition of \( a_1, a_2, \ldots, a_k \)). In other words, \( \deg(a_{i-t}) < n-k+(i-t) \). Hence, the polynomial \( e_t (x_1, x_2, \ldots, x_{i-1}) a_{i-t} \) has degree

\[
\deg(e_t (x_1, x_2, \ldots, x_{i-1}) a_{i-t}) = \deg(e_t (x_1, x_2, \ldots, x_{i-1})) + \deg(a_{i-t}) \leq t + (n-k+(i-t)) = n-k+i.
\]

In other words, the polynomial \( e_t (x_1, x_2, \ldots, x_{i-1}) a_{i-t} \) has degree \( < n-k+i \). Qed.

\(^4\)because the \( i \)-th entry of the Gröbner basis \( G \) has head term \( x_i^{n-k+i} \).
Substituting $i - n + k$ for $i$ in this statement, we obtain the following: For each $i \in \{n - k + 1, n - k + 2, \ldots, n\}$, we have
\[
\deg (a_{n-k+i}) < n - k + (i - n + k) = i.
\] (14)

Let $I$ be the ideal of $S$ generated by the $k$ differences $\{2\}$. Hence, these differences belong to $I$. Thus,
\[
h_{n-k+j} \equiv a_j \mod I \quad \text{for each } j \in \{1, 2, \ldots, k\}.
\] (15)

Renaming the index $j$ as $i - n + k$ in this statement, we obtain
\[
h_i \equiv a_{i-n+k} \mod I \quad \text{for each } i \in \{n - k + 1, n - k + 2, \ldots, n\}.
\] (16)

\[\textbf{Lemma 5.2.} \quad \text{Let } A \text{ be a commutative } k\text{-algebra. Let } B \text{ be a commutative } A\text{-algebra. Assume that the } A\text{-module } B \text{ is spanned by the family } (b_u)_{u \in U} \in B^U. \text{ Let } I \text{ be an ideal of } A. \text{ Let } (a_v)_{v \in V} \in A^V \text{ be a family of elements of } A \text{ such that the } k\text{-module } A/I \text{ is spanned by the family } (\pi_v)_{v \in V} \in (A/I)^V. \text{ Then, the } k\text{-module } B/(IB) \text{ is spanned by the family } (\overline{a_v b_u})_{(u,v) \in U \times V} \in (B/(IB))^{U \times V}.\]

\[\textbf{Proof of Lemma 5.2.} \quad \text{Easy. Here is the proof under the assumption that the set } U \text{ is finite.}\]

Let $x \in B/(IB)$. Thus, $x = \overline{b}$ for some $b \in B$. Consider this $b$. Recall that the $A$-module $B$ is spanned by the family $(b_u)_{u \in U}$. Hence, $b = \sum_{u \in U} p_u b_u$ for some family $(p_u)_{u \in U} \in A^U$ of elements of $A$. Consider this family $(p_u)_{u \in U}$.

Recall that the $k$-module $A/I$ is spanned by the family $(\pi_v)_{v \in V} \in (A/I)^V$. Thus, for each $u \in U$, there exists a family $(q_{u,v})_{v \in V} \in k^V$ of elements of $k$ such that $\overline{p_u} = \sum_{v \in V} q_{u,v} \pi_v$ (and such that all but finitely many $v \in V$ satisfy $q_{u,v} = 0$).

Consider this family $(q_{u,v})_{v \in V}$.

Now, recall that $B/(IB)$ is an $A/I$-module (since $B$ is an $A$-module, but each $i \in I$ clearly acts as 0 on $B/(IB)$). Now,
\[
x = \overline{b} = \sum_{u \in U} p_u b_u = \left(\text{since } b = \sum_{u \in U} p_u b_u\right) = \sum_{u \in U} \sum_{v \in V} q_{u,v} \pi_v \overline{a_v b_u} = \sum_{(u,v) \in U \times V} q_{u,v} a_v \pi_v \overline{b_u}.
\]

\[\text{The case when } U \text{ is infinite needs only minor modifications. But we shall only use the case when } U \text{ is finite.}\]
Thus, $x$ belongs to the $k$-submodule of $B/(IB)$ spanned by the family
\[
(\alpha_u \beta_v)_{(u,v) \in U \times V}.
\] Since we have proven this for all $x \in B/(IB)$, we thus conclude that the $k$-module $B/(IB)$ is spanned by the family $(\alpha_u \beta_v)_{(u,v) \in U \times V} \in (B/(IB))^{U \times V}$. This proves Lemma 5.2. 

**Lemma 5.3.** Let $M$ be a free $k$-module with a finite basis $(b_s)_{s \in S}$. Let $(a_u)_{u \in U}$ be a family that spans $M$. Assume that $|U| = |S|$. Then, $(a_u)_{u \in U}$ is a basis of the $k$-module $M$. (In other words: A spanning family of $M$ whose size equals the size of a basis must itself be a basis, as long as the sizes are finite.)

**Proof of Lemma 5.3.** Well-known (see, e.g., [GriRei18, Exercise 2.5.18 (b)]).

**Lemma 5.4.** Let $i$ be an integer such that $i > n - k$. Then,
\[
h_i \equiv \text{(some symmetric polynomial of degree < } i \text{)} \mod I.
\]

**Proof of Lemma 5.4 (sketched).** We shall prove Lemma 5.4 by strong induction on $i$. Thus, we assume (as the induction hypothesis) that
\[
h_j \equiv \text{(some symmetric polynomial of degree < } j \text{)} \mod I \quad (17)
\]
for every $j \in \{n - k + 1, n - k + 2, \ldots, i - 1\}$.

If $i \leq n$, then (16) yields $h_i \equiv a_{i-n+k} \mod I$ (since $i \in \{n - k + 1, n - k + 2, \ldots, n\}$), which clearly proves Lemma 5.4 (since $a_{i-n+k}$ is a symmetric polynomial of degree < $i$).

Thus, for the rest of this proof, we WLOG assume that $i > n$. Hence, each $t \in \{1, 2, \ldots, k\}$ satisfies $i - t \in \{n - k + 1, n - k + 2, \ldots, i - 1\}$ (since $i > n$ and $i - t \geq 1$).

Therefore
\[
h_{i-t} \equiv \text{(some symmetric polynomial of degree < } i - t \text{)} \mod I \quad (18)
\]
(by (17), applied to $j = i - t$).

But $i$ is a positive integer (since $i > n \geq 0$). Hence, Corollary 3.3 (applied to $p = i$) yields
\[
h_i = - \sum_{t=1}^{k} (-1)^t e_t \equiv \text{(some symmetric polynomial of degree < } i \text{)} \mod I 
\]
(by (18))
\[
\equiv - \sum_{t=1}^{k} (-1)^t e_t \cdot \text{(some symmetric polynomial of degree < } i - t \text{)}
\]
\[
= \text{(some symmetric polynomial of degree < } i \text{)} \mod I.
\]

This completes the induction step. Thus, Lemma 5.4 is proven.
**Definition 5.5.** The **size** of a partition \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots) \) is defined as \( \lambda_1 + \lambda_2 + \lambda_3 + \cdots \), and is denoted by \(|\lambda|\).

**Definition 5.6.** Let \( P_k \) denote the set of all partitions with at most \( k \) parts. Thus, the elements of \( P_k \) are weakly decreasing \( k \)-tuples of nonnegative integers.

**Proposition 5.7.** Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) be a partition in \( P_k \). Then:

(a) We have
\[
s_\lambda = \det \left( (h_{\lambda_1, u, v})_{1 \leq u \leq k, 1 \leq v \leq k} \right).
\]

(b) Let \( p \in \{0, 1, \ldots, k\} \) be such that \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \). Then,
\[
s_\lambda = \det \left( (h_{\lambda_1, u, v})_{1 \leq u \leq p, 1 \leq v \leq p} \right).
\]

**Proof of Proposition 5.7** (b) Proposition 5.7 (b) is the well-known Jacobi-Trudi identity, and is proven in various places. (For instance, [GriRei18, (2.4.9)] states a similar formula for skew Schur functions; if we set \( \mu = \emptyset \) in it and apply both sides to the variables \( x_1, x_2, \ldots, x_k \), then we recover the claim of Proposition 5.7 (b).)

(a) We have \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \). Hence, Proposition 5.7 (a) is the particular case of Proposition 5.7 (b) for \( p = k \).

**Lemma 5.8.** Let \( \lambda \in P_k \) be a partition such that \( \lambda / \in P_{k,n} \). Then,
\[
s_\lambda \equiv (\text{some symmetric polynomial of degree } < |\lambda|) \mod I.
\]

**Proof of Lemma 5.8** (sketched). Write the partition \( \lambda \) as \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \). (This can be done, since \( \lambda \in P_k \).) Note that \( k > 0 \) (since otherwise, \( \lambda \in P_k \) would lead to \( \lambda = \emptyset \in P_{k,n} \), which would contradict \( \lambda \notin P_{k,n} \)).

From \( \lambda \in P_k \) and \( \lambda \notin P_{k,n} \), we conclude that not all parts of the partition \( \lambda \) are \( \leq n - k \). Thus, the first entry \( \lambda_1 \) of \( \lambda \) is \( > n - k \) (since \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \)). But \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \). Thus, Proposition 5.7 (a) yields
\[
s_\lambda = \det \left( (h_{\lambda_1, u, v})_{1 \leq u \leq k, 1 \leq v \leq k} \right) = \sum_{j=1}^{k} h_{\lambda_1-1+j} \cdot C_j,
\]
where \( C_j \) denotes the \((1,j)\)-th cofactor of the \( k \times k\)-matrix \( (h_{\lambda_1, u, v})_{1 \leq u \leq k, 1 \leq v \leq k} \) (here, we have expanded the determinant of the matrix \( (h_{\lambda_1, u, v})_{1 \leq u \leq k, 1 \leq v \leq k} \) along its first row).
For each $j \in \{1, 2, \ldots, k\}$, the cofactor $C_j$ is a symmetric polynomial of degree
\[
\sum_{u \in \{1, 2, \ldots, k\}; \ u \neq 1} (\lambda_u - u) + \sum_{v \in \{1, 2, \ldots, k\}; \ u \neq j} v
= \sum_{u \in \{1, 2, \ldots, k\}} (\lambda_u - u) - (\lambda_1 - 1) + \sum_{v \in \{1, 2, \ldots, k\}} v - j
= \sum_{u \in \{1, 2, \ldots, k\}} \lambda_u - \sum_{u \in \{1, 2, \ldots, k\}} u - (\lambda_1 - 1) + \sum_{u \in \{1, 2, \ldots, k\}} u - j
= \sum_{u \in \{1, 2, \ldots, k\}} \lambda_u - (\lambda_1 - 1) - j = |\lambda| - (\lambda_1 - 1) - j = |\lambda| - (\lambda_1 - 1 + j).
\]

Therefore, (19) becomes
\[
S_\lambda = \sum_{j=1}^{k} \underbrace{h_{\lambda_1 - 1 + j}}_{\text{(some symmetric polynomial of degree } < \lambda_1 - 1 + j\text{) mod } I} \equiv (\text{some symmetric polynomial of degree } |\lambda| - (\lambda_1 - 1 + j))
\]
\[
= \sum_{j=1}^{k} (\text{some symmetric polynomial of degree } < \lambda_1 - 1 + j)
\]
\[
= \text{(some symmetric polynomial of degree } |\lambda| - (\lambda_1 - 1 + j)\text{) mod } I.
\]

This proves Lemma 5.8.

Recall Definition 5.6.

**Lemma 5.9.** Let $N \in \mathbb{N}$. Let $f \in S$ be a symmetric polynomial of degree $< N$. Then, there exists a family $(c_\kappa)_{\kappa \in P_k; |\kappa| < N}$ of elements of $k$ such that $f = \sum_{\kappa \in P_k; |\kappa| < N} c_\kappa S_\kappa$.

**Proof of Lemma 5.9.** For each $d \in \mathbb{N}$, we let $S_{\text{deg}=d}$ be the $d$-th graded part of the graded $k$-module $S$. This is the $k$-submodule of $S$ consisting of all homogeneous elements of $S$ of degree $d$ (including the zero vector 0, which is homogeneous of every degree).
Recall that the family \((s_\lambda)_{\lambda \in P_k}\) is a graded basis of the graded \(k\)-module \(S\). In other words, for each \(d \in \mathbb{N}\), the family \((s_\lambda)_{\lambda \in P_k; |\lambda|=d}\) is a basis of the \(k\)-submodule \(S_{\text{deg}=d}\) of \(S\). Hence, for each \(d \in \mathbb{N}\), we have

\[
S_{\text{deg}=d} = \left( \text{the } k\text{-linear span of the family } (s_\lambda)_{\lambda \in P_k; |\lambda|=d} \right) = \sum_{\lambda \in P_k; |\lambda|=d} ks_\lambda.
\]  

(20)

The polynomial \(f\) has degree \(< N\). Hence, we can write \(f\) in the form \(f = \sum_{d=0}^{N-1} f_d\) for some \(f_0, f_1, \ldots, f_{N-1} \in P\), where each \(f_d\) is a homogeneous polynomial of degree \(d\). Consider these \(f_0, f_1, \ldots, f_{N-1}\). These \(N\) polynomials \(f_0, f_1, \ldots, f_{N-1}\) are the first \(N\) homogeneous components of \(f\), and thus are symmetric (since \(f\) is symmetric); in other words, \(f_0, f_1, \ldots, f_{N-1}\) are elements of \(S\). Thus, for each \(d \in \{0, 1, \ldots, N-1\}\), the polynomial \(f_d\) is an element of \(S\) and is homogeneous of degree \(d\) (as we already know). In other words, for each \(d \in \{0, 1, \ldots, N-1\}\), we have

\[
f_d \in S_{\text{deg}=d}.
\]  

(21)

Now,

\[
f = \sum_{d=0}^{N-1} f_d \quad \in \sum_{d=0}^{N-1} S_{\text{deg}=d} \quad = \sum_{d=0}^{N-1} \sum_{\lambda \in P_k; |\lambda|=d} ks_\lambda \quad = \sum_{\kappa \in P_k; |\kappa|<N} \sum_{\lambda \in P_k; |\lambda|=d} ks_\lambda
\]

(by (21))

(here, we have renamed the summation index \(\lambda\) as \(\kappa\) in the sum). In other words, there exists a family \((c_\kappa)_{\kappa \in P_k; |\kappa|<N}\) of elements of \(k\) such that \(f = \sum_{\kappa \in P_k; |\kappa|<N} c_\kappa s_\kappa\). This proves Lemma 5.9.

\[
\square
\]

**Lemma 5.10.** For each \(\mu \in P_k\), the element \(\overline{\mu} \in S/I\) belongs to the \(k\)-submodule of \(S/I\) spanned by the family \((\overline{s_\lambda})_{\lambda \in P_k,n}\).

**Proof of Lemma 5.10.** Let \(M\) be the \(k\)-submodule of \(S/I\) spanned by the family \((\overline{s_\lambda})_{\lambda \in P_k,n}\). We thus must prove that \(\overline{\mu} \in M\) for each \(\mu \in P_k\).

We shall prove this by strong induction on \(|\mu|\). Thus, we fix some \(N \in \mathbb{N}\), and we assume (as induction hypothesis) that

\[
\overline{\kappa} \in M \quad \text{for each } \kappa \in P_k \text{ satisfying } |\kappa| < N.
\]  

(22)
Now, let \( \mu \in P_k \) be such that \(|\mu| = N\). We then must show that \( \overline{s_\mu} \in M \).

If \( \mu \in P_{k,n} \), then this is obvious (since \( \overline{s_\mu} \) then belongs to the family that spans \( M \)). Thus, for the rest of this proof, we WLOG assume that \( \mu \notin P_{k,n} \). Hence, Lemma 5.8 (applied to \( \lambda = \mu \)) yields

\[
s_\mu \equiv (\text{some symmetric polynomial of degree } < |\mu|) \mod I.
\]

In other words, there exists some symmetric polynomial \( f \in S \) of degree \( < |\mu| \) such that \( s_\mu \equiv f \mod I \). Consider this \( f \).

The polynomial \( f \) is a symmetric polynomial of degree \( < |\mu| \). In other words, \( f \) is a symmetric polynomial of degree \( < N \) (since \(|\mu| = N\)). Hence, Lemma 5.9 shows that there exists a family \( (c_\lambda)_{\lambda \in P_k; |\lambda| < N} \) of elements of \( k \) such that

\[
f = \sum_{\lambda \in P_k; |\lambda| < N} c_\lambda s_\lambda.
\]

Consider this family. From \( f = \sum_{\lambda \in P_k; |\lambda| < N} c_\lambda s_\lambda \), we obtain

\[
\overline{f} = \sum_{\lambda \in P_k; |\lambda| < N} \sum_{\kappa \in P_\kappa; |\kappa| < N} \kappa f = \sum_{\kappa \in P_\kappa; |\kappa| < N} \kappa \overline{f} \in \sum_{\kappa \in P_\kappa; |\kappa| < N} \kappa M \subseteq M \quad (\text{since } M \text{ is a } k\text{-module}).
\]

But from \( s_\mu \equiv f \mod I \), we obtain \( \overline{s_\mu} = \overline{f} \in M \). This completes our induction step. Thus, we have proven by strong induction that \( \overline{s_\mu} \in M \) for each \( \mu \in P_k \). This proves Lemma 5.10.

Proof of Theorem 2.7 (sketched). Proposition 2.1 yields that \( (x^a)_{a \in N^k; a_i < i \text{ for each } i} \) is a spanning set of the \( S \)-module \( P \).

Recall Definition 5.6. It is well-known that \( (s_\lambda)_{\lambda \in P_k} \) is a basis of the \( k \)-module \( S \). Hence, \( (\overline{s_\lambda})_{\lambda \in P_k} \) is a spanning set of the \( k \)-module \( S/I \). Thus, \( (\overline{s_\lambda})_{\lambda \in P_{k,n}} \) is also a spanning set of the \( k \)-module \( S/I \) (because Lemma 5.10 shows that every element of the first spanning set belongs to the span of the second). It remains to prove that this spanning set is also a basis.

In order to do so, we consider the family \( (\overline{s_\lambda x^a})_{\lambda \in P_{k,n}; a \in N^k; a_i < i \text{ for each } i} \) in the \( k \)-module \( P/J \). This family spans \( P/J \) (by Lemma 5.2), because the family \( (\overline{s_\lambda})_{\lambda \in P_{k,n}} \) spans \( S/I \) whereas the family \( (x^a)_{a \in N^k; a_i < i \text{ for each } i} \) spans \( P \) over \( S \) (and because \( IP = J \)). Moreover, this family \( (\overline{s_\lambda x^a})_{\lambda \in P_{k,n}; a \in N^k; a_i < i \text{ for each } i} \) has size

\[
\left\lfloor \frac{|P_{k,n}|}{n^k} \right\rfloor \cdot \left\lfloor \frac{\sum_{a \in N^k; a_i < i \text{ for each } i} 1}{n^k} \right\rfloor = \binom{n}{k} k!
\]

\[
= n (n - 1) \cdots (n - k + 1),
\]

which is exactly the size of the basis \( (x^a)_{a \in N^k; a_i < n-k+i \text{ for each } i} \) of the \( k \)-module \( P/J \) (this is a basis by Theorem 2.2). Thus, this family \( (\overline{s_\lambda x^a})_{\lambda \in P_{k,n}; a \in N^k; a_i < i \text{ for each } i} \)
must be a basis of the \( k \)-module \( \mathcal{P} / I \) (by Lemma 5.3), and hence is \( k \)-linearly independent. Thus, its subfamily \( (\overline{s}_\lambda)_{\lambda \in P_{k,n}} \) is also \( k \)-linearly independent.

The canonical \( k \)-linear map \( S / I \rightarrow \mathcal{P} / I \) (obtained as a quotient of the inclusion \( S \rightarrow \mathcal{P} \)) is injective (because it sends the spanning set \( (\overline{s}_\lambda)_{\lambda \in P_{k,n}} \) of \( S / I \) to the \( k \)-linearly independent family \( (\overline{s}_\lambda)_{\lambda \in P_{k,n}} \) in \( \mathcal{P} / I \)). Hence, the \( k \)-linear independence of the family \( (\overline{s}_\lambda)_{\lambda \in P_{k,n}} \) in \( \mathcal{P} / I \) yields the \( k \)-linear independency of the family \( (\overline{s}_\lambda)_{\lambda \in P_{k,n}} \) in \( S / I \). Thus, the family \( (\overline{s}_\lambda)_{\lambda \in P_{k,n}} \) in \( S / I \) is a basis of \( S / I \) (since it is \( k \)-linearly independent and spans \( S / I \)). This proves Theorem 2.7 \( \square \)

6. Symmetry of the multiplicative structure constants

**Convention 6.1.** For the rest of Section 6 we assume that \( a_1, a_2, \ldots, a_k \) belong to \( k \).

If \( m \in S \), then the notation \( \overline{m} \) shall always mean the projection of \( m \in S \) onto the quotient \( S / I \) (and not the projection of \( m \in \mathcal{P} \) onto the quotient \( \mathcal{P} / I \)).

**Definition 6.2.**

(a) Let \( \omega \) be the partition \((n-k, n-k, \ldots, n-k)\) with \( k \) entries equal to \( n-k \). (This is the largest partition in \( P_{k,n} \).)

(b) Let \( I \) be the ideal of \( S \) generated by the \( k \) differences \((2)\). For each \( \mu \in P_{k,n} \), let \( \text{coeff}_\mu : S / I \rightarrow k \) be the \( k \)-linear map that sends \( \overline{s}_\mu \) to 1 while sending all other \( \overline{s}_\lambda \) (with \( \lambda \in P_{k,n} \)) to 0. (This is well-defined by Theorem 2.7.)

Actually, \( (\text{coeff}_\mu)_{\mu \in P_{k,n}} \) is the dual basis to the basis \( (\overline{s}_\lambda)_{\lambda \in P_{k,n}} \) of \( S / I \).

(c) If \( \lambda \) is any partition and if \( p \) is a positive integer, then \( \lambda_p \) shall always denote the \( p \)-th entry of \( \lambda \). Thus, \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots) \) for every partition \( \lambda \).

(d) For every partition \( \nu = (\nu_1, \nu_2, \ldots, \nu_k) \in P_{k,n} \), we let \( \nu^\vee \) denote the partition \((n-k-v_k, n-k-v_{k-1}, \ldots, n-k-v_1) \in P_{k,n} \). This partition \( \nu^\vee \) is called the complement of \( \nu \).

We can now make a more substantial claim:

**Theorem 6.3.** Each \( \nu \in P_{k,n} \) and \( f \in S / I \) satisfy \( \text{coeff}_\nu (\overline{s}_\nu f) = \text{coeff}_{\nu^\vee} (f) \).

The proof of this theorem requires some preliminary work.

We first recall some basic notations from [GriRei18, Chapter 2]. If \( \lambda \) and \( \mu \) are two partitions, then we say that \( \mu \subseteq \lambda \) if and only if each positive integer \( p \) satisfies \( \mu_p \leq \lambda_p \). A skew partition means a pair \((\lambda, \mu)\) of two partitions satisfying \( \mu \subseteq \lambda \); such a pair is denoted by \( \lambda / \mu \). We refer to [GriRei18 §2.7] for the definition of a vertical i-strip (where \( i \in \mathbb{N} \)).

Let \( \Lambda \) be the ring of symmetric functions in infinitely many indeterminates \( x_1, x_2, x_3, \ldots \) over \( k \). If \( f \in \Lambda \) is a symmetric function, then \( f (x_1, x_2, \ldots, x_k) \) is a
symmetric polynomial in $S$; the map

$$\Lambda \to S, \quad f \mapsto f(x_1, x_2, \ldots, x_k)$$

is a surjective $k$-algebra homomorphism. We shall use boldfaced notations for symmetric functions in $\Lambda$ in order to distinguish them from symmetric polynomials in $S$. In particular:

- For any $i \in \mathbb{Z}$, we let $h_i$ be the $i$-th complete homogeneous symmetric function in $\Lambda$. (This is called $h_i$ in [GriRei18, Definition 2.2.1].)

- For any $i \in \mathbb{Z}$, we let $e_i$ be the $i$-th elementary symmetric function in $\Lambda$. (This is called $e_i$ in [GriRei18, Definition 2.2.1].)

- For any partition $\lambda$, we let $e_\lambda$ be the corresponding elementary symmetric function in $\Lambda$. (This is called $e_\lambda$ in [GriRei18, Definition 2.2.1].)

- For any partition $\lambda$, we let $s_\lambda$ be the corresponding Schur function in $\Lambda$. (This is called $s_\lambda$ in [GriRei18, Definition 2.2.1].)

- For any partitions $\lambda$ and $\mu$, we let $s_{\lambda/\mu}$ be the corresponding skew Schur function in $\Lambda$. (This is called $s_{\lambda/\mu}$ in [GriRei18, §2.3]. Note that $s_{\lambda/\mu} = 0$ unless $\mu \subseteq \lambda$.)

Also, we shall use the skewing operators as defined (e.g.) in [GriRei18, §2.8]. We recall their main properties:

- For each $f \in \Lambda$, the skewing operator $f^\perp$ is a $k$-linear map $\Lambda \to \Lambda$. It depends $k$-linearly on $f$ (that is, we have $(\alpha f + \beta g)^\perp = \alpha f^\perp + \beta g^\perp$ for any $\alpha, \beta \in k$ and $f, g \in \Lambda$).

- For any partitions $\lambda$ and $\mu$, we have

$$\left(s_\mu\right)^\perp (s_\lambda) = s_{\lambda/\mu}. \quad (23)$$

(This is [GriRei18, (2.8.2)].)

- For any $f, g \in \Lambda$, we have

$$\left(fg\right)^\perp = g^\perp \circ f^\perp. \quad (24)$$

(This is [GriRei18, Proposition 2.8.2(ii)], applied to $A = \Lambda$.)

- We have $1^\perp = \text{id}.$

For each partition $\lambda$, let $\lambda^t$ denote the conjugate partition of $\lambda$; see [GriRei18, Definition 2.2.8] for its definition.

Recall the second Jacobi-Trudi identity ([GriRei18, (2.4.10)]):
Proposition 6.4. Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \) be two partitions. Then,
\[
s_{\lambda'}/\mu' = \det \left( \left( e_{\lambda_i - \mu_j - i + j} \right)_{1 \leq i, j \leq \ell} \right).
\]

Corollary 6.5. Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \) be a partition. Then,
\[
s_{\lambda'} = \det \left( \left( e_{\lambda_i - i + j} \right)_{1 \leq i, j \leq \ell} \right).
\]

Proof of Corollary 6.5. This follows from Proposition 6.4 applied to \( \mu = \emptyset \) (since \( \emptyset' = \emptyset \) and thus \( s_{\lambda'}/\emptyset' = s_{\lambda'}/\emptyset = s_{\lambda'} \)).

We also recall one of the Pieri rules ([GriRei18, (2.7.2)]):

Proposition 6.6. Let \( \lambda \) be a partition, and let \( i \in \mathbb{N} \). Then,
\[
s_{\lambda} e_i = \sum_{\mu \text{ is a partition}; \mu/\lambda \text{ is a vertical } i\text{-strip}} s_{\mu}.
\]

From this, we can easily derive the following:

Corollary 6.7. Let \( \lambda \) be a partition, and let \( i \in \mathbb{N} \). Then,
\[
(e_i)^{\perp} s_{\lambda} = \sum_{\lambda/\mu \text{ is a vertical } i\text{-strip}} s_{\mu}.
\]

Corollary 6.7 is also proven in [GriRei18, version with solutions (ancillary file), Lemma 12.83.3(b)].

The next proposition is the claim of [GriRei18 Exercise 2.9.1(b)]:

Proposition 6.8. Let \( \lambda \) be a partition. Let \( m \in \mathbb{Z} \) be such that \( m \geq \lambda_1 \). Then,
\[
\sum_{i \in \mathbb{N}} (-1)^i h_{m+i} \left( e_i \right)^\perp s_{\lambda} = s_{(m, \lambda_2, \lambda_3, \ldots)}.
\]

We shall use this to derive the following corollary:

Corollary 6.9. Let \( \lambda \) be a partition with at most \( k \) parts. Let \( \overline{\lambda} \) be the partition \( (\lambda_2, \lambda_3, \lambda_4, \ldots) \). Then,
\[
s_{\lambda} = \sum_{i=0}^{k-1} (-1)^i h_{\lambda_1+i} \sum_{\overline{\mu} \text{ is a partition}; \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}} s_{\mu}.
\]
Proof of Corollary 6.9] The partition $\overline{\lambda}$ is obtained from $\lambda$ by removing the first part. Hence, this partition $\overline{\lambda}$ has at most $k - 1$ parts (since $\lambda$ has at most $k$ parts). Thus, if $i \in \mathbb{N}$ satisfies $i \geq k$, then

there exists no partition $\mu$ such that $\overline{\lambda}/\mu$ is a vertical $i$-strip. \hfill (25)

We have $\overline{\lambda} = (\lambda_2, \lambda_3, \lambda_4, \ldots)$, so that $(\lambda_2, \lambda_3, \lambda_4, \ldots) = \overline{\lambda} = (\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \ldots)$.

Hence,

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots) = (\lambda_1, \overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \ldots).$$

Also, clearly, $\lambda_1 \geq \overline{\lambda}_1$ (since $\lambda_1 \geq \lambda_2 = \overline{\lambda}_1$). Hence, Proposition 6.8 (applied to $\overline{\lambda}$ and $\lambda_1$ instead of $\lambda$ and $m$) yields

$$\sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{\lambda_1+i}(\mathbf{e}_i) \mathbf{s}_{\overline{\lambda}} = \sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{\lambda_1+i} \sum_{\mu \text{ is a partition}; \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}} \mathbf{s}_{\mu}$$

(since $(\lambda_1, \overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \ldots) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots) = \lambda$). Therefore,

$$\mathbf{s}_{\lambda} = \sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{\lambda_1+i} \sum_{\mu \text{ is a partition}; \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}} \mathbf{s}_{\mu} + \sum_{i \geq k} (-1)^i \mathbf{h}_{\lambda_1+i} \sum_{\mu \text{ is a partition}; \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}} \mathbf{s}_{\mu}$$

$$= \sum_{i=0}^{k-1} (-1)^i \mathbf{h}_{\lambda_1+i} \sum_{\mu \text{ is a partition}; \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}} \mathbf{s}_{\mu}.$$

This proves Corollary 6.9 \hfill \Box

[Convention 6.10] We WLOG assume that $k > 0$ for the rest of Section 6 (since otherwise, Theorem 6.3 is trivial).

Next, we define a filtration on the $k$-module $S/I$:

[Definition 6.11] For each $p \in \mathbb{Z}$, we let $Q_p$ denote the $k$-submodule of $S/I$ spanned by the $S_{\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\lambda_k \leq p$.

Thus, $0 = Q_{-1} \subseteq Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \cdots$. Theorem 2.7 shows that the $k$-module $S/I$ is free with basis $(\mathbf{s}_{\lambda})_{\lambda \in P_{k,n}}$; hence, $S/I = Q_{n-k}$ (since each $\lambda \in P_{k,n}$ satisfies $\lambda_k \leq n-k$).

Note that $(Q_0, Q_1, Q_2, \ldots)$ is a filtration of the $k$-module $S/I$, but not (in general) of the $k$-algebra $S/I$. 

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Lemma 6.12. We have \( \text{coeff}_\omega (Q_{n-k-1}) = 0 \).

Proof of Lemma 6.12. The map \( \text{coeff}_\omega \) is \( k \)-linear; thus, it suffices to prove that \( \text{coeff}_\omega (s_\lambda) = 0 \) for each \( \lambda \in P_{k,n} \) satisfying \( \lambda_k \leq n-k-1 \) (because the \( k \)-module \( Q_{n-k-1} \) is spanned by the \( s_\lambda \) with \( \lambda \in P_{k,n} \) satisfying \( \lambda_k \leq n-k-1 \)). So let us fix some \( \lambda \in P_{k,n} \) satisfying \( \lambda_k \leq n-k-1 \). We must then prove that \( \text{coeff}_\omega (s_\lambda) = 0 \).

We have \( \lambda_k \leq n-k-1 < n-k = \omega_k \). Thus, \( \lambda_k \neq \omega_k \), so that \( \lambda \neq \omega \).

The definition of the map \( \text{coeff}_\omega \) yields \( \text{coeff}_\omega (s_\lambda) = \begin{cases} 1, & \text{if } \lambda = \omega; \\ 0, & \text{if } \lambda \neq \omega \end{cases} \) (since \( \lambda \neq \omega \)). This completes our proof of Lemma 6.12.

Lemma 6.13. Let \( \lambda \) be a partition with at most \( k \) parts. Assume that \( \lambda_1 = n-k+1 \). Let \( \bar{\lambda} \) be the partition \((\lambda_2, \lambda_3, \lambda_4, \ldots)\). Then,

\[
\bar{s}_{\lambda} = \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\mu \text{ is a partition;} \atop \bar{\lambda}/\mu \text{ is a vertical } i\text{-strip}} s_\mu.
\]

Proof of Lemma 6.13. Corollary 6.9 yields

\[
s_{\lambda} = \sum_{i=0}^{k-1} (-1)^i h_{\lambda_1+i} \sum_{\mu \text{ is a partition;} \atop \bar{\lambda}/\mu \text{ is a vertical } i\text{-strip}} s_\mu.
\]

This is an identity in \( \Lambda \). Evaluating both of its sides at the \( k \) variables \( x_1, x_2, \ldots, x_k \), we obtain

\[
s_{\lambda} = \sum_{i=0}^{k-1} (-1)^i \frac{h_{\lambda_1+i}}{\lambda_1+i} \sum_{\mu \text{ is a partition;} \atop \bar{\lambda}/\mu \text{ is a vertical } i\text{-strip}} s_\mu
\]

\[
= \sum_{i=0}^{k-1} (-1)^i \frac{h_{n-k+1+i}}{n-k+1+i} \sum_{\mu \text{ is a partition;} \atop \bar{\lambda}/\mu \text{ is a vertical } i\text{-strip}} s_\mu
\]

\[
\equiv \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\mu \text{ is a partition;} \atop \bar{\lambda}/\mu \text{ is a vertical } i\text{-strip}} s_\mu \mod I.
\]

Projecting both sides of this equality from \( S \) to \( S/I \), we obtain

\[
\bar{s}_{\lambda} = \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\mu \text{ is a partition;} \atop \bar{\lambda}/\mu \text{ is a vertical } i\text{-strip}} s_\mu = \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\mu \text{ is a partition;} \atop \bar{\lambda}/\mu \text{ is a vertical } i\text{-strip}} \bar{s}_\mu.
\]
This proves Lemma 6.13.

**Lemma 6.14.** Let \( \lambda \) be a partition with at most \( k \) parts. Assume that \( \lambda_1 = n - k + 1 \). Then, \( \overline{s}_{\lambda} \in Q_0 \).

**Proof of Lemma 6.14.** We shall prove Lemma 6.14 by strong induction on \( |\lambda| \).

Thus, we fix some \( N \in \mathbb{N} \), and we assume (as induction hypothesis) that Lemma 6.14 is already proven whenever \( |\lambda| < N \). We now must proveLemma 6.14 in the case when \( |\lambda| = N \).

So let \( \lambda \) be as in Lemma 6.14 and assume that \( |\lambda| = N \). Let \( \overline{\lambda} \) be the partition \((\lambda_2, \lambda_3, \lambda_4, \ldots)\). Then, Lemma 6.13 yields

\[
\overline{s}_{\lambda} = \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\mu \text{ is a partition; } \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}} \overline{s}_{\mu}.
\]

But if \( \mu \) is a partition such that \( \overline{\lambda}/\mu \) is a vertical \( i \)-strip, then

\[
\overline{s}_{\mu} \in Q_0.
\]

[Proof of (27): The partition \( \lambda \) has at most \( k \) parts; thus, the partition \( \overline{\lambda} \) has at most \( k - 1 \) parts.

Now, let \( \mu \) be a partition such that \( \overline{\lambda}/\mu \) is a vertical \( i \)-strip. Then, \( \mu \subseteq \overline{\lambda} \), so that \( \mu \) has at most \( k - 1 \) parts (since \( \overline{\lambda} \) has at most \( k - 1 \) parts). Thus, \( \mu_k = 0 \leq 0 \). Also, \( \mu \) has at most \( k \) parts (since \( \mu \) has at most \( k - 1 \) parts). If \( \mu_1 \leq n - k \), then this yields that \( \mu \in P_{k,n} \) and therefore \( \overline{s}_{\mu} \in Q_0 \) (since \( \mu \in P_{k,n} \) and \( \mu_k \leq 0 \)). Thus, (27) is proven if \( \mu_1 \leq n - k \). Hence, for the rest of this proof, we WLOG assume that we don’t have \( \mu_1 \leq n - k \). Hence, \( \mu_1 > n - k \).

But \( \mu \subseteq \overline{\lambda} \), so that \( \mu_1 \leq \lambda_1 = \lambda_2 \leq \lambda_1 = n - k + 1 \). Combining this with \( \mu_1 > n - k \), we obtain \( \mu_1 = n - k + 1 \). Also, \( \mu \subseteq \overline{\lambda} \), so that

\[
|\mu| \leq |\overline{\lambda}| = |\lambda| - \sum_{\lambda_1 = n-k+1}^{\lambda_1 \geq 1} 1 < |\lambda| = N.
\]

Hence, we can apply Lemma 6.14 to \( \mu \) instead of \( \lambda \) (by the induction hypothesis). We thus obtain \( \overline{s}_{\mu} \in Q_0 \). This completes the proof of (27).]

Now, (26) becomes

\[
\overline{s}_{\lambda} = \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\mu \text{ is a partition; } \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}} \overline{s}_{\mu} \in Q_0.
\]

Thus, we have proven Lemma 6.14 for our \( \lambda \). This completes the induction step; thus, Lemma 6.14 is proven. 

\[\square\]
Lemma 6.15. Let $i \in \mathbb{N}$ and $\lambda \in P_{k,n}$. Then,

$$s_i s_\lambda \equiv \sum_{\mu \in P_{k,n} \mid \mu/\lambda \text{ is a vertical } i\text{-strip}} s_\mu \mod Q_0.$$ 

Proof of Lemma 6.15. If $\mu$ is a partition such that $\mu/\lambda$ is a vertical $i$-strip and $\mu \notin P_{k,n}$, then

$$s_\mu \equiv 0 \mod Q_0. \tag{28}$$

[Proof of (28): Let $\mu$ be a partition such that $\mu/\lambda$ is a vertical $i$-strip and $\mu \notin P_{k,n}$. We must prove (28).

If the partition $\mu$ has more than $k$ parts, then (28) easily follows. Hence, for the rest of this proof, we WLOG assume that the partition $\mu$ has at most $k$ parts.

Since $\mu/\lambda$ is a vertical strip, we have $\mu_1 \leq \lambda_1 + 1$. But $\lambda_1 \leq n - k$ (since $\lambda \in P_{k,n}$). If $\mu_1 = n - k + 1$, then (28) easily follows. Hence, for the rest of this proof, we WLOG assume that $\mu_1 \neq n - k + 1$. Combining this with $\mu_1 \leq \lambda_1 + 1 \leq n - k + 1$, we obtain $\mu_1 < n - k + 1$, so that $\mu_1 \leq n - k$.

Hence, $\mu \in P_{k,n}$ (since $\mu$ has at most $k$ parts). This contradicts $\mu \notin P_{k,n}$. Thus, $s_\mu \equiv 0 \mod Q_0$ (because ex falso quodlibet). Hence, (28) is proven.]

Proposition 6.6 yields

$$s_\lambda e_i = \sum_{\mu \text{ is a partition}; \mu/\lambda \text{ is a vertical } i\text{-strip}} s_\mu.$$ 

This is an identity in $\Lambda$. Evaluating both of its sides at the $k$ variables $x_1, x_2, \ldots, x_k$, we obtain

$$s_\lambda e_i = \sum_{\mu \text{ is a partition}; \mu/\lambda \text{ is a vertical } i\text{-strip}} s_\mu.$$ 

\[\text{Proof. Assume that the partition } \mu \text{ has more than } k \text{ parts. Thus, (3) (applied to } \mu \text{ instead of } \lambda) \text{ yields } s_\mu = 0. \text{ Thus, } s_\mu \equiv 0 \equiv 0 \mod Q_0. \text{ Thus, (28) holds.}\]

\[\text{Proof. Assume that } \mu_1 = n - k + 1. \text{ Then, Lemma 6.14 (applied to } \mu \text{ instead of } \lambda) \text{ yields } s_\mu \in Q_0. \text{ Hence, } s_\mu \equiv 0 \mod Q_0. \text{ Thus, (28) holds.}\]
Projecting both sides of this equality from $S$ to $S/I$, we obtain

$$s^i e^j = \sum_{\mu \text{ is a partition; } \mu / \lambda \text{ is a vertical } i\text{-strip}} s_{\mu} = \sum_{\mu \text{ is a partition; } \mu / \lambda \text{ is a vertical } i\text{-strip}} \overline{s_{\mu}}$$

$$= \sum_{\mu \text{ is a partition; } \mu / \lambda \text{ is a vertical } i\text{-strip; } \mu \in P_{k,n}} \overline{s_{\mu}} + \sum_{\mu \text{ is a partition; } \mu / \lambda \text{ is a vertical } i\text{-strip; } \mu \notin P_{k,n}} \overline{s_{\mu}} \equiv 0 \mod Q_0 \quad \text{(by \eqref{eq:28})}$$

$$\equiv \sum_{\mu \text{ is a partition; } \mu / \lambda \text{ is a vertical } i\text{-strip; } \mu \in P_{k,n}} \overline{s_{\mu}} = \sum_{\mu \text{ is a partition; } \mu / \lambda \text{ is a vertical } i\text{-strip}} \overline{s_{\mu}} \mod Q_0.$$ 

Thus, $\overline{e^i s^j} = \overline{s^i e^j} \equiv \sum_{\mu \text{ is a partition; } \mu / \lambda \text{ is a vertical } i\text{-strip}} \overline{s_{\mu}} \mod Q_0$. This proves Lemma \ref{lem:6.15} \qed}

**Lemma 6.16.** Let $i \in \mathbb{N}$ and $p \in \mathbb{Z}$. Then, $\overline{e^i Q_p} \subseteq Q_{p+1}$.

**Proof of Lemma 6.16.** Due to the definition of $Q_p$, it suffices that every $\lambda \in P_{k,n}$ satisfying $\lambda_k \leq p$ satisfies $\overline{e^i s^j} \in Q_{p+1}$. So let us fix $\lambda \in P_{k,n}$ satisfying $\lambda_k \leq p$. We must prove that $\overline{e^i s^j} \in Q_{p+1}$.

If $\mu \in P_{k,n}$ is such that $\mu / \lambda$ is a vertical $i$-strip, then

$$s_{\mu} \equiv 0 \mod Q_{p+1}. \tag{29}$$

[Proof of \eqref{eq:29}: Let $\mu \in P_{k,n}$ be such that $\mu / \lambda$ is a vertical $i$-strip. We must prove \eqref{eq:29}.

Since $\mu / \lambda$ is a vertical strip, we have $\mu_k \leq \lambda_k + 1 \leq p + 1$. From $\mu \in P_{k,n}$ and $\mu_k \leq p + 1$, we obtain $s_{\mu} \in Q_{p+1}$. In other words, $s_{\mu} \equiv 0 \mod Q_{p+1}$. Thus, \eqref{eq:29} is proven.]

Lemma \ref{lem:6.15} yields

$$\overline{e^i s^j} \equiv \sum_{\mu \text{ is a partition; } \mu / \lambda \text{ is a vertical } i\text{-strip}} \overline{s_{\mu}} \mod Q_0.$$ 

Hence,

$$\overline{e^i s^j} - \sum_{\mu \text{ is a partition; } \mu / \lambda \text{ is a vertical } i\text{-strip}} \overline{s_{\mu}} \equiv 0 \subseteq Q_{p+1}.$$ 

Thus,

$$\overline{e^i s^j} \equiv \sum_{\mu \text{ is a partition; } \mu / \lambda \text{ is a vertical } i\text{-strip}} \overline{s_{\mu}} \equiv 0 \mod Q_{p+1}. \quad \text{(by \eqref{eq:29})}$$

In other words, $\overline{e^i s^j} \in Q_{p+1}$. This completes our proof of Lemma \ref{lem:6.16} \qed
The next fact that we use from the theory of symmetric functions are some basic properties of the Littlewood-Richardson coefficients. For any partitions \( \lambda, \mu, \nu \), we let \( c_{\mu, \nu}^\lambda \) be the Littlewood-Richardson coefficient as defined in [GriRei18, Definition 2.5.8]. Then, we have the following fact (part of [GriRei18, Remark 2.5.9]):

**Proposition 6.17.** Let \( \lambda \) and \( \mu \) be two partitions.

(a) We have

\[
 s_{\lambda/\mu} = \sum_{\nu \text{ is a partition}} c_{\mu, \nu}^\lambda s_\nu.
\]

(b) If \( \nu \) is a partition, then \( c_{\mu, \nu}^\lambda = 0 \) unless \( \nu \subseteq \lambda \).

(c) If \( \nu \) is a partition, then \( c_{\mu, \nu}^\lambda = 0 \) unless \( |\mu| + |\nu| = |\lambda| \).

Next, let \( Z \) be the \( k \)-submodule of \( \Lambda \) spanned by the \( s_\lambda \) with \( \lambda \in P_{k,n} \). Then, \( (s_\lambda)_{\lambda \in P_{k,n}} \) is a basis of the \( k \)-module \( Z \) (since \( (s_\lambda)_{\lambda \in P_{k,n}} \) is a partition is a basis of the \( k \)-module \( \Lambda \)). We thus can define a \( k \)-linear map \( \delta : Z \to S/I \) by setting

\[
 \delta (s_\lambda) = s_{\lambda'/\nu} \quad \text{for every } \lambda \in P_{k,n}.
\]

Notice that a partition \( \lambda \) satisfies \( \lambda \in P_{k,n} \) if and only if \( \lambda \subseteq \omega \).

**Lemma 6.18.** We have \( f^\perp (Z) \subseteq Z \) for each \( f \in \Lambda \).

**Proof of Lemma 6.18.** Since \( f^\perp \) depends \( k \)-linearly on \( f \), it suffices to check that \( (s_\mu)^\perp (Z) \subseteq Z \) for each partition \( \mu \). So let us fix a partition \( \mu \); we then must prove that \( (s_\mu)^\perp (Z) \subseteq Z \).

Recall that \( Z \) is the \( k \)-module spanned by the \( s_\lambda \) with \( \lambda \in P_{k,n} \). Hence, in order to prove that \( (s_\mu)^\perp (Z) \subseteq Z \), it suffices to check that \( (s_\mu)^\perp (s_\lambda) \in Z \) for each \( \lambda \in P_{k,n} \). So let us fix \( \lambda \in P_{k,n} \); we must then prove that \( (s_\mu)^\perp (s_\lambda) \in Z \).

From (23), we obtain

\[
 (s_\mu)^\perp (s_\lambda) = s_{\lambda/\mu} = \sum_{\nu \text{ is a partition}} c_{\mu, \nu}^\lambda s_\nu \quad \text{(by Proposition 6.17 (a))}
\]

\[
 = \sum_{\nu \text{ is a partition; } \nu \subseteq \lambda} c_{\mu, \nu}^\lambda s_\nu + \sum_{\nu \text{ is a partition; } \nu \not\subseteq \lambda} c_{\mu, \nu}^\lambda s_\nu \quad \text{(by Proposition 6.17 (b))}
\]

\[
 = \sum_{\nu \text{ is a partition; } \nu \subseteq \lambda} c_{\mu, \nu}^\lambda s_\nu \quad \text{(because } \nu \subseteq \lambda \text{ and } \lambda \in P_{k,n} \text{ lead to } \nu \in P_{k,n} \text{)}
\]

This completes our proof of Lemma 6.18. \( \square \)
Lemma 6.19. Let \( i \in \mathbb{Z} \) and \( f \in \mathbb{Z} \). Then,

\[
\delta \left( (e_i) \perp f \right) \equiv \overline{e_i} \delta (f) \mod Q_0.
\]

(Note that \( \delta \left( (e_i) \perp f \right) \) is well-defined, since Lemma 6.18 yields \((e_i) \perp f \in \mathbb{Z}\).)

Proof of Lemma 6.19. Both sides of the claim are \( k \)-linear in \( f \). Hence, we can WLOG assume that \( f = s_\lambda \) for some \( \lambda \in P_{k,n} \) (since \( (s_\lambda) \lambda \in P_{k,n} \) is a basis of the \( k \)-module \( \mathbb{Z} \)). Assume this, and consider this \( \lambda \).

It is easy to see that if \( \mu \in P_{k,n} \), then we have the following equivalence of statements:

\[
(\lambda / \mu \text{ is a vertical } i\text{-strip}) \iff (\mu^\vee / \lambda^\vee \text{ is a vertical } i\text{-strip}). \quad (30)
\]

(Indeed, the skew Young diagram of \( \mu^\vee / \lambda^\vee \) is obtained from the skew Young diagram of \( \lambda / \mu \) by a rotation by 180°.)

We must prove that \( \delta \left( (e_i) \perp f \right) \equiv \overline{e_i} \delta (f) \mod Q_0 \). If \( i < 0 \), then this is obvious (because if \( i < 0 \), then both \( e_i \) and \( e_i \) equal 0, and therefore both sides of the congruence \( \delta \left( (e_i) \perp f \right) \equiv \overline{e_i} \delta (f) \mod Q_0 \) are equal to 0). Hence, for the rest of this proof, we WLOG assume that we don't have \( i < 0 \). Thus, \( i \geq 0 \), so that \( i \in \mathbb{N} \).

From \( f = s_\lambda \), we obtain

\[
(e_i) \perp f = (e_i) \perp s_\lambda = \sum_{\substack{\mu \text{ a partition;} \\
\lambda / \mu \text{ is a vertical } i\text{-strip}}} s_\mu \quad \text{(by Corollary 6.7)}
\]

(because if \( \mu \) is a partition such that \( \lambda / \mu \) is a vertical \( i \)-strip, then \( \mu \in P_{k,n} \) (since \( \mu \subseteq \lambda \) and \( \lambda \in P_{k,n} \)). Applying the map \( \delta \) to both sides of this equality, we find

\[
\delta \left( (e_i) \perp f \right) = \delta \left( \sum_{\substack{\mu \in P_{k,n}; \\
\lambda / \mu \text{ is a vertical } i\text{-strip}}} s_\mu \right) = \sum_{\substack{\mu \in P_{k,n}; \\
\lambda / \mu \text{ is a vertical } i\text{-strip}}} \overline{\delta (s_\mu)} \quad = \overline{\delta (s_\mu)} \quad \text{(by the definition of } \delta) \]

\[
= \sum_{\substack{\mu \in P_{k,n}; \\
\mu^\vee / \lambda^\vee \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \quad = \sum_{\substack{\mu \in P_{k,n}; \\
\mu / \lambda^\vee \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \quad (31)
\]
(here, we have substituted $\mu$ for $\mu^\vee$ in the sum, since the map $P_{k,n} \to P_{k,n}$, $\mu \mapsto \mu^\vee$ is a bijection).

On the other hand, from $f = s_\lambda$, we obtain $\delta(f) = \delta(s_\lambda) = s_\lambda^\vee$ (by the definition of $\delta$) and thus

$$\overline{e_i} \delta(f) = \overline{e_i} s_\lambda^\vee \equiv \sum_{\mu \in P_{k,n}; \mu / \lambda^\vee \text{is a vertical } i\text{-strip}} \overline{s_\mu} \mod Q_0 \quad (\text{by Lemma 6.15, applied to } \lambda^\vee \text{ instead of } \lambda).$$

Comparing this with (31), we obtain $\delta\left(\left(\overline{e_i}\right)^\perp f\right) \equiv \overline{e_i} \delta(f) \mod Q_0$. This proves Lemma 6.19.

**Lemma 6.20.** Let $p \in \mathbb{N}$. Let $i_1, i_2, \ldots, i_p \in \mathbb{Z}$ and $f \in \mathbb{Z}$. Then,

$$\delta\left(\left(\overline{e_i e_{i_2} \cdots e_{i_p}}\right)^\perp f\right) \equiv \overline{e_i e_{i_2} \cdots e_{i_p}} \delta(f) \mod Q_{p-1}. \quad (32)$$

**Proof of Lemma 6.20.** We proceed by induction on $p$.

The **induction base** (the case $p = 0$) is obvious (since $1^\perp = \text{id}$ and thus $1^\perp f = f$).

**Induction step:** Let $q \in \mathbb{N}$. Assume (as the induction hypothesis) that Lemma 6.20 holds for $p = q$. We must now prove that Lemma 6.20 holds for $p = q + 1$. In other words, we must prove that every $i_1, i_2, \ldots, i_{q+1} \in \mathbb{Z}$ and $f \in \mathbb{Z}$ satisfy

$$\delta\left(\left(\overline{e_i e_{i_2} \cdots e_{i_{q+1}}}\right)^\perp f\right) \equiv \overline{e_i e_{i_2} \cdots e_{i_{q+1}}} \delta(f) \mod Q_q. \quad (32)$$

So let $i_1, i_2, \ldots, i_{q+1} \in \mathbb{Z}$ and $f \in \mathbb{Z}$. We must prove (32).

Lemma 6.16 (applied to $i_{q+1}$ and $q - 1$ instead of $i$ and $p$) yields $\overline{e_{i_{q+1}}} Q_{q-1} \subseteq Q_q$.

The induction hypothesis yields

$$\delta\left(\left(\overline{e_i e_{i_2} \cdots e_{i_q}}\right)^\perp f\right) \equiv \overline{e_i e_{i_2} \cdots e_{i_q}} \delta(f) \mod Q_{q-1}. \quad (32)$$

Multiplying both sides of this congruence by $\overline{e_{i_{q+1}}}$, we obtain

$$\overline{e_{i_{q+1}}} \delta\left(\left(\overline{e_i e_{i_2} \cdots e_{i_q}}\right)^\perp f\right) \equiv \overline{e_{i_{q+1}}} \overline{e_i e_{i_2} \cdots e_{i_q}} \delta(f) \mod Q_q \quad (33)$$

(since $\overline{e_{i_{q+1}}} Q_{q-1} \subseteq Q_q$).

Applying Lemma 6.18 to $f = e_i e_{i_2} \cdots e_{i_q}$, we obtain $\left(e_i e_{i_2} \cdots e_{i_q}\right)^\perp (\mathbb{Z}) \subseteq \mathbb{Z}$.

Hence, $\left(e_i e_{i_2} \cdots e_{i_q}\right)^\perp f \in \mathbb{Z}$ (since $f \in \mathbb{Z}$).
But (24) (applied to $f = e_{i_1}e_{i_2} \cdots e_{i_q}$ and $g = e_{i_{q+1}}$) yields

$$\left( e_{i_1}e_{i_2} \cdots e_{i_{q+1}} \right)^\perp = \left( e_{i_{q+1}} \right)^\perp \circ \left( e_{i_1}e_{i_2} \cdots e_{i_q} \right)^\perp.$$ 

Hence,

$$\left( e_{i_1}e_{i_2} \cdots e_{i_{q+1}} \right)^\perp f = \left( \left( e_{i_{q+1}} \right)^\perp \circ \left( e_{i_1}e_{i_2} \cdots e_{i_q} \right)^\perp \right) f = \left( e_{i_{q+1}} \right)^\perp \left( \left( e_{i_1}e_{i_2} \cdots e_{i_q} \right)^\perp f \right).$$

Applying the map $\delta$ to both sides of this equality, we find

$$\delta \left( \left( e_{i_1}e_{i_2} \cdots e_{i_{q+1}} \right)^\perp f \right) = \delta \left( \left( e_{i_{q+1}} \right)^\perp \left( \left( e_{i_1}e_{i_2} \cdots e_{i_q} \right)^\perp f \right) \right) = e_{i_{q+1}}^q \delta \left( e_{i_1}e_{i_2} \cdots e_{i_q} \right)^\perp f \mod Q_0$$

(by Lemma 6.19, applied to $i_{q+1}$ and $\left( e_{i_1}e_{i_2} \cdots e_{i_q} \right)^\perp f$ instead of $i$ and $f$). Since $Q_0 \subseteq Q_q$, this yields

$$\delta \left( \left( e_{i_1}e_{i_2} \cdots e_{i_{q+1}} \right)^\perp f \right) \equiv e_{i_{q+1}}^q \delta \left( e_{i_1}e_{i_2} \cdots e_{i_q} \right)^\perp f \mod Q_q.$$ 

Thus, (32) is proven. This completes the induction step. Thus, Lemma 6.20 is proven.

**Lemma 6.21.** Let $\lambda \in P_{k,n}$ and $f \in \mathcal{Z}$. Then,

$$\delta \left( \left( s_\lambda \right)^\perp f \right) \equiv s_\lambda^\perp \delta (f) \mod Q_{n-k-1}.$$ 

**Proof of Lemma 6.21.** Let $\ell = n - k$. From $\lambda \in P_{k,n}$, we have $\lambda_1 \leq n - k = \ell$.

Consider the conjugate partition $\lambda^t$ of $\lambda$. Then, $\lambda^t$ has exactly $\lambda_1$ parts. Thus, $\lambda^t$ has $\leq \ell$ parts (since $\lambda_1 \leq \ell$). Therefore, $
abla = (\lambda^t)_1, (\lambda^t)_2, \ldots, (\lambda^t)_\ell$. Hence, Corollary 6.5 (applied to $\lambda^t$ instead of $\lambda$) yields

$$s_\lambda^\perp \lambda^t = \det \left( (e_{\lambda^t_1-i+j})_{1 \leq i \leq \ell, 1 \leq j \leq \ell} \right).$$

In view of $(\lambda^t)^t = \lambda$, this rewrites as

$$s_\lambda = \det \left( (e_{(\lambda^t)_1-i+j})_{1 \leq i \leq \ell, 1 \leq j \leq \ell} \right) = \sum_{\sigma \in S_\ell} (-1)^\sigma \prod_{i=1}^\ell e_{(\lambda^t)_1-i+\sigma(i)}$$

(34)
(where $S_\ell$ denotes the symmetric group of the set \{1, 2, \ldots, \ell\}, and where \((-1)^\sigma\) denotes the sign of a permutation $\sigma \in S_\ell$). Hence,

$$
(s_\lambda) \perp f = \left( \sum_{\sigma \in S_\ell} (-1)^\sigma \prod_{i=1}^\ell e_{(\lambda_i')_{i-i+\sigma(i)}} \right) \perp f = \sum_{\sigma \in S_\ell} (-1)^\sigma \left( \prod_{i=1}^\ell e_{(\lambda_i')_{i-i+\sigma(i)}} \right) \perp f.
$$

Applying the map $\delta$ to this equality, we obtain

$$
\delta \left( (s_\lambda) \perp f \right) = \delta \left( \sum_{\sigma \in S_\ell} (-1)^\sigma \left( \prod_{i=1}^\ell e_{(\lambda_i')_{i-i+\sigma(i)}} \right) \perp f \right) = \sum_{\sigma \in S_\ell} (-1)^\sigma \delta \left( \left( \prod_{i=1}^\ell e_{(\lambda_i')_{i-i+\sigma(i)}} \right) \perp f \right) \quad (\text{since } \delta \text{ is } k\text{-linear})
$$

$$
\equiv \prod_{i=1}^\ell e_{(\lambda_i')_{i-i+\sigma(i)}} \delta(f) \mod Q_{\ell-1} \quad \text{(by Lemma 6.20 applied to } p=\ell \text{ and } i_j=(\lambda_i')_{i-j+\sigma(i)})
$$

$$
\equiv \sum_{\sigma \in S_\ell} (-1)^\sigma \prod_{i=1}^\ell e_{(\lambda_i')_{i-i+\sigma(i)}} \delta(f) \mod Q_{\ell-1}. \quad (35)
$$

On the other hand, (34) is an identity in $\Lambda$. Evaluating both of its sides at the $k$ variables $x_1, x_2, \ldots, x_k$, we obtain

$$s_\lambda = \sum_{\sigma \in S_\ell} (-1)^\sigma \prod_{i=1}^\ell e_{(\lambda_i')_{i-i+\sigma(i)}}.
$$

Hence,

$$\overline{s_\lambda} \delta(f) = \sum_{\sigma \in S_\ell} (-1)^\sigma \prod_{i=1}^\ell e_{(\lambda_i')_{i-i+\sigma(i)}} \delta(f) = \sum_{\sigma \in S_\ell} (-1)^\sigma \prod_{i=1}^\ell e_{(\lambda_i')_{i-i+\sigma(i)}} \delta(f).
$$

Thus, (35) rewrites as $\delta \left( (s_\lambda) \perp f \right) \equiv \overline{s_\lambda} \delta(f) \mod Q_{\ell-1}$. In other words, $\delta \left( (s_\lambda) \perp f \right) \equiv \overline{s_\lambda} \delta(f) \mod Q_{n-k-1}$ (since $\ell = n - k$). This proves Lemma 6.21.

**Lemma 6.22.** Let $\lambda \in P_{k,n}$ and $\mu \in P_{k,n}$. Then,

$$\text{coeff}_\omega \left( \overline{s_\lambda s_\mu} \right) = \begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee. \end{cases}
$$
Proof of Lemma 6.22. From \( \mu \in P_{k,n} \), we obtain \( \mu^\vee \in P_{k,n} \). Hence, \( s_{\mu^\vee} \in \mathcal{Z} \) and

\[
\delta \left( s_{\mu^\vee} \right) = \overline{s_{(\mu^\vee)^\vee}} \quad \text{(by the definition of } \delta) \\
= \overline{s_{\mu}} \quad \text{(since } (\mu^\vee)^\vee = \mu \).
\]

Also, Lemma 6.21 (applied to \( f = s_{\mu^\vee} \)) yields

\[
\delta \left( (s_{\lambda})^\perp s_{\mu^\vee} \right) \equiv \overline{s_{\lambda}} \delta \left( s_{\mu^\vee} \right) \mod Q_{n-k-1}
\]

(since \( s_{\mu^\vee} \in \mathcal{Z} \)). In other words, \( \delta \left( (s_{\lambda})^\perp s_{\mu^\vee} \right) - \overline{s_{\lambda}} \delta \left( s_{\mu^\vee} \right) \in \mathcal{O}_{Q_{n-k-1}} \). Hence,

\[
\text{coeff}_\omega \left( \delta \left( (s_{\lambda})^\perp s_{\mu^\vee} \right) - \overline{s_{\lambda}} \delta \left( s_{\mu^\vee} \right) \right) \in \text{coeff}_\omega \left( Q_{n-k-1} \right) = 0
\]

(by Lemma 6.12). Thus,

\[
\text{coeff}_\omega \left( \delta \left( (s_{\lambda})^\perp s_{\mu^\vee} \right) \right) = \text{coeff}_\omega \left( \overline{s_{\lambda}} \delta \left( s_{\mu^\vee} \right) \right) = \text{coeff}_\omega \left( \overline{s_{\lambda} s_{\mu}} \right)
\]

(by Lemma 6.12). Thus,

\[
\text{coeff}_\omega \left( \delta \left( (s_{\lambda})^\perp s_{\mu^\vee} \right) \right) = \text{coeff}_\omega \left( \overline{s_{\lambda} s_{\mu}} \right).
\] (36)

Applying (23) to \( \lambda \) and \( \mu^\vee \) instead of \( \mu \) and \( \lambda \), we obtain \( (s_{\lambda})^\perp s_{\mu^\vee} = s_{\mu^\vee/\lambda} \). Thus, (36) rewrites as

\[
\text{coeff}_\omega \left( \delta \left( s_{\mu^\vee/\lambda} \right) \right) = \text{coeff}_\omega \left( \overline{s_{\lambda} s_{\mu}} \right).
\] (37)

We are in one of the following three cases:

Case 1: We have \( \lambda = \mu^\vee \).

Case 2: We have \( \lambda \subseteq \mu^\vee \) but not \( \lambda = \mu^\vee \).

Case 3: We don’t have \( \lambda \subseteq \mu^\vee \).

Let us first consider Case 1. In this case, we have \( \lambda = \mu^\vee \). Thus, \( s_{\mu^\vee/\lambda} = s_{\mu^\vee/\mu^\vee} = 1 = s_{\emptyset} \) and thus

\[
\delta \left( s_{\mu^\vee/\lambda} \right) = \delta \left( s_{\emptyset} \right) = \overline{s_{\emptyset}} \vee \quad \text{(by the definition of } \delta) \\
= \overline{s_{\emptyset}} \quad \text{(since } \emptyset^\vee = \omega \).
\]

Therefore, \( \text{coeff}_\omega \left( \delta \left( s_{\mu^\vee/\lambda} \right) \right) = \text{coeff}_\omega \left( \overline{s_{\emptyset}} \right) = 1 \) (by the definition of \( \text{coeff}_\omega \)). Comparing this with

\[
\begin{cases} 
1, & \text{if } \lambda = \mu^\vee; \\
0, & \text{if } \lambda \neq \mu^\vee
\end{cases}
\]

(since \( \lambda = \mu^\vee \)).
we obtain \( \text{coeff}_\omega (\overline{s_\lambda s_\mu}) = \begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee. \end{cases} \) Hence, Lemma 6.22 is proven in Case 1.

Let us next consider Case 2. In this case, we have \( \lambda \subseteq \mu^\vee \) but not \( \lambda = \mu^\vee \).
Hence, \( |\lambda| < |\mu^\vee| \) and \( \lambda \neq \mu^\vee \).
Now, every partition \( \nu \) satisfying \( |\lambda| + |\nu| = |\mu^\vee| \) and \( \nu \subseteq \mu^\vee \) must satisfy \( \nu \in P_{k,n} \) and \( \text{coeff}_\omega (\delta (s_\nu)) = 0. \) (38)

[Proof of (38): Let \( \nu \) be a partition satisfying \( |\lambda| + |\nu| = |\mu^\vee| \) and \( \nu \subseteq \mu^\vee \). We must prove (38).
First of all, from \( \nu \subseteq \mu^\vee \) and \( \mu^\vee \in P_{k,n} \), we obtain \( \nu \in P_{k,n} \). It thus remains to show that \( \text{coeff}_\omega (\delta (s_\nu)) = 0. \)

The definition of \( \delta \) yields \( \delta (s_\nu) = \overline{s_\nu} \) (since \( \nu \in P_{k,n} \)). But \( |\lambda| + |\nu| = |\mu^\vee| \) yields \( |\nu| = |\mu^\vee| - |\lambda| > 0 \) (since \( |\lambda| < |\mu^\vee| \)).

But every partition \( \kappa \in P_{k,n} \) satisfies \( |\kappa^\vee| = k (n-k) - |\kappa| = |\omega| - |\kappa| \). Applying this to \( \kappa = \nu \), we obtain
\[
|\nu^\vee| = |\omega| - \underbrace{|\nu|}_{>0} < |\omega|.
\]
Hence, \( |\nu^\vee| \neq |\omega| \), so that \( \nu^\vee \neq \omega \).

But the definition of \( \text{coeff}_\omega \) yields \( \text{coeff}_\omega (\overline{s_\nu}) = \begin{cases} 1, & \text{if } \nu^\vee = \omega; \\ 0, & \text{if } \nu^\vee \neq \omega \end{cases} \) (since \( \nu^\vee \neq \omega \)). In view of \( \delta (s_\nu) = \overline{s_\nu} \), this rewrites as \( \text{coeff}_\omega (\delta (s_\nu)) = 0. \) This completes the proof of (38).]
Proposition 6.17 (a) (applied to $\mu^\vee$ and $\lambda$ instead of $\lambda$ and $\mu$) yields

$$s_{\mu^\vee/\lambda} = \sum_{\nu \text{ is a partition; } \nu \subseteq \mu^\vee; |\lambda| + |\nu| = |\mu^\vee|} c_{\lambda,\nu}^{\mu^\vee} s_{\nu}$$

(by Proposition 6.17 (b), applied to $\mu^\vee$ and $\lambda$ instead of $\lambda$ and $\mu$)

$$= \sum_{\nu \text{ is a partition; } \nu \subseteq \mu^\vee; |\lambda| + |\nu| = |\mu^\vee|} c_{\lambda,\nu}^{\mu^\vee} s_{\nu} + \sum_{\nu \text{ is a partition; } \nu \subseteq \mu^\vee; |\lambda| + |\nu| = |\mu^\vee|} 0$$

(by Proposition 6.17 (c), applied to $\mu^\vee$ and $\lambda$ instead of $\lambda$ and $\mu$)

Applying the map $\delta$ to this equality, we find

$$\delta \left( s_{\mu^\vee/\lambda} \right) = \sum_{\nu \text{ is a partition; } \nu \subseteq \mu^\vee; |\lambda| + |\nu| = |\mu^\vee|} c_{\lambda,\nu}^{\mu^\vee} \delta \left( s_{\nu} \right)$$

Since every partition $\nu$ satisfying $\nu \subseteq \mu^\vee$ and $|\lambda| + |\nu| = |\mu^\vee|$ must satisfy $\nu \in P_{k,n}$ (by (38)) and thus $s_{\nu} \in \mathbb{Z}$.

Applying the map $\text{coeff}_\omega$ to this equality, we find

$$\text{coeff}_\omega \left( \delta \left( s_{\mu^\vee/\lambda} \right) \right) = \sum_{\nu \text{ is a partition; } \nu \subseteq \mu^\vee; |\lambda| + |\nu| = |\mu^\vee|} c_{\lambda,\nu}^{\mu^\vee} \text{coeff}_\omega \left( \delta \left( s_{\nu} \right) \right)$$

(by (38))

Comparing this with

$$\begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee \end{cases}$$

(since $\lambda \neq \mu^\vee$),

35
we obtain $\text{coeff}_\omega (s_\lambda s_\mu) = \begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee. \end{cases}$ Hence, Lemma 6.22 is proven in Case 2.

Let us finally consider Case 3. In this case, we don’t have $\lambda \subseteq \mu^\vee$. Hence, we don’t have $\lambda = \mu^\vee$ either. Thus, $\lambda \neq \mu^\vee$.

Also, $s_{\mu^\vee / \lambda} = 0$ (since we don’t have $\lambda \subseteq \mu^\vee$). Thus,

$$\text{coeff}_\omega \left( \delta \left( \frac{s_{\mu^\vee / \lambda}}{=0} \right) \right) = \text{coeff}_\omega (\delta (0)) = 0.$$  

Comparing this with

$$\begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee \end{cases}$$

we obtain $\text{coeff}_\omega (s_\lambda s_\mu) = \begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee. \end{cases}$ Hence, Lemma 6.22 is proven in Case 3.

We have now proven Lemma 6.22 in all three Cases 1, 2 and 3. Thus, Lemma 6.22 always holds. $\square$

**Proof of Theorem 6.3.** Write $f \in S/I$ in the form $f = \sum_{\lambda \in P_{k,n}} \alpha_\lambda s_\lambda$ with $\alpha_\lambda \in k$. (This is possible, since $(s_\lambda)_{\lambda \in P_{k,n}}$ is a basis of the $k$-module $S/I$.) Then, the definition of $\text{coeff}_{\nu^\vee}$ yields $\text{coeff}_{\nu^\vee} (f) = \alpha_{\nu^\vee}$.

On the other hand,

$$\text{coeff}_\omega \left( \frac{f}{=0} \right) = \text{coeff}_\omega \left( \frac{s_\nu \sum_{\lambda \in P_{k,n}} \alpha_\lambda s_\lambda}{=0} \right) = \sum_{\lambda \in P_{k,n}} \alpha_\lambda \text{coeff}_\omega \left( \frac{s_\lambda s_\nu}{=0} \right)$$

(by Lemma 6.22 applied to $\mu=\nu$)

$$= \sum_{\lambda \in P_{k,n}} \alpha_\lambda \begin{cases} 1, & \text{if } \lambda = \nu^\vee; \\ 0, & \text{if } \lambda \neq \nu^\vee \end{cases}$$

(since $\nu^\vee \in P_{k,n}$). Comparing this with $\text{coeff}_{\nu^\vee} (f) = \alpha_{\nu^\vee}$, we obtain $\text{coeff}_\omega (s_\nu f) = \text{coeff}_{\nu^\vee} (f)$. This proves Theorem 6.3. $\square$
Definition 6.23. For any three partitions $\alpha, \beta, \gamma \in P_{k,n}$, let $g_{\alpha, \beta, \gamma} = \text{coeff}_{\gamma} (\overline{s_\alpha s_\beta}) \in k$.

These scalars $g_{\alpha, \beta, \gamma}$ are thus the structure constants of the $k$-algebra $S/I$ in the basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ (although slightly reindexed). As a consequence of Theorem 6.3, we obtain the following $S_3$-property of these structure constants:

Corollary 6.24. We have

$$g_{\alpha, \beta, \gamma} = g_{\alpha, \gamma, \beta} = g_{\beta, \alpha, \gamma} = g_{\gamma, \alpha, \beta} = g_{\gamma, \beta, \alpha} = \text{coeff}_{\omega} (\overline{s_\alpha s_\beta s_\gamma})$$

for any $\alpha, \beta, \gamma \in P_{k,n}$.

Proof of Corollary 6.24. Let $\alpha, \beta, \gamma \in P_{k,n}$. It clearly suffices to prove $g_{\alpha, \beta, \gamma} = \text{coeff}_{\omega} (\overline{s_\alpha s_\beta s_\gamma})$, since the rest of the claim then follows by analogy.

Theorem 6.3 (applied to $\nu = \gamma$ and $f = s_\alpha s_\beta$) yields

$$\text{coeff}_{\omega} (\overline{s_\gamma s_\alpha s_\beta}) = \text{coeff}_{\gamma} (\overline{s_\alpha s_\beta}) = g_{\alpha, \beta, \gamma}$$

(by the definition of $g_{\alpha, \beta, \gamma}$). Thus, $g_{\alpha, \beta, \gamma} = \text{coeff}_{\omega} (\overline{s_\gamma s_\alpha s_\beta}) = \text{coeff}_{\omega} (\overline{s_\alpha s_\beta s_\gamma})$.

This completes our proof of Corollary 6.24. \qed

7. Complete homogeneous symmetric polynomials

In this section, we shall further explore the projections $\overline{h_i}$ of complete homogeneous symmetric polynomials $h_i$ onto $S/I$. This exploration will culminate in a second proof of Theorem 6.3.

Convention 7.1. Convention 6.1 remains in place for the whole Section 7.

We shall also use all the notations introduced in Section 6.

If $j \in \mathbb{N}$, then the expression “$1^j$” in a tuple stands for $j$ consecutive entries equal to 1 (that is, 1, 1, . . ., 1). Thus, $(m, 1^j) = \left( \begin{array}{c} m, 1, 1, \ldots, 1 \\ j \text{ times} \end{array} \right)$ for any $m \in \mathbb{N}$ and $j \in \mathbb{N}$.

7.1. A reduction formula for $h_{n+m}$

The following result helps us reduce complete homogeneous symmetric polynomials $h_{n+m}$ modulo the ideal $I$:

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Proposition 7.2. Let $m$ be a positive integer. Then,

$$h_{n+m} \equiv \sum_{j=0}^{k-1} (-1)^j a_{k-j} s_{(m,1^j)} \mod I.$$ 

We shall derive Proposition 7.2 from the following identity between symmetric functions in $\Lambda$:

Proposition 7.3. Let $m$ be a positive integer. Then,

$$h_{n+m} = \sum_{j=0}^n (-1)^j h_{n-j} s_{(m,1^j)}.$$ 

Proof of Proposition 7.3. Let $j \in \mathbb{N}$. In [GriRei18, Exercise 2.9.14(b)], it is shown that

$$\sum_{i=0}^b (-1)^i h_{a+i+1} e_{b-i} = s_{(a+1,1^b)}$$ (39)

for all $a, b \in \mathbb{N}$. Applying this equality to $a = m - 1$ and $b = j$, we obtain

$$\sum_{i=0}^j (-1)^i h_{m+i} e_{j-i} = s_{(m,1^j)}.$$ (40)

Now, forget that we fixed $j$. We thus have proven (40) for each $j \in \mathbb{N}$. Also, for any $N \in \mathbb{N}$, we have

$$\sum_{(i,j) \in \mathbb{N}^2; i+j=N} (-1)^i e_i h_j = \delta_{0,N}$$

(where $\delta_{0,N}$ is a Kronecker delta). (This is [GriRei18, (2.4.4)], with $n$ renamed as $N$.) Thus, for any $N \in \mathbb{N}$, we have

$$\delta_{0,N} = \sum_{(i,j) \in \mathbb{N}^2; i+j=N} (-1)^i e_i h_j = \sum_{(i,j) \in \mathbb{N}^2; i+j=N} (-1)^j h_j e_i$$

$$= \sum_{i=0}^N (-1)^i h_{N-i} e_i \quad \left(\text{here, we have substituted } (i, N-i) \text{ for } (i, j) \text{ in the sum}\right)$$

$$= \sum_{j=0}^N (-1)^j h_{N-j} e_j \quad \left(\text{here, we have renamed the summation index } i \text{ as } j\right).$$
Thus, for any $N \in \mathbb{N}$, we have

$$\sum_{j=0}^{N} (-1)^j h_{N-j} e_j = \delta_{0,N}. \quad (41)$$

For each $i \in \{0, 1, \ldots, n\}$, we have

$$\sum_{j=0}^{n} (-1)^{i-j} h_{n-j} e_{j-i}$$

$$= \sum_{j=0}^{n-i} (-1)^j h_{n-j} e_j \quad \text{(here, we have substituted } j \text{ for } j - i \text{ in the sum)}$$

$$= \delta_{0,n-i} \quad \text{(by (41), applied to } n - i \text{ instead of } N)$$

$$= \delta_{i,n}. \quad (42)$$

Now,

$$\sum_{j=0}^{n} (-1)^j h_{n-j} s_{(m,1^j)}$$

$$= \sum_{i=0}^{j} (-1)^i h_{m+i} e_{j-i} \quad \text{(by (40))}$$

$$= \sum_{j=0}^{n} \sum_{i=0}^{j} (-1)^i h_{n-j} e_{j-i} = \sum_{j=0}^{n} \sum_{i=0}^{j} (-1)^{i-j} h_{n-j} e_{j-i}$$

$$= \sum_{j=0}^{n} \sum_{i=0}^{m+i} (-1)^{j-i} h_{n-j} e_{j-i} = \sum_{i=0}^{n} h_{m+i} \sum_{j=0}^{n} (-1)^{j-i} h_{n-j} e_{j-i}$$

$$= \sum_{i=0}^{n} h_{m+i} \sum_{j=0}^{n} (-1)^{j-i} h_{n-j} e_{j-i} = \sum_{i=0}^{n} h_{m+i} \delta_{i,n} = h_{m+n} = h_{n+m}. \quad \text{(by (42))}$$

This proves Proposition 7.3. \hfill \Box

**Proof of Proposition 7.2.** For each integer $j \geq k$, we have

$$s_{(m,1^j)} = 0. \quad (43)$$

[Proof of (43): Let $j \geq k$ be an integer. Then, the partition $(m, 1^j)$ has $j + 1$ parts; thus, this partition has more than $k$ parts (since $j + 1 > j \geq k$). Thus, (3) (applied to $\lambda = (m, 1^j)$) yields $s_{(m,1^j)} = 0$. This proves (43).]
Proposition 7.3 yields

\[ h_{n+m} = \sum_{j=0}^{n} (-1)^j h_{n-j}s_{(m,1^j)}. \]

This is an identity in \( \Lambda \). Evaluating both of its sides at the \( k \) variables \( x_1, x_2, \ldots, x_k \), we obtain

\[
\begin{align*}
  h_{n+m} &= \sum_{j=0}^{k-1} (-1)^j h_{n-j}s_{(m,1^j)} + \sum_{j=k}^{n} (-1)^j h_{n-j}s_{(m,1^j)} \\
  &= \sum_{j=0}^{k-1} (-1)^j h_{n-j}s_{(m,1^j)} + \sum_{j=k}^{n} (-1)^j h_{n-j}s_{(m,1^j)} \quad \text{(by (43))} \\
  &= \sum_{j=0}^{k-1} (-1)^j h_{n-j}s_{(m,1^j)} + \sum_{j=0}^{k-1} (-1)^j a_{k-j}s_{(m,1^j)} \quad \text{mod } I.
\end{align*}
\]

This proves Proposition 7.2. \( \square \)

### 7.2. Lemmas on free modules

Next, we state a basic lemma from commutative algebra:

#### Lemma 7.4

Let \( r \in \mathbb{N} \). Let \( X \) and \( Y \) be two free \( k \)-modules of rank \( r \). Then, every surjective \( k \)-linear map from \( X \) to \( Y \) is a \( k \)-module isomorphism.

**Proof of Lemma 7.4** Let \( f : X \to Y \) be a surjective \( k \)-linear map from \( X \) to \( Y \). We must prove that \( f \) is a \( k \)-module isomorphism.

There is clearly a \( k \)-module isomorphism \( j : Y \to X \) (since \( X \) and \( Y \) are free \( k \)-modules of the same rank). Consider this \( j \). Then, the composition \( j \circ f \) is surjective (since \( j \) and \( f \) are surjective), and thus is a surjective endomorphism of the finitely generated \( k \)-module \( X \). But [GriRei18, Exercise 2.5.18(a)] shows that any surjective endomorphism of a finitely generated \( k \)-module is a \( k \)-module isomorphism. Hence, we conclude that \( j \circ f \) is a \( k \)-module isomorphism. Thus, \( f \) is a \( k \)-module isomorphism (since \( j \) is a \( k \)-module isomorphism). This proves Lemma 7.4. \( \square \)

#### Lemma 7.5

Let \( Z \) be a \( k \)-module. Let \( U, X \) and \( Y \) be \( k \)-submodules of \( Z \) such that \( Z = X \oplus Y \) and \( X \subseteq U \). Let \( r \in \mathbb{N} \). Assume that the \( k \)-module \( X \) has a basis with \( r \) elements, whereas the \( k \)-module \( U \) can be spanned by \( r \) elements. Then, \( X = U \).
Proof of Lemma 7.5. Let $\pi : Z \to X$ be the canonical projection from the direct sum $Z = X \oplus Y$ onto its addend $X$. Let $i : X \to U$ be the canonical injection. Then, the composition

$$X \xrightarrow{i} U \xrightarrow{\pi|U} X$$

is just $\text{id}_X$ (since $\pi|X = \text{id}_X$). Hence, the map $\pi|U$ is surjective.

We assumed that the $k$-module $U$ can be spanned by $r$ elements. Thus, there is a surjective $k$-module homomorphism $u : k^r \to U$. Consider this $u$.

Both $k$-modules $k^r$ and $X$ are free of rank $r$ (since $X$ has a basis with $r$ elements). The composition

$$k^r \xrightarrow{u} U \xrightarrow{\pi|U} X$$

is surjective (since both $u$ and $\pi|U$ are surjective), and thus is a $k$-module isomorphism (by Lemma 7.4, applied to $k^r$ and $X$ instead of $X$ and $Y$). Hence, it is injective. Thus, $u$ is injective. Since $u$ is also surjective, we thus conclude that $u$ is bijective, and therefore a $k$-module isomorphism. Since both $u$ and the composition $k^r \xrightarrow{u} U \xrightarrow{\pi|U} X$ are $k$-module isomorphisms, we now conclude that the map $\pi|U$ is a $k$-module isomorphism. Hence, it has an inverse. But this inverse must be $i$ (since the composition $X \xrightarrow{i} U \xrightarrow{\pi|U} X$ is $\text{id}_X$). Thus, $i$ is a $k$-module isomorphism, too. Thus, in particular, $i$ is surjective. Therefore, $U = i(X) = X$. This proves Lemma 7.5. \hfill \square

7.3. The symmetric polynomials $h_v$

Definition 7.6. Let $\ell \in \mathbb{N}$, and let $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathbb{Z}^\ell$ be any $\ell$-tuple of integers. Then, we define the symmetric polynomial $h_\nu \in S$ as follows:

$$h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_\ell}.$$

Note that the polynomial $h_\nu$ does not change if we permute the entries of the $\ell$-tuple $\nu$. If an $\ell$-tuple $\nu$ of integers contains any negative entries, then $h_\nu = 0$ (since $h_i = 0$ for any $i < 0$). Also, if an $\ell$-tuple $\nu$ of integers contains any entry $= 0$, then we can remove this entry without changing $h_\nu$ (since $h_0 = 1$).

7.4. The submodules $L_p$ and $H_p$ of $S/I$

It is time to define two further filtrations of the $k$-module $S/I$ (in addition to the filtration $(Q_p)_{p \in \mathbb{Z}}$ from Definition 6.11):

Definition 7.7. (a) If $\lambda$ is a partition, then $\ell(\lambda)$ shall denote the length of $\lambda$; this is defined as the number of positive entries of $\lambda$. Note that $\ell(\lambda) \leq k$ for each $\lambda \in P_{k,n}$.\n
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(b) For each $p \in \mathbb{Z}$, we let $L_p$ denote the $k$-submodule of $S/I$ spanned by the $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq p$.

(c) For each $p \in \mathbb{Z}$, we let $H_p$ denote the $k$-submodule of $S/I$ spanned by the $\overline{h_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq p$.

The only partition $\lambda$ satisfying $\ell(\lambda) \leq 0$ is the empty partition $\emptyset = ()$; it belongs to $P_{k,n}$ and satisfies $\overline{s_\lambda} = 1$. Hence, $L_0$ is the $k$-submodule of $S/I$ spanned by 1. Similarly, $H_0$ is the same $k$-submodule.

Also, $L_k$ is the $k$-submodule of $S/I$ spanned by all $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ (because each $\lambda \in P_{k,n}$ satisfies $\ell(\lambda) \leq k$). But the latter $k$-submodule is $S/I$ itself (by Theorem 2.7). Thus, we conclude that $L_k$ is $S/I$ itself. In other words,

$$L_k = S/I.$$ 

Clearly, $L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$ and $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$. We shall soon see that the families $(L_p)_{p \in \mathbb{Z}}$ and $(H_p)_{p \in \mathbb{Z}}$ are identical (Proposition 7.11) and are filtrations of the $k$-algebra $S/I$ (Proposition 7.15). First let us show a basic fact:

**Lemma 7.8.** Let $p \in \mathbb{N}$ be such that $p \leq k$. Let $v = (v_1, v_2, \ldots, v_p) \in \mathbb{Z}^p$.

Assume that $v_i \leq n$ for each $i \in \{1, 2, \ldots, p\}$. Then, $\overline{h_v} \in H_p$.

(The condition "$p \leq k$" can be removed from this lemma, but we aren’t yet at the point where this is easy to see. We will show this in Proposition 7.14 below.)

**Proof of Lemma 7.8** We WLOG assume that $v_1 \geq v_2 \geq \cdots \geq v_p$ (since otherwise, we can just permute the entries of $v$ to achieve this). Let $j$ be the number of $i \in \{1, 2, \ldots, p\}$ satisfying $v_i > n - k$. Then,

$$v_1 \geq v_2 \geq \cdots \geq v_j > n - k \geq v_{j+1} \geq v_{j+2} \geq \cdots \geq v_p.$$ 

We WLOG assume that all of the $v_1, v_2, \ldots, v_p$ are nonnegative (since otherwise, we have $h_v = 0$ and thus $\overline{h_v} = 0 \in H_p$).

Now,

$$\overline{h_v} \in k \quad \text{for each } i \in \{1, 2, \ldots, j\}.$$  \hspace{1cm} (44)

[Proof of (44): Let $i \in \{1, 2, \ldots, j\}$. Then, $v_i > n - k$ (since $v_1 \geq v_2 \geq \cdots \geq v_j > n - k$), but also $v_i \leq n$ (by the assumptions of Lemma 7.8). Thus, $n - k < v_i \leq n$, so that $v_i \in \{n - k + 1, n - k + 2, \ldots, n\}$ and thus $v_i - (n - k) \in \{1, 2, \ldots, k\}$. Hence, $v_i - (n - k) \in \{1, 2, \ldots, k\}$. Thus, $v_i - (n - k) \in \{1, 2, \ldots, k\}$. Hence, $\overline{h_v} = \overline{a_{v_i - (n - k)}} \in k$. This proves (44).] 

Furthermore, $(v_{j+1}, v_{j+2}, \ldots, v_p)$ is a partition (since $v_{j+1} \geq v_{j+2} \geq \cdots \geq v_p$ and since all of the $v_1, v_2, \ldots, v_p$ are nonnegative) with at most $k$ entries (indeed, its number of entries is $\leq p - j \leq p \leq k$), and all of its entries are $\leq n - k$ (since $n - k \geq v_{j+1} \geq v_{j+2} \geq \cdots \geq v_p$). Hence, $(v_{j+1}, v_{j+2}, \ldots, v_p)$ belongs to $P_{k,n}$. 

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From \((v_{j+1}, v_{j+2}, \ldots, v_p) \in P_{k,n}\) and \(\ell (v_{j+1}, v_{j+2}, \ldots, v_p) \leq p - j \leq p\), we obtain
\[
\overline{h}_{(v_{j+1}, v_{j+2}, \ldots, v_p)} \in H_p \quad (by \ the \ definition \ of \ H_p).
\]

Now, the definition of \(h_v\) yields \(h_v = h_{v_1}h_{v_2} \cdots h_{v_p}\), so that
\[
\overline{h}_v = \overline{h}_{v_1}h_{v_2} \cdots \overline{h}_{v_p} = \overline{h}_{v_1}h_{v_2} \cdots \overline{h}_{v_p} \in \frac{L_\lambda}{\nu} \quad (by \ (44)).
\]
\[
\overline{h} = \overline{h}_{v_1}h_{v_2} \cdots \overline{h}_{v_p} = \overline{h}_{v_1}h_{v_2} \cdots \overline{h}_{v_p} \in H_p
\]
\[
\in kH_p \subseteq H_p.
\]

This proves Lemma 7.8. \hfill \Box

**Lemma 7.9.** Let \(p \in \mathbb{Z}\). Then, the family \((\overline{s}_\lambda)_{\lambda \in P_{k,n}; \ \ell(\lambda) \leq p}\) is a basis of the \(k\)-module \(L_p\).

**Proof of Lemma 7.9.** Theorem 2.7 yields that \((\overline{s}_\lambda)_{\lambda \in P_{k,n}}\) is a basis of the \(k\)-module \(S/1\). Hence, this family \((\overline{s}_\lambda)_{\lambda \in P_{k,n}}\) is \(k\)-linearly independent. Thus, its subfamily \((\overline{s}_\lambda)_{\lambda \in P_{k,n}; \ \ell(\lambda) \leq p}\) is \(k\)-linearly independent as well. Moreover, this subfamily \((\overline{s}_\lambda)_{\lambda \in P_{k,n}; \ \ell(\lambda) \leq p}\) spans the \(k\)-module \(L_p\) (by the definition of \(L_p\)). Hence, this subfamily \((\overline{s}_\lambda)_{\lambda \in P_{k,n}; \ \ell(\lambda) \leq p}\) is a basis of the \(k\)-module \(L_p\). This proves Lemma 7.9. \hfill \Box

**Lemma 7.10.** Let \(p \in \{0, 1, \ldots, k\}\). Then, \(L_p = H_p\). (This lemma holds more generally for all \(p \in \mathbb{Z}\), as we shall see in Lemma 7.11 below.)

**Proof of Lemma 7.10.** Let \(\lambda \in P_{k,n}\) be such that \(\ell (\lambda) \leq p\). We shall show that \(\overline{s}_\lambda \in H_p\).

Indeed, let \(S_p\) denote the group of permutations of \(\{1, 2, \ldots, p\}\). For each \(\sigma \in S_p\), let \((-1)^{\sigma}\) denote the sign of \(\sigma\).

For each \(\sigma \in S_p\), we have
\[
\prod_{i=1}^{p} h_{\lambda_i - i + \sigma(i)} \in H_p. \quad (45)
\]

[Proof of (45): Let \(\sigma \in S_p\). Then, each \(i \in \{1, 2, \ldots, p\}\) satisfies
\[
\frac{\lambda_i}{\lambda} - \frac{i}{n-k} + \frac{\sigma(i)}{p-k} \leq n-k+0+k=n.
\]
Thus, Lemma 7.8 (applied to \((\lambda_1 - 1 + \sigma(1), \lambda_2 - 2 + \sigma(2), \ldots, \lambda_p - p + \sigma(p))\) and \(\lambda_i - i + \sigma(i)\) instead of \(v\) and \(v_i\)) yields
\[
\overline{h}_{(\lambda_1-1+\sigma(1),\lambda_2-2+\sigma(2),\ldots,\lambda_p-p+\sigma(p))} \in H_p
\]
(since \( p \leq k \)). In view of

\[
\ell \left( \lambda_1-1+\sigma(1), \lambda_2-2+\sigma(2), \ldots, \lambda_p-p+\sigma(p) \right) = \prod_{i=1}^{p} h_{\lambda_i-i+\sigma(i)},
\]

this rewrites as \( \prod_{i=1}^{p} h_{\lambda_i-i+\sigma(i)} \in H_p \). Thus, \([45]\) is proven.]

We have \( \ell (\lambda) \leq p \) and thus \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \). Hence, Proposition \([5.7]\) (b) yields

\[
s_{\lambda} = \det \left( (h_{\lambda_u-u+v})_{1 \leq u \leq p, 1 \leq v \leq p} \right) = \sum_{\sigma \in S_p} (-1)^\sigma \prod_{i=1}^{p} h_{\lambda_i-i+\sigma(i)}
\]

(by the definition of a determinant). Projecting both sides of this equality onto \( S/I \), we obtain

\[
\overline{s}_{\lambda} = \sum_{\sigma \in S_p} (-1)^\sigma \prod_{i=1}^{p} h_{\lambda_i-i+\sigma(i)} = \sum_{\sigma \in S_p} (-1)^\sigma \prod_{i=1}^{p} h_{\lambda_i-i+\sigma(i)} \in H_p, \quad \text{[by} \text{45]}\]

Now, forget that we fixed \( \lambda \). We thus have proven that \( \overline{s}_{\lambda} \in H_p \) for each \( \lambda \in P_{k,n} \) satisfying \( \ell (\lambda) \leq p \).

Therefore, \( L_p \subseteq H_p \) (since \( L_p \) is the \( k \)-submodule of \( S/I \) spanned by the \( \overline{s}_{\lambda} \) with \( \lambda \in P_{k,n} \) satisfying \( \ell (\lambda) \leq p \)).

Lemma \([7.9]\) yields that the family \( (\overline{s}_{\lambda})_{\lambda \in P_{k,n}; \ell(\lambda) \leq p} \) is a basis of the \( k \)-module \( L_p \).

Now, let \( L'_p \) be the \( k \)-submodule of \( S/I \) spanned by the \( \overline{s}_{\lambda} \) with \( \lambda \in P_{k,n} \) satisfying \( \ell (\lambda) > p \). Recall (from Theorem \([2.7]\)) that \( (\overline{s}_{\lambda})_{\lambda \in P_{k,n}} \) is a basis of the \( k \)-module \( S/I \). Hence, \( S/I = L_p \oplus L'_p \) (since each \( \lambda \in P_{k,n} \) satisfies either \( \ell (\lambda) \leq p \) or \( \ell (\lambda) > p \) but not both). Let \( r \) be the number of all \( \lambda \in P_{k,n} \) satisfying \( \ell (\lambda) \leq p \). Then, the \( k \)-module \( H_p \) can be spanned by \( r \) elements (namely, by the \( h_{\lambda} \) with \( \lambda \in P_{k,n} \) satisfying \( \ell (\lambda) \leq p \)), whereas the \( k \)-module \( L_p \) has a basis with \( r \) elements (namely, the family \( (\overline{s}_{\lambda})_{\lambda \in P_{k,n}; \ell(\lambda) \leq p} \)). Thus, Lemma \([7.5]\) (applied to \( Z = S/I, X = L_p, Y = L'_p \) and \( U = H_p \)) yields \( L_p = H_p \). This proves Lemma \([7.10]\) \( \square \)

**Proposition 7.11.** Let \( p \in \mathbb{Z} \). Then, \( L_p = H_p \).

**Proof of Proposition \([7.11]\)** If \( p \) is negative, then both \( L_p \) and \( H_p \) equal 0 (since there exists no \( \lambda \in P_{k,n} \) satisfying \( \ell (\lambda) \leq p \) in this case). Thus, if \( p \) is negative, then
$L_p = H_p$ is obviously true. Hence, for the rest of this proof, we WLOG assume that $p > 0$ is not true. Thus, $p \in \mathbb{N}$.

If $p \in \{0, 1, \ldots, k\}$, then $L_p = H_p$ follows from Lemma 7.10. Hence, for the rest of this proof, we WLOG assume that $p \notin \{0, 1, \ldots, k\}$. Thus, $p > k$ (since $p \in \mathbb{N}$). Hence, $k < p$, so that $H_k \subseteq H_p$ (since $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$). But Lemma 7.10 (applied to $k$ instead of $p$) yields $L_k = H_k$.

But recall that $L_k = S/I$. Thus, $S/I = L_k = H_k \subseteq H_p$. Thus, $H_p \supseteq S/I \supseteq L_p$. On the other hand, $k < p$ and thus $L_k \subseteq L_p$ (since $L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$). Hence, $L_p \supseteq L_k = S/I \supseteq H_p$. Combining this with $H_p \supseteq L_p$, we obtain $L_p = H_p$. This proves Proposition 7.11.

**Corollary 7.12.** Let $p \in \mathbb{Z}$. Then, the family $(\overline{h_\lambda})_{\lambda \in P_k \cap \ell; \ell(\lambda) \leq p}$ is a basis of the $k$-module $L_p$.

**Proof of Corollary 7.12.** Lemma 7.9 yields that the family $(\overline{\lambda})_{\lambda \in P_k \cap \ell; \ell(\lambda) \leq p}$ is a basis of the $k$-module $L_p$. On the other hand, the family $(\overline{h_\lambda})_{\lambda \in P_k \cap \ell; \ell(\lambda) \leq p}$ spans the $k$-module $H_p$ (by the definition of $H_p$). In other words, the family $(\overline{h_\lambda})_{\lambda \in P_k \cap \ell; \ell(\lambda) \leq p}$ spans the $k$-module $L_p$ (since Proposition 7.11 yields $L_p = H_p$).

Since $|\{\lambda \in P_k \cap \ell; \ell(\lambda) \leq p\}| = |\{\lambda \in P_k \cap \ell; \ell(\lambda) \leq p\}|$, we can therefore apply Lemma 5.3 to $L_p$, $(\overline{\lambda})_{\lambda \in P_k \cap \ell; \ell(\lambda) \leq p}$ and $(\overline{h_\lambda})_{\lambda \in P_k \cap \ell; \ell(\lambda) \leq p}$ instead of $M_s (b_s)_{s \in S}$ and $(a_u)_{u \in U}$. We thus conclude that $(\overline{h_\lambda})_{\lambda \in P_k \cap \ell; \ell(\lambda) \leq p}$ is a basis of the $k$-module $L_p$. This proves Corollary 7.12.

**Theorem 7.13.** The family $(\overline{h_\lambda})_{\lambda \in P_k}$ is a basis of the $k$-module $S/I$.

**Proof of Theorem 7.13.** Corollary 7.12 (applied to $p = k$) shows that the family $(\overline{h_\lambda})_{\lambda \in P_k; \ell(\lambda) \leq k}$ is a basis of the $k$-module $L_k$. In view of $(\overline{h_\lambda})_{\lambda \in P_k; \ell(\lambda) \leq k} = (\overline{h_\lambda})_{\lambda \in P_k}$ (since each $\lambda \in P_k$ satisfies $\ell(\lambda) \leq k$) and $L_k = S/I$, this rewrites as follows: The family $(\overline{h_\lambda})_{\lambda \in P_k}$ is a basis of the $k$-module $S/I$. This proves Theorem 7.13.

**Proposition 7.14.** Let $p \in \mathbb{N}$. Let $v = (v_1, v_2, \ldots, v_p) \in \mathbb{Z}^p$. Assume that $v_i \leq n$ for each $i \in \{1, 2, \ldots, p\}$. Then, $\overline{h_v} \in H_p$.

**Proof of Proposition 7.14.** If $p \leq k$, then this follows from Lemma 7.8. Thus, for the rest of this proof, we WLOG assume that $p > k$. Hence, $k < p$, so that $H_k \subseteq H_p$ (since $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$). But Proposition 7.11 (applied to $k$ instead of $p$) yields $H_k = L_k = S/I$. Now, $\overline{h_v} \in S/I = H_k \subseteq H_p$. This proves Proposition 7.14.
We recall that the \( k \)-submodules of a given \( k \)-algebra \( A \) form a monoid under multiplication: The product \( XY \) of two \( k \)-submodules \( X \) and \( Y \) of \( A \) is defined as the \( k \)-linear span of all products \( xy \) with \( x \in X \) and \( y \in Y \). The neutral element of this monoid is \( k \cdot 1_A \). We shall specifically use this monoid in the case when \( A = S/I \).

### Proposition 7.15

The family \( (L_p)_{p \in \mathbb{N}} \) is a filtration of the \( k \)-algebra \( S/I \); that is, we have

\[
L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots, \quad \bigcup_{p \in \mathbb{N}} L_p = S/I,
\]

\[
1 \in L_0, \quad \text{and} \quad L_aL_b \subseteq L_{a+b} \quad \text{for every } a, b \in \mathbb{N}.
\]

**Proof of Proposition 7.15** We already know that \( L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots \). Also, \( 1 \in L_0 \) (since \( L_0 \) is the \( k \)-submodule of \( S/I \) spanned by 1). Also, \( L_k = S/I \), so that \( S/I = L_k \subseteq \bigcup_{p \in \mathbb{N}} L_p \). Combining this with \( \bigcup_{p \in \mathbb{N}} L_p \subseteq S/I \), we obtain \( \bigcup_{p \in \mathbb{N}} L_p = S/I \).

Hence, it remains to prove that \( L_aL_b \subseteq L_{a+b} \) for every \( a, b \in \mathbb{N} \). Let us fix \( a, b \in \mathbb{N} \). We must prove that \( L_aL_b \subseteq L_{a+b} \).

If \( a + b \geq k \), then this is obvious (because if \( a + b \geq k \), then \( k \leq a + b \), hence \( L_k \subseteq L_{a+b} \) (since \( L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots \)), hence \( L_aL_b \subseteq S/I = L_k \subseteq L_{a+b} \)). Hence, we WLOG assume that \( a + b < k \).

We must prove that \( L_aL_b \subseteq L_{a+b} \). It clearly suffices to show that \( fg \in L_{a+b} \) for each \( f \in L_a \) and \( g \in L_b \). So let us fix \( f \in L_a \) and \( g \in L_b \); we must prove that \( fg \in L_{a+b} \).

Proposition 7.11 yields that \( L_a = H_a \). Thus, \( f \in L_a = H_a \), so that \( f \) is a \( k \)-linear combination of the \( \overline{h_\lambda} \) with \( \lambda \in P_{k,n} \) satisfying \( \ell(\lambda) \leq a \) (because \( H_a \) is the \( k \)-submodule of \( S/I \) spanned by these \( \overline{h_\lambda} \)). Since the claim we are proving (that is, \( fg \in L_{a+b} \)) depends \( k \)-linearly on \( f \), we can thus WLOG assume that \( f \) is one of those \( \overline{h_\lambda} \). In other words, we can WLOG assume that \( f = \overline{h_\alpha} \) for some \( \alpha \in P_{k,n} \) satisfying \( \ell(\alpha) \leq a \). Assume this, and consider this \( \alpha \). For similar reasons, we WLOG assume that \( g = \overline{h_\beta} \) for some \( \beta \in P_{k,n} \) satisfying \( \ell(\beta) \leq b \). Consider this \( \beta \).

Note that each entry of \( \alpha \) is \( \leq n - k \) (since \( \alpha \in P_{k,n} \)), and therefore \( \leq n \). Thus, we can consider \( \alpha \) as an \( a \)-tuple of elements of \( \{0,1,\ldots,n\} \) (since \( \ell(\alpha) \leq a \)). Likewise, consider \( \beta \) as a \( b \)-tuple of elements of \( \{0,1,\ldots,n\} \).

Let \( \gamma \) be the concatenation of the \( a \)-tuple \( \alpha \) with the \( b \)-tuple \( \beta \). Thus, \( \gamma \) is an \((a+b)\)-tuple of elements of \( \{0,1,\ldots,n\} \) (since \( \alpha \) is an \( a \)-tuple of elements of \( \{0,1,\ldots,n\} \) and since \( \beta \) is a \( b \)-tuple of elements of \( \{0,1,\ldots,n\} \)), and satisfies \( h_\gamma = h_\alpha h_\beta \). (But \( \gamma \) is not necessarily a partition.) Moreover, \( a + b \leq k \) (since \( a + b < k \)). Finally, write \( \gamma \) in the form \( \gamma = (\gamma_1,\gamma_2,\ldots,\gamma_{a+b}) \); then, we have \( \gamma_i \leq n \) for each \( i \in \{1,2,\ldots,a+b\} \) (because \( \gamma \) is an \((a+b)\)-tuple of elements of \( \{0,1,\ldots,n\} \)).
\{0,1,\ldots,n\}). Hence, Lemma \ref{lem:composition} (applied to \(p = a + b, \nu = \gamma\) and \(\nu_i = \gamma_i\)) yields \(h_\gamma \in H_{a+b}\). But Proposition \ref{prop:sum} yields that \(L_{a+b} = H_{a+b}\).

From \(f = h_a\) and \(g = h_\beta\), we obtain \(fg = h_a h_\beta = h_a h_\beta = h_\gamma\) (since \(h_a h_\beta = h_\gamma\)).

Thus, \(fg = h_\gamma \in H_{a+b} = L_{a+b}\) (since \(L_{a+b} = H_{a+b}\)). This completes our proof of Proposition \ref{prop:main}

\begin{corollary}
We have \((L_1)^m \subseteq L_m\) for each \(m \in \mathbb{N}\).
\end{corollary}

\begin{proof}[Proof of Corollary \ref{cor:main}] This follows by induction on \(m\), using the facts (which we proved in Proposition \ref{prop:main}) that \(1 \in L_0\) and that \(L_a L_b \subseteq L_{a+b}\) for every \(a, b \in \mathbb{N}\).
\end{proof}

\section{A formula for hook-shaped Schur functions}

\begin{lemma}
Let \(m\) be a positive integer. Let \(j \in \mathbb{N}\). Then,

\[ s_{(m,1^j)} = \sum_{i=1}^{m} (-1)^{i-1} h_{m-i} e_{j+i}. \]

\end{lemma}

\begin{proof}[Proof of Lemma \ref{lem:hook}] For each \(N \in \mathbb{N}\), we have

\[ \sum_{p=0}^{N} (-1)^{p} h_{N-p} e_{p} = \delta_{0,N}. \]  

(47)

(This is just the equality \ref{eq:2}, with \(j\) renamed as \(p\).)

From \(m > 0\) and \(j \geq 0\), we obtain \(m+j > 0\), so that \(\delta_{0,m+j} = 0\). The equality \ref{eq:2} (applied to \(N = m+j\)) becomes

\[ \sum_{p=0}^{m+j} (-1)^{p} h_{m+j-p} e_{p} = \delta_{0,m+j} = 0. \]
Thus,
\[
0 = \sum_{p=0}^{m+j} (-1)^p h_{m+j-p} e_p = \sum_{p=0}^{m+j} (-1)^p h_{m+j-p} e_p
\]
\[
= \sum_{i=-m}^{j} (-1)^{j-i} h_{m+i} e_{j-i}
\]
\[
(\text{here, we have substituted } j - i \text{ for } p \text{ in the sum})
\]
\[
= \sum_{i=-m}^{-1} (-1)^{j-i} h_{m+i} e_{j-i} + \sum_{i=0}^{j} (-1)^{j-i} h_{m+i} e_{j-i}
\]
\[
= \sum_{i=1}^{m} (-1)^{j+i} h_{m-i} e_{j+i} + (-1)^j \sum_{i=0}^{j} (-1)^i h_{m+i} e_{j-i}
\]
\[
= \sum_{i=1}^{m} (-1)^{j+i} h_{m-i} e_{j+i} + (-1)^j s_{(m,1)}.
\]
Solving this equality for \(s_{(m,1)}\), we obtain
\[
s_{(m,1)} = -\frac{1}{(-1)^j} \sum_{i=1}^{m} (-1)^{j+i} h_{m-i} e_{j+i} = \sum_{i=1}^{m} (-1)^{j-i} h_{m-i} e_{j+i}.
\]
This proves Lemma 7.17.

7.6. The submodules \(C\) and \(R_p\) of \(S/I\)

Next, we introduce some more \(k\)-submodules of \(S/I\):

**Definition 7.18.** (a) Let \(C\) be the \(k\)-submodule of \(S/I\) spanned by the \(e_i\) with \(i \in \mathbb{N}\).

(b) For each \(p \in \mathbb{Z}\), we let \(R_p\) be the \(k\)-submodule of \(S/I\) spanned by the \(h_i\) with \(i \in \mathbb{N}\) satisfying \(i \leq p\).

We recall that \(e_i = 0\) for every \(i > k\). Thus, \(e_i = 0\) for every \(i > k\). Hence, the \(k\)-module \(C\) is spanned by \(e_0, e_1, \ldots, e_k\) (because all the other among its designated generators \(e_i\) are 0). Also, the definition of \(C\) yields \(e_0 \in C\), so that \(1 = e_0 \in C\). Thus, each \(i \in \mathbb{N}\) satisfies \(C^i = \bigcup_{C \in \mathcal{C}} C^i \subseteq \mathbb{C} C^i = C^{i+1}\). In other words, \(C^0 \subseteq C^1 \subseteq C^2 \subseteq \cdots\).

Note that \(R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots\). Also:
\[ \textbf{Proposition 7.19.} \text{ We have } R_{n-k} = L_1. \]

\[ \text{Proof of Proposition } 7.19 \text{ We WLOG assume that } k \neq 0, \text{ because the case when } k = 0 \text{ is trivial for its own reasons.\footnote{Proof. Assume that } k = 0. \text{ Then, } S = k \text{ and } I = 0, \text{ whence } S/I = k \cdot 1. \text{ Both } k\text{-submodules } R_{n-k} \text{ and } L_1 \text{ contain } 1 \text{ (since } 1 = h_0 \text{ and since } 1 = \overline{h_0}); \text{ hence, both of these } k\text{-submodules must be the whole } S/I \text{ (since } S/I = k \cdot 1) \text{ and therefore must be equal. So we have proven } R_{n-k} = L_1. \text{ In other words, we have proven Proposition } 7.19 \text{ under the assumption that } k = 0. \]

So the family \( \overline{h_i} \) for each \( i \in \{0, 1, \ldots, n-k\} \).

Recall that \( L_1 \) was defined as the \( k \)-submodule of \( S/I \) spanned by the \( \overline{s(\lambda)} \) with \( \lambda \in P_{k,n} \) satisfying \( \ell(\lambda) \leq 1 \). But the \( \lambda \in P_{k,n} \) satisfying \( \ell(\lambda) \leq 1 \) are exactly the partitions of the form \( (i) \) for \( i \in \{0, 1, \ldots, n-k\} \). Hence, \( L_1 \) is the \( k \)-submodule of \( S/I \) spanned by the \( \overline{s(i)} \) with \( i \in \{0, 1, \ldots, n-k\} \). Since we have \( s(i) = h_i \) for each \( i \in \{0, 1, \ldots, n-k\} \), we can rewrite this as follows: \( L_1 \) is the \( k \)-submodule of \( S/I \) spanned by the \( \overline{h_i} \) with \( i \in \{0, 1, \ldots, n-k\} \). In other words, \( L_1 \) is the \( k \)-submodule of \( S/I \) spanned by the \( \overline{h_i} \) with \( i \in \mathbb{N} \) satisfying \( i \leq n-k \). But this is precisely the definition of the \( k \)-submodule \( R_{n-k} \). Hence, \( L_1 = R_{n-k} \). This proves Proposition 7.19. \( \square \)

It is easy to see that \( R_{n-k} = R_{n-k+1} = \cdots = R_n \), but the sequence \( (R_0, R_1, R_2, \ldots) \) may and may not grow after its \( n \)-th term depending on the choice of \( a_1, a_2, \ldots, a_k \). So the family \( \{R_p\}_{p \in \mathbb{Z}} \) is a filtration of some \( k \)-submodule of \( S/I \), but it isn’t easy to say which specific \( k \)-submodule it is.

\[ \textbf{Lemma 7.20.} \text{ We have } R_p \subseteq C^p \text{ for each } p \in \mathbb{N}. \]

\[ \text{Proof of Lemma } 7.20 \text{ We have } \overline{h_i} \in C \text{ for each } i \in \mathbb{N} \]

(by the definition of \( C \)).

Let \( p \in \mathbb{N} \). Recall that \( R_p \) is the \( k \)-submodule of \( S/I \) spanned by the \( \overline{h_i} \) with \( i \in \mathbb{N} \) satisfying \( i \leq p \). Hence, in order to prove that \( R_p \subseteq C^p \), it suffices to show that \( \overline{h_i} \in C^p \) for each \( i \in \mathbb{N} \) satisfying \( i \leq p \).

We first claim that \( \overline{h_i} \in C^i \) for each \( i \in \mathbb{N} \).

\[ \text{Proof of (49): We shall prove (49) by strong induction on } i. \text{ So we fix } j \in \mathbb{N} \text{, and we assume (as induction hypothesis) that (49) holds for all } i < j. \text{ We must now prove that (49) holds for } i = j \text{. In other words, we must prove that } \overline{h_j} \in C^j. \]

If \( j = 0 \), then this is obvious (because in this case, we have \( \overline{h_0} = \overline{h_0} = \overline{1} = 1 \in C^0 \)). Thus, we WLOG assume that \( j \neq 0 \). Hence, \( j \) is a positive integer. Thus, Corollary 3.3 (applied to \( j \) instead of \( p \)) yields

\[ h_j = -\sum_{t=1}^{k} (-1)^t e_t h_{j-t}. \]
Hence,

\[ h_j = - \sum_{t=1}^{k} (-1)^t e_t h_{j-t} = - \sum_{t=1}^{k} (-1)^t e_t \underbrace{h_{j-t}}_{\in \mathbb{C}^j} \quad \text{(by (48))} \]

\[ \subseteq - \sum_{t=1}^{k} (-1)^t \mathbb{C}^{j-t} \subseteq - \sum_{t=1}^{k} (-1)^t \mathbb{C}^j \subseteq \mathbb{C}^i. \]

In other words, (49) holds for \( i = j \). This completes the induction step. Thus, (49) is proven.

Now, let us fix \( i \in \mathbb{N} \) satisfying \( i \leq p \). Then, \( \mathbb{C}^i \subseteq \mathbb{C}^p \) (since \( i \leq p \) and \( \mathbb{C}^0 \subseteq \mathbb{C}^1 \subseteq \mathbb{C}^2 \subseteq \cdots \)). But (49) yields \( \overline{h}_i \in \mathbb{C}^i \subseteq \mathbb{C}^p \).

Now, forget that we fixed \( i \). We thus have shown that \( \overline{h}_i \in \mathbb{C}^p \) for each \( i \in \mathbb{N} \) satisfying \( i \leq p \). As we have said, this proves Lemma 7.20.

**Lemma 7.21.** Let \( m \) be a positive integer. Let \( j \in \mathbb{N} \). Then, \( \overline{s}_{(m,1)} \in R_{m-1} \mathbb{C}^j \).

**Proof of Lemma 7.21.** Lemma 7.17 yields

\[ \overline{s}_{(m,1)} = \sum_{i=1}^{m} (-1)^{i-1} h_{m-i} e_{j+i}. \]

This is an equality in \( \Lambda \). If we evaluate both of its sides at \( x_1, x_2, \ldots, x_k \), then we obtain

\[ s_{(m,1)} = \sum_{i=1}^{m} (-1)^{i-1} h_{m-i} e_{j+i}. \]

Thus,

\[ \overline{s}_{(m,1)} = \sum_{i=1}^{m} (-1)^{i-1} h_{m-i} e_{j+i} = \sum_{i=1}^{m} (-1)^{i-1} \overline{h}_{m-i} \overline{e}_{j+i} \quad \text{(by the definition of} \ R_{m-1}, \text{by the definition of} \ C) \]

\[ \subseteq \sum_{i=1}^{m} (-1)^{i-1} R_{m-1} \mathbb{C} \subseteq R_{m-1} \mathbb{C}. \]

This proves Lemma 7.21.

**Corollary 7.22.** Let \( m \) be a positive integer. Then, \( \overline{h}_{n+m} \in R_{m-1} \mathbb{C} \).
Proof of Corollary 7.22 Proposition 7.2 yields

\[ h_{n+m} \equiv \sum_{j=0}^{k-1} (-1)^j a_{k-j}s_{(m,1^j)} \mod I. \]

Thus,

\[ \overline{h_{n+m}} = \sum_{j=0}^{k-1} (-1)^j a_{k-j}s_{(m,1^j)} = \sum_{j=0}^{k-1} (-1)^j a_{k-j} \frac{s_{(m,1^j)}}{e \in R_{m-1}C} \]

(by Lemma 7.21)

\[ \in \sum_{j=0}^{k-1} (-1)^j a_{k-j}R_{m-1}C \subseteq R_{m-1}C. \]

This proves Corollary 7.22.

\[\square\]

Lemma 7.23. Let \( j \in \mathbb{N} \) be such that \( j \leq n \).

(a) We have \( \overline{h_j} \in L_1 \).

(b) Assume that \( n > k \) and \( j \neq n - k \). Then, \( \overline{h_j} \in R_{n-k-1} \).

Proof of Lemma 7.23 (a) We are in one of the following two cases:

Case 1: We have \( j \leq n - k \).

Case 2: We have \( j > n - k \).

Let us first consider Case 1. In this case, we have \( j \leq n - k \). Recall that \( R_{n-k} \)
was defined as the \( k \)-submodule of \( S/I \) spanned by the \( \overline{h_i} \) with \( i \in \mathbb{N} \) satisfying
\( i \leq n - k \). Hence, \( \overline{h_j} \in R_{n-k} \) (since \( j \in \mathbb{N} \) and \( j \leq n - k \)). Thus, \( \overline{h_j} \in R_{n-k} = L_1 \)
(by Proposition 7.19). Thus, Lemma 7.23 (a) is proven in Case 1.

Let us now consider Case 2. In this case, we have \( j > n - k \). Hence, \( n - k < j \leq n \), so that \( j \in \{n-k+1,n-k+2,\ldots,n\} \) and therefore \( j - (n - k) \in \{1,2,\ldots,k\} \). Hence, \( \{15\} \) (applied to \( j - (n - k) \) instead of \( j \)) yields \( h_j \equiv a_{j-(n-k)} \mod I \).

Hence, \( h_j = a_{j-(n-k)} \in \mathbb{k} \).

But \( 0 \leq n - k \) and thus \( \overline{h_0} \in R_{n-k} \) (by the definition of \( R_{n-k} \)). Hence, \( 1 = \overline{h_0} \in R_{n-k} \), so that \( \mathbb{k} \subseteq R_{n-k} \) and thus \( \overline{h_j} \in \mathbb{k} \subseteq R_{n-k} = L_1 \) (by Proposition 7.19).

Thus, Lemma 7.23 (a) is proven in Case 2.

We have now proven Lemma 7.23 (a) in each of the two Cases 1 and 2. Thus, Lemma 7.23 (a) is proven.

(b) We are in one of the following two cases:

Case 1: We have \( j \leq n - k \).

Case 2: We have \( j > n - k \).

Let us first consider Case 1. In this case, we have \( j \leq n - k \). Thus, \( j < n - k \)
(since \( j \neq n - k \)), so that \( j \leq n - k - 1 \). Thus, \( n - k - 1 \geq j \geq 0 \), so that
\( n - k - 1 \in \mathbb{N} \). Recall that \( R_{n-k-1} \) is defined as the \( k \)-submodule of \( S/I \) spanned
by the \( \overline{h_i} \) with \( i \in \mathbb{N} \) satisfying \( i \leq n - k - 1 \). Hence, \( \overline{h_j} \in R_{n-k-1} \) (since \( j \in \mathbb{N} \) and \( j \leq n - k - 1 \)). Thus, Lemma 7.23 (b) is proven in Case 1.
Let us now consider Case 2. In this case, we have \( j > n - k \). Hence, \( n - k < j \leq n \), so that \( j \in \{n - k + 1, n - k + 2, \ldots, n\} \) and therefore \( j - (n - k) \in \{1, 2, \ldots, k\} \). Hence, \((15)\) (applied to \( j - (n - k) \) instead of \( j \)) yields \( h_j \equiv a_{j - (n - k)} \mod I \).

Hence, \( h_j = a_j - (n - k) \in k \).

But \( n - k > 0 \) (since \( n > k \)), and thus \( 1 \leq n - k \), so that \( 0 \leq n - k - 1 \). Hence, \( h_0 \in R_{n-k-1} \) (by the definition of \( R_{n-k-1} \)). Hence, \( 1 = h_0 \in R_{n-k-1} \), so that \( k \subseteq R_{n-k-1} \) and thus \( h_j \in k \subseteq R_{n-k-1} \). Thus, Lemma 7.23 (b) is proven in Case 2.

We have now proven Lemma 7.23 (b) in each of the two Cases 1 and 2. Thus, Lemma 7.23 (b) is proven.

### 7.7. Connection to the \( Q_p \)

**Convention 7.24.** We WLOG assume that \( k > 0 \) from now on.

Now, let us recall Definition 6.11.

**Proposition 7.25.** We have \( L_{k-1} = Q_0 \).

**Proof of Proposition 7.25**. Recall the following:

- We have defined \( L_{k-1} \) as the \( k \)-submodule of \( S/I \) spanned by the \( \bar{s}_\lambda \) with \( \lambda \in P_{k,n} \) satisfying \( \ell(\lambda) \leq k - 1 \).

- We have defined \( Q_0 \) as the \( k \)-submodule of \( S/I \) spanned by the \( \bar{s}_\lambda \) with \( \lambda \in P_{k,n} \) satisfying \( \lambda_k \leq 0 \).

Comparing these two definitions, we conclude that \( L_{k-1} = Q_0 \) (because for any \( \lambda \in P_{k,n} \), the statement \( \ell(\lambda) \leq k - 1 \) is equivalent to the statement \( \lambda_k \leq 0 \)). This proves Proposition 7.25.

**Lemma 7.26.** We have \((L_1)^{k-1} \subseteq Q_0 \).

**Proof of Lemma 7.26**. Corollary 7.16 yields \((L_1)^{k-1} \subseteq L_{k-1} = Q_0 \) (by Proposition 7.25). This proves Lemma 7.26.

**Lemma 7.27.** Let \( p \in \mathbb{Z} \). Then, \( CQ_p \subseteq Q_{p+1} \).

**Proof of Lemma 7.27**. Lemma 6.16 shows that \( \bar{e}_i Q_p \subseteq Q_{p+1} \) for each \( i \in \mathbb{N} \). Thus, \( CQ_p \subseteq Q_{p+1} \) (since the \( k \)-module \( C \) is spanned by the \( \bar{e}_i \) with \( i \in \mathbb{N} \)). This proves Lemma 7.27.

**Corollary 7.28.** Let \( p \in \mathbb{Z} \) and \( q \in \mathbb{N} \). Then, \( C^q Q_p \subseteq Q_{p+q} \).

**Proof of Corollary 7.28**. This follows by induction on \( q \), where the induction step uses Lemma 7.27.
7.8. Criteria for \( \text{coeff}_\omega \left( \overline{h_V} \right) = 0 \)

We shall now show two sufficient criteria for when a \( p \)-tuple \( v \in \mathbb{Z}^p \) satisfies \( \text{coeff}_\omega \left( \overline{h_V} \right) = 0 \).

**Theorem 7.29.** Let \( p \in \mathbb{N} \) be such that \( p \leq k \). Let \( v = (v_1, v_2, \ldots, v_p) \in \mathbb{Z}^p \) be an \( p \)-tuple of integers. Let \( q \in \{1, 2, \ldots, p\} \) be such that

\[
v_1 \geq v_2 \geq \cdots \geq v_q > n \geq v_{q+1} \geq v_{q+2} \geq \cdots \geq v_p
\]

and \( v_q \leq 2n - k - q \).

Assume also that

\[
v_i \leq 2n - k + 1 \quad \text{for each } i \in \{1, 2, \ldots, p\}. \quad \quad (50)
\]

Then, \( \text{coeff}_\omega \left( \overline{h_V} \right) = 0 \).

**Proof of Theorem 7.29** From \( v_q \leq 2n - k - q \), we obtain \( 2n - k - q \geq v_q > n \), so that \( n - k - q > 0 \). Thus, \( n - k - q - 1 \in \mathbb{N} \).

If any of the entries \( v_1, v_2, \ldots, v_p \) of \( v \) is negative, then Theorem 7.29 holds for easy reasons.\(^{10}\) Hence, we WLOG assume that none of the entries \( v_1, v_2, \ldots, v_p \) of \( v \) is negative. Thus, all of the entries \( v_1, v_2, \ldots, v_p \) are nonnegative integers.

From \( p \leq k \), we obtain \( p - 1 \leq k - 1 \) and thus \( L_{p-1} \subseteq L_{k-1} \) (since \( L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots \)). Thus,

\[
L_{p-1} \subseteq L_{k-1} = Q_0 \quad \quad (51)
\]

(by Proposition 7.25).

From \( n \geq v_{q+1} \geq v_{q+2} \geq \cdots \geq v_p \), we conclude that \( v_j \leq n \) for each \( j \in \{q + 1, q + 2, \ldots, p\} \). In other words, \( v_{q+i} \leq n \) for each \( i \in \{1, 2, \ldots, p - q\} \). Hence, Proposition 7.14 (applied to \( p - q \), \( (v_{q+1}, v_{q+2}, \ldots, v_p) \)) and \( v_{q+i} \) instead of \( p, v \) and \( v_i \)) yields \( h_{(v_{q+1}, v_{q+2}, \ldots, v_p)} \in H_{p-q} \). But Proposition 7.11 (applied to \( p - q \) instead of \( p \)) yields \( L_{p-q} = H_{p-q} \). Thus,

\[
\overline{h_{(v_{q+1}, v_{q+2}, \ldots, v_p)}} \in H_{p-q} = L_{p-q}. \quad \quad (52)
\]

Next, we claim that

\[
\overline{h_{v_i}} \in L_{1}C \quad \text{for each } i \in \{1, 2, \ldots, q - 1\}. \quad \quad (53)
\]

\(^{10}\)Indeed, in this case we have \( v_i < 0 \) for some \( i \in \{1, 2, \ldots, p\} \), and therefore \( h_{v_i} = 0 \) for this \( i \), and therefore

\[
h_v = h_{v_1} h_{v_2} \cdots h_{v_p} = (h_{v_1} h_{v_2} \cdots h_{v_{q-1}}) \overline{h_{v_q}} \left( h_{v_{q+1}} h_{v_{q+2}} \cdots h_{v_p} \right) = 0,
\]

and therefore \( \text{coeff}_\omega \left( \overline{h_V} \right) = 0 \), qed.
Thus, $\text{coeff}_\omega(h_v) \in \text{coeff}_\omega(Q_{n-k-1}) = 0$ (by Lemma 6.12), and thus $\text{coeff}_\omega(h_v) = 0$. This proves Theorem 7.29. \hfill $\square$
**Theorem 7.30.** Assume that \( n > k \). Let \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{Z}^k \) be a \( k \)-tuple of integers such that \( \gamma \neq \omega \).

Assume that

\[
\gamma_i \leq 2n - k - i \quad \text{for each } i \in \{1, 2, \ldots, k\}. \tag{54}
\]

Then, \( \text{coeff}_\omega \left( \frac{h_\gamma}{\gamma} \right) = 0 \).

**Proof of Theorem 7.30** We have \( k \neq 0 \) \(^{11}\). Thus, \( k > 0 \); hence, \( \gamma_1 \) is well-defined.

If any of the entries \( \gamma_1, \gamma_2, \ldots, \gamma_k \) of \( \gamma \) is negative, then Theorem 7.30 holds for easy reasons.\(^{12}\) Hence, we WLOG assume that none of the entries \( \gamma_1, \gamma_2, \ldots, \gamma_k \) of \( \gamma \) is negative. Thus, all of the entries \( \gamma_1, \gamma_2, \ldots, \gamma_k \) are nonnegative integers. In other words, \( (\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{N}^k \).

Let \( \nu = (v_1, v_2, \ldots, v_k) \in \mathbb{Z}^k \) be the weakly decreasing permutation of the \( k \)-tuple \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \). Thus, \( h_{v_1} h_{v_2} \cdots h_{v_k} = h_{\gamma_1} h_{\gamma_2} \cdots h_{\gamma_k} \). Hence, \( h_v = h_{v_1} h_{v_2} \cdots h_{v_k} = h_{\gamma_1} h_{\gamma_2} \cdots h_{\gamma_k} = h_\gamma \).

Recall that \( (v_1, v_2, \ldots, v_k) \) is a permutation of \( (\gamma_1, \gamma_2, \ldots, \gamma_k) \). In other words, there exists a permutation \( \sigma \in S_k \) such that

\[
\left( v_i = \gamma_{\sigma(i)} \text{ for each } i \in \{1, 2, \ldots, k\} \right). \tag{55}
\]

Consider this \( \sigma \).

Recall that \( (v_1, v_2, \ldots, v_k) \) is weakly decreasing. Thus, \( v_1 \geq v_2 \geq \cdots \geq v_k \). Also, \( (v_1, v_2, \ldots, v_k) \in \mathbb{N}^k \) (since \( (v_1, v_2, \ldots, v_k) \) is a permutation of \( (\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{N}^k \)).

For each \( i \in \{1, 2, \ldots, k\} \), we have

\[
v_i = \gamma_{\sigma(i)} \quad \text{(by } 55)\]

\[
\leq 2n - k - \sigma(i) \tag{56}
\]

(by 54, applied to \( \sigma(i) \) instead of \( i \)).

We are in one of the following two cases:

- **Case 1:** We have \( v_1 \leq n \).
- **Case 2:** We have \( v_1 > n \).

\(^{11}\)Proof. Assume the contrary. Thus, \( k = 0 \). Now, \( \gamma, \omega \in \mathbb{Z}^0 = \mathbb{Z}^0 \) (since \( k = 0 \)), whence \( \gamma = \omega \).

But \( k = 0 \) also leads to \( \omega = \), and thus \( \gamma = \) \( \omega \). But this contradicts \( \gamma \neq \omega \). This contradiction shows that our assumption was false. Qed.

\(^{12}\)Indeed, in this case we have \( \gamma_i < 0 \) for some \( i \in \{1, 2, \ldots, k\} \), and therefore \( h_{\gamma_i} = 0 \) for this \( i \), and therefore

\[
h_\gamma = h_{\gamma_1} h_{\gamma_2} \cdots h_{\gamma_k} = (h_{\gamma_1} h_{\gamma_2} \cdots h_{\gamma_{i-1}}) \underbrace{h_{\gamma_i} (h_{\gamma_{i+1}} h_{\gamma_{i+2}} \cdots h_{\gamma_k})}_{=0} = 0,
\]

and therefore \( \text{coeff}_\omega \left( \frac{h_\gamma}{\gamma} \right) = 0 \), qed.
Let us first consider Case 1. In this case, we have \( v_1 \leq n \). But recall that \( \gamma \neq \omega \). Hence, there exists at least one \( q \in \{1, 2, \ldots, k\} \) satisfying \( v_q \neq n - k \). Consider such a \( q \).

Next, we claim that
\[
\overline{h_{v_i}} \in L_1 \quad \text{for each } i \in \{1, 2, \ldots, k\}.
\]  
(57)

[Proof of (57): Let \( i \in \{1, 2, \ldots, k\} \). We have \( v_1 \geq v_2 \geq \cdots \geq v_k \), thus \( v_i \leq v_1 \leq n \). Now, \( v_i \leq n \) and \( v_i \in \mathbb{N} \) (since \((v_1, v_2, \ldots, v_k) \in \mathbb{N}^k\)). Hence, Lemma 7.23 (a) (applied to \( j = v_i \)) yields \( \overline{h_{v_i}} \in L_1 \). This proves (57).]

Also, \( v_1 \geq v_2 \geq \cdots \geq v_k \), thus \( v_q \leq v_1 \leq n \). Also, \( n > k \) and \( v_q \in \mathbb{N} \) (since \((v_1, v_2, \ldots, v_k) \in \mathbb{N}^k\)) and \( v_q \neq n - k \). Hence, Lemma 7.23 (b) (applied to \( j = v_q \)) yields \( \overline{h_{v_q}} \in R_{n-k-1} \). From \( n > k \), we obtain \( n - k > 0 \), so that \( n - k \geq 1 \), and thus \( n - k - 1 \in \mathbb{N} \).

Now, \( h_v = h_{v_1}h_{v_2}\cdots h_{v_k} = \prod_{i=1}^{k} h_{v_i} \), so that
\[
\overline{h_v} = \overline{\prod_{i=1}^{k} h_{v_i}} = \left( \prod_{i \neq q} \overline{h_{v_i}} \right) \overline{h_{v_q}} \in L_{n-k-1}
\]  
(by Corollary 7.28, applied to \( n - k - 1 \) and 0 instead of \( q \) and \( p \)). In other words, \( h_v \in Q_{n-k-1} \). In view of \( h_v = h_{v_1} \), this rewrites as \( \overline{h_\gamma} \in Q_{n-k-1} \). Hence, \( \text{coeff}_\omega (\overline{h_\gamma}) \in \text{coeff}_\omega (Q_{n-k-1}) = 0 \) (by Lemma 6.12), and thus \( \text{coeff}_\omega (\overline{h_\gamma}) = 0 \). Thus, Theorem 7.30 is proven in Case 1.

Let us now consider Case 2. In this case, we have \( v_1 > n \). Hence, there exists at least one \( r \in \{1, 2, \ldots, k\} \) such that \( v_r > n \) (namely, \( r = 1 \)). Let \( q \) be the largest such \( r \). Thus, \( v_q > n \), but each \( r > q \) satisfies \( v_r \leq n \). Hence,
\[
v_1 \geq v_2 \geq \cdots \geq v_q > n \geq v_{q+1} \geq v_{q+2} \geq \cdots \geq v_k
\]  

\[15\text{Proof. Assume the contrary. Thus, } v_i = n - k \text{ for each } i \in \{1, 2, \ldots, k\}. \text{Now, let } j \in \{1, 2, \ldots, k\} \text{ be arbitrary. Then, } v_{\sigma^{-1}(j)} = n - k \text{ (since } v_i = n - k \text{ for each } i \in \{1, 2, \ldots, k\}). \text{ But (55)} \text{ (applied to } i = \sigma^{-1}(j)) \text{ yields } v_{\sigma^{-1}(j)} = \gamma_{\sigma(\sigma^{-1}(j))} = \gamma_j. \text{ Hence, } \gamma_j = v_{\sigma^{-1}(j)} = n - k. \text{ Now, forget that we fixed } j. \text{ We thus have proven that } \gamma_j = n - k \text{ for each } j \in \{1, 2, \ldots, k\}. \text{ Hence, } \gamma = (n - k, n - k, \ldots, n - k) = \omega. \text{ This contradicts } \gamma \neq \omega. \text{ This contradiction shows that our assumption was false, qed.} \]
(since \(v_1 \geq v_2 \geq \cdots \geq v_k\)). Also, \(v_q \leq 2n - k - q\). Furthermore, \((56)\) shows that
\[
v_i \leq 2n - k - \sigma(i) \leq 2n - k + 1
\]
for each \(i \in \{1, 2, \ldots, k\}\). Hence, Theorem \ref{thm:coeff} \((\text{applied to } p = k)\) yields \(\text{coeff}_\omega\left(h_{v_i}\right) = 0\). In view of \(h_{v_i} = h_{\gamma_i}\), this rewrites as \(\text{coeff}_\omega\left(h_{\gamma_i}\right) = 0\). Thus, Theorem \ref{thm:coeff} is proven in Case 2.

We have now proven Theorem \ref{thm:coeff} in both Cases 1 and 2. Hence, Theorem \ref{thm:coeff} always holds. \(\square\)

7.9. A criterion for \(\text{coeff}_\omega\left(S_\lambda\right) = 0\)

\begin{theorem}
Let \(\lambda\) be a partition with at most \(k\) parts. Assume that \(\lambda_1 \leq 2(n - k)\) and \(\lambda \neq \omega\). Then, \(\text{coeff}_\omega\left(S_\lambda\right) = 0\).
\end{theorem}

\begin{proof}
We have \(n > k\). We have \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)\) \((\text{since the partition } \lambda \text{ has at most } k \text{ parts})\). Proposition \ref{prop:partition} \((a)\) yields
\[
s_\lambda = \text{det}\left((h_{\lambda_i-u+v})_{1 \leq u \leq k, \ 1 \leq v \leq k}\right) = \sum_{\sigma \in S_k} (-1)^\sigma \prod_{i=1}^k h_{\lambda_i-i+\sigma(i)}
\]
(by the definition of a determinant). Hence,
\[
S_\lambda = \sum_{\sigma \in S_k} (-1)^\sigma \prod_{i=1}^k h_{\lambda_i-i+\sigma(i)} = \sum_{\sigma \in S_k} (-1)^\sigma \prod_{i=1}^k h_{\lambda_i-i+\sigma(i)}.
\]
\end{proof}

\begin{proof}
Assume the contrary. Thus, \(v_q > 2n - k - q\).

The map \(\sigma\) is a permutation, and thus injective. Hence, \(|\sigma\left(\{1, 2, \ldots, q\}\right)| = |\{1, 2, \ldots, q\}| = q\). Thus, \(\sigma\left(\{1, 2, \ldots, q\}\right)\) cannot be a subset of \(\{1, 2, \ldots, q - 1\}\) \((\text{because this would lead to } |\sigma\left(\{1, 2, \ldots, q\}\right)| \leq |\{1, 2, \ldots, q - 1\}| = q - 1 < q\), which would contradict \(|\sigma\left(\{1, 2, \ldots, q\}\right)| = q\). In other words, not every \(i \in \{1, 2, \ldots, q\}\) satisfies \(\sigma(i) \in \{1, 2, \ldots, q - 1\}\). In other words, there exists some \(i \in \{1, 2, \ldots, q\}\) that satisfies \(\sigma(i) \notin \{1, 2, \ldots, q - 1\}\). Consider such an \(i\).

From \(i \in \{1, 2, \ldots, q\}\), we obtain \(i \leq q\) and thus \(v_i \geq v_q\) \((\text{since } v_1 \geq v_2 \geq \cdots \geq v_k)\).
From \(\sigma(i) \notin \{1, 2, \ldots, q - 1\}\), we obtain \(\sigma(i) \geq q - 1\), so that \(\sigma(i) \geq q\). Now, \((56)\) yields \(v_i \leq 2n - k - \sigma(i) \leq 2n - k - q\); but this contradicts \(v_i \geq v_q > 2n - k - q\). This contradiction shows that our assumption was false, qed.

\begin{proof}
Assume the contrary. Thus, \(n \leq k\) and therefore \(n = k\) \((\text{since } n \geq k)\). Hence, \(n - k = 0\). Thus, \(\lambda_1 \leq 2(n - k) = 0\), so that \(\lambda_1 = 0\) and thus \(\lambda = \emptyset\) \((\text{since } \lambda \text{ is a partition})\). But from \(n - k = 0\), we also obtain \(\omega = \emptyset\) \((\text{since } \omega = (n - k, n - k, \ldots, n - k))\). Thus, \(\lambda = \emptyset = \omega\). But this contradicts \(\lambda \neq \omega\). This contradiction shows that our assumption was wrong, qed.
\end{proof}
Now, we claim that each $\sigma \in S_k$ satisfies
\[
\text{coeff}_{\omega} \left( \prod_{i=1}^{k} h_{\lambda_i - i + \sigma(i)} \right) = 0.
\] (59)

[Proof of (59): Let $\sigma \in S_k$. Define a $k$-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{Z}^k$ of integers by
\[
(\gamma_i = \lambda_i - i + \sigma(i)) \quad \text{for each } i \in \{1, 2, \ldots, k\}.
\] (60)
Then, $\gamma \neq \omega$. Moreover,
\[
\gamma_i \leq 2n - k - i \quad \text{for each } i \in \{1, 2, \ldots, k\}
\]
Hence, Theorem 7.30 yields $\text{coeff}_{\omega} \left( \overline{h_\gamma} \right) = 0$. In view of
\[
h_\gamma = h_{\gamma_1} h_{\gamma_2} \cdots h_{\gamma_k} = \prod_{i=1}^{k} h_{\gamma_i} = \prod_{i=1}^{k} h_{\lambda_i - i + \sigma(i)},
\]
this rewrites as $\text{coeff}_{\omega} \left( \prod_{i=1}^{k} h_{\lambda_i - i + \sigma(i)} \right) = 0$. Thus, (59) is proven.]

16Proof. Assume the contrary. Thus, $\gamma = \omega$.

Let $i \in \{1, 2, \ldots, k-1\}$. From $\gamma = \omega$, we obtain $\gamma_i = \omega_i = n - k$. Comparing this with (60), we find $\lambda_i - i + \sigma(i) = n - k$. The same argument (applied to $i+1$ instead of $i$) yields $\lambda_{i+1} - (i+1) + \sigma(i+1) = n - k$. But $\lambda_i \geq \lambda_{i+1}$ (since $\lambda$ is a partition). Hence,
\[
\lambda_i - i + \sigma(i) > \lambda_{i+1} - (i+1) + \sigma(i+1) = n - k = \lambda_i - i + \sigma(i)
\]
(since $\lambda_i - i + \sigma(i) = n - k$). If we subtract $\lambda_i - i$ from this inequality, we obtain $\sigma(i+1) > \sigma(i)$. In other words, $\sigma(i) < \sigma(i+1)$.

Now, forget that we fixed $i$. We thus have shown that each $i \in \{1, 2, \ldots, k-1\}$ satisfies $\sigma(i) < \sigma(i+1)$. In other words, we have $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$. Hence, $\sigma$ is a strictly increasing map from $\{1, 2, \ldots, k\}$ to $\{1, 2, \ldots, k\}$. But the only such map is id. Thus, $\sigma = id$. Hence, for each $i \in \{1, 2, \ldots, k\}$, we have
\[
\gamma_i = \lambda_i - i + \underbrace{\sigma(i)}_{=id} \quad \text{(by (60))}
\]
\[
= \lambda_i - i + id(i) = \lambda_i - i + i = \lambda_i.
\]
Thus, $\gamma = \lambda$. Comparing this with $\gamma = \omega$, we obtain $\lambda = \omega$. This contradicts $\lambda \neq \omega$. This contradiction shows that our assumption was wrong. qed.

17Proof. Let $i \in \{1, 2, \ldots, k\}$. Then, $\lambda_1 \geq \lambda_i$ (since $\lambda$ is a partition), so that $\lambda_i \leq \lambda_1 \leq 2(n-k)$. Now, (60) yields
\[
\gamma_i = \underbrace{\lambda_i}_{\leq 2(n-k)} - i + \sigma(i) \leq 2(n-k) - i + k = 2n - k - i,
\]
qed.
From (58), we obtain
\[
\text{coeff}_\omega (s^\lambda) = \text{coeff}_\omega \left( \sum_{\sigma \in S_k} (-1)^{\sigma} \prod_{i=1}^{k} h_{\lambda_i - i + \sigma(i)} \right) = 0.
\]

This proves Theorem 7.31.

8. Another proof of Theorem 6.3

We can use Theorem 7.31 to obtain a second proof of Theorem 6.3. To that end, we shall use a few more basic facts about Littlewood-Richardson coefficients. First we introduce a few notations (only for this section):

**Convention 8.1.** Convention 6.1 remains in place for the whole Section 8. We shall also use all the notations introduced in Section 6.

8.1. Some basics on Littlewood-Richardson coefficients

**Definition 8.2.** Let \( a \in \mathbb{N} \).
- \( (a) \) We let \( \text{Par}_a \) denote the set of all partitions with size \( a \). (That is, \( \text{Par}_a = \{ \lambda \text{ is a partition } | |\lambda| = a \} \).)
- \( (b) \) If \( \lambda \) and \( \mu \) are two partitions with size \( a \), then we write \( \lambda \triangleright \mu \) if and only if we have

\[
\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \quad \text{for each } i \in \{1, 2, \ldots, a\}.
\]

This defines a binary relation \( \triangleright \) on \( \text{Par}_a \). This relation is the smaller-or-equal relation of a partial order on \( \text{Par}_a \), which is called the dominance order.

Here is another way to describe the dominance order:

**Remark 8.3.** Let \( a \in \mathbb{N} \). Let \( \lambda \) and \( \mu \) be two partitions with size \( a \). Then, we have \( \lambda \triangleright \mu \) if and only if we have

\[
\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \quad \text{for each } i \geq 1. \quad (61)
\]

**Proof of Remark 8.3** \( \iff \): Assume that we have (61). We must prove that \( \lambda \triangleright \mu \).
For each \( i \in \{1,2,\ldots,a\} \), we have \( i \geq 1 \) and therefore \( \lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \) (by (61)). In other words, we have

\[
\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \quad \text{for each } i \in \{1,2,\ldots,a\}.
\]

In other words, we have \( \lambda \triangleright \mu \) (by the definition of the relation \( \triangleright \)). This proves the “\( \rightarrow \)” direction of Remark 8.3.

\( \Rightarrow: \) Assume that \( \lambda \triangleright \mu \). We must prove that we have (61).

We have assumed that \( \lambda \triangleright \mu \). In other words, we have

\[
\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \quad \text{for each } i \in \{1,2,\ldots,a\} \quad (62)
\]

(by the definition of the relation \( \triangleright \)).

Now, let \( i \geq 1 \). Our goal is to show that \( \lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \). If \( i \in \{1,2,\ldots,a\} \), then this follows from (62). Hence, for the rest of this proof, we WLOG assume that we don’t have \( i \in \{1,2,\ldots,a\} \). Hence, \( i \geq a + 1 \) (because \( i \geq 1 \)), so that \( a + 1 \leq i < i + 1 \). But \( \lambda \) is a partition; thus, \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \).

Now, recall that \( \lambda \) is a partition of size \( a \); hence, \( |\lambda| = a \). Thus,

\[
a = |\lambda| = \lambda_1 + \lambda_2 + \lambda_3 + \cdots = \sum_{p=1}^{i+1} \lambda_p = \sum_{p=1}^{i+1} \left( \sum_{\lambda_p \geq \lambda_{i+1}} \lambda_p \right) \quad \text{(since } p \leq i+1 \text{ and } \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots)\]

\[
\geq \sum_{p=1}^{i+1} \lambda_{i+1} = (i + 1) \lambda_{i+1}.
\]

Hence, \( \lambda_{i+1} \leq \frac{a}{i+1} < 1 \) (since \( a < a + 1 < i + 1 \)). Thus, \( \lambda_{i+1} = 0 \) (since \( \lambda_{i+1} \in \mathbb{N} \)). Furthermore, from \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \), we conclude that each \( p \in \{i+1,i+2,i+3,\ldots\} \) satisfies \( \lambda_{i+1} \geq \lambda_p \) and thus \( \lambda_p \leq \lambda_{i+1} = 0 \) and therefore \( \lambda_p = 0 \) (since \( \lambda_p \in \mathbb{N} \)). Hence, \( \sum_{p=i+1}^{\infty} \lambda_p = \sum_{p=i+1}^{\infty} 0 = 0 \). Thus,

\[
a = \sum_{p=1}^{i} \lambda_p = \sum_{p=1}^{i} \lambda_p + \sum_{p=i+1}^{\infty} \lambda_p = \sum_{p=1}^{i} \lambda_p = \lambda_1 + \lambda_2 + \cdots + \lambda_i.
\]

The same argument (applied to the partition \( \mu \) instead of \( \lambda \)) yields

\[
a = \mu_1 + \mu_2 + \cdots + \mu_i.
\]

Comparing these two equalities, we find \( \lambda_1 + \lambda_2 + \cdots + \lambda_i = \mu_1 + \mu_2 + \cdots + \mu_i \).

Hence, \( \lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \).

Now, forget that we fixed \( i \). We thus have shown that

\[
\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \quad \text{for each } i \geq 1.
\]

In other words, (61). This proves the “\( \rightarrow \)” direction of Remark 8.3. \( \square \)
Definition 8.4. Let $\mu$ and $\nu$ be two partitions. Then, we define two new partitions $\mu + \nu$ and $\mu \sqcup \nu$ as follows:

- The partition $\mu + \nu$ is defined as $(\mu_1 + \nu_1, \mu_2 + \nu_2, \mu_3 + \nu_3, \ldots)$.
- The partition $\mu \sqcup \nu$ is defined as the result of sorting the list $(\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}, \nu_1, \nu_2, \ldots, \nu_{\ell(\nu)})$ in decreasing order.

We shall use the following fact:

Proposition 8.5. Let $a \in \mathbb{N}$ and $b \in \mathbb{N}$ be such that $a \leq b$. Let $\mu \in \text{Par}_a$, $\nu \in \text{Par}_{b-a}$ and $\lambda \in \text{Par}_b$ be such that $c_{\mu,\nu}^\lambda \neq 0$. Then, $\mu + \nu \triangleright \lambda \triangleright \mu \sqcup \nu$.

Proposition 8.5 is precisely [GriRei18, Exercise 2.9.17(c)] (with $k$ and $n$ renamed as $a$ and $b$).

Corollary 8.6. Let $\lambda$, $\mu$, and $\nu$ be three partitions such that $\lambda_1 > \mu_1 + \nu_1$. Then, $c_{\mu,\nu}^\lambda = 0$.

Proof of Corollary 8.6. Assume the contrary. Thus, $c_{\mu,\nu}^\lambda \neq 0$.

Let $a = |\mu|$; thus, $\mu \in \text{Par}_a$. Let $b = |\lambda|$; thus, $\lambda \in \text{Par}_b$.

Proposition 6.17 (c) shows that $c_{\mu,\nu}^\lambda = 0$ unless $|\mu| + |\nu| = |\lambda|$. Hence, $|\mu| + |\nu| = |\lambda|$ (since $c_{\mu,\nu}^\lambda \neq 0$). Thus, $|\nu| = \lfloor |\lambda| - |\mu| \rfloor = b - a$. Hence, $b - a = |\nu| \geq 0$,

so that $a \leq b$. Also, from $|\nu| = b - a$, we obtain $\nu \in \text{Par}_{b-a}$. Thus, Proposition 8.5 yields $\mu + \nu \triangleright \lambda \triangleright \mu \sqcup \nu$.

But $b = |\lambda| \geq \lambda_1 > \mu_1 + \nu_1 \geq 0$, so that $1 \in \{1, 2, \ldots, b\}$.

Now, from $\mu + \nu \triangleright \lambda$, we conclude that

$$(\mu + \nu)_1 + (\mu + \nu)_2 + \cdots + (\mu + \nu)_i \geq \lambda_1 + \lambda_2 + \cdots + \lambda_i$$

for each $i \in \{1, 2, \ldots, b\}$

(by the definition of the relation $\triangleright$, since $\mu + \nu$ and $\lambda$ are two partitions of size $b$). Applying this to $i = 1$, we obtain $(\mu + \nu)_1 \geq \lambda_1$ (since $1 \in \{1, 2, \ldots, b\}$). But the definition of $\mu + \nu$ yields $(\mu + \nu)_1 = \mu_1 + \nu_1 < \lambda_1$ (since $\lambda_1 > \mu_1 + \nu_1$). This contradicts $(\mu + \nu)_1 \geq \lambda_1$. This contradiction shows that our assumption was false. Hence, Corollary 8.6 is proven.

Next, we recall the Littlewood-Richardson rule itself:

Proposition 8.7. Let $\lambda$ and $\mu$ be two partitions. Then,

$$s_\lambda s_\mu = \sum_{\rho \text{ is a partition}} c_{\lambda,\mu}^\rho s_\rho.$$
Proposition 8.7 is precisely [GriRei18, (2.5.6)] (with λ, μ and ν renamed as ρ, λ and μ).

Corollary 8.8. Let λ ∈ P_k,n and μ ∈ P_k,n. Then,

\[ s_{\lambda}s_{\mu} = \sum_{\rho \text{ is a partition with at most } k \text{ parts; } \rho_1 \leq 2(n-k)} c_{\lambda,\mu}^{\rho} s_{\rho}. \]

Proof of Corollary 8.8. If ρ is a partition satisfying \( \rho_1 > 2(n-k) \), then

\[ c_{\lambda,\mu}^{\rho} = 0. \]  

(63)

[Proof of (63): Let ρ be a partition satisfying \( \rho_1 > 2(n-k) \).

We have \( \lambda \in P_{k,n} \), thus, each part of \( \lambda \) is \( \leq n-k \). Thus, \( \lambda_1 \leq n-k \). Similarly, \( \mu_1 \leq n-k \). Hence, \( \lambda_1 + \mu_1 \leq 2(n-k) < \rho_1 \). In other words, \( \rho_1 > \lambda_1 + \mu_1 \).

Hence, Corollary 8.6 (applied to \( \rho, \lambda \) and \( \mu \) instead of \( \lambda, \mu \) and \( \nu \)) yields \( c_{\lambda,\mu}^{\rho} = 0 \). This proves (63).

Proposition 8.7 yields

\[ s_{\lambda}s_{\mu} = \sum_{\rho \text{ is a partition}} c_{\lambda,\mu}^{\rho} s_{\rho}. \]

This is an equality in Λ. Evaluating both of its sides at the \( k \) indeterminates \( x_1, x_2, \ldots, x_k \), we find

\[
\begin{align*}
\sum_{\rho \text{ is a partition with at most } k \text{ parts; } \rho_1 \leq 2(n-k)} c_{\lambda,\mu}^{\rho} s_{\rho} & + \sum_{\rho \text{ is a partition; } \rho_1 > 2(n-k)} c_{\lambda,\mu}^{\rho} s_{\rho} \\
& = \sum_{\rho \text{ is a partition; } \rho_1 \leq 2(n-k)} c_{\lambda,\mu}^{\rho} s_{\rho} \\
& + \sum_{\rho \text{ is a partition with more than } k \text{ parts; } \rho_1 \leq 2(n-k)} c_{\lambda,\mu}^{\rho} s_{\rho} \\
& = \sum_{\rho \text{ is a partition with at most } k \text{ parts; } \rho_1 \leq 2(n-k)} c_{\lambda,\mu}^{\rho} s_{\rho}.
\end{align*}
\]

This proves Corollary 8.8. □
Next, let us recall another known fact on skew Schur functions:

**Proposition 8.9.** Let $\lambda$ be any partition. Then, $s_{\omega/\lambda^\vee} = s_\lambda$.

*Proof of Proposition 8.9.* From [GriRei18, Exercise 2.9.15(a)] (applied to $n-k$ and $\emptyset$ instead of $m$ and $\mu$), we obtain $s_{\lambda/\emptyset} = s_{\emptyset/\lambda^\vee}$. In view of $\emptyset^\vee = \omega$, this rewrites as $s_{\lambda/\emptyset} = s_{\omega/\lambda^\vee}$. Thus, $s_{\omega/\lambda^\vee} = s_{\lambda/\emptyset} = s_\lambda$. This proves Proposition 8.9.

**Corollary 8.10.** Let $\lambda$ and $\mu$ be two partitions. Then,

$$c_{\omega/\lambda, \mu} = \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \mu = \lambda^\vee; \\ 0, & \text{else} \end{cases}$$

*Proof of Corollary 8.10.* Proposition 6.17 (applied to $\omega$ and $\lambda$ instead of $\lambda$ and $\mu$) shows that

$$s_{\omega/\lambda} = \sum_{\nu \text{ is a partition}} c_{\omega/\lambda, \nu} s_\nu.$$  \hfill (64)

On the other hand, it is easy to see that

$$s_{\omega/\lambda} = \sum_{\nu \text{ is a partition}} \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^\vee; \\ 0, & \text{else} \end{cases} \ s_\nu. \hfill (65)$$

*Proof of (65):* We are in one of the following two cases:

**Case 1:** We have $\lambda \in P_{k,n}$.

**Case 2:** We have $\lambda \not\in P_{k,n}$.

Let us first consider Case 1. In this case, we have $\lambda \in P_{k,n}$. Thus, $\lambda^\vee$ is well-defined, and we have $(\lambda^\vee)^\vee = \lambda$. Hence, Proposition 8.9 (applied to $\lambda^\vee$ instead of $\lambda$) yields

$$s_{\omega/(\lambda^\vee)^\vee} = s_{\lambda^\vee} = \sum_{\nu \text{ is a partition}} \begin{cases} 1, & \text{if } \nu = \lambda^\vee; \\ 0, & \text{else} \end{cases} \ s_\nu$$

$$= \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^\vee; \\ 0, & \text{else} \end{cases} \ s_\nu.$$

In view of $(\lambda^\vee)^\vee = \lambda$, this rewrites as

$$s_{\omega/\lambda} = \sum_{\nu \text{ is a partition}} \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^\vee; \\ 0, & \text{else} \end{cases} \ s_\nu.$$
Thus, (65) is proven in Case 1.

Now, let us consider Case 2. In this case, we have $\lambda \notin P_{k,n}$. Hence, $\lambda \not\subseteq \omega$ (since $\lambda \subseteq \omega$ holds if and only if $\lambda \in P_{k,n}$). Thus, $s_{\omega/\lambda} = 0$. Comparing this with

$$\sum_{\nu \text{ is a partition}} \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^\vee; \\ 0, & \text{else} \end{cases} s_{\nu} = 0,$$

we obtain

$$s_{\omega/\lambda} = \sum_{\nu \text{ is a partition}} \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^\vee; \\ 0, & \text{else} \end{cases} s_{\nu}.$$

Thus, (65) is proven in Case 2.

We have now proven (65) in each of the two Cases 1 and 2. Thus, (65) always holds.

Now, comparing (65) with (64), we obtain

$$\sum_{\nu \text{ is a partition}} c_{\lambda,\mu}^{\omega} s_{\nu} = \sum_{\nu \text{ is a partition}} \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^\vee; \\ 0, & \text{else} \end{cases} s_{\nu}.$$

Since the family $(s_{\nu})_{\nu \text{ is a partition}}$ is a basis of the $k$-module $\Lambda$, we can compare the coefficients of $s_\mu$ on both sides of this equality. We thus obtain

$$c_{\lambda,\mu}^{\omega} = \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \mu = \lambda^\vee; \\ 0, & \text{else} \end{cases}.$$

This proves Corollary 8.10. \hfill \Box

### 8.2. Another proof of Theorem 6.3

We are now ready to prove Theorem 6.3 again. More precisely, we shall prove Lemma 6.22 (as we know that Theorem 6.3 quickly follows from Lemma 6.22).

**Second proof of Lemma 6.22** If $k = 0$, then Lemma 6.22 holds\(^{18}\). Hence, for the rest of this proof, we WLOG assume that $k \neq 0$. Thus, $k > 0$. Hence, $\omega_1 = 1$.

\(^{18}\)Proof. Assume that $k = 0$. Then, $P_{k,n} = \{\emptyset\}$, so that $\lambda \in P_{k,n} = \{\emptyset\}$ and thus $\lambda = \emptyset$. Similarly, $\mu = \emptyset$. Therefore, $\lambda = \mu^\vee$ holds. Also, $\omega = \emptyset$. Moreover, from $\lambda = \emptyset$, we obtain $s_\lambda = s_\emptyset = 1$; similarly, $s_\mu = 1$. Thus, $s_\lambda$ and $s_\mu$ are both 1. Therefore, $\omega_1 = 1$. Comparing this with \begin{align*}
\text{coeff}_\omega(s_\lambda s_\mu) &= \begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee \end{cases} = 1 \quad \text{(since } \lambda = \mu^\vee) \\
\text{coeff}_\omega(s_{\emptyset} s_{\emptyset}) &= \begin{cases} 1, & \text{if } \mu = \mu^\vee; \\ 0, & \text{if } \mu \neq \mu^\vee \end{cases} = 1 \quad \text{(since } \mu = \mu^\vee)
\end{align*}
holds, we obtain $\text{coeff}_\omega(s_\lambda s_\mu) = \begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee \end{cases}$. Thus, Lemma 6.22 holds. Qed.
\( n - k \leq 2(n - k) \). Thus, \( \omega \) is a partition \( \rho \) with at most \( k \) parts that satisfies \( \rho_1 \leq 2(n - k) \) (since \( \omega_1 \leq 2(n - k) \)).

Corollary 8.8 yields

\[
S_\lambda S_\mu = \sum_{\rho \text{ is a partition with at most } k \text{ parts; } \rho_1 \leq 2(n - k)} c_{\lambda, \mu}^\rho S_\rho.
\]

Hence,

\[
\frac{S_\lambda S_\mu}{\rho \text{ is a partition with at most } k \text{ parts; } \rho_1 \leq 2(n - k)} = \sum_{\rho \text{ is a partition with at most } k \text{ parts; } \rho_1 \leq 2(n - k)} c_{\lambda, \mu}^\rho S_\rho.
\]

Thus,

\[
\operatorname{coeff}_\omega \left( \frac{S_\lambda S_\mu}{\rho \text{ is a partition with at most } k \text{ parts; } \rho_1 \leq 2(n - k)} \right) = \operatorname{coeff}_\omega \left( \sum_{\rho \text{ is a partition with at most } k \text{ parts; } \rho_1 \leq 2(n - k)} c_{\lambda, \mu}^\rho S_\rho \right) = \sum_{\rho \text{ is a partition with at most } k \text{ parts; } \rho_1 \leq 2(n - k); \rho \neq \omega} c_{\lambda, \mu}^\rho \operatorname{coeff}_\omega (S_\rho).
\]

(here, we have split off the addend for \( \rho = \omega \) from the sum, since \( \omega \) is a partition \( \rho \) with at most \( k \) parts that satisfies \( \rho_1 \leq 2(n - k) \))

\[
= c_{\lambda, \mu}^\omega \operatorname{coeff}_\omega (S_\omega) + \sum_{\rho \text{ is a partition with at most } k \text{ parts; } \rho_1 \leq 2(n - k); \rho \neq \omega} c_{\lambda, \mu}^\rho \operatorname{coeff}_\omega (S_\rho) = 0
\]

(by Theorem 7.31, applied to \( \rho \) instead of \( \lambda \))

\[
= \begin{cases} 1, & \text{if } \lambda \in P_{k, n} \text{ and } \mu = \lambda \vee; \\
0, & \text{else} \end{cases}
\]

(by Corollary 8.10)

\[
= \begin{cases} 1, & \text{if } \mu = \lambda \vee; \\
0, & \text{if } \mu \neq \lambda \vee \quad \text{(since } \lambda \in P_{k, n} \text{ holds)}
\end{cases}
\]

\[
= \begin{cases} 1, & \text{if } \lambda = \mu \vee; \\
0, & \text{if } \lambda \neq \mu \vee \quad \text{(since } \mu = \lambda \vee \text{ holds if and only if } \lambda = \mu \vee). \end{cases}
\]

Thus, Lemma 6.22 is proven again. \( \square \)
9. The $h$-basis and the $m$-basis

**Convention 9.1.** For the rest of Section 9, we assume that $a_1, a_2, \ldots, a_k$ belong to $S$.

9.1. A lemma on the $s$-basis

For future use, we shall show a technical lemma, which improves on Lemma 5.10:

**Lemma 9.2.** Let $N \in \mathbb{N}$. Let $f \in S$ be a symmetric polynomial of degree $< N$. Then, in $S/I$, we have

$$f \in \sum_{\kappa \in P_k; |\kappa| < N} k s_\kappa.$$

**Proof of Lemma 9.2.** We shall prove Lemma 9.2 by strong induction on $N$. Thus, we fix some $M \in \mathbb{N}$, and we assume (as the induction hypothesis) that Lemma 9.2 holds whenever $N < M$. We now must prove that Lemma 9.2 holds for $N = M$.

Let $f \in S$ be a symmetric polynomial of degree $< M$. Then, in $S/I$, we shall show that $f \in \sum_{\kappa \in P_k; |\kappa| < M} k s_\kappa$.

Indeed, let $U$ be the $k$-submodule $\sum_{\kappa \in P_k; |\kappa| < M} k s_\kappa$ of $S/I$. Hence, $U$ is the $k$-submodule of $S/I$ spanned by the family $(s_\kappa)_{\kappa \in P_k; |\kappa| < M}$. Hence,

$$s_\kappa \in U \quad \text{for each } \kappa \in P_k, n \text{ satisfying } |\kappa| < M. \quad (66)$$

We are going to show that $f \in U$.

Lemma 5.9 (applied to $N = M$) shows that there exists a family $(c_\kappa)_{\kappa \in P_k; |\kappa| < M}$ of elements of $k$ such that $f = \sum_{\kappa \in P_k; |\kappa| < M} c_\kappa s_\kappa$. Consider this family. Thus,

$$f = \sum_{\kappa \in P_k; |\kappa| < M} c_\kappa s_\kappa = \sum_{\mu \in P_k; |\mu| < M} c_\mu s_\mu \quad (67)$$

(here, we have renamed the summation index $\kappa$ as $\mu$).

Now, let $\mu \in P_k$ satisfy $|\mu| < M$. We shall show that $s_\mu \in U$.

[Proof: If $\mu \in P_{k,n}$, then this follows directly from (66) (applied to $\kappa = \mu$). Hence, for the rest of this proof, we WLOG assume that $\mu \notin P_{k,n}$. Thus, Lemma 5.8 (applied to $\mu$ instead of $\lambda$) shows that $s_\mu \equiv (\text{some symmetric polynomial of degree } < |\mu|) \mod I$.]
In other words, there exists a symmetric polynomial \( g \in S \) of degree \( < |\mu| \) such that \( s_\mu \equiv g \mod I \). Consider this \( g \). We have \( |\mu| < M \). Hence, Lemma 9.2 holds for \( N = |\mu| \) (by our induction hypothesis). Thus, we can apply Lemma 9.2 to \( g \) and \( |\mu| \) instead of \( f \) and \( N \). We thus conclude that

\[
\bar{g} \equiv \sum_{\kappa \in \mathcal{P}_{k,n} ; |\kappa| < |\mu|} k\bar{s}_\kappa.
\]

But from \( s_\mu \equiv g \mod I \), we obtain

\[
s_\mu \equiv g \in \sum_{\kappa \in \mathcal{P}_{k,n} ; |\kappa| < |\mu|} k\bar{s}_\kappa \subseteq \sum_{\kappa \in \mathcal{P}_{k,n} ; |\kappa| < |\mu|} k\bar{s}_\kappa
\]

(since each \( \kappa \in \mathcal{P}_{k,n} \) satisfying \( |\kappa| < |\mu| \) must also satisfy \( |\kappa| < M \) because \( |\kappa| < |\mu| < M \)), and therefore the sum \( \sum_{\kappa \in \mathcal{P}_{k,n} ; |\kappa| < |\mu|} k\bar{s}_\kappa \) is a subsum of the sum \( \sum_{\kappa \in \mathcal{P}_{k,n} ; |\kappa| < M} k\bar{s}_\kappa \).

Hence,

\[
s_\mu \in \sum_{\kappa \in \mathcal{P}_{k,n} ; |\kappa| < M} k\bar{s}_\kappa = U \quad \left( \text{since } U \text{ is defined as } \sum_{\kappa \in \mathcal{P}_{k,n} ; |\kappa| < M} k\bar{s}_\kappa \right),
\]

qed.

Forget that we fixed \( \mu \). We thus have shown that

\[
s_\mu \in U \quad \text{for each } \mu \in \mathcal{P}_k \text{ satisfying } |\mu| < M.
\]  

(68)

Now, (67) yields

\[
\bar{f} = \sum_{\mu \in \mathcal{P}_k ; |\mu| < M} c_\mu s_\mu = \sum_{\mu \in \mathcal{P}_k ; |\mu| < M} c_\mu \overline{s_\mu} \in \sum_{\mu \in \mathcal{P}_k ; |\mu| < M} c_\mu U \subseteq U
\]

(by (68))

(since \( U \) is a \( k \)-module).

Forget that we fixed \( f \). We thus have shown that if \( f \in S \) is a symmetric polynomial of degree \( < M \), then \( \bar{f} \in \sum_{\kappa \in \mathcal{P}_{k,n} ; |\kappa| < M} k\bar{s}_\kappa \). In other words, Lemma 9.2 holds for \( N = M \). This completes the induction step. Hence, Lemma 9.2 is proven by induction.

9.2. The \( h \)-basis

In Theorem 7.13, we have shown that the family \( \left( \overline{h_\lambda^\alpha} \right)_{\lambda \in \mathcal{P}_{k,n}} \) is a basis of the \( k \)-module \( S/I \) under the condition that \( a_1, a_2, \ldots, a_k \in k \). We shall soon prove this
Definition 9.3. We define a binary relation $\geq^*$ on the set $P_{k,n}$ as follows: For two partitions $\lambda \in P_{k,n}$ and $\mu \in P_{k,n}$, we set $\lambda \geq^* \mu$ if and only if

- either $|\lambda| > |\mu|$ or
- $|\lambda| = |\mu|$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i$ for all $i \geq 1$.

It is clear that this relation $\geq^*$ is the greater-or-equal relation of a partial order on $P_{k,n}$. This order will be called the size-then-antidominance order.

Note that the condition “$|\lambda| = |\mu|$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i$ for all $i \geq 1$” in Definition 9.3 can also be restated as “$\mu \triangleright \lambda$”, where $\triangleright$ means the dominance relation (defined in Definition 8.2 (b)). Indeed, this follows easily from Remark 8.3 (applied to $\mu$ and $\lambda$ instead of $\lambda$ and $\mu$).

For future reference, we need two simple criteria for the $\geq^*$ relation:

Remark 9.4. Let $\lambda \in P_{k,n}$ and $\mu \in P_{k,n}$.

(a) If $|\lambda| > |\mu|$, then $\lambda \geq^* \mu$.

(b) Let $a \in \mathbb{N}$. If both $\lambda$ and $\mu$ are partitions of size $a$ and satisfy $\mu \triangleright \lambda$, then $\lambda \geq^* \mu$. (See Definition 8.2 (b) for the meaning of “$\triangleright$”.)

Proof of Remark 9.4 (a) This follows immediately from the definition of the relation $\geq^*$.

(b) Assume that both $\lambda$ and $\mu$ are partitions of size $a$ and satisfy $\mu \triangleright \lambda$. Now, both partitions $\lambda$ and $\mu$ have size $a$; in other words, $|\lambda| = a$ and $|\mu| = a$. Hence, $|\lambda| = a = |\mu|$. We have $\mu \triangleright \lambda$. In other words, we have

$$\mu_1 + \mu_2 + \cdots + \mu_i \geq \lambda_1 + \lambda_2 + \cdots + \lambda_i$$

for each $i \geq 1$ (by Remark 8.3, applied to $\mu$ and $\lambda$ instead of $\lambda$ and $\mu$). In other words, $\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i$ for all $i \geq 1$. Hence, we have $|\lambda| = |\mu|$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i$ for all $i \geq 1$. Therefore, $\lambda \geq^* \mu$ (by the definition of the relation $\geq^*$). This proves Remark 9.4 (b).

Now, we can put the size-then-antidominance order to use. Recall that Theorem 2.7 yields that the family $(\overline{s}_\lambda)^{\lambda \in P_{k,n}}$ is a basis of the $k$-module $S/1$. 

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A basis for a quotient of symmetric polynomials (draft)  June 29, 2019
Theorem 9.5. The family \((\overline{h}_\lambda)_{\lambda \in P_{k,n}}\) expands unitriangularly in the family \((\overline{s}_\lambda)_{\lambda \in P_{k,n}}\). Here, the word “expands unitriangularly” is understood according to [GriRei18, Definition 11.1.16(c)], with the poset structure on \(P_{k,n}\) being given by the size-then-antidominance order.

Example 9.6. For this example, let \(n = 5\) and \(k = 3\). Assume that \(a_1, a_2 \in \mathbb{k}\). Then, the expansion of the \(h_\emptyset\) in the basis \((\overline{s}_\lambda)_{\lambda \in P_{k,n}}\) looks as follows:

\[
\begin{align*}
\overline{h}_\emptyset &= \overline{s}_\emptyset; \\
\overline{h}_{(1)} &= \overline{s}_{(1)}; \\
\overline{h}_{(2)} &= \overline{s}_{(2)}; \\
\overline{h}_{(1,1)} &= \overline{s}_{(2)} + \overline{s}_{(1,1)}; \\
\overline{h}_{(2,1)} &= a_1 \overline{s}_{(1)} + \overline{s}_{(2,1)}; \\
\overline{h}_{(1,1,1)} &= a_1 \overline{s}_\emptyset + \overline{s}_{(1,1)} + 2\overline{s}_{(2,1)}; \\
\overline{h}_{(2,2)} &= a_1 \overline{s}_{(1)} + \overline{s}_{(2,2)}; \\
\overline{h}_{(2,1,1)} &= -a_2 \overline{s}_\emptyset + 2a_1 \overline{s}_{(1)} + \overline{s}_{(2,1,1)} + \overline{s}_{(2,2)}; \\
\overline{h}_{(2,2,1)} &= -a_2 \overline{s}_{(1)} + a_1 \overline{s}_{(1,1)} + 2a_1 \overline{s}_{(2)} + \overline{s}_{(2,2,1)}; \\
\overline{h}_{(2,2,2)} &= a_1^2 \overline{s}_\emptyset - a_2 \overline{s}_{(1,1)} + 2a_1 \overline{s}_{(2,1)} + \overline{s}_{(2,2,2)}.
\end{align*}
\]

These equalities hold for arbitrary \(a_1, a_2 \in \mathbb{S}\), not only for \(a_1, a_2 \in \mathbb{k}\); but in the general case they are not expansions in the basis \((\overline{s}_\lambda)_{\lambda \in P_{k,n}}\), since \(a_1, a_2\) themselves can be expanded further.

Our proof of Theorem 9.5 will use the concept of Kostka numbers. Let us recall their definition:

Definition 9.7. (a) See [GriRei18, §2.2] for the definition of a column-strict tableau of shape \(\lambda\) (where \(\lambda\) is a partition), and also for a definition of \(\text{cont}(T)\) where \(T\) is such a tableau.

(b) Let \(\lambda\) and \(\mu\) be two partitions. Then, the Kostka number \(K_{\lambda, \mu}\) is defined to be the number of all column-strict tableaux \(T\) of shape \(\lambda\) having \(\text{cont}(T) = \mu\).

This definition of \(K_{\lambda, \mu}\) is a particular case of the definition of \(K_{\lambda, \mu}\) in [GriRei18, Exercise 2.2.13].

We shall use the following properties of Kostka numbers:

Lemma 9.8. (a) If \(a \in \mathbb{N}\), then we have \(K_{\lambda, \mu} = 0\) for any partitions \(\lambda \in \text{Par}_a\) and \(\mu \in \text{Par}_a\) that don’t satisfy \(\lambda \triangleright \mu\).

(b) If \(a \in \mathbb{N}\), then we have \(K_{\lambda, \lambda} = 1\) for any \(\lambda \in \text{Par}_a\).

(c) If \(\lambda\) and \(\mu\) are two partitions such that \(|\lambda| \neq |\mu|\), then \(K_{\lambda, \mu} = 0\).
(d) For any partition $\mu$, we have

$$h_\mu = \sum_{\lambda \in \text{Par}} K_{\lambda,\mu} s_\lambda,$$

where $\text{Par}$ denotes the set of all partitions.

(e) For any $a \in \mathbb{N}$ and any $\lambda \in \text{Par}_a$, we have

$$h_\lambda = \sum_{\mu \in \text{Par}_a} K_{\mu,\lambda} s_\mu.$$

(f) For any $a \in \mathbb{N}$ and any $\lambda \in \text{Par}_a$, we have

$$h_\lambda = \sum_{\mu \in \text{Par}_a; \mu \approx \lambda} K_{\mu,\lambda} s_\mu.$$

Proof of Lemma 9.8 (a) This is [GriRei18, Exercise 2.2.13(d)], applied to $a$ instead of $n$.

(b) This is [GriRei18, Exercise 2.2.13(e)], applied to $a$ instead of $n$.

(c) Let $\lambda$ and $\mu$ be two partitions such that $|\lambda| \neq |\mu|$. Let $T$ be a column-strict tableau of shape $\lambda$ having $\text{cont}(T) = \mu$. We shall derive a contradiction.

Indeed, the tableau $T$ has shape $\lambda$, and thus has $|\lambda|$ many cells. Hence,

$$|\lambda| = (\text{the number of cells of } T) = (\text{the number of entries of } T)$$

$$= |\text{cont}(T)| = |\mu| \quad \text{(since } \text{cont}(T) = \mu).$$

This contradicts $|\lambda| \neq |\mu|$.

Now, forget that we fixed $T$. We thus have found a contradiction whenever $T$ is a column-strict tableau of shape $\lambda$ having $\text{cont}(T) = \mu$. Hence, there exist no such tableau. In other words, the number of such tableaux is 0. In other words, $K_{\lambda,\mu} = 0$ (since $K_{\lambda,\mu}$ is defined to be the number of such tableaux). This proves Lemma 9.8 (c).

(d) This is [GriRei18, Exercise 2.7.10(a)].

(e) Let $\text{Par}$ denote the set of all partitions. Then, Lemma 9.8 (d) yields that

$$h_\mu = \sum_{\lambda \in \text{Par}} K_{\lambda,\mu} s_\lambda$$

for any partition $\mu$.

Hence, for any partition $\mu$, we have

$$h_\mu = \sum_{\lambda \in \text{Par}} K_{\lambda,\mu} s_\lambda = \sum_{\lambda \in \text{Par}; |\lambda| = |\mu|} K_{\lambda,\mu} s_\lambda + \sum_{\lambda \in \text{Par}; |\lambda| \neq |\mu|} K_{\lambda,\mu} s_\lambda$$

(by Lemma 9.8 (e))

$$= \sum_{\lambda \in \text{Par}; |\lambda| = |\mu|} K_{\lambda,\mu} s_\lambda + \sum_{\lambda \in \text{Par}; |\lambda| \neq |\mu|} 0 s_\lambda = \sum_{\lambda \in \text{Par}; |\lambda| = |\mu|} K_{\lambda,\mu} s_\lambda.$$
Renaming \( \mu \) and \( \lambda \) as \( \lambda \) and \( \mu \) in this equality, we obtain the following: For any partition \( \lambda \), we have

\[
h_\lambda = \sum_{\mu \in \text{Par}; |\mu| = |\lambda|} K_{\mu, \lambda} s_\mu \quad \text{for any partition } \lambda. \tag{69}
\]

Now, let \( a \in \mathbb{N} \) and \( \lambda \in \text{Par}_a \). Then, \( |\lambda| = a \). Now, (69) becomes

\[
h_\lambda = \sum_{\mu \in \text{Par}; |\mu| = a} K_{\mu, \lambda} s_\mu = \sum_{\mu \in \text{Par}_a} K_{\mu, \lambda} s_\mu \quad \text{(since } |\lambda| = a) = \sum_{\mu \in \text{Par}_a} K_{\mu, \lambda} s_\mu.
\]

This proves Lemma 9.8 (e).

(f) Let \( a \in \mathbb{N} \) and \( \lambda \in \text{Par}_a \). Lemma 9.8 (e) yields \( h_\lambda = \sum_{\mu \in \text{Par}_a} K_{\mu, \lambda} s_\mu \). This is an identity in \( \Lambda \). Evaluating both of its sides at the \( k \) variables \( x_1, x_2, \ldots, x_k \), we obtain

\[
h_\lambda = \sum_{\mu \in \text{Par}_a} K_{\mu, \lambda} s_\mu = \sum_{\mu \in \text{Par}_a; \mu \text{ has at most } k \text{ parts}} K_{\mu, \lambda} s_\mu + \sum_{\mu \in \text{Par}_a; \mu \text{ has more than } k \text{ parts}} K_{\mu, \lambda} s_\mu = \sum_{\mu \in \text{Par}_a; \mu \text{ has at most } k \text{ parts}} K_{\mu, \lambda} s_\mu + \sum_{\mu \in \text{Par}_a; \mu \text{ has more than } k \text{ parts}} K_{\mu, \lambda} s_\mu + 0 = \sum_{\mu \in \text{Par}_a; \mu \text{ has more than } k \text{ parts}} K_{\mu, \lambda} s_\mu + 0 = \sum_{\mu \in \text{Par}_a; \mu \text{ has more than } k \text{ parts}} K_{\mu, \lambda} s_\mu + 0 = \sum_{\mu \in \text{Par}_a; \mu \text{ has more than } k \text{ parts}} K_{\mu, \lambda} s_\mu + 0 = \sum_{\mu \in \text{Par}_a; \mu \text{ has more than } k \text{ parts}} K_{\mu, \lambda} s_\mu.
\]

This proves Lemma 9.8 (f).

Proof of Theorem 9.5. Let \( <^* \) denote the smaller relation of the size-then-antidominance order on \( P_{k,n} \). Thus, two partitions \( \lambda \) and \( \mu \) satisfy \( \mu <^* \lambda \) if and only if \( \mu \neq \lambda \) and \( \lambda \geq^* \mu \).

Our goal is to show that the family \( \{ h_{\lambda} \}_{\lambda \in P_{k,n}} \) expands unitriangularly in the family \( \{ s_\lambda \}_{\lambda \in P_{k,n}} \). In other words, our goal is to show that each \( \lambda \in P_{k,n} \) satisfies

\[
\overline{h_\lambda} = \overline{s_\lambda} + (\text{a } k\text{-linear combination of the elements } \overline{s_\mu} \text{ for } \mu \in P_{k,n} \text{ satisfying } \mu <^* \lambda) \tag{70}
\]

(because [GriRei18] Remark 11.1.17(c)] shows that the family \( \{ \overline{h_\lambda} \}_{\lambda \in P_{k,n}} \) expands unitriangularly in the family \( \{ \overline{s_\lambda} \}_{\lambda \in P_{k,n}} \) if and only if every \( \lambda \in P_{k,n} \) satisfies (70). So let us prove (70).
Fix $\lambda \in P_{k,n}$. Define $a \in \mathbb{N}$ by $a = |\lambda|$. Thus, $\lambda \in \text{Par}_a$. Hence, Lemma 9.8 (f) yields
\[
h_{\lambda} = \sum_{\substack{\mu \in P_k; \mu \in \text{Par}_a}} K_{\mu, \lambda}s_\mu = K_{\lambda, \lambda}s_\lambda + \sum_{\substack{\mu \in P_k; \mu \in \text{Par}_a; \mu \neq \lambda}} K_{\mu, \lambda}s_\mu
\] (71)
(here, we have split off the addend for $\mu = \lambda$, since $\lambda \in P_{k,n} \subseteq P_k$ and $\lambda \in \text{Par}_a$).

Now, let $M$ be the $k$-submodule of $S/I$ spanned by the elements $\overline{s_\mu}$ for $\mu \in P_{k,n}$ satisfying $\mu <^* \lambda$. Thus, we have
\[
\overline{s_\mu} \in M \quad \text{for each } \mu \in P_{k,n} \text{ satisfying } \mu <^* \lambda.
\] (72)

Also, $0 \in M$ (since $M$ is a $k$-submodule of $S/I$).

We shall next show that
\[
K_{\mu, \lambda}\overline{s_\mu} \in M \quad \text{for each } \mu \in P_k \text{ satisfying } \mu \in \text{Par}_a \text{ and } \mu \neq \lambda.
\] (73)

[Proof of (73): Let $\mu \in P_k$ be such that $\mu \in \text{Par}_a$ and $\mu \neq \lambda$. We must prove that $K_{\mu, \lambda}\overline{s_\mu} \in M$.

If $K_{\mu, \lambda} = 0$, then this follows immediately from $K_{\mu, \lambda}\overline{s_\mu} = 0\overline{s_\mu} = 0 \in M$. Hence, for the rest of this proof, we WLOG assume that $K_{\mu, \lambda} \neq 0$.

If $\mu$ and $\lambda$ would not satisfy $\mu \triangleright \lambda$, then we would have $K_{\mu, \lambda} = 0$ (by Lemma 9.8 (a), applied to $\mu$ and $\lambda$ instead of $\lambda$ and $\mu$), which would contradict $K_{\mu, \lambda} \neq 0$. Hence, $\mu$ and $\lambda$ must satisfy $\mu \triangleright \lambda$. Both $\lambda$ and $\mu$ are partitions of size $a$ (since $\lambda \in \text{Par}_a$ and $\mu \in \text{Par}_a$). Thus, $|\lambda| = a$ and $|\mu| = a$.

Now, we are in one of the following two cases:

Case 1: We have $\mu \in P_{k,n}$.

Case 2: We have $\mu \notin P_{k,n}$.

Let us first consider Case 1. In this case, we have $\mu \in P_{k,n}$. Thus, $\lambda \geq^* \mu$ (by Remark 9.4 (b)) and thus $\mu <^* \lambda$ (since $\mu \neq \lambda$). Hence, (72) shows that $\overline{s_\mu} \in M$. Thus, $K_{\mu, \lambda}\overline{s_\mu} \in K_{\mu, \lambda}M \subseteq M$ (since $M$ is a $k$-submodule of $S/I$). Thus, (73) is proven in Case 1.

Let us next consider Case 2. In this case, we have $\mu \notin P_{k,n}$. Hence, Lemma 5.8 (applied to $\mu$ instead of $\lambda$) shows that
\[
s_\mu \equiv (\text{some symmetric polynomial of degree } < |\mu|) \mod I.
\]
In other words, there exists some symmetric polynomial $f \in S$ of degree $< |\mu|$ such that $s_\mu \equiv f \mod I$. Consider this $f$. Lemma 9.2 (applied to $N = |\mu|$) yields that in $S/I$, we have
\[
\overline{f} \in \sum_{\substack{\kappa \in P_{k,n}; |\kappa| < |\mu|}} k\overline{s_\kappa}.
\] (74)
Now, let \( \kappa \in P_{k,n} \) be such that \(|\kappa| < |\mu|\). Then, \(|\kappa| < |\mu| = a = |\lambda|\), so that \(|\lambda| > |\kappa|\). Thus, Remark 9.4 (a) (applied to \( \kappa \) instead of \( \mu \)) yields \( \lambda \geq^* \kappa \). Also, \(|\kappa| \neq |\lambda|\) (since \(|\kappa| < |\lambda|\)) and thus \( \kappa \neq \lambda \). Combining this with \( \lambda \geq^* \kappa \), we obtain \( \kappa \leq^* \lambda \). Hence, (72) (applied to \( \kappa \) instead of \( \mu \)) yields \( \overline{s}_\kappa \in M \). Hence, \( k \overline{s}_\kappa \subseteq kM \subseteq M \) (since \( M \) is a \( k \)-module).

Forget that we fixed \( \kappa \). We thus have shown that \( k \overline{s}_\kappa \subseteq M \) for each \( \kappa \in P_{k,n} \) satisfying \(|\kappa| < |\mu|\). Hence, \( \sum_{\kappa \in P_{k,n}; \quad |\kappa| < |\mu|} k \overline{s}_\kappa \subseteq \sum_{\kappa \in P_{k,n}; \quad |\kappa| < |\mu|} M \subseteq M \) (since \( M \) is a \( k \)-module).

Thus, (74) becomes \( \bar{f} \in \sum_{\kappa \in P_{k,n}; \quad |\kappa| < |\mu|} k \overline{s}_\kappa = M \). But \( s_\mu \equiv f \mod I \) and thus \( \overline{s}_\mu = \bar{f} \in M \).

Thus, \( k_{\mu,\lambda} \overline{s}_\mu \in k_{\mu,\lambda} M \subseteq M \) (since \( M \) is a \( k \)-submodule of \( S/I \)). Hence, (73) is proven in Case 2.

We have now proven (73) in both Cases 1 and 2. Hence, (73) always holds.

Now, from (71), we obtain

\[
\overline{h}_\lambda = K_{\lambda,\lambda} \overline{s}_\lambda + \sum_{\mu \in P_k; \quad \mu \in \text{Par}_a; \quad \mu \neq \lambda} K_{\mu,\lambda} \overline{s}_\mu = K_{\lambda,\lambda} \overline{s}_\lambda + \sum_{\mu \in P_k; \quad \mu \in \text{Par}_a; \quad \mu \neq \lambda} K_{\mu,\lambda} \overline{s}_\mu \quad \quad \text{(by Lemma 9.8 (b))}
\]

\[
\in \overline{s}_\lambda + \sum_{\mu \in P_k; \quad \mu \in \text{Par}_a; \quad \mu \neq \lambda} M \subseteq \overline{s}_\lambda + M.
\]

(since \( M \) is a \( k \)-module)

In other words, \( \overline{h}_\lambda - \overline{s}_\lambda \in M \). In other words, \( \overline{h}_\lambda - \overline{s}_\lambda \) is a \( k \)-linear combination of the elements \( \overline{s}_\mu \) for \( \mu \in P_{k,n} \) satisfying \( \mu <^* \lambda \) (since \( M \) was defined as the \( k \)-submodule of \( S/I \) spanned by these elements). In other words,

\[
\overline{h}_\lambda = \overline{s}_\lambda + \left( \text{a \( k \)-linear combination of the elements} \overline{s}_\mu \text{ for } \mu \in P_{k,n} \text{ satisfying } \mu <^* \lambda \right).
\]

Thus, (70) is proven. As we already have explained, this completes the proof of Theorem 9.5. \( \square \)

We can now prove Theorem 7.13 again. Better yet, we can prove the following more general fact:

\textbf{Theorem 9.9.} The family \( \langle \overline{h}_\lambda \rangle_{\lambda \in P_{k,n}} \) is a basis of the \( k \)-module \( S/I \).

Theorem 9.9 makes the exact same claim as Theorem 7.13, but is nevertheless more general because we have stated it in a more general context (namely, \( a_1, a_2, \ldots, a_k \in S \) rather than \( a_1, a_2, \ldots, a_k \in k \)).
Proof of Theorem 9.9. Consider the finite set $P_{k,n}$ as a poset (using the size-then-
 antidominance order).

Theorem 9.5 says that the family $(h_\lambda)_{\lambda \in P_{k,n}}$ expands unitriangularly in the
family $(s_\lambda)_{\lambda \in P_{k,n}}$. Hence, the family $(h_\lambda)_{\lambda \in P_{k,n}}$ expands invertibly triangularly in the family $(s_\lambda)_{\lambda \in P_{k,n}}$. Thus, [GriRei18, Corollary 11.1.19(e)] (applied to $S/I$, $P_{k,n}$, $(h_\lambda)_{\lambda \in P_{k,n}}$ and $(s_\lambda)_{\lambda \in P_{k,n}}$ instead of $M$, $S$, $(e_s)_{s \in S}$ and $(f_s)_{s \in S}$) shows that the family $(h_\lambda)_{\lambda \in P_{k,n}}$ is a basis of the $k$-module $S/I$ if and only if the family $(s_\lambda)_{\lambda \in P_{k,n}}$ is a basis of the $k$-module $S/I$. Hence, the family $(h_\lambda)_{\lambda \in P_{k,n}}$ is a basis of the $k$-module $S/I$ (since the family $(s_\lambda)_{\lambda \in P_{k,n}}$ is a basis of the $k$-module $S/I$). Thus, Theorem 9.9 is proven. (And therefore, Theorem 7.13 is proven again.) □

9.3. The $m$-basis

Next, we recall another well-known family of symmetric polynomials:

**Definition 9.10.** For any partition $\lambda$, we let $m_\lambda$ denote the monomial symmetric polynomial in $x_1, x_2, \ldots, x_k$ corresponding to the partition $\lambda$. This monomial symmetric polynomial is what is called $m_\lambda (x_1, x_2, \ldots, x_k)$ in [GriRei18, Chapter 2]. Note that

$$m_\lambda = 0 \quad \text{if } \lambda \text{ has more than } k \text{ parts.} \quad (75)$$

If $\lambda$ is any partition, then the monomial symmetric polynomial $m_\lambda = m_\lambda (x_1, x_2, \ldots, x_k)$ is symmetric and thus belongs to $S$.

We now claim the following:

**Theorem 9.11.** The family $(\overline{m}_\lambda)_{\lambda \in P_{k,n}}$ is a basis of the $k$-module $S/I$.

In order to prove this, we again need the concept of unitriangularity and a partial order on the set $P_{k,n}$. The partial order, this time, is not the size-then-antidominance order, but a simpler one (the “graded dominance order”):

**Definition 9.12.** We define a binary relation $\geq_*$ on the set $P_{k,n}$ as follows: For two partitions $\lambda \in P_{k,n}$ and $\mu \in P_{k,n}$, we set $\lambda \geq_* \mu$ if and only if

- $|\lambda| = |\mu|$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$ for all $i \geq 1$.

It is clear that this relation $\geq_*$ is the greater-or-equal relation of a partial order on $P_{k,n}$. This order will be called the graded dominance order.

\[\text{See } [\text{GriRei18, Definition 11.1.6(b)}] \text{ for the meaning of this word.}\]
Note that the condition “$|\lambda| = |\mu|$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$ for all $i \geq 1$” in Definition 9.12 can also be restated as “$\lambda \triangleright \mu$”, where $\triangleright$ means the dominance relation (defined in Definition 8.2(b)). Indeed, this follows easily from Remark 8.3.

For future reference, we need a simple criterion for the relation $\geq^*$:

**Remark 9.13.** Let $\lambda \in P_{k,n}$ and $\mu \in P_{k,n}$.

Let $a \in \mathbb{N}$. If both $\lambda$ and $\mu$ are partitions of size $a$ and satisfy $\lambda \triangleright \mu$, then $\lambda \geq^* \mu$. (See Definition 8.2(b) for the meaning of “$\triangleright$”.)

**Proof of Remark 9.13.** Assume that both $\lambda$ and $\mu$ are partitions of size $a$ and satisfy $\lambda \triangleright \mu$. Now, both partitions $\lambda$ and $\mu$ have size $a$; in other words, $|\lambda| = a$ and $|\mu| = a$. Hence, $|\lambda| = a = |\mu|$.

We have $\lambda \triangleright \mu$. In other words, we have

$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$ for each $i \geq 1$ (by Remark 8.3). Hence, we have $|\lambda| = |\mu|$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i$ for all $i \geq 1$. Therefore, $\lambda \geq^* \mu$ (by the definition of the relation $\geq^*$).

This proves Remark 9.13. $\square$

Now, we can put the graded dominance order to use. Recall that Theorem 2.7 yields that the family $(s_\lambda)_{\lambda \in P_{k,n}}$ is a basis of the $k$-module $S/I$.

**Theorem 9.14.** The family $(s_\lambda)_{\lambda \in P_{k,n}}$ expands unitriangularly in the family $(m_\lambda)_{\lambda \in P_{k,n}}$. Here, the word “expands unitriangularly” is understood according to [GriRei18, Definition 11.1.16(c)], with the poset structure on $P_{k,n}$ being given by the graded dominance order.

**Example 9.15.** For this example, let $n = 5$ and $k = 3$. Then, the expansion of $s_\lambda$ in the basis $(m_\lambda)_{\lambda \in P_{k,n}}$ looks as follows:

$s_\emptyset = m_\emptyset$;  
$s_{(1)} = m_{(1)}$;  
$s_{(2)} = m_{(1,1)} + m_{(2)}$;  
$s_{(1,1)} = m_{(1,1)}$;  
$s_{(2,1)} = 2m_{(1,1,1)} + m_{(2,1)}$;  
$s_{(1,1,1)} = m_{(1,1,1)}$;  
$s_{(2,2)} = m_{(2,1,1)} + m_{(2,2)}$;  
$s_{(2,1,1)} = m_{(2,1,1)}$;  
$s_{(2,2,1)} = m_{(2,2,1)}$;  
$s_{(2,2,2)} = m_{(2,2,2)}$.

The coefficients in these expansions are Kostka numbers; the $a_1, a_2, \ldots, a_k$ do not appear in them. (This will become clear in the proof of Theorem 9.14)
To prove Theorem 9.14, we shall use the monomial symmetric functions \( m_\lambda \):

- For any partition \( \lambda \), we let \( m_\lambda \) be the corresponding monomial symmetric function in \( \Lambda \). (This is called \( m_\lambda \) in [GriRei18, (2.1.1)].)

We shall furthermore use the following property of the dominance order:

**Lemma 9.16.** Let \( a \in \mathbb{N} \). Let \( \lambda \in \text{Par}_a \) and \( \mu \in \text{Par}_a \) be such that \( \lambda \triangleright \mu \). Assume that \( \lambda \in P_{k,n} \) and \( \mu \in P_k \). Then, \( \mu \in P_{k,n} \).

**Proof of Lemma 9.16.** We have \( \lambda \triangleright \mu \). In other words, we have \( \lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \) for each \( i \geq 1 \) (by Remark 8.3). Applying this to \( i = 1 \), we obtain \( \lambda_1 \geq \mu_1 \). Hence, \( \mu_1 \leq \lambda_1 \leq n - k \) (since \( \lambda \in P_{k,n} \)). Thus, all parts of the partition \( \mu \) are \( \leq n - k \) (since \( \mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots \)). Hence, \( \mu \in P_{k,n} \) (since \( \mu \in P_k \)). This proves Lemma 9.16.

Also, we shall again use Kostka numbers, specifically their following properties:

**Lemma 9.17.** (a) For any \( a \in \mathbb{N} \) and any \( \lambda \in \text{Par}_a \), we have

\[
    s_{\lambda} = \sum_{\mu \in \text{Par}_a} K_{\lambda,\mu} m_\mu.
\]

(b) For any \( a \in \mathbb{N} \) and \( \lambda \in \text{Par}_a \), we have

\[
    s_{\lambda} = \sum_{\substack{\mu \in P_k; \mu \in \text{Par}_a}} K_{\lambda,\mu} m_\mu.
\]

(c) For any \( a \in \mathbb{N} \) and \( \lambda \in \text{Par}_a \) satisfying \( \lambda \in P_{k,n} \), we have

\[
    s_{\lambda} = \sum_{\substack{\mu \in P_{k,n}; \mu \in \text{Par}_a}} K_{\lambda,\mu} m_\mu.
\]

**Proof of Lemma 9.17.** (a) This is [GriRei18, Exercise 2.2.13(c)].

(b) Let \( a \in \mathbb{N} \) and \( \lambda \in \text{Par}_a \). Lemma 9.17 (a) yields \( s_{\lambda} = \sum_{\mu \in \text{Par}_a} K_{\lambda,\mu} m_\mu \). This is an identity in \( \Lambda \). Evaluating both of its sides at the \( k \) variables \( x_1, x_2, \ldots, x_k \), we
This proves Lemma 9.17 \((b)\).

\((c)\) Let \(a \in \mathbb{N}\) and \(\lambda \in \text{Par}_a\) satisfy \(\lambda \in P_{k,n}\).

Fix some \(\mu \in P_k\) such that \(\mu \in \text{Par}_a\) and \(\mu \not\in P_{k,n}\). Then, we don’t have \(\lambda \triangleright \mu\) (since otherwise, Lemma 9.16 would yield that \(\mu \in P_{k,n}\); but this would contradict \(\mu \not\in P_{k,n}\)). Hence, Lemma 9.8 \((a)\) yields \(K_{\lambda,\mu} = 0\).

Forget that we fixed \(\mu\). We thus have shown that

\[
K_{\lambda,\mu} = 0 \quad \text{for every } \mu \in P_k \text{ satisfying } \mu \in \text{Par}_a \text{ and } \mu \not\in P_{k,n}. \tag{76}
\]

Now, Lemma 9.17 \((b)\) yields

\[
s_\lambda = \sum_{\mu \in P_k; \mu \in \text{Par}_a} K_{\lambda,\mu} m_\mu = \underbrace{\sum_{\mu \in P_k; \mu \in \text{Par}_a; \mu \in P_{k,n}} K_{\lambda,\mu} m_\mu}_\text{\((by (75), applied to } \mu \text{ instead of } \lambda\)} + \sum_{\mu \in P_k; \mu \in \text{Par}_a; \mu \not\in P_{k,n}} K_{\lambda,\mu} m_\mu = 0.
\]

This proves Lemma 9.17 \((c)\). \hfill \Box

**Proof of Theorem 9.14** Let \(<_s\) denote the smaller relation of the graded dominance order on \(P_{k,n}\). Thus, two partitions \(\lambda\) and \(\mu\) satisfy \(\mu <_s \lambda\) if and only if \(\mu \not= \lambda\) and \(\lambda \geq_s \mu\).

Our goal is to show that the family \((\overline{s}_\lambda)_{\lambda \in P_{k,n}}\) expands unitriangularly in the family \((\overline{m_\lambda})_{\lambda \in P_{k,n}}\). In other words, our goal is to show that each \(\lambda \in P_{k,n}\) satisfies

\[
\overline{s_\lambda} = \overline{m_\lambda} + \text{ (a } k\text{-linear combination of the elements } \overline{m_\mu} \text{ for } \mu \in P_{k,n} \text{ satisfying } \mu <_s \lambda) \tag{77}
\]
Both \(\lambda \in P_{k,n}\) yields \(|\lambda| \geq 0\) (by (77)). So let us prove (77).

Fix \(\lambda \in P_{k,n}\). Define \(a \in \mathbb{N}\) by \(a = |\lambda|\). Thus, \(\lambda \in \text{Par}_a\). Hence, Lemma 9.17(c) yields

\[
s_{\lambda} = \sum_{\substack{\mu \in P_{k,m}; \\
\mu \in \text{Par}_a; \\
\mu \neq \lambda}} K_{\lambda,\mu} m_{\mu} = K_{\lambda,\lambda} m_{\lambda} + \sum_{\substack{\mu \in P_{k,m}; \\
\mu \in \text{Par}_a; \\
\mu \neq \lambda}} K_{\lambda,\mu} m_{\mu}
\]  

(78)

(here, we have split off the addend for \(\mu = \lambda\), since \(\lambda \in P_{k,n}\) and \(\lambda \in \text{Par}_a\)).

Now, let \(M\) be the \(k\)-submodule of \(S/I\) spanned by the elements \(m_{\mu}\) for \(\mu \in P_{k,n}\) satisfying \(\mu <_{s} \lambda\). Thus, we have

\[
m_{\mu} \in M \quad \text{for each } \mu \in P_{k,n} \text{ satisfying } \mu <_{s} \lambda.
\]  

(79)

Also, \(0 \in M\) (since \(M\) is a \(k\)-submodule of \(S/I\)).

We shall next show that

\[
K_{\lambda,\mu} m_{\mu} \in M \quad \text{for each } \mu \in P_{k,n} \text{ satisfying } \mu \in \text{Par}_a \text{ and } \mu \neq \lambda.
\]  

(80)

[Proof of (80): Let \(\mu \in P_{k,n}\) be such that \(\mu \in \text{Par}_a\) and \(\mu \neq \lambda\). We must prove that \(K_{\lambda,\mu} m_{\mu} \in M\).

If \(K_{\lambda,\mu} = 0\), then this follows immediately from \(K_{\lambda,\mu} m_{\mu} = 0m_{\mu} = 0 \in M\).

Hence, for the rest of this proof, we WLOG assume that \(K_{\lambda,\mu} \neq 0\).

If \(\lambda\) and \(\mu\) would not satisfy \(\lambda \triangleright \mu\), then we would have \(K_{\lambda,\mu} = 0\) (by Lemma 9.8(a)), which would contradict \(K_{\lambda,\mu} \neq 0\). Hence, \(\lambda\) and \(\mu\) must satisfy \(\lambda \triangleright \mu\).

Both \(\lambda\) and \(\mu\) are partitions of size \(a\) (since \(\lambda \in \text{Par}_a\) and \(\mu \in \text{Par}_a\)). Thus, \(|\lambda| = a\) and \(|\mu| = a\). Thus, \(\lambda \geq_{s} \mu\) (by Remark 9.13) and thus \(\mu <_{s} \lambda\) (since \(\mu \neq \lambda\)). Hence, (79) shows that \(m_{\mu} \in M\). Thus, \(K_{\lambda,\mu} m_{\mu} \in K_{\lambda,\mu} M \subseteq M\) (since \(M\) is a \(k\)-submodule of \(S/I\)). Thus, (80) is proven.]

Now, from (78), we obtain

\[
s_{\lambda} = K_{\lambda,\lambda} m_{\lambda} + \sum_{\substack{\mu \in P_{k,m}; \\
\mu \in \text{Par}_a; \\
\mu \neq \lambda}} K_{\lambda,\mu} m_{\mu} = K_{\lambda,\lambda} m_{\lambda} + \sum_{\substack{\mu \in P_{k,m}; \\
\mu \in \text{Par}_a; \\
\mu \neq \lambda}} K_{\lambda,\mu} m_{\mu} = \underbrace{K_{\lambda,\lambda} m_{\lambda}}_{=1} + \sum_{\substack{\mu \in P_{k,m}; \\
\mu \in \text{Par}_a; \\
\mu \neq \lambda}} K_{\lambda,\mu} m_{\mu}
\]

\[
\in m_{\lambda} + \sum_{\substack{\mu \in P_{k,m}; \\
\mu \in \text{Par}_a; \\
\mu \neq \lambda}} M \subseteq m_{\lambda} + M.
\]

(since \(M\) is a \(k\)-module)

In other words, \(s_{\lambda} - m_{\lambda} \in M\). In other words, \(s_{\lambda} - m_{\lambda}\) is a \(k\)-linear combination of the elements \(m_{\mu}\) for \(\mu \in P_{k,n}\) satisfying \(\mu <_{s} \lambda\) (since \(M\) was defined as the
\( k \)-submodule of \( S/I \) spanned by these elements). In other words,

\[
s_\lambda = m_\lambda + \left( \text{a } k \text{-linear combination of the elements } m_\mu \text{ for } \mu \in P_{k,n} \text{ satisfying } \mu <_s \lambda \right).
\]

Thus, (77) is proven. As we already have explained, this completes the proof of Theorem 9.14.

Proof of Theorem 9.11. Consider the finite set \( P_{k,n} \) as a poset (using the graded dominance order).

Theorem 9.14 says that the family \( (s_\lambda)_{\lambda \in P_{k,n}} \) expands unitriangularly in the family \( (m_\lambda)_{\lambda \in P_{k,n}} \). Hence, the family \( (s_\lambda)_{\lambda \in P_{k,n}} \) expands invertibly triangularly in \( (m_\lambda)_{\lambda \in P_{k,n}} \). Thus, [GriRei18, Corollary 11.1.19(e)] (applied to \( S/I, P_{k,n}, (s_\lambda)_{\lambda \in P_{k,n}} \) and \( (m_\lambda)_{\lambda \in P_{k,n}} \) instead of \( M, S, (e_s)_{s \in S} \) and \( (f_s)_{s \in S} \)) shows that the family \( (s_\lambda)_{\lambda \in P_{k,n}} \) is a basis of the \( k \)-module \( S/I \) if and only if the family \( (m_\lambda)_{\lambda \in P_{k,n}} \) is a basis of the \( k \)-module \( S/I \). Hence, the family \( (m_\lambda)_{\lambda \in P_{k,n}} \) is a basis of the \( k \)-module \( S/I \). Thus, Theorem 9.11 is proven.

9.4. The \( p \)-not-basis

What other known families of symmetric functions give rise to bases of \( S/I \)? Here is an example of a family that does not lead to such a basis (at least not in an obvious way):

Remark 9.18. Let \( n = 4 \) and \( k = 2 \). Let \( a_1, a_2 \in k \). For each partition \( \lambda \), let \( p_\lambda \) be the corresponding power sum symmetric polynomial, i.e., the \( p_\lambda(x_1, x_2, \ldots, x_k) \) from [GriRei18, Definition 2.2.1]. Then, the family \( (p_\lambda)_{\lambda \in P_{k,n}} \) is not a basis of the \( k \)-module \( S/I \) (unless \( k = 0 \)).

Proof of Remark 9.18. Straightforward computations yield the following expansions of the \( p_\lambda \) in the basis \( (s_\lambda)_{\lambda \in P_{k,n}} \) of \( S/I \):

\[
\begin{align*}
p_\emptyset &= s_\emptyset; \\
p_{(1)} &= s_{(1)}; \\
p_{(2)} &= -s_{(1,1)} + s_{(2)}; \\
p_{(1,1)} &= s_{(1,1)} + s_{(2)}; \\
p_{(2,1)} &= a_1 s_\emptyset; \\
p_{(2,2)} &= 2a_2 s_\emptyset - a_1 s_{(1)} + 2s_{(2,2)}.
\end{align*}
\]

Thus, \( p_{(2,1)} - a_1 p_\emptyset = 0 \). Hence, the family \( (p_\lambda)_{\lambda \in P_{k,n}} \) fails to be \( k \)-linearly independent, and thus cannot be a basis of \( S/I \). This proves Remark 9.18.  

\(^{20}\) See [GriRei18, Definition 11.1.6(b)] for the meaning of this word.
10. Pieri rules for multiplying by $h_j$

**Convention 10.1.** Convention 6.1 remains in place for the whole Section 10. We shall also use all the notations introduced in Section 6.

In this section, we shall explore formulas for expanding products of the form $s_\lambda h_j$ in the basis $(s_\mu)_{\mu \in P_{k,n}}$. We begin with the simplest case – that of $j = 1$:

10.1. Multiplying by $h_1$

**Proposition 10.2.** Let $\lambda \in P_{k,n}$. Assume that $k > 0$.

(a) If $\lambda_1 < n - k$, then

$$s_\lambda h_1 = \sum_{\mu \in P_{k,n}; \mu/\lambda \text{ is a single box}} s_\mu.$$

(b) Let $\lambda$ be the partition $(\lambda_2, \lambda_3, \lambda_4, \ldots)$. If $\lambda_1 = n - k$, then

$$s_\lambda h_1 = \sum_{\mu \in P_{k,n}; \mu/\lambda \text{ is a single box}} s_\mu + \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\mu \in P_{k,n}; \lambda/\mu \text{ is a vertical } i\text{-strip}} s_\mu.$$

**Proof of Proposition 10.2** We have $h_1 = e_1$, thus

$$s_\lambda h_1 = s_\lambda e_1 = \sum_{\mu \text{ is a partition; } \mu/\lambda \text{ is a vertical 1-strip}} s_\mu$$

(by Proposition 6.6, applied to $i = 1$). Evaluating both sides of this identity at the $k$ variables $x_1, x_2, \ldots, x_k$, we find

$$s_\lambda h_1 = \sum_{\mu \text{ is a partition; } \mu/\lambda \text{ is a vertical 1-strip}} s_\mu = \sum_{\mu \text{ is a partition; } \mu/\lambda \text{ is a single box}} s_\mu.$$
(because a skew diagram $\mu/\lambda$ is a vertical 1-strip if and only if it is a single box). This becomes

$$s_\lambda h_1 = \sum_{\mu \text{ is a partition; } \mu/\lambda \text{ is a single box}} s_\mu = \sum_{\mu \text{ is a partition; } \mu/\lambda \text{ is a single box; } \mu \text{ has at most } k \text{ parts}} s_\mu + \sum_{\mu \text{ is a partition; } \mu/\lambda \text{ is a single box; } \mu \text{ has more than } k \text{ parts}} s_\mu \ (\text{by } (8))$$

(81)

(a) Assume that $\lambda_1 < n-k$. Then, each partition $\mu$ satisfying

$$(\mu/\lambda \text{ is a single box}) \land (\mu \text{ has at most } k \text{ parts})$$  

must satisfy

$$\mu \in P_{k,n}. \tag{83}$$

[Proof of (83): Let $\mu$ be a partition satisfying (82). We must prove that $\mu \in P_{k,n}$. We have $\mu_1 \leq \lambda_1 + 1$ (since $\mu/\lambda$ is a single box) and thus $\mu_1 \leq \lambda_1 + 1 \leq n-k$ (since $\lambda_1 < n-k$). Hence, each part of $\mu$ is $\leq n-k$ (since $\mu$ is a partition). Thus, $\mu \in P_{k,n}$ (since $\mu$ has at most $k$ parts). This proves (83).]

Now, (81) becomes

$$s_\lambda h_1 = \sum_{\mu \text{ is a partition; } \mu/\lambda \text{ is a single box; } \mu \text{ has at most } k \text{ parts}} s_\mu = \sum_{\mu \in P_{k,n}; \mu/\lambda \text{ is a single box}} s_\mu$$

(b) Assume that $\lambda_1 = n-k$. Let $\nu$ be the partition $(\lambda_1 + 1, \lambda_2, \lambda_3, \ldots)$. Then, $\nu/\lambda$ is a single box, which lies in the first row. The definition of $\nu$ yields $\nu_1 = \lambda_1 + 1 = n-k+1$ (since $\lambda_1 = n-k$) and thus $\nu_1 > n-k$; hence, not all parts of $\nu$ are $\leq n-k$. Thus, $\nu \notin P_{k,n}$. 

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Clearly, \( \lambda \in P_{k,n} \). Hence, if \( i \in \mathbb{N} \), and if \( \mu \) is any partition such that \( \lambda / \mu \) is a vertical \( i \)-strip, then \( \mu \in P_{k,n} \) (since \( \mu \subseteq \lambda \)). Thus, for each \( i \in \mathbb{N} \), we have the following equality of summation signs:

\[
\sum_{\mu \text{ is a partition}; \lambda / \mu \text{ is a vertical } i\text{-strip}} = \sum_{\mu \in P_{k,n}; \lambda / \mu \text{ is a vertical } i\text{-strip}}.
\] (84)

The partition \( \nu \) has at most \( k \) parts (since \( \lambda \) has at most \( k \) parts, and since \( k > 0 \)). The definition of \( \nu \) yields \( \nu_1 = \lambda_1 + 1 = n - k + 1 \) (since \( \lambda_1 = n - k \)) and \( (\nu_2, \nu_3, \nu_4, \ldots) = (\lambda_2, \lambda_3, \lambda_4, \ldots) = \lambda \). Hence, Lemma 6.13 (applied to \( \nu \) and \( \nu_i \) instead of \( \lambda \) and \( \lambda_i \)) yields

\[
\overline{s}_\nu = k - 1 \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\mu \text{ is a partition}; \lambda / \mu \text{ is a vertical } i\text{-strip}} \overline{s}_\mu = k - 1 \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\mu \in P_{k,n}; \lambda / \mu \text{ is a vertical } i\text{-strip}} \overline{s}_\mu
\] (by (84)).

Each partition \( \mu \) satisfying

\[
(\mu / \lambda \text{ is a single box}) \land (\mu \text{ has at most } k \text{ parts}) \land (\mu \neq \nu)
\] (86)

must satisfy

\[
\mu \in P_{k,n}.
\] (87)

[Proof of (87): Let \( \mu \) be a partition satisfying (86). We must prove that \( \mu \in P_{k,n} \). We know that \( \mu / \lambda \) is a single box. If we had \( \mu_1 > \lambda_1 \), then this box would lie in the first row, which would yield that \( \mu = \nu \) (because \( \nu \) is the partition obtained from \( \lambda \) by adding a box in the first row); but this would contradict \( \mu \neq \nu \). Hence, we cannot have \( \mu_1 > \lambda_1 \). Thus, we have \( \mu_1 \leq \lambda_1 = n - k \). Hence, each part of \( \mu \) is \( \leq n - k \) (since \( \mu \) is a partition). Thus, \( \mu \in P_{k,n} \) (since \( \mu \) has at most \( k \) parts). This proves (87).]

Conversely, each \( \mu \in P_{k,n} \) satisfies \( \mu \neq \nu \) (because \( \nu \notin P_{k,n} \)) and has at most \( k \) parts. Combining this with (87), we obtain the following equality of summation signs:

\[
\sum_{\mu \text{ is a partition}; \mu / \lambda \text{ is a single box}; \mu \text{ has at most } k \text{ parts}; \mu \neq \nu} = \sum_{\mu \in P_{k,n}; \mu / \lambda \text{ is a single box}}.
\] (88)
Now, (81) becomes
\[ s_\lambda h_1 = \sum_{\mu \text{ is a partition; } \mu/\lambda \text{ is a single box; } \mu \text{ has at most } k \text{ parts} } s_\mu = s_v + \sum_{\mu \text{ is a partition; } \mu/\lambda \text{ is a single box; } \mu \text{ has at most } k \text{ parts} } s_\mu \]

(here, we have split off the addend for \( \mu = v \) from the sum)
\[ = s_v + \sum_{\mu \in P_{k,n}; } s_\mu \]
(by (88)).

Projecting both sides of this equality onto \( S/I \), we obtain
\[ \overline{s_\lambda h_1} = s_v + \sum_{\mu \text{ is a single box; } \mu/\lambda \text{ is a single box} } s_\mu = \overline{s_v} + \sum_{\mu \text{ is a single box; } \mu/\lambda \text{ is a single box} } \overline{s_\mu} + \overline{s_v} \]

\[ = \sum_{\mu \in P_{k,n}; } \overline{s_\mu} + \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\mu \in P_{k,n}; } \overline{s_\mu} \]
(\( \lambda/\mu \) is a vertical \( i \)-strip)

(by (85)). This proves Proposition 10.2 (b).

10.2. Multiplying by \( h_{n-k} \)

On the other end of the spectrum is the case of \( j = n - k \); this case also turns out to have a simple answer:

**Proposition 10.3.** Let \( \lambda \in P_{k,n} \). Assume that \( k > 0 \).

(a) We have
\[ \overline{s_\lambda h_{n-k}} = \overline{s_{(n-k,\lambda_1,\lambda_2,\ldots)}} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\mu \in P_{k,n}; } \overline{s_\mu}, \]
(\( \lambda/\mu \) is a vertical \( i \)-strip)

(b) If \( \lambda_k > 0 \), then
\[ \overline{s_\lambda h_{n-k}} = -\sum_{i=1}^{k} (-1)^i a_i \sum_{\mu \in P_{k,n}; } \overline{s_\mu}, \]
(\( \lambda/\mu \) is a vertical \( i \)-strip)

**Proof of Proposition 10.3** We have \( \lambda \in P_{k,n} \), thus \( \lambda_1 \leq n - k \). Hence, \( n - k \geq \lambda_1 \). Thus, \((n - k, \lambda_1, \lambda_2, \lambda_3, \ldots)\) is a partition.

(a) We have
\[ (e_i) \perp s_\lambda = 0 \quad \text{for every integer } i > k. \] (89)
[Proof of (89): Let $i > k$ be an integer. The partition $\lambda$ has at most $k$ parts (since $\lambda \in \mathcal{P}_{k,\mu}$). In other words, the Young diagram of $\lambda$ contains at most $k$ rows. Hence, this diagram contains no vertical $i$-strip (since a vertical $i$-strip would involve more than $k$ rows (because $i > k$)). Thus, there exists no partition $\mu$ such that $\lambda/\mu$ is a vertical $i$-strip. Hence,\[ \sum_{\mu \text{ is a partition; } \lambda/\mu \text{ is a vertical } i\text{-strip}} s_\mu = (\text{empty sum}) = 0. \]

But Corollary 6.7 yields $(e_i)^\perp s_\lambda = \sum_{\mu \text{ is a partition; } \lambda/\mu \text{ is a vertical } i\text{-strip}} s_\mu = 0$. This proves (89).]

Recall that $e_0 = 1$ and thus $(e_0)^\perp = 1^\perp = \text{id}$. Hence, $(e_0)^\perp s_\lambda = \text{id} s_\lambda = s_\lambda$.

But $n - k \geq \lambda_1$. Hence, Proposition 6.8 (applied to $m = n - k$) yields

\[
\sum_{i \in \mathbb{N}} (-1)^i h_{n-k+i} (e_i)^\perp s_\lambda = s_{(n-k,\lambda_1,\lambda_2,\lambda_3,...)}.
\]

Hence,

\[
s_{(n-k,\lambda_1,\lambda_2,\lambda_3,...)} = \sum_{i \in \mathbb{N}} (-1)^i h_{n-k+i} (e_i)^\perp s_\lambda
\]

\[
= \sum_{i=0}^{k} (-1)^i h_{n-k+i} (e_i)^\perp s_\lambda + \sum_{i=k+1}^{\infty} (-1)^i h_{n-k+i} (e_i)^\perp s_\lambda = 0
\]

(by (89))

\[
= \sum_{i=0}^{k} (-1)^i h_{n-k+i} (e_i)^\perp s_\lambda
\]

\[
= (-1)^0 h_{n-k} (e_0)^\perp s_\lambda + \sum_{i=1}^{k} (-1)^i h_{n-k+i} (e_i)^\perp s_\lambda
\]

(by Corollary 6.7)

\[
= h_{n-k} s_\lambda + \sum_{i=1}^{k} (-1)^i h_{n-k+i} s_\lambda
\]

(by Corollary 6.7)

\[
= s_\lambda h_{n-k} + \sum_{i=1}^{k} (-1)^i h_{n-k+i} s_\lambda
\]

(by Corollary 6.7)

so that

\[
s_\lambda h_{n-k} = s_{(n-k,\lambda_1,\lambda_2,\lambda_3,...)} - \sum_{i=1}^{k} (-1)^i h_{n-k+i} s_\lambda.
\]

This is an equality in $\Lambda$. If we evaluate both of its sides at $x_1, x_2, \ldots, x_k$, then we
obtain
\[ s_{\lambda} h_{n-k} = s_{(n-k, \lambda_1, \lambda_2, \lambda_3, \ldots)} - \sum_{i=1}^{k} (-1)^i h_{n-k+i} \equiv \sum_{\mu \text{ is a partition;}}^{\lambda / \mu \text{ is a vertical } i\text{-strip}} s_{\mu} \]
(by (15))
\[ \equiv s_{(n-k, \lambda_1, \lambda_2, \lambda_3, \ldots)} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\mu \in P_{k,n};}^{\lambda / \mu \text{ is a vertical } i\text{-strip}} s_{\mu} \mod I. \]

In other words,
\[ s_{\lambda} h_{n-k} \equiv s_{(n-k, \lambda_1, \lambda_2, \lambda_3, \ldots)} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\mu \in P_{k,n};}^{\lambda / \mu \text{ is a vertical } i\text{-strip}} s_{\mu} \mod I. \]

This proves Proposition 10.3 (a).

(b) Assume that \( \lambda_k > 0 \). Hence, the partition \((n-k, \lambda_1, \lambda_2, \lambda_3, \ldots)\) has more than \(k\) parts (since its \((k+1)\)-st entry is \(\lambda_k > 0\)). Thus, (3) (applied to \((n-k, \lambda_1, \lambda_2, \lambda_3, \ldots)\) instead of \(\lambda\)) yields \(s_{(n-k, \lambda_1, \lambda_2, \lambda_3, \ldots)} = 0\). Hence, \(s_{(n-k, \lambda_1, \lambda_2, \lambda_3, \ldots)} = 0 = 0\). Now, Proposition 10.3 (a) yields
\[ s_{\lambda} h_{n-k} = \sum_{i=1}^{k} (-1)^i a_i \sum_{\mu \in P_{k,n};}^{\lambda / \mu \text{ is a vertical } i\text{-strip}} s_{\mu} \]
\[ = - \sum_{i=1}^{k} (-1)^i a_i \sum_{\mu \in P_{k,n};}^{\lambda / \mu \text{ is a vertical } i\text{-strip}} s_{\mu}. \]

This proves Proposition 10.3 (b). \(\square\)

10.3. Multiplying by \(\overline{h}_j\)

At last, let us give an explicit expansion for \(s_{\lambda} h_{n-k}\) in the basis \(\{s_{\mu}\}_{\mu \in P_{k,n}}\) that holds for all \(j \in \{0, 1, \ldots, n-k\}\). Before we state it, we need a notation:
Definition 10.4. Let \( f \in \Lambda \) be any symmetric function. Then, \( \overline{f} \in S/I \) is defined to be \( \overline{f} \), where \( f \in S \) is the result of evaluating the symmetric function \( f \in \Lambda \) at the \( k \) variables \( x_1, x_2, \ldots, x_k \). Thus, for every partition \( \lambda \), we have \( \overline{s_\lambda} = \overline{s_\lambda} \). Likewise, for any \( m \in \mathbb{N} \), we have \( \overline{h_m} = h_m \) and \( \overline{e_m} = e_m \).

Theorem 10.5. Let \( \lambda \in P_{k,n} \). Let \( j \in \{0,1,\ldots,n-k\} \). Then,

\[
\overline{s_\lambda h_j} = \sum_{\mu/\lambda \text{ is a horizontal } j\text{-strip}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \left( s_{(n-k-j+1,i-1)} \right) \perp s_\lambda.
\]

Example 10.6. If \( n = 7 \) and \( k = 3 \), then

\[
\overline{s_{(4,3,2)} h_2} = \overline{s_{(4,4,3)}} + a_1 \left( \overline{s_{(4,2)}} + \overline{s_{(3,2,1)}} + \overline{s_{(3,3)}} \right) - a_2 \left( \overline{s_{(4,1)}} + \overline{s_{(2,2,1)}} + \overline{s_{(3,1,1)}} + 2\overline{s_{(3,2)}} \right) \\
+ a_3 \left( \overline{s_{(2,2)}} + \overline{s_{(2,1,1)}} + \overline{s_{(3,1)}} \right).
\]

It is not hard to reveal Propositions 10.2 and 10.3 as particular cases of Theorem 10.5 (by setting \( j = 1 \) or \( j = n - k \), respectively). Likewise, one can see that Theorem 10.5 generalizes \[BeCiFu99, (22)\]. Indeed, \[BeCiFu99, (22)\] says that if \( a_1 = a_2 = \cdots = a_{k-1} = 0 \), then every \( \lambda \in P_{k,n} \) and \( j \in \{0,1,\ldots,n-k\} \) satisfy

\[
\overline{s_\lambda h_j} = \sum_{\mu/\lambda \text{ is a horizontal } j\text{-strip}} \overline{s_\mu} - \sum_{\nu} (-1)^k a_k \sum_{\nu} \overline{s_\nu},
\]

where the second sum runs over all \( \nu \in P_{k,n} \) satisfying

\[
(\lambda_i - 1 \geq \nu_i \text{ for all } i \in \{1,2,\ldots,k\}) \quad \text{and} \quad (\nu_i \geq \lambda_{i+1} - 1 \text{ for all } i \in \{1,2,\ldots,k-1\}) \quad \text{and} \quad |\nu| = |\lambda| + j - n.
\]

Note, however, that the sums in Theorem 10.5 contain multiplicities (see the “\( 2\overline{s_{(3,2)}} \)” in Example 10.6), unlike those in \[BeCiFu99, (22)\].

We shall prove Theorem 10.5 by deriving it from an identity between genuine symmetric functions (in \( \Lambda \), not in \( S \) or \( S/I \)):

Theorem 10.7. Let \( \lambda \in P_{k,n} \). Let \( j \in \{0,1,\ldots,n-k\} \). Then,

\[
s_\lambda h_j = \sum_{\mu \text{ is a partition}; \mu \leq n-k; \mu/\lambda \text{ is a horizontal } j\text{-strip}} s_\mu - \sum_{i=1}^{k} (-1)^i h_{n-k+i} \left( s_{(n-k-j+1,i-1)} \right) \perp s_\lambda.
\]
Before we prove this theorem, we need several auxiliary results. First, we recall one of the Pieri rules ([GriRei18, (2.7.1)]):

**Proposition 10.8.** Let $\lambda$ be a partition, and let $i \in \mathbb{N}$. Then,

$$s_\lambda h_i = \sum_{\mu \text{ is a partition}; \mu/\lambda \text{ is a horizontal } i\text{-strip}} s_\mu.$$

From this, we can easily derive the following:

**Corollary 10.9.** Let $\lambda$ be a partition, and let $i \in \mathbb{N}$. Then,

$$(h_i) \perp s_\lambda = \sum_{\mu \text{ is a partition}; \mu/\lambda \text{ is a horizontal } i\text{-strip}} s_\mu.$$

Corollary 10.9 is also proven in [GriRei18, version with solutions (ancillary file), Lemma 12.83.3(a)].

Next, let us show some further lemmas:

**Lemma 10.10.** Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \ldots, n - k\}$. Let $g$ be a positive integer. Then,

$$\sum_{\mu \text{ is a partition}; \mu_1 = n - k + g; \mu/\lambda \text{ is a horizontal } j\text{-strip}} s_\mu = \sum_{w \geq 1} (-1)^{w-g} h_{n-k+w} (h_{n-k+g-j} e_{w-g}) \perp s_\lambda.$$

*Proof of Lemma 10.10 (sketched).* First, we observe that $\lambda_1 \leq n - k$ (since $\lambda \in P_{k,n}$). Now, every partition $\mu$ satisfying $\mu_1 = n - k + g$ must automatically satisfy $\mu_1 \geq \lambda_1$ (because $\mu_1 = n - k + g \geq n - k \geq \lambda_1$).

Let $A$ be the set of all partitions $\mu$ such that $\mu_1 = n - k + g$ and such that $\mu/\lambda$ is a horizontal $j$-strip. Let $B$ be the set of all partitions $\nu$ such that $\lambda/\nu$ is a horizontal $(n - k + g - j)$-strip. Then,

$$A = \{ \mu \text{ is a partition } | \mu_1 = n - k + g \text{ and } |\mu| - |\lambda| = j \text{ and } \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \mu_3 \geq \lambda_3 \geq \cdots \}$$

$$= \{ \mu \text{ is a partition } | \mu_1 = n - k + g \text{ and } |\mu| - |\lambda| = j \text{ and } \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \mu_3 \geq \lambda_3 \geq \mu_4 \geq \cdots \}$$

21 We are using Definition 6.2 (c) here.
(since every partition \( \mu \) satisfying \( \mu_1 = n - k + g \) must automatically satisfy \( \mu_1 \geq \lambda_1 \)) and

\[
B = \{ \nu \text{ is a partition} \mid |\lambda| - |\nu| = n - k - j \quad \text{and} \quad \lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \lambda_3 \geq \nu_3 \geq \cdots \}. 
\]

Hence, it is easy to check that the map

\[
B \rightarrow A, \quad \nu \mapsto (n - k + g, \nu_1, \nu_2, \nu_3, \ldots)
\]

is well-defined (because every \( \nu \in B \) satisfies \( \lambda_1 \geq \nu_1 \) and thus \( n - k + g \geq 0 \)) and is a bijection (its inverse map just sends each \( \mu \in A \) to \( (\mu_2, \mu_3, \mu_4, \ldots) \in B \)). Thus, we can substitute \( (n - k + g, \nu_1, \nu_2, \nu_3, \ldots) \) for \( \mu \) in the sum \( \sum_{\mu \in A} s_\mu \). We thus obtain

\[
\sum_{\mu \in A} s_\mu = \sum_{\nu \in B} s_{(n-k+g,\nu_1,\nu_2,\nu_3,\ldots)}. \tag{90}
\]

But each \( \nu \in B \) satisfies \( n - k + g \geq 0 \geq n - k \geq \lambda_1 \geq \nu_1 \) and thus

\[
\sum_{i \in \mathbb{N}} (-1)^i h_{n-k+g+i}(e_i)^\perp s_\nu = s_{(n-k+g,\nu_1,\nu_2,\nu_3,\ldots)} \tag{91}
\]

(by Proposition 6.8, applied to \( \nu \) and \( n - k + g \) instead of \( \lambda \) and \( m \)). Hence, (90) becomes

\[
\sum_{\mu \in A} s_\mu = \sum_{\nu \in B} s_{(n-k+g,\nu_1,\nu_2,\nu_3,\ldots)} = \sum_{\nu \in B} \sum_{i \in \mathbb{N}} (-1)^i h_{n-k+g+i}(e_i)^\perp s_\nu
\]

(by (91))

\[
= \sum_{i \in \mathbb{N}} (-1)^i h_{n-k+g+i}(e_i)^\perp \left( \sum_{\nu \in B} s_\nu \right). \tag{92}
\]

But Corollary 10.9 (applied to \( i = n - k + g - j \)) yields\(^{22}\)

\[
(h_{n-k+g-j})^\perp s_\lambda = \sum_{\mu \text{ is a partition}; \lambda/\mu \text{ is a horizontal } (n-k+g-j)-\text{strip}} s_\mu = \sum_{\mu \in B} s_\mu
\]

(by the definition of \( B \))

\[
= \sum_{\nu \in B} s_\nu. \tag{93}
\]

\(^{22}\)More precisely: This follows from Corollary 10.9 (applied to \( i = n - k + g - j \)) when \( n - k + g - j \in \mathbb{N} \). But otherwise, it is obvious for trivial reasons (\( 0 = 0 \).
Hence, (92) becomes

\[
\sum_{\mu \in A} s_{\mu} = \sum_{i \in \mathbb{N}} (-1)^i h_{n-k+g+i} (e_i) \left( \sum_{v \in B} s_v \right) = (h_{n-k+g-1})^\perp \ s_\lambda
\]

\[
= \sum_{i \in \mathbb{N}} (-1)^i h_{n-k+g+i} \left( e_i \right) \left( h_{n-k+g-j} \right) \left( s_\lambda \right) = (e_i)^\perp \circ (h_{n-k+g-j})^\perp \ s_\lambda
\]

\[
= \sum_{i \in \mathbb{N}} (-1)^i h_{n-k+g+i} \left( h_{n-k+g-j} e_i \right) \left( s_\lambda \right)
\]

\[
= \sum_{w \geq g} \left( -1 \right)^{w-g} h_{n-k+w} \left( h_{n-k+g-j} e_{w-g} \right) \left( s_\lambda \right)
\]

(here, we have substituted \( w - g \) for \( i \) in the sum).

Comparing this with

\[
\sum_{w \geq 1} \left( -1 \right)^{w-g} h_{n-k+w} \left( h_{n-k+g-j} e_{w-g} \right) \left( s_\lambda \right)
\]

\[
= \sum_{w=1}^{g-1} \left( -1 \right)^{w-g} h_{n-k+w} \left( h_{n-k+g-j} e_{w-g} \right) \left( s_\lambda \right)
\]

\[
+ \sum_{w \geq g} \left( -1 \right)^{w-g} h_{n-k+w} \left( h_{n-k+g-j} e_{w-g} \right) \left( s_\lambda \right)
\]

(since \( g \) is a positive integer)

\[
= \sum_{w=1}^{g-1} \left( -1 \right)^{w-g} h_{n-k+w} \left( h_{n-k+g-j} e_{w-g} \right) \left( s_\lambda \right) + \sum_{w \geq g} \left( -1 \right)^{w-g} h_{n-k+w} \left( h_{n-k+g-j} e_{w-g} \right) \left( s_\lambda \right)
\]

\[
= \sum_{w \geq g} \left( -1 \right)^{w-g} h_{n-k+w} \left( h_{n-k+g-j} e_{w-g} \right) \left( s_\lambda \right)
\]
we obtain
\[ \sum_{\mu \in A} s_{\mu} = \sum_{w \geq 1} (-1)^{w-g} h_{n-k+w} (h_{n-k+g-j}^{e_{w-g}^j})^\perp s_{\lambda}. \] (94)

In view of
\[ \sum_{\mu \in A} = \sum_{\substack{\mu \text{ is a partition;} \\mu_1=n-k+g; \\mu/\lambda \text{ is a horizontal } j\text{-strip}}} \]

this rewrites as
\[ \sum_{\substack{\mu \text{ is a partition;} \\mu_1=n-k+g; \\mu/\lambda \text{ is a horizontal } j\text{-strip}}} s_{\mu} = \sum_{w \geq 1} (-1)^{w-g} h_{n-k+w} (h_{n-k+g-j}^{e_{w-g}^j})^\perp s_{\lambda}. \]

This proves Lemma \[ \text{10.10} \]

Our next lemma will be a slight generalization of Lemma \[ \text{7.17} \] but first we extend our definition of \( s_{(m,1)} \):

**Convention 10.11.** Let \( m \in \mathbb{N} \), and let \( j \) be a negative integer. Then, we shall understand the (otherwise undefined) expression \( s_{(m,1)} \) to mean \( 0 \in \Lambda \).

We can now generalize Lemma \[ \text{7.17} \] as follows:

**Lemma 10.12.** Let \( m \) be a positive integer. Let \( j \in \mathbb{Z} \) be such that \( m+j > 0 \). Then,
\[ s_{(m,1)} = \sum_{i=1}^{m} (-1)^{i-1} h_{m-i} e_{j+i}. \]

**Proof of Lemma 10.12** If \( j \in \mathbb{N} \), then this follows directly from Lemma \[ \text{7.17} \]. Hence, for the rest of this proof, we WLOG assume that \( j \notin \mathbb{N} \). Hence, \( j < 0 \).

Now, the proof of Lemma \[ \text{10.12} \] is the same as our above proof of Lemma \[ \text{7.17} \] with two changes:

- The inequality \( m+j > 0 \) no longer follows from \( m > 0 \) and \( j \geq 0 \), but rather comes straight from the assumptions.

- The equality \( \sum_{i=0}^{j} (-1)^i h_{m+i} e_{j-i} = s_{(m,1)} \) no longer follows from (40), but rather comes from comparing \( \sum_{i=0}^{j} (-1)^i h_{m+i} e_{j-i} = (\text{empty sum}) = 0 \) with \( s_{(m,1)} = 0 \).

Thus, Lemma \[ \text{10.12} \] is proven.
Lemma 10.13. Let \( j \in \{0, 1, \ldots, n-k\} \), and let \( w \) be a positive integer. Then,
\[
\sum_{g=1}^{j} (-1)^{w-g} h_{n-k+g-j} e_{w-g} = (-1)^{w-j} s_{(n-k+1, w-j-1)} - (-1)^{w} s_{(n-k+j+1, w-1)}.
\]

Proof of Lemma 10.13. From \( j \in \{0, 1, \ldots, n-k\} \), we obtain \( 0 \leq j \leq n-k \leq n-k+1 \).

We have \( n \geq k+1 \geq k-k+1 = 1 \); thus, \( n-k+1 \) is a positive integer. Also,
\[
(n-k+1) + \left( w - j - 1 \right) \geq (n-k+1) + (w - (n-k) - 1) = w > 0 \text{ (since } w \text{ is a positive integer)}.
\]

Hence, Lemma 10.12 (applied to \( n-k+1 \) and \( w-j-1 \) instead of \( m \) and \( j \)) yields
\[
s_{(n-k+1, w-j-1)} = \sum_{i=1}^{n-k+1} (-1)^{i-1} h_{n-k+1-i} e_{w-j-1+i}
\]
\[
= \sum_{i=1}^{j} (-1)^{i-1} h_{n-k+1-i} e_{w-j-1+i} + \sum_{i=j+1}^{n-k+1} (-1)^{i-1} h_{n-k+1-i} e_{w-j-1+i}
\]
(95)

(since \( 0 \leq j \leq n-k + 1 \)). Also, \( n-k-\underbrace{j+1}_{\leq n-k} \geq n-k-(n-k) + 1 = 1 \); thus, \( n-k-j+1 \) is a positive integer. Also,
\[
(n-k-\underbrace{j+1}_{\leq n-k}) + (w-1) \geq (n-k-(n-k)+1) + (w-1) = w > 0.
\]

Hence, Lemma 7.17 (applied to \( n-k-j+1 \) and \( w-1 \) instead of \( m \) and \( j \)) yields
\[
s_{(n-k-j+1, w-1)} = \sum_{i=1}^{n-k-j+1} (-1)^{i-1} h_{n-k-j+1-i} e_{w-1+i}
\]
\[
= \sum_{i=j+1}^{n-k+1} \frac{(-1)^{i-j-1}}{(-1)^{i-1}} h_{n-k-j+1-(i-j)} e_{w-1+i-j} = h_{n-k+1-i} e_{w-j-1+i}
\]
(here, we have substituted \( i-j \) for \( i \) in the sum)
\[
= (-1)^{j} \sum_{i=j+1}^{n-k+1} (-1)^{i-1} h_{n-k+1-i} e_{w-j-1+i}.
\]
We have Proof of Theorem 10.7.

Multiplying this equality by \((-1)^j\), we find

\[
(-1)^j s_{(n-k-j+1,1^{w-1})} = \sum_{i=j+1}^{n-k+1} (-1)^{i-1} h_{n-k+1-i} e_{w-j+1+i}.
\]

Subtracting this equality from (95), we obtain

\[
s_{(n-k+1,1^{w-j-1})} - (-1)^j s_{(n-k-j+1,1^{w-1})} = \left(\sum_{i=1}^{j} (-1)^{i-1} h_{n-k+1-i} e_{w-j+1+i} + \sum_{i=j+1}^{n-k+1} (-1)^{i-1} h_{n-k+1-i} e_{w-j+1+i}\right)
\]

\[
- \sum_{i=j+1}^{n-k+1} (-1)^{i-1} h_{n-k+1-i} e_{w-j+1+i}
\]

\[
= \sum_{i=1}^{j} (-1)^{i-1} h_{n-k+1-i} e_{w-j+1+i}. \tag{96}
\]

On the other hand,

\[
\sum_{g=1}^{j} (-1)^{w-g} h_{n-k+1-g} e_{w-g}
\]

\[
= \sum_{i=1}^{j} (-1)^{w-(j+1-i)} h_{n-k+(j+1-i)-i} e_{w-(j+1-i)}
\]

\[
= (-1)^{w-j} \sum_{i=1}^{j} (-1)^{i-1} h_{n-k+1-i} e_{w-j+1+i}
\]

\[
= s_{(n-k+1,1^{w-j-1})} - (-1)^j s_{(n-k-j+1,1^{w-1})} \tag{by 96}
\]

\[
= (-1)^{w-j} s_{(n-k+1,1^{w-j-1})} - (-1)^{w-j} (-1)^j s_{(n-k-j+1,1^{w-1})}
\]

\[
= (-1)^{w-j} s_{(n-k+1,1^{w-j-1})} - (-1)^{w-j} s_{(n-k-j+1,1^{w-1})}
\]

\[
= (-1)^{w-j} s_{(n-k+1,1^{w-j-1})} - (-1)^w s_{(n-k-j+1,1^{w-1})}.\]

This proves Lemma 10.13

\[\square\]

**Proof of Theorem 10.7** We have \(j \in \{0, 1, \ldots, n - k\}\), thus \(0 \leq j \leq n - k\). Also, we have \(\lambda \in P_{k,n}\); thus, the partition \(\lambda\) has at most \(k\) parts and satisfies \(\lambda_1 \leq n - k\).
Let $g$ be an integer such that $g \geq j + 1$. If $\mu$ is a partition such that $\mu/\lambda$ is a horizontal $j$-strip, then $\mu_1 \leq \lambda_1 + j \leq n - k + j < n - k + g$ and thus $\mu_1 \neq n - k + g$. Thus, there exists no partition $\mu$ such that $\mu_1 = n - k + g$ and such that $\mu/\lambda$ is a horizontal $j$-strip. Hence,

$$\sum_{\mu \text{ is a partition; } \mu_1 = n - k + g; \mu/\lambda \text{ is a horizontal } j\text{-strip}} s_{\mu} = (\text{empty sum}) = 0. \quad (97)$$

Now, forget that we fixed $g$. We thus have proven the equality (97) for every integer $g$ satisfying $g \geq j + 1$.

On the other hand, let $g \in \{1, 2, \ldots, j\}$. Thus, $g \leq j \leq n - k$. If $w$ is an integer satisfying $w \geq n + 1$, then $w - g \geq (n + 1) - (n - k) = k + 1 > k$, and thus the partition $(1^w - g)$ does not satisfy $(1^w - g) \subseteq \lambda$ (because the partition $\lambda$ has at most $k$ parts, whereas the partition $(1^w - g)$ has $w - g > k$ parts), and therefore we have

$$\begin{pmatrix} e_{1^w - g} \\ e_{n + 1 - w} \end{pmatrix}_{(1^w - g)} \perp (s_{1^w - g}) = (s_{1^w - g}) \perp (s_\lambda) = s_{\lambda/1^w - g} \quad (by \ (23))$$

$$= 0 \quad (since \ we \ don’t \ have \ (1^w - g) \subseteq \lambda). \quad (98)$$

Hence, if $w$ is an integer satisfying $w \geq n + 1$, then

$$\begin{pmatrix} h_{n - k + g - j} \\ e_{1^w - g} \end{pmatrix}_{(1^w - g)} \perp s_\lambda = \left( e_{1^w - g} h_{n - k + g - j} \right) \perp s_\lambda = \left( h_{n - k + g - j} \right) \perp \left( e_{1^w - g} \right) \perp (s_\lambda)$$

$$= \left( h_{n - k + g - j} \right) \perp \left( e_{1^w - g} \right) \perp (s_\lambda) = 0. \quad (99)$$

(by 98)
Now,

\[
\sum_{\mu \text{ is a partition}; \mu_1 = n - k + g; \mu / \lambda \text{ is a horizontal } j\text{-strip}} s_\mu = \sum_{w \geq 1} (-1)^{w-g} h_{n-k+w} (h_{n-k+g-j} e_{w-g}) \perp s_\lambda
\]

(by Lemma 10.10)

\[
= \sum_{w=1}^{n} (-1)^{w-g} h_{n-k+w} (h_{n-k+g-j} e_{w-g}) \perp s_\lambda
\]

\[
+ \sum_{w \geq n+1} (-1)^{w-g} h_{n-k+w} (h_{n-k+g-j} e_{w-g}) \perp s_\lambda
\] (by \ref{eq:lemma10.10})

\[
= \sum_{w=1}^{n} (-1)^{w-g} h_{n-k+w} (h_{n-k+g-j} e_{w-g}) \perp s_\lambda. \tag{100}
\]

Now, forget that we fixed \(g\). We thus have proven the equality (100) for each \(g \in \{1, 2, \ldots, j\}\).

Proposition 10.8 (applied to \(i = j\)) yields

\[
s_\lambda h_j = \sum_{\mu \text{ is a partition}; \mu / \lambda \text{ is a horizontal } j\text{-strip}} s_\mu = \sum_{\mu \text{ is a partition}; \mu_1 = n - k; \mu / \lambda \text{ is a horizontal } j\text{-strip}} s_\mu + \sum_{\mu \text{ is a partition}; \mu_1 > n - k; \mu / \lambda \text{ is a horizontal } j\text{-strip}} s_\mu
\]
(since each partition $\mu$ satisfies either $\mu_1 \leq n - k$ or $\mu_1 > n - k$). Hence,

\[
\begin{align*}
    s_\lambda h_j &= \sum_{\mu \text{ is a partition; } \mu_1 \leq n - k; \mu/\lambda \text{ is a horizontal } j\text{-strip}} s_\mu \\
    &= \sum_{\mu \text{ is a partition; } \mu_1 \leq n - k; \mu/\lambda \text{ is a horizontal } j\text{-strip}} s_\mu \\
    &= \sum_{\mu \text{ is a partition; } \mu_1 = n - k + g; \mu/\lambda \text{ is a horizontal } j\text{-strip}} s_\mu \\
    &= \sum_{\mu \text{ is a partition; } \mu_1 = n - k + g; \mu/\lambda \text{ is a horizontal } j\text{-strip}} s_\mu \\
    &= \sum_{g=1}^{j} \sum_{w=1}^{n} (-1)^{w-g} h_{n-k+w} \left(h_{n-k+g-j} e^{w-g}\right)^{\perp} s_\lambda \\
    &= \sum_{w=1}^{n} \sum_{g=1}^{j} (-1)^{w-g} h_{n-k+w} \left(h_{n-k+g-j} e^{w-g}\right)^{\perp} s_\lambda \\
    &= \sum_{w=1}^{n} \sum_{g=1}^{j} (-1)^{w-g} h_{n-k+w} \left(h_{n-k+g-j} e^{w-g}\right)^{\perp} s_\lambda \\
    &= \sum_{w=1}^{n} h_{n-k+w} \left(\sum_{g=1}^{j} (-1)^{w-g} h_{n-k+g-j} e^{w-g}\right) s_\lambda \\
    &= \sum_{w=1}^{n} h_{n-k+w} \left((-1)^{w-j} s_{n-k+1,1^w-j-1} - (-1)^{w} s_{n-k-j+1,1^w-1}\right) s_\lambda \\
    &= \sum_{w=1}^{n} h_{n-k+w} \left((-1)^{w-j} s_{n-k+1,1^w-j-1} - (-1)^{w} s_{n-k-j+1,1^w-1}\right) s_\lambda \\
    &= \sum_{w=1}^{n} h_{n-k+w} \left((-1)^{w-j} s_{n-k+1,1^w-j-1}\right) s_\lambda \\
    &- \sum_{w=1}^{n} h_{n-k+w} \left((-1)^{w} s_{n-k-j+1,1^w-1}\right) s_\lambda.
\end{align*}
\]
Next, we claim that
\[
\left( \mathbf{s}_{(n-k+1,1^w-1)} \right) \perp \mathbf{s}_\lambda = 0 \quad \text{for each } w \in \{1, 2, \ldots, n\}. \quad (102)
\]

[Proof of (102): Let \( w \in \{1, 2, \ldots, n\} \). If \( w - j - 1 \) is a negative integer, then \( \mathbf{s}_{(n-k+1,1^w-1)} = 0 \) (by Convention 10.11), and thus (102) holds in this case. Hence, for the rest of this proof of (102), we WLOG assume that \( w - j - 1 \) is not a negative integer. Thus, \( w - j - 1 \in \mathbb{N} \). Now, the partition \( (n-k+1,1^w-1) \) has a bigger first entry than the partition \( \lambda \) (since its first entry is \( n-k+1 > n-k \geq \lambda_1 \)). Thus, we do not have \( (n-k+1,1^w-1) \subseteq \lambda \). Hence, \( \mathbf{s}_\lambda / (n-k+1,1^w-1) = 0 \). But (23) yields \( \left( \mathbf{s}_{(n-k+1,1^w-1)} \right) \perp \mathbf{s}_\lambda = \mathbf{s}_\lambda / (n-k+1,1^w-1) = 0. \) This proves (102).]

Next, we claim that
\[
\left( \mathbf{s}_{(n-k-j+1,1^w)} \right) \perp \mathbf{s}_\lambda = 0 \quad \text{for each } w \in \{k+1, k+2, \ldots, n\}. \quad (103)
\]

[Proof of (103): Let \( w \in \{k+1, k+2, \ldots, n\} \). Then, \( w \geq k+1 \). Now, the number of parts of the partition \( (n-k-j+1,1^w) \) is \( 1 + (w-1) = w \geq k+1 > k \), which is bigger than the number of parts of \( \lambda \) (since \( \lambda \) has at most \( k \) parts). Hence, we don’t have \( (n-k-j+1,1^w) \subseteq \lambda \). Thus, \( \mathbf{s}_\lambda / (n-k-j+1,1^w) = 0 \). But (23) yields \( \left( \mathbf{s}_{(n-k-j+1,1^w)} \right) \perp \mathbf{s}_\lambda = \mathbf{s}_\lambda / (n-k-j+1,1^w) = 0. \) This proves (103).]
Now, (101) becomes
\[ s_\lambda h_j - \sum_{\mu \text{ is a partition; } \mu_1 \leq n-k; \mu/\lambda \text{ is a horizontal } j\text{-strip}} \mu \]
\[ = \sum_{w=1}^{n} h_{n-k+w} (-1)^{w-j} \left( s_{(n-k+1,w-j-1)} \right) \perp s_\lambda \]
(by (102))
\[ - \sum_{w=1}^{n} h_{n-k+w} (-1)^{w} \left( s_{(n-k-j+1,w-1)} \right) \perp s_\lambda \]
\[ = - \sum_{w=1}^{n} h_{n-k+w} (-1)^{w} \left( s_{(n-k-j+1,w-1)} \right) \perp s_\lambda \]
\[ = - \left( \sum_{w=1}^{k} h_{n-k+w} (-1)^{w} \left( s_{(n-k-j+1,w-1)} \right) \perp s_\lambda \right) \]
\[ + \sum_{w=k+1}^{n} h_{n-k+w} (-1)^{w} \left( s_{(n-k-j+1,w-1)} \right) \perp s_\lambda \]
(by (103))

(since $0 \leq k \leq n$)
\[ = - \sum_{w=1}^{k} h_{n-k+w} (-1)^{w} \left( s_{(n-k-j+1,w-1)} \right) \perp s_\lambda \]
\[ = \sum_{w=1}^{k} (-1)^{w} h_{n-k+w} \left( s_{(n-k-j+1,w-1)} \right) \perp s_\lambda \]
\[ = - \sum_{i=1}^{k} (-1)^{i} h_{n-k+i} \left( s_{(n-k-j+1,i-1)} \right) \perp s_\lambda \]

(here, we have renamed the summation index $w$ as $i$). Hence,
\[ s_\lambda h_j = \sum_{\mu \text{ is a partition; } \mu_1 \leq n-k; \mu/\lambda \text{ is a horizontal } j\text{-strip}} \mu - \sum_{i=1}^{k} (-1)^{i} h_{n-k+i} \left( s_{(n-k-j+1,i-1)} \right) \perp s_\lambda. \]

This proves Theorem 10.7. \hfill \square

Proof of Theorem 10.5 Theorem 10.7 yields
\[ s_\lambda h_j = \sum_{\mu \text{ is a partition; } \mu_1 \leq n-k; \mu/\lambda \text{ is a horizontal } j\text{-strip}} \mu - \sum_{i=1}^{k} (-1)^{i} h_{n-k+i} \left( s_{(n-k-j+1,i-1)} \right) \perp s_\lambda. \]
Both sides of this equality are symmetric functions in $\Lambda$. If we evaluate them at $x_1, x_2, \ldots, x_k$ and project the resulting symmetric polynomials onto $S/I$, then we obtain

$$\overline{s_\lambda h_j} = \sum_{\mu \text{ is a partition;} \atop \mu_1 \leq n-k; \atop \mu/\lambda \text{ is a horizontal } j\text{-strip}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i \frac{h_{n-k+i}}{a_i} \sum_{\nu_1 \leq n-k; \atop \nu/\lambda \text{ is a horizontal } j\text{-strip}} \overline{s_\nu} \left( s_{(n-k-j+1,1,1^{i-1})} \right)^\perp s_\lambda$$

(since (15) yields $h_{n-k+i} \equiv a_i \mod I$)

$$= \sum_{\mu \text{ is a partition;} \atop \mu_1 \leq n-k; \atop \mu/\lambda \text{ is a horizontal } j\text{-strip}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \left( s_{(n-k-j+1,1^{i-1})} \right)^\perp s_\lambda. \quad (104)$$

But every partition $\mu$ has either at most $k$ parts or more than $k$ parts. Hence,

$$\sum_{\mu \text{ is a partition;} \atop \mu_1 \leq n-k; \atop \mu/\lambda \text{ is a horizontal } j\text{-strip}} \overline{s_\mu} = \sum_{\mu \text{ is a partition;} \atop \mu_1 \leq n-k; \atop \mu \text{ has at most } k \text{ parts;} \atop \mu/\lambda \text{ is a horizontal } j\text{-strip}} \overline{s_\mu} + \sum_{\mu \text{ is a partition;} \atop \mu_1 \leq n-k; \atop \mu \text{ has more than } k \text{ parts;} \atop \mu/\lambda \text{ is a horizontal } j\text{-strip}} \overline{s_\mu} = \sum_{\mu \in P_{k,n}; \atop \mu/\lambda \text{ is a horizontal } j\text{-strip}} \overline{s_\mu}$$

(because the partitions $\mu$ such that $\mu_1 \leq n-k$ and such that $\mu$ has at most $k$ parts are precisely the partitions $\mu \in P_{k,n}$)

Hence, (104) becomes

$$\overline{s_\lambda h_j} = \sum_{\mu \text{ is a partition;} \atop \mu_1 \leq n-k; \atop \mu/\lambda \text{ is a horizontal } j\text{-strip}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i \left( s_{(n-k-j+1,1^{i-1})} \right)^\perp s_\lambda$$

This proves Theorem 10.5.
Let us again use the notation $c_{\alpha,\beta}^{\gamma}$ for a Littlewood–Richardson coefficient (defined as in [GriRei18, Definition 2.5.8], for example). Then, we can restate Theorem 10.5 as follows:

**Theorem 10.14.** Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \ldots, n - k\}$. Then,

$$s_{\lambda} h_j = \sum_{\mu \in P_{k,n}}^\perp s_{\mu} - \sum_{i=1}^{k} \sum_{\nu \subseteq \lambda} (-1)^i a_i \sum_{v \subseteq \lambda} c_{\nu}^{\lambda} (n-k-j+1,1^{i-1}) \nu s_v,$$

where the last sum ranges over all partitions $\nu$ satisfying $\nu \subseteq \lambda$.

**Proof of Theorem 10.14.** Let $\mu$ be a partition. Then, (23) yields

$$(s_{\mu}) \perp s_{\lambda} = s_{\lambda/\mu} = \sum_{\nu \text{ is a partition}} c_{\mu,\nu}^{\lambda} s_{\nu},$$

$$= \sum_{\nu \subseteq \lambda} c_{\mu,\nu}^{\lambda} s_{\nu} + \sum_{\nu \text{ is a partition; we don’t have } \nu \subseteq \lambda} c_{\mu,\nu}^{\lambda} s_{\nu} \quad \text{(by Proposition 6.17(b))},$$

$$= \sum_{\nu \text{ is a partition; } \nu \subseteq \lambda} c_{\mu,\nu}^{\lambda} s_{\nu} = \sum_{\nu \subseteq \lambda} c_{\mu,\nu}^{\lambda} s_{\nu}. \quad (105)$$

Both sides of this equality are symmetric functions in $\Lambda$. If we evaluate them at $x_1, x_2, \ldots, x_k$ and project the resulting symmetric polynomials onto $S/I$, then we obtain

$$(s_{\mu}) \perp s_{\lambda} = \sum_{\nu \subseteq \lambda} c_{\mu,\nu}^{\lambda} \pi_v. \quad (106)$$

Now, forget that we fixed $\mu$. We thus have proven (106) for each partition $\mu$. Theorem 10.5 yields

$$s_{\lambda} h_j = \sum_{\mu \in P_{k,n}; \mu/\lambda \text{ is a horizontal } j\text{-strip}}^\perp s_{\mu} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\nu \subseteq \lambda} c_{\nu}^{\lambda} (n-k-j+1,1^{i-1}) \nu \pi_v,$$

(by 106, applied to $\mu = (n-k-j+1,1^{i-1})$)

$$= \sum_{\mu \in P_{k,n}; \mu/\lambda \text{ is a horizontal } j\text{-strip}}^\perp s_{\mu} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\nu \subseteq \lambda} c_{\nu}^{\lambda} (n-k-j+1,1^{i-1}) \nu \pi_v.$$

This proves Theorem 10.14. 

Note that Theorem 10.14 can also be used to prove Theorem 9.5.
10.4. Positivity?

Let us recall some background about the quantum cohomology ring $\text{QH}^*(\text{Gr}_{kn})$ discussed in [Postni05]. The structure constants of the $\mathbb{Z}[q]$-algebra $\text{QH}^*(\text{Gr}_{kn})$ are polynomials in the indeterminate $q$, whose coefficients are the famous Gromov-Witten invariants $C^{d}_{\lambda,\mu,\nu}$. These Gromov-Witten invariants $C^{d}_{\lambda,\mu,\nu}$ are nonnegative integers (as follows from their geometric interpretation, but also from the “Quantum Littlewood-Richardson Rule” [BKPT16 Theorem 2]). This appears to generalize to the general case of $S/I$:

**Conjecture 10.15.** Let $b_i = (-1)^{n-k-1}a_i$ for each $i \in \{1, 2, \ldots, k\}$. Let $\lambda$, $\mu$ and $\nu$ be three partitions in $P_{k,n}$. Then, $(-1)^{|\lambda|+|\mu|-|\nu|}$ coeff$_{\nu}(\bar{s}_{\lambda}s_{\mu})$ is a polynomial in $b_1, b_2, \ldots, b_k$ with nonnegative integer coefficients. (See Definition 6.2 (b) for the meaning of coeff$_{\nu}$.)

We have verified this conjecture for all $n \leq 8$ using SageMath.

11. The “rim hook algorithm”

We shall next take aim at a recursive formula for “straightening” a Schur polynomial – i.e., representing an $s_{\mu}$, where $\mu$ is a partition that does not belong to $P_{k,n}$ as a $k$-linear combination of “smaller” $s_{\lambda}$’s. However, before we can state this formula, we will have to introduce several new notations.

11.1. Schur polynomials for non-partitions

Recall Definition 5.6. Thus, the elements of $P_k$ are weakly decreasing $k$-tuples in $\mathbb{N}^k$. For each $\lambda \in P_k$, a Schur polynomial $s_{\lambda} \in S$ is defined. Let us extend this definition by defining $s_{\lambda}$ for each $\lambda \in \mathbb{Z}^k$:

**Definition 11.1.** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{Z}^k$. Then, we define a symmetric polynomial $s_{\lambda} \in S$ by

$$s_{\lambda} = \det \left( (h_{\lambda, u+v})_{1 \leq u \leq k, 1 \leq v \leq k} \right). \quad (107)$$

This new definition does not clash with the previous use of the notation $s_{\lambda}$, because when $\lambda \in P_k$, both definitions yield the same result (because of Proposition 5.7 (a)).

This definition is similar to the definition of $\bar{s}_{(a_1, a_2, \ldots, a_n)}$ in [GriRei18 Exercise 2.9.1 (c)], but we are working with symmetric polynomials rather than symmetric functions here.

Definition 11.1 does not really open the gates to a new world of symmetric polynomials; indeed, each $s_{\alpha}$ (with $\alpha \in \mathbb{Z}^k$) defined in Definition 11.1 is either
0 or can be rewritten in the form \( \pm s_\lambda \) for some \( \lambda \in P_k \). Here is a more precise statement of this:

**Proposition 11.2.** Let \( \alpha \in \mathbb{Z}^k \). Define a \( k \)-tuple \( \beta = (\beta_1, \beta_2, \ldots, \beta_k) \) by

\[
(\beta_i = \alpha_i + k - i \quad \text{for each} \ i \in \{1, 2, \ldots, k\}).
\]

(a) If \( \beta \) has at least one negative entry, then \( s_\alpha = 0 \).

(b) If \( \beta \) has two equal entries, then \( s_\alpha = 0 \).

(c) Assume that \( \beta \) has no negative entries and no two equal entries. Let \( \sigma \in S_k \) be the permutation such that \( \beta_{\sigma(1)} > \beta_{\sigma(2)} > \cdots > \beta_{\sigma(k)} \). (Such a permutation \( \sigma \) exists and is unique, since \( \beta \) has no two equal entries.) Define a \( k \)-tuple \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{Z}^k \) by

\[
(\lambda_i = \beta_{\sigma(i)} - k + i \quad \text{for each} \ i \in \{1, 2, \ldots, k\}.)
\]

Then, \( \lambda \in P_k \) and \( s_\alpha = (-1)^\sigma s_\lambda \).

**Proof of Proposition 11.2.** For each \( u \in \{1, 2, \ldots, k\} \), we have \( \beta_u = \alpha_u + k - u \) (by the definition of \( \beta_u \)) and thus

\[
\beta_u = (\alpha_u + k - u) - k = \alpha_u - u.
\]

The definition of \( s_\alpha \) yields

\[
s_\alpha = \det \left( \begin{array}{c c c c c}
& & & & h_{\alpha_u - u + v} \\
& & & & = h_{\beta_u - k + v} \\
& & & & \text{(since \( (108) \) yields \( a_{u-u} = \beta_u - k \))} \\
& & & & 1 \leq u \leq k, 1 \leq v \leq k \\
& & & & \end{array} \right) \]

\[
= \det \left( (h_{\beta_u - k + v})_{1 \leq u \leq k, 1 \leq v \leq k} \right). \quad (109)
\]

**Proof of (b)** Assume that \( \beta \) has two equal entries. In other words, there are two distinct elements \( i \) and \( j \) of \( \{1, 2, \ldots, k\} \) such that \( \beta_i = \beta_j \). Consider these \( i \) and \( j \). The \( i \)-th and \( j \)-th rows of the matrix \( (h_{\beta_u - k + v})_{1 \leq u \leq k, 1 \leq v \leq k} \) are equal (since \( \beta_i = \beta_j \)). Hence, this matrix has two equal rows. Thus, its determinant is 0. In other words, \( \det \left( (h_{\beta_u - k + v})_{1 \leq u \leq k, 1 \leq v \leq k} \right) = 0 \). Now, (109) becomes \( s_\alpha = \det \left( (h_{\beta_u - k + v})_{1 \leq u \leq k, 1 \leq v \leq k} \right) = 0 \). This proves Proposition 11.2 (b).

**Proof of (a)** Assume that \( \beta \) has at least one negative entry. In other words, there exists some \( i \in \{1, 2, \ldots, k\} \) such that \( \beta_i < 0 \). Consider this \( i \). For each \( v \in \{1, 2, \ldots, k\} \), we have \( \beta_i - k + \sum_{\nu \leq k} \nu \leq \beta_i - k + k = \beta_i < 0 \) and thus \( h_{\beta_i - k + v} = 0 \).
Hence, all entries of the \(i\)-th row of the matrix \((h_{b_u-k+v})_{1 \leq u \leq k, 1 \leq v \leq k}\) are 0. Hence, this matrix has a zero row. Thus, its determinant is 0. In other words, \(\det \left((h_{b_u-k+v})_{1 \leq u \leq k, 1 \leq v \leq k}\right) = 0\). Now, (109) becomes \(s_{h} = \det \left((h_{b_u-k+v})_{1 \leq u \leq k, 1 \leq v \leq k}\right) = 0\). This proves Proposition 11.2 (a).

(c) It is well-known that if we permute the rows of a \(k \times k\)-matrix using a permutation \(\tau\), then the determinant of the matrix gets multiplied by \((-1)^\tau\). In other words, every \(k \times k\)-matrix \((b_{u,v})_{1 \leq u \leq k, 1 \leq v \leq k}\) and every \(\tau \in S_k\) satisfy

\[
\det \left((b_{\tau(u),v})_{1 \leq u \leq k, 1 \leq v \leq k}\right) = (-1)^\tau \det \left((b_{u,v})_{1 \leq u \leq k, 1 \leq v \leq k}\right).
\]

Applying this to \((b_{u,v})_{1 \leq u \leq k, 1 \leq v \leq k} = (h_{b_u-k+v})_{1 \leq u \leq k, 1 \leq v \leq k}\) and \(\tau = \sigma\), we obtain

\[
\det \left((h_{b_{\sigma(u)}-k+v})_{1 \leq u \leq k, 1 \leq v \leq k}\right) = (-1)^\sigma \det \left((h_{b_u-k+v})_{1 \leq u \leq k, 1 \leq v \leq k}\right).
\]

Multiplying both sides of this equality by \((-1)^\sigma\), we find

\[
(-1)^\sigma \det \left((h_{b_{\sigma(u)}-k+v})_{1 \leq u \leq k, 1 \leq v \leq k}\right) = (-1)^\sigma (-1)^\sigma \det \left((h_{b_u-k+v})_{1 \leq u \leq k, 1 \leq v \leq k}\right)
= (-1)^{\sigma^2} = 1
= \det \left((h_{b_{\sigma(u)}-k+v})_{1 \leq u \leq k, 1 \leq v \leq k}\right).
\]

(110)

For each \(u \in \{1, 2, \ldots, k\}\), we have \(\lambda_u = \beta_{\sigma(u)}-k+u\) (by the definition of \(\lambda_u\)) and thus

\[
\lambda_u - u = \beta_{\sigma(u)} - k.
\]

(111)

Now, (109) becomes

\[
s_{h} = \det \left((h_{b_u-k+v})_{1 \leq u \leq k, 1 \leq v \leq k}\right)
= (-1)^\sigma \det \left(h_{\beta_{\sigma(u)}-k+v}ight)_{1 \leq u \leq k, 1 \leq v \leq k}
= \det \left(h_{\lambda_u-u+v}ight)_{1 \leq u \leq k, 1 \leq v \leq k}
= \det \left(h_{\lambda_{i+1}}\right)_{1 \leq u \leq k, 1 \leq v \leq k}
= \det \left(h_{\lambda_{i+1}}\right)_{1 \leq u \leq k, 1 \leq v \leq k}
= (-1)^\sigma s_{\lambda}.
\]

(by (110))

It remains to prove that \(\lambda \in P_k\).

Let \(i \in \{1, 2, \ldots, k-1\}\). Then, \(\beta_{\sigma(i)} > \beta_{\sigma(i+1)}\) (since \(\beta_{\sigma(1)} > \beta_{\sigma(2)} > \cdots > \beta_{\sigma(k)}\)) and thus \(\beta_{\sigma(i)} \geq \beta_{\sigma(i+1)} + 1\) (since \(\beta_{\sigma(i)}\) and \(\beta_{\sigma(i+1)}\) are integers). The definition of \(\lambda_{i+1}\) yields \(\lambda_{i+1} = \beta_{\sigma(i+1)} - k + (i + 1)\). The definition of \(\lambda_i\) yields

\[
\lambda_i = \beta_{\sigma(i)} - k + i \geq \beta_{\sigma(i+1)} + 1 - k + i = \beta_{\sigma(i+1)} - k + (i + 1) = \lambda_{i+1}.
\]
Now, forget that we fixed $i$. We thus have proven that $\lambda_i \geq \lambda_{i+1}$ for each $i \in \{1, 2, \ldots, k-1\}$. In other words, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$.

Let $i \in \{1, 2, \ldots, k\}$. Then, $1 \leq i \leq k$ and thus $k \geq 1$, so that $\lambda_k$ is well-defined. Furthermore, from $i \leq k$, we obtain $\lambda_i \geq \lambda_k$ (since $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$). But the definition of $\lambda_k$ yields $\lambda_k = \beta_{\sigma(k)} - k + k = \beta_{\sigma(k)} \geq 0$ (since all entries of $\beta$ are nonnegative (since $\beta$ has no negative entries)). Thus, $\lambda_i \geq \lambda_k \geq 0$.

Now, forget that we fixed $i$. We thus have proven that $\lambda_i \geq 0$ for each $i \in \{1, 2, \ldots, k\}$. In other words, $\lambda_1, \lambda_2, \ldots, \lambda_k$ are nonnegative integers (since they are clearly integers). Hence, $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{N}^k$. Combining this with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, we obtain $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in P_k$. Hence, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in P_k$.

This completes the proof of Proposition 11.2(c).

Let us next recall the bialternant formula for Schur polynomials. We need a few definitions first:

**Definition 11.3.** (a) Let $\rho$ denote the $k$-tuple $(k-1, k-2, \ldots, 0) \in \mathbb{N}^k$.

(b) We regard $\mathbb{Z}^k$ as a $\mathbb{Z}$-module in the obvious way: Addition is defined entrywise (i.e., we set $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_k + \beta_k)$ for any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{Z}^k$ and any $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathbb{Z}^k$). This also defines subtraction on $\mathbb{Z}^k$ (which, too, works entrywise). We let $0$ denote the $k$-tuple $\left(0, 0, \ldots, 0\right) \in \mathbb{N}^k \subset \mathbb{Z}^k$; this is the zero vector of $\mathbb{Z}^k$.

**Definition 11.4.** Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$. Then, we define the alternant $a_\alpha \in \mathcal{P}$ by

$$a_\alpha = \det \left( x_i^{a_j} \right)_{1 \leq i \leq k, 1 \leq j \leq k}.$$

The two definitions we have just made match the notations in [GriRei18 §2.6], except that we are using $k$ instead of $n$ for the number of indeterminates.

Note that the element $a_\rho$ of $\mathcal{P}$ is the Vandermonde determinant

$$\det \left( x_i^{k-j} \right)_{1 \leq i \leq k, 1 \leq j \leq k} = \prod_{1 \leq i < j \leq k} (x_i - x_j);$$

it is a regular element of $\mathcal{P}$ (that is, a non-zero-divisor).

We recall the bialternant formula for Schur polynomials ([GriRei18 Corollary 2.6.6]):

**Proposition 11.5.** For any $\lambda \in P_k$, we have $s_\lambda = a_{\lambda+\rho}/a_\rho$ in $\mathcal{P}$.

Let us extend this fact to arbitrary $\lambda \in \mathbb{Z}^k$ satisfying $\lambda + \rho \in \mathbb{N}^k$ (and rename $\lambda$ as $a$):
Proposition 11.6. Let $\alpha \in \mathbb{Z}^k$ be such that $\alpha + \rho \in \mathbb{N}^k$. Then, $s_\alpha = a_{\alpha + \rho} / a_\rho$ in $\mathcal{P}$.

Proof of Proposition 11.6. We have $\rho = (k - 1, k - 2, \ldots, 0)$. Thus,

$$\rho_i = k - i \quad \text{for each } i \in \{1, 2, \ldots, k\}.$$  \hfill (112)

Define a $k$-tuple $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ as in Proposition 11.2. Thus, for each $i \in \{1, 2, \ldots, k\}$, we have

$$\beta_i = \alpha_i + k - i = \alpha_i + \rho_i = (\alpha + \rho)_i.$$  \hfill (by (112))

In other words, $\beta = \alpha + \rho$. Hence, $\beta = \alpha + \rho \in \mathbb{N}^k$. Thus, the $k$-tuple $\beta$ has no negative entries.

Moreover, from $\alpha + \rho = \beta$, we obtain

$$a_{\alpha + \rho} = a_\beta = \det \left( x_{ij}^{\beta} \right)_{1 \leq i, j \leq k} \quad \text{(by the definition of } a_\beta)$$

$$= \det \left( x_{uv}^{\beta} \right)_{1 \leq u, v \leq k} \quad \text{(113)}$$

(here, we have renamed the indices $i$ and $j$ as $u$ and $v$). Now, we are in one of the following two cases:

Case 1: The $k$-tuple $\beta$ has two equal entries.

Case 2: The $k$-tuple $\beta$ has no two equal entries.

Let us first consider Case 1. In this case, the $k$-tuple $\beta$ has two equal entries. In other words, there are two distinct elements $i$ and $j$ of $\{1, 2, \ldots, k\}$ such that $\beta_i = \beta_j$. Consider these $i$ and $j$. The $i$-th and $j$-th columns of the matrix $\left( x_{ij}^{\beta} \right)_{1 \leq u, v \leq k}$ are equal (since $\beta_i = \beta_j$). Hence, this matrix has two equal columns. Thus, its determinant is 0. In other words, $\det \left( x_{uv}^{\beta} \right)_{1 \leq u, v \leq k} = 0$. Now, (113) becomes $a_{\alpha + \rho} = \det \left( x_{uv}^{\beta} \right)_{1 \leq u, v \leq k} = 0$. Hence, $a_{\alpha + \rho} / a_\rho = 0 / a_\rho = 0$. Comparing this with $s_\alpha = 0$ (which follows from Proposition 11.2(b)), we obtain $s_\alpha = a_{\alpha + \rho} / a_\rho$. Thus, Proposition 11.6 is proven in Case 1.

Let us next consider Case 2. In this case, the $k$-tuple $\beta$ has no two equal entries. Thus, there is a unique permutation $\sigma \in S_k$ that sorts this $k$-tuple into strictly decreasing order. In other words, there is a unique permutation $\sigma \in S_k$ such that $\beta_{\sigma(1)} > \beta_{\sigma(2)} > \cdots > \beta_{\sigma(k)}$. Consider this $\sigma$. Define a $k$-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{Z}^k$ by

$$\lambda_i = \beta_{\sigma(i)} - k + i \quad \text{for each } i \in \{1, 2, \ldots, k\}.$$
Then, Proposition \([11.2(c)](\text{c})\) yields \(\lambda \in P_k\) and \(s_\lambda = (-1)^{\sigma} s_\lambda\).

It is well-known that if we permute the columns of a \(k \times k\)-matrix using a permutation \(\tau\), then the determinant of the matrix gets multiplied by \((-1)^{\tau}\).

In other words, every \(k\) permutation \(\tau\) \(\det\left(\left(b_{u,v}\right)_{1 \leq u \leq k, 1 \leq v \leq k}\right) = (-1)^{\tau} \det\left(\left(b_{u,v}\right)_{1 \leq u \leq k, 1 \leq v \leq k}\right)\). Applying this to \(\det\left(\left(b_{u,v}\right)_{1 \leq u \leq k, 1 \leq v \leq k}\right) = \left(x_u^{\beta_{\sigma(v)}}\right)_{1 \leq u \leq k, 1 \leq v \leq k}\) and \(\tau = \sigma\), we obtain

\[
\det\left(\left(x_u^{\beta_{\sigma(v)}}\right)_{1 \leq u \leq k, 1 \leq v \leq k}\right) = (-1)^{\sigma} \det\left(\left(x_u^{\beta_v}\right)_{1 \leq u \leq k, 1 \leq v \leq k}\right).
\] (114)

But each \(v \in \{1, 2, \ldots, k\}\) satisfies

\[
(\lambda + \rho)_v = \begin{cases} 
\lambda_v & \text{by the definition of } \lambda_v \\
\rho_v & \text{by (112)} \\
= \beta_{\sigma(v)} - k + \nu & \text{(by (112))} \\
= \beta_{\sigma(v)}. & \text{(115)}
\end{cases}
\]

Now, the definition of \(a_{\lambda+\rho}\) yields

\[
a_{\lambda+\rho} = \det\left(\left(x_i^{(\lambda+\rho)}\right)_{1 \leq i \leq k, 1 \leq j \leq k}\right) = \det\left(\left(x_u^{(\lambda+\rho)}\right)_{1 \leq u \leq k, 1 \leq v \leq k}\right)
\]

(here, we have renamed the indices \(i\) and \(j\) as \(u\) and \(v\))

\[
= \det\left(\left(x_u^{\beta_{\sigma(v)}}\right)_{1 \leq u \leq k, 1 \leq v \leq k}\right)
\]

(by (114))

\[
= (-1)^{\sigma} a_{\lambda+\rho}.
\]

But \(\lambda \in P_k\). Hence, Proposition \([11.5]\) yields

\[
s_\lambda = a_{\lambda+\rho} / a_\rho = (-1)^{\sigma} a_{\lambda+\rho} / a_\rho.
\]

Hence,

\[
s_\lambda = (-1)^{\sigma} s_\lambda = (-1)^{\sigma} (-1)^{\sigma} a_{\lambda+\rho} / a_\rho = a_{\lambda+\rho} / a_\rho.
\]

Hence,

\[
s_{\lambda} = (-1)^{\sigma} s_\lambda = (-1)^{\sigma} (-1)^{\sigma} a_{\lambda+\rho} / a_\rho = a_{\lambda+\rho} / a_\rho.
\]

\[
((-1)^{\sigma})^2 = 1
\]
Thus, Proposition 11.6 is proven in Case 2.

We have now proven Proposition 11.6 in both Cases 1 and 2. Thus, Proposition 11.6 is proven.

**11.2. The uncancelled Pieri rule**

Having defined $s_{\lambda}$ for all $\lambda \in \mathbb{Z}^k$ (rather than merely for partitions), we can state a nonstandard version of the Pieri rule for products of the form $s_{\lambda} h_i$, which will turn out rather useful:

**Theorem 11.7.** Let $\lambda \in \mathbb{Z}^k$ be such that $\lambda + \rho \in \mathbb{N}^k$. Let $m \in \mathbb{N}$. Then,

$$s_{\lambda} h_m = \sum_{\nu \in \mathbb{N}^k; |\nu| = m} s_{\lambda + \nu}.$$

**Example 11.8.** For this example, let $k = 3$ and $\lambda = (-2, 2, 1)$. Then, $\lambda + \rho = (-2, 2, 1) + (2, 1, 0) = (0, 3, 1)$. It is easy to see (using Proposition 11.2(c)) that $s_{\lambda} = s_{(1)}$.

Furthermore, set $m = 2$. Then, the $\nu \in \mathbb{N}^k$ satisfying $|\nu| = m$ are the six 3-tuples $(2, 0, 0), (0, 2, 0), (0, 0, 2), (1, 1, 0), (1, 0, 1), (0, 1, 1)$.

Hence, Theorem 11.7 yields

$$s_{(-2,2,1)} h_2 = \sum_{\nu \in \mathbb{N}^k; |\nu| = m} s_{(-2,2,1) + \nu}$$

$$= s_{(-2,2,1) + (2,0,0)} + s_{(-2,2,1) + (0,2,0)} + s_{(-2,2,1) + (0,0,2)}$$

$$= s_{(-2,2,1) + (2,0,0)} + s_{(-2,2,1) + (0,2,0)} + s_{(-2,2,1) + (0,0,2)}$$

$$= s_{(-2,2,1) + (2,0,0)} + s_{(-2,2,1) + (0,2,0)} + s_{(-2,2,1) + (0,0,2)}$$

$$= s_{(1,1,1)} + s_{(3)} + 0 + s_{(1,1,1)} + s_{(2,1)} = s_{(2,1)} + s_{(3)}.$$

In view of $s_{(-2,2,1)} = s_{(1)}$, this rewrites as $s_{(1)} h_2 = s_{(2,1)} + s_{(3)}$, which is exactly what the usual Pieri rule would yield. Note that the expression we obtained from Theorem 11.7 involves both vanishing addends (here, $s_{(-2,2,1) + (0,0,2)}$ and $s_{(-2,2,1) + (1,1,0)}$) and mutually cancelling addends (here, $s_{(-2,2,1) + (2,0,0)}$ and $s_{(-2,2,1) + (1,0,1)}$); this is why I call it the “uncancelled Pieri rule”.

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We note that the idea of such an “uncancelled Pieri rule” as our Theorem 11.7 is not new (similar things appeared in [LakTho07, §2] and [Tamvak13]), but we have not seen it stated in this exact form anywhere in the literature. Thus, let us give a proof:

Proof of Theorem 11.7 Define $\beta \in \mathbb{N}^k$ by $\beta = \lambda + \rho$. (This is well-defined, since $\lambda + \rho \in \mathbb{N}^k$.)

From (1), we obtain

$$h_m = \sum_{\alpha \in \mathbb{N}^k, |\alpha| = m} \prod_{i=1}^{k} x_{\alpha_i}^1 x_{\alpha_i}^2 \cdots x_{\alpha_i}^k = \prod_{i=1}^{k} x_{\beta_i}^i.$$

(116)

For each permutation $\sigma \in S_k$, we have

$$h_m = h_m \left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}\right) \quad \text{(since the polynomial $h_m$ is symmetric)}$$

$$= \sum_{\alpha \in \mathbb{N}^k, |\alpha| = m} \prod_{i=1}^{k} x_{\sigma(i)}^{\alpha_i} = \prod_{i=1}^{k} x_{\beta_i}^i.$$

(117)

(here, we have substituted $x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}$ for $x_1, x_2, \ldots, x_k$ in the equality (116)).

But Proposition 11.6 (applied to $\alpha = \lambda$) yields $s_\lambda = a_{\lambda+\rho}/a_\rho$ in $\mathcal{P}$. Thus,

$$a_\rho s_\lambda = a_{\lambda+\rho} = a_\beta \quad \text{(since $\lambda + \rho = \beta$)}$$

$$= \det \begin{pmatrix} x_{\beta_i}^j \end{pmatrix}_{1 \leq i \leq k, 1 \leq j \leq k} \quad \text{(by the definition of $a_\beta$)}$$

$$= \det \begin{pmatrix} x_{\beta_i}^j \end{pmatrix}_{1 \leq i \leq k, 1 \leq j \leq k} \quad \text{(since the determinant of a matrix equals the determinant of its transpose)}$$

$$= \sum_{\sigma \in S_k} (-1)^\sigma \prod_{i=1}^{k} x_{\sigma(i)}^{\beta_i} \quad \text{(by the definition of a determinant)}.$$

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Multiplying both sides of this equality with $h_m$, we find

$$a_\rho s_\lambda h_m = \left( \sum_{\sigma \in S_k} (-1)^\sigma \prod_{i=1}^k x_{\sigma(i)}^{\beta_i} \right) h_m = \sum_{\sigma \in S_k} (-1)^\sigma \left( \prod_{i=1}^k x_{\sigma(i)}^{\beta_i} \right)$$

$$= \sum_{\sigma \in S_k} (-1)^\sigma \left( \prod_{i=1}^k x_{\sigma(i)}^{\beta_i} \right) \sum_{\alpha \in \mathbb{N}^k; |\alpha| = m} \prod_{i=1}^k x_{\sigma(i)}^{\alpha_i}$$

$$= \sum_{\alpha \in \mathbb{N}^k; \sigma \in S_k} |\alpha| = m (-1)^\sigma \left( \prod_{i=1}^k x_{\sigma(i)}^{\beta_i} \right) \prod_{i=1}^k x_{\sigma(i)}^{\alpha_i}$$

$$= \sum_{\alpha \in \mathbb{N}^k; \sigma \in S_k} |\alpha| = m (-1)^\sigma \prod_{i=1}^k \left( x_{\sigma(i)}^{\beta_i} x_{\sigma(i)}^{\alpha_i} \right)$$

$$= \sum_{\alpha \in \mathbb{N}^k; \sigma \in S_k} |\alpha| = m (-1)^\sigma \prod_{i=1}^k x_{\sigma(i)}^{\beta_i + \alpha_i}$$

$$= \sum_{\nu \in \mathbb{N}^k; \sigma \in S_k} |\nu| = m (-1)^\sigma \prod_{i=1}^k x_{\sigma(i)}^{\beta_i + \nu_i}$$

(118)

(here, we have renamed the summation index $\alpha$ as $\nu$).

On the other hand, let $\nu \in \mathbb{N}^k$. Then, $(\lambda + \nu) + \rho = (\lambda + \rho) + \nu \in \mathbb{N}^k$. Thus, Proposition 11.6 (applied to $\alpha = \lambda + \nu$) yields $s_{\lambda+\nu} = a_{(\lambda+\nu)+\rho}/a_\rho$ in $\mathcal{P}$. 

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Thus,

\[ a_\rho s_{\lambda + \nu} = a_{(\lambda + \nu) + \rho} = a_{\beta + \nu} \quad \left( \text{since } (\lambda + \nu) + \rho = \lambda + \rho + \nu = \beta + \nu \right) \]

\[ = \det \left( \begin{pmatrix} x_i^{(\beta + \nu)} \\ 1 \leq i \leq k, 1 \leq j \leq k \end{pmatrix} \right) \quad \left( \text{by the definition of } a_{\beta + \nu} \right) \]

\[ = \det \left( \begin{pmatrix} x_j^{(\beta + \nu)} \\ 1 \leq i \leq k, 1 \leq j \leq k \end{pmatrix} \right) \quad \left( \text{since the determinant of a matrix equals the determinant of its transpose} \right) \]

\[ = \sum_{\sigma \in S_k} (-1)^\sigma \prod_{i=1}^{k} x_{\sigma(i)}^{(\beta + \nu)} \quad (119) \]

(by the definition of a determinant).

Now, forget that we fixed \( \nu \). We thus have proven (119) for each \( \nu \in \mathbb{N}^k \).

Now, (118) becomes

\[ a_\rho s_{\lambda} h_m = \sum_{\nu \in \mathbb{N}^k; |\nu| = m} (-1)^\sigma \prod_{i=1}^{k} x_{\sigma(i)}^{(\beta + \nu)} = \sum_{\nu \in \mathbb{N}^k; |\nu| = m} a_\rho s_{\lambda + \nu} = a_\rho \sum_{\nu \in \mathbb{N}^k; |\nu| = m} s_{\lambda + \nu}. \]

We can cancel \( a_\rho \) from this equality (since \( a_\rho \) is a regular element of \( \mathcal{P} \)), and thus obtain

\[ s_{\lambda} h_m = \sum_{\nu \in \mathbb{N}^k; |\nu| = m} s_{\lambda + \nu}. \]

This proves Theorem 11.7.

11.3. The “rim hook algorithm”

For the rest of this section, we assume that \( k > 0 \).

We need one more weird definition:

**Definition 11.9.** Let \( V \) be the set of all \( k \)-tuples \((-n, \tau_2, \tau_3, \ldots, \tau_k) \in \mathbb{Z}^k \) satisfying

\[ (\tau_i \in \{0, 1\} \quad \text{for each } i \in \{2, 3, \ldots, k\}). \] (120)
Example 11.10. If $n = 6$ and $k = 3$, then
\[ V = \{ (-6,0,0), (-6,0,1), (-6,1,0), (-6,1,1) \} . \] (121)

Proposition 11.11. Let $\tau \in V$. Then, $-|\tau| \in \{ n - k + 1, n - k + 2, \ldots, n \}$.

Proof of Proposition 11.11. We have $\tau \in V$. Thus, $\tau$ has the form $\tau = (n, \tau_2, \tau_3, \ldots, \tau_k) \in \mathbb{Z}^k$ for some $\tau_2, \tau_3, \ldots, \tau_k$ satisfying (120) (by the definition of $V$). Consider these $\tau_2, \tau_3, \ldots, \tau_k$. We have $\tau = (-n, \tau_2, \tau_3, \ldots, \tau_k)$ and thus
\[ |\tau| = (-n) + \tau_2 + \tau_3 + \cdots + \tau_k = (-n) + \sum_{i=2}^{k} \tau_i \]
and thus
\[ -|\tau| = -\left( -n + \sum_{i=2}^{k} \tau_i \right) = n - \sum_{i=2}^{k} \tau_i \geq n - \sum_{i=2}^{k} 1 \]
\[ = n - (k - 1) = n - k + 1. \]
Combining this with
\[ -|\tau| = n - \sum_{i=2}^{k} \tau_i \geq n - \sum_{i=2}^{k} 0 = n, \]
we obtain $n - k + 1 \leq -|\tau| \leq n$. Thus, $-|\tau| \in \{ n - k + 1, n - k + 2, \ldots, n \}$ (since $-|\tau|$ is an integer). This proves Proposition 11.11.

We are now ready to state the main theorem of this section: a generalization of the “rim hook algorithm” from [BeCiFu99, §2, Main Lemma]:

Theorem 11.12. Assume that $a_1, a_2, \ldots, a_k$ belong to $k$.

Let $\mu \in P_k$ be such that $\mu_1 > n - k$. Then,
\[ s_\mu = \sum_{j=1}^{k} (-1)^{k-j} a_j \sum_{\tau \in V; -|\tau| = n - k + j} s_{\mu + \tau}. \]
Example 11.13. For this example, set \( n = 6 \) and \( k = 3 \) and \( \mu = (5, 4, 2) \). Then, Theorem 11.12 yields

\[
\overline{s}_{(5,4,2)} = \sum_{j=1}^{k} (-1)^{k-j} a_j \sum_{\tau \in \mathcal{V}; \quad -|\tau| = n-k+j} \overline{s}_{(5,4,2)+\tau}
\]

\[
= (-1)^{3-1} a_1 \overline{s}_{(5,4,1) + (-6,1,1)} + (-1)^{3-2} a_2 \left( \overline{s}_{(5,4,1) + (-6,0,1)} + \overline{s}_{(5,4,1) + (-6,1,0)} \right)
\]

\[
+ (-1)^{3-3} a_3 \overline{s}_{(5,4,1) + (-6,0,0)}
\]

(by (121))

\[
= a_1 \overline{s}_{(5,4,1) + (-6,1,1)} - a_2 \underbrace{\overline{s}_{(5,4,1) + (-6,0,1)}}_{=\overline{s}_{(-1,5,2)}} + \underbrace{\overline{s}_{(5,4,1) + (-6,1,0)}}_{=\overline{s}_{(-1,4,3)}}
\]

(by Proposition 11.2 (c))

\[
+ a_3 \underbrace{\overline{s}_{(5,4,1) + (-6,0,0)}}_{=\overline{s}_{(-1,4,1)}}
\]

(by Proposition 11.2 (b))

\[
= a_1 \overline{s}_{(4,1,1)} - a_2 \overline{s}_{(3,1,1)}.
\]

Note that this is not yet an expansion of \( \overline{s}_\mu \) in the basis \( (\overline{s}_\lambda)_{\lambda \in P_k} \). Indeed, we still have a term \( \overline{s}_{(4,1,1)} \) on the right hand side which has \( (4,1,1) \notin P_{k,n} \). But this term can, in turn, be rewritten using Theorem 11.12, and so on until we end up with an expansion of \( \overline{s}_\mu \) in the basis \( (\overline{s}_\lambda)_{\lambda \in P_k} \), namely

\[
\overline{s}_{(5,4,2)} = -a_2 \overline{s}_{(3,1,1)} + a_1 \overline{s}_{(1,1)} - a_1 a_2 \overline{s}_{(1)} + a_1 a_3 \overline{s}_{(1)}.
\]

As we saw in this example, when we apply Theorem 11.12, some of the \( \overline{s}_{\mu+\tau} \) addends on the right hand side may be 0 (by Proposition 11.2 (b)). Once these addends are removed, the remaining addends can be rewritten in the form \( \pm \overline{s}_\lambda \) for some \( \lambda \in P_k \) satisfying \( |\lambda| < |\mu| \) (using Proposition 11.2 (c)). The resulting sum is multiplicity-free – in the sense that no \( \overline{s}_\lambda \) occurs more than once in it. (This is not difficult to check, but would take us too far afield.) However, this sum is (in general) not an expansion of \( \overline{s}_\mu \) in the basis \( (\overline{s}_\lambda)_{\lambda \in P_k} \) yet, because it often contains terms \( \overline{s}_\lambda \) with \( \lambda \notin P_{k,n} \). If we keep applying Theorem 11.12 multiple times until we reach an expansion of \( \overline{s}_\mu \) in the basis \( (\overline{s}_\lambda)_{\lambda \in P_k} \), then this latter expansion may contain multiplicities: For example, for \( n = 6 \) and \( k = 3 \),
we have
\[
\overline{s}_{(4,4,3)} = -a_2 s_{(3,3)} + a_3 s_{(3,2)} + a_1^2 s_{(3)} - 2a_1 a_2 s_{(2)} + a_2^2 s_{(1)}.
\]

We owe the reader an explanation of why we call Theorem 11.12 a “rim hook algorithm”. It owes this name to the fact that it generalizes the “rim hook algorithm” for quantum cohomology [BeCiFu99, §2, Main Lemma] (which can be obtained from it with some work by setting \(a_i = 0\) for all \(i < k\)). Nevertheless, it does not visibly involve any rim hooks itself. I am, in fact, unaware of a way to restate it in the language of Young diagrams; the operation \(\mu \mapsto \mu + \tau\) for \(\tau \in V\) resembles both the removal of an \(n\)-rim hook (since it lowers the first entry by \(n\)) and the addition of a vertical strip (since it increases each of the remaining entries by 0 or 1), but it cannot be directly stated as one of these operations followed by the other.

We shall prove Theorem 11.12 by deriving it from an identity in \(S\):

**Theorem 11.14.** Let \(\mu \in P_k\) be such that \(\mu_1 > n - k\). Then,
\[
s_\mu = \sum_{j=1}^{k} (-1)^{k-j} h_{n-k+j} \sum_{\substack{\tau \in V; \\
-|\tau| = n-k+j}} s_{\mu + \tau}.
\]

Our proof of this identity, in turn, will rely on the following combinatorial lemmas:

**Lemma 11.15.** Let \(j \in \{2, 3, \ldots, k\}\). Let \(\Delta\) be the vector \((0, 0, \ldots, 0, 1, 0, 0, \ldots, 0) \in \mathbb{Z}^k\), where 1 is the \(j\)-th entry.

(a) If \(\tau \in V\) satisfies \(\tau_j = 0\), then \(\tau + \Delta \in V\) and \((\tau + \Delta)_j = 1\).

(b) If \(\tau \in V\) satisfies \(\tau_j = 1\), then \(\tau - \Delta \in V\) and \((\tau - \Delta)_j = 0\).

(c) If \(v \in \mathbb{N}^k\) satisfies \(v_j \neq 0\), then \(v - \Delta \in \mathbb{N}^k\).

(d) If \(v \in \mathbb{N}^k\), then \(v + \Delta \in \mathbb{N}^k\).

**Proof of Lemma 11.15** We have \(j \in \{2, 3, \ldots, k\}\), thus \(j \neq 1\).

We have \(\Delta = (0, 0, \ldots, 0, 1, 0, 0, \ldots, 0) \in \mathbb{N}^k\). Thus, \(\Delta_j = 1\) and
\[
(\Delta_i = 0 \quad \text{for each} \; i \in \{1, 2, \ldots, k\} \; \text{satisfying} \; i \neq j). \quad (122)
\]

Applying (122) to \(i = 1\), we obtain \(\Delta_1 = 0\) (since \(1 \neq j\)).

(a) Let \(\tau \in V\) be such that \(\tau_j = 0\).

We have \(\tau \in V\). According to the definition of \(V\), this means that \(\tau\) is a \(k\)-tuple \((-n, \tau_2, \tau_3, \ldots, \tau_k) \in \mathbb{Z}^k\) satisfying (120). In other words, \(\tau \in \mathbb{Z}^k\) and \(\tau_1 = -n\) and
\[
(\tau_i \in \{0, 1\} \quad \text{for each} \; i \in \{2, 3, \ldots, k\}). \quad (123)
\]
Define a $k$-tuple $\sigma \in \mathbb{Z}^k$ by $\sigma = \tau + \Delta$. Thus, $\sigma_1 = (\tau + \Delta)_1 = \tau_1 + \underbrace{\Delta_1 = \tau_1 = -n}_{=0}$.

Furthermore, from $\sigma = \tau + \Delta$, we obtain $\sigma_j = (\tau + \Delta)_j = \underbrace{\tau_j + \Delta_j = 1}_{=0}$ for each $i \in \{0, 1\}$.

Next, we have $\sigma_i \in \{0, 1\}$ for each $i \in \{2, 3, \ldots, k\}$. Altogether, we thus have shown that $\sigma \in \mathbb{Z}^k$ and $\sigma_1 = -n$ and

$$(\sigma_i \in \{0, 1\} \quad \text{for each } i \in \{2, 3, \ldots, k\}). \quad (124)$$

In other words, $\sigma$ is a $k$-tuple $(-n, \sigma_2, \sigma_3, \ldots, \sigma_k) \in \mathbb{Z}^k$ satisfying (124). In other words, $\sigma \in V$ (by the definition of $V$). Thus, $\tau + \Delta = \sigma \in V$. So we have proven that $\tau + \Delta \in V$ and $(\tau + \Delta)_i = 1$. Thus, Lemma 11.15 (a) is proven.

(b) The proof of Lemma 11.15 (b) is analogous to the above proof of Lemma 11.15 (a), and is left to the reader.

(c) Let $v \in \mathbb{N}^k$ be such that $v_j \neq 0$. We must prove that $v - \Delta \in \mathbb{N}^k$.

We have $v_j \in \mathbb{N}$ (since $v \in \mathbb{N}^k$). Hence, from $v_j \neq 0$, we conclude that $v_j \geq 1$.

Thus, $v_j - 1 \in \mathbb{N}$. Also, the entries $v_1, v_2, \ldots, v_{j-1}, v_{j+1}, v_{j+2}, \ldots, v_k$ of $v$ belong to $\mathbb{N}$ (since $v \in \mathbb{N}^k$).

Recall that $\Delta$ is the vector $(0, 0, \ldots, 0, 1, 0, 0, \ldots, 0) \in \mathbb{Z}^k$, where 1 is the $j$-th entry.

Hence,

$$v - \Delta = v - (0, 0, \ldots, 0, 1, 0, 0, \ldots, 0)$$

$$= (v_1, v_2, \ldots, v_{j-1}, v_j - 1, v_{j+1}, v_{j+2}, \ldots, v_k) \in \mathbb{N}^k$$

(since $v_j - 1 \in \mathbb{N}$ and since the entries $v_1, v_2, \ldots, v_{j-1}, v_{j+1}, v_{j+2}, \ldots, v_k$ of $v$ belong to $\mathbb{N}$). This proves Lemma 11.15 (c).

(d) Let $v \in \mathbb{N}^k$. Also, $\Delta \in \mathbb{N}^k$. Thus, $\underbrace{v + \Delta}_{\in \mathbb{N}^k} \in \mathbb{N}^k$. This proves Lemma 11.15 (d).

Lemma 11.16. Let $\gamma \in \mathbb{Z}^k$. Then,

$$\sum_{\tau \in V} \sum_{v \in \mathbb{N}^k: \atop \nu \in \mathbb{N}^k: \atop |\nu| = -|\tau|; \atop \nu + \tau = \gamma} (-1)^{n+|\tau|} = \begin{cases} 1, & \text{if } \gamma = 0; \\ 0, & \text{if } \gamma \neq 0. \end{cases}$$

(Recall that 0 denotes the vector $\underbrace{0, 0, \ldots, 0}_{k \text{ zeroes}} \in \mathbb{Z}^k$.)

Proof. Let $i \in \{2, 3, \ldots, k\}$. We must prove $\sigma_i \in \{0, 1\}$.

If $i = j$, then this follows from $\sigma_i \in \{0, 1\}$. Hence, for the rest of this proof, we WLOG assume that $i \neq j$. Thus, (122) yields $\Delta_i = 0$. Now, from $\sigma = \tau + \Delta$, we obtain $\sigma_i = (\tau + \Delta)_i = \tau_i + \Delta_i = \tau_i \in \{0, 1\}$ (by (123)). Qed.

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Proof of Lemma 11.16. Let $Q$ be the set of all pairs $(\tau, \nu) \in V \times \mathbb{N}^k$ satisfying $|\nu| = -|\tau|$ and $\nu + \tau = \gamma$. We have the following equality of summation signs:

$$
\sum_{\tau \in V} \sum_{\nu \in \mathbb{N}^k; |\nu| = -|\tau|; \nu + \tau = \gamma} = \sum_{(\tau, \nu) \in Q} (\tau, \nu) = \sum_{(\tau, \nu) \in Q}
$$

(since $Q$ is the set of all pairs $(\tau, \nu) \in V \times \mathbb{N}^k$ satisfying $|\nu| = -|\tau|$ and $\nu + \tau = \gamma$).

We are in one of the following three cases:

Case 1: We have $(\gamma_2, \gamma_3, \ldots, \gamma_k) \neq (0, 0, \ldots, 0)$.

Case 2: We have $(\gamma_2, \gamma_3, \ldots, \gamma_k) = (0, 0, \ldots, 0)$ and $\gamma_1 \neq 0$.

Case 3: We have $(\gamma_2, \gamma_3, \ldots, \gamma_k) = (0, 0, \ldots, 0)$ and $\gamma_1 = 0$.

Let us first consider Case 1. In this case, we have $(\gamma_2, \gamma_3, \ldots, \gamma_k) \neq (0, 0, \ldots, 0)$. In other words, there exists a $j \in \{2, 3, \ldots, k\}$ such that $\gamma_j \neq 0$. Consider such a $j$. Clearly, $\gamma \neq 0$ (since $\gamma_j \neq 0$). Hence, $\begin{cases} 1, & \text{if } \gamma = 0; \\ 0, & \text{if } \gamma \neq 0 \end{cases} = 0$.

Let $\Delta$ be the vector $(0, 0, \ldots, 0, 1, 0, 0, \ldots, 0) \in \mathbb{Z}^k$, where 1 is the $j$-th entry. Clearly, $\Delta \in \mathbb{N}^k$ and $|\Delta| = 1$.

Let $Q_0$ be the set of all $(\tau, \nu) \in Q$ satisfying $\tau_j = 0$. (Recall that $\tau_j$ denotes the $j$-th entry of the $k$-tuple $\tau \in V \subseteq \mathbb{Z}^k$.) Let $Q_1$ be the set of all $(\tau, \nu) \in Q$ satisfying $\tau_j = 1$. Each $(\tau, \nu) \in Q$ satisfies $(\tau, \nu) \in V \times \mathbb{N}^k$ (by the definition of $Q$) and $\nu + \tau \in V$ and thus $\tau_j \in \{0, 1\}$ (by (126), applied to $i = j$). In other words, each $(\tau, \nu) \in Q$ satisfies either $\tau_j = 0$ or $\tau_j = 1$ (but not both at the same time). In other words, each $(\tau, \nu) \in Q$ belongs to either $Q_0$ or $Q_1$ (but not both at the same time).

For each $(\tau, \nu) \in Q_0$, we have $(\tau + \Delta, \nu - \Delta) \in Q_1$. Thus, the map

$$
Q_0 \to Q_1, \quad (\tau, \nu) \mapsto (\tau + \Delta, \nu - \Delta)
$$

Proof: Let $(\tau, \nu) \in Q_0$. According to the definition of $Q_0$, this means that $(\tau, \nu) \in Q$ and $\tau_j = 0$.

We have $(\tau, \nu) \in Q$. According to the definition of $Q$, this means that $(\tau, \nu) \in V \times \mathbb{N}^k$ and $|\nu| = -|\tau|$ and $\nu + \tau = \gamma$.

From $(\tau, \nu) \in V \times \mathbb{N}^k$, we obtain $\tau \in V$ and $\nu \in \mathbb{N}^k$.

From $\nu + \tau = \gamma$, we obtain $\nu = \gamma_j$. Hence, $\gamma_j = \nu + \tau_j = \nu_j$. Thus, $\nu_j = \gamma_j \neq 0$. Thus, Lemma 11.15(e) yields $\nu - \Delta \in \mathbb{N}^k$. Also, Lemma 11.15(a) yields that $\tau + \Delta \in V$ and $\tau + \Delta_j = 1$. Also, any two $k$-tuples $\alpha \in \mathbb{N}^k$ and $\beta \in \mathbb{N}^k$ satisfy $|\alpha + \beta| = |\alpha| + |\beta|$ and $|\alpha - \beta| = |\alpha| - |\beta|$. Thus, $|\tau + \Delta| = |\tau| + |\Delta|$ and $|\nu - \Delta| = |\nu| - |\Delta| = -|\tau| - |\Delta| = -(|\tau| + |\Delta|) = -|\tau + \Delta|$. Also, $(\nu - \Delta) + (\tau + \Delta) = \nu + \tau = \gamma$.

From $\tau + \Delta \in V$ and $\nu - \Delta \in \mathbb{N}^k$ and $|\nu - \Delta| = -|\tau + \Delta|$ and $(\nu - \Delta) + (\tau + \Delta) = \gamma$, we obtain $(\tau + \Delta, \nu - \Delta) \in Q$ (by the definition of $Q$). Combining this with $\tau_j = 1$, we obtain $(\tau + \Delta, \nu - \Delta) \in Q_1$ (by the definition of $Q_1$), qed.
is well-defined.

For each \((\tau, \nu) \in Q_1\), we have \((\tau - \Delta, \nu + \Delta) \in Q_0\) \(^{25}\) Thus, the map

\[
Q_1 \to Q_0, \quad (\tau, \nu) \mapsto (\tau - \Delta, \nu + \Delta)
\]

is well-defined.

The two maps \((126)\) and \((127)\) are mutually inverse (this is clear from their definitions), and thus are bijections. Hence, in particular, the map \((126)\) is a bijection.

Also, each \(\tau \in \mathbb{Z}^k\) satisfies

\[
|\tau + \Delta| = |\tau| + \sum_{i=1}^{\Delta} |\Delta|
\]

\[
= |\tau| + 1
\]

and thus

\[
(-1)^{n+|\tau+\Delta|} = (-1)^{n+|\tau|+1} = -(-1)^{n+|\tau|}.
\]

Now, recall that \(Q_0\) and \(Q_1\) are two subsets of \(Q\) such that each \((\tau, \nu) \in Q\) belongs to either \(Q_0\) or \(Q_1\) (but not both at the same time). In other words, \(Q_0\) and \(Q_1\) are two disjoint subsets of \(Q\) whose union is the whole set \(Q\). Hence, we

\(^{25}\text{Proof. Let } (\tau, \nu) \in Q_1. \text{ According to the definition of } Q_1, \text{ this means that } (\tau, \nu) \in Q \text{ and } \tau_1 = 1. \text{ We have } (\tau, \nu) \in Q. \text{ According to the definition of } Q, \text{ this means that } (\tau, \nu) \in V \times \mathbb{N}^k \text{ and } |\nu| = -|\tau| \text{ and } \nu + \tau = \gamma. \text{ From } (\tau, \nu) \in V \times \mathbb{N}^k, \text{ we obtain } \tau \in V \text{ and } \nu \in \mathbb{N}^k. \text{ Lemma } [11.15 \text{ (d)} \text{ yields } \nu + \Delta \in \mathbb{N}^k. \text{ Also, Lemma } [11.15 \text{ (b)} \text{ yields that } \tau - \Delta \in V \text{ and } (\tau - \Delta)_j = 0. \text{ Also, any two } k\text{-tuples } \alpha \in \mathbb{N}^k \text{ and } \beta \in \mathbb{N}^j \text{ satisfy } |\alpha + \beta| = |\alpha| + |\beta| \text{ and } |\alpha - \beta| = |\alpha| - |\beta|. \text{ Thus, } |\tau - \Delta| = |\tau| - |\Delta| \text{ and } |\nu + \Delta| = |\nu| + |\Delta| = |\tau| + |\Delta| = -|\nu|. \text{ Also, } \nu + \Delta + (\tau - \Delta) = \nu + \tau = \gamma. \text{ From } \tau - \Delta \in V \text{ and } \nu + \Delta \in \mathbb{N}^k \text{ and } |\nu + \Delta| = -|\tau - \Delta| \text{ and } \nu + \Delta + (\tau - \Delta) = \gamma, \text{ we obtain } (\tau - \Delta, \nu + \Delta) \in Q \text{ (by the definition of } Q). \text{ Combining this with } (\tau - \Delta)_j = 0, \text{ we obtain } (\tau - \Delta, \nu + \Delta) \in Q_0 \text{ (by the definition of } Q_0), \text{ qed.} \)
can split the sum \(\sum_{(\tau, \nu) \in Q} (-1)^{n+|\tau|}\) as follows:

\[
\sum_{(\tau, \nu) \in Q} (-1)^{n+|\tau|} = \sum_{(\tau, \nu) \in Q_0} (-1)^{n+|\tau|} + \sum_{(\tau, \nu) \in Q_1} (-1)^{n+|\tau|}
\]

\[
= \sum_{(\tau, \nu) \in Q_0} (-1)^{n+|\tau|} + \sum_{(\tau, \nu) \in Q_0} (-1)^{n+|\tau+\Delta|}
\]

(here, we have substituted \((\tau+\Delta, \nu-\Delta)\) for \((\tau, \nu)\) in the sum, since the map (126) is a bijection)

\[
= \sum_{(\tau, \nu) \in Q_0} (-1)^{n+|\tau|} = (-1)^{n+|\tau|} \quad \text{(by (128))}
\]

\[
= \sum_{(\tau, \nu) \in Q_0} (-1)^{n+|\tau|} - \sum_{(\tau, \nu) \in Q_0} (-1)^{n+|\tau|} = 0.
\]

Now, (125) yields

\[
\sum_{\tau \in V} \sum_{\nu \in \mathbb{N}^k; \left|\nu\right| = -|\tau|; \nu+\tau = \gamma} (-1)^{n+|\tau|} = \sum_{(\tau, \nu) \in Q} (-1)^{n+|\tau|} = 0 = \begin{cases} 1, & \text{if } \gamma = 0; \\ 0, & \text{if } \gamma \neq 0 \end{cases}
\]

(since \(\begin{cases} 1, & \text{if } \gamma = 0; \\ 0, & \text{if } \gamma \neq 0 \end{cases} = 0\)). Thus, Lemma 11.16 is proven in Case 1.

Let us now consider Case 2. In this case, we have \((\gamma_2, \gamma_3, \ldots, \gamma_k) = (0, 0, \ldots, 0)\) and \(\gamma_1 \neq 0\). From \(\gamma_1 \neq 0\), we obtain \(\gamma \neq 0\) and thus \(\begin{cases} 1, & \text{if } \gamma = 0; \\ 0, & \text{if } \gamma \neq 0 \end{cases} = 0\).
Now, $Q = \emptyset$. But (125) yields

$$\sum_{\tau \in V} \sum_{v \in \mathbb{N}^k; |v| = -|\tau|; v + \tau = \gamma} (-1)^{|v| + |\tau|} = \sum_{(\tau, v) \in Q} (-1)^{|v| + |\tau|} = \text{empty sum} \quad \text{(since } Q = \emptyset)$$

Hence, $\nu_1 = 0$. Also, $\nu_j = 0$ for each $j$. Therefore, $\nu = 0$.

Thus, Lemma 11.16 is proven in Case 2.

Let us finally consider Case 3. In this case, we have $(\gamma_2, \gamma_3, \ldots, \gamma_k) = (0, 0, \ldots, 0)$.

26Proof. Let $(\tau, v) \in Q$. We shall derive a contradiction.

Indeed, we have $(\tau, v) \in Q$. According to the definition of $Q$, this means that $(\tau, v) \in V \times \mathbb{N}^k$ and $|v| = -|\tau|$ and $v + \tau = \gamma$.

From $(\tau, v) \in V \times \mathbb{N}^k$, we obtain $\tau \in V$ and $v \in \mathbb{N}^k$.

We have $\tau \in V$. According to the definition of $V$, this means that $\tau$ is a $k$-tuple $(-n, \tau_2, \tau_3, \ldots, \tau_k) \in \mathbb{Z}^k$ satisfying (120). In other words, $\tau \in \mathbb{Z}^k$ and $\tau_1 = -n$ and the condition (120) holds.

Now, fix $j \in \{2, 3, \ldots, k\}$. Then, $\tau_j \in \{0, 1\}$ (by (120), applied to $i = j$). Hence, $\tau_j \geq 0$.

Also, $v_j \in \mathbb{N}$ (since $v \in \mathbb{N}^k$), so that $v_j \geq 0$. But $(\gamma_2, \gamma_3, \ldots, \gamma_k) = (0, 0, \ldots, 0)$, and thus $\gamma_j = 0$ (since $j \in \{2, 3, \ldots, k\}$). But $\gamma = v + \tau$, and thus $\gamma_j = (v + \tau)_j = v_j + \tau_j$. Hence, $\nu_j + \tau_j = \gamma_j = 0$, so that $\nu_j = -\tau_j \leq 0$. Combining this with $v_j \geq 0$, we obtain $v_j = 0$.

Hence, $v_j = -\tau_j$ rewrites as $0 = -\tau_j$, so that $\tau_j = 0$.

Now, forget that we fixed $j$. Thus, we have shown that each $j \in \{2, 3, \ldots, k\}$ satisfies

$$v_j = 0 \quad \text{(129)}$$

and

$$\tau_j = 0. \quad \text{(130)}$$

Now,

$$|\tau| = \tau_1 + \tau_2 + \cdots + \tau_k = \sum_{j=1}^k \tau_j = \tau_1 + \sum_{j=2}^k \tau_j = \tau_1 = -n, \quad \text{(by (130))}$$

so that $-|\tau| = n$. Furthermore,

$$|v| = v_1 + v_2 + \cdots + v_k = \sum_{j=1}^k v_j = v_1 + \sum_{j=2}^k v_j = v_1, \quad \text{(by (129))}$$

so that $v_1 = |v| = -|\tau| = n$.

Now, from $\gamma = v + \tau$, we obtain $\gamma_1 = (v + \tau)_1 = v_1 + \tau_1 = n + (-n) = 0$. This contradicts $\gamma_1 \neq 0$.

Now, forget that we fixed $(\tau, v)$. We thus have found a contradiction for each $(\tau, v) \in Q$. Thus, there exists no $(\tau, v) \in Q$. In other words, $Q = \emptyset$. 

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and $\gamma_1 = 0$. Combining these two equalities, we obtain $\gamma_i = 0$ for all $i \in \{1,2,\ldots,k\}$. In other words, $\gamma = 0$. Hence, \[
\begin{cases} 
1, & \text{if } \gamma = 0; \\
0, & \text{if } \gamma \neq 0.
\end{cases}
\]

Now, define two $k$-tuples $\tau_0 \in \mathbb{Z}^k$ and $v_0 \in \mathbb{Z}^k$ by

$$
\tau_0 = (-n,0,0,\ldots,0) \quad \text{and} \quad v_0 = (n,0,0,\ldots,0).
$$

Clearly, $\tau_0 \in V$ (by the definition of $V$) and $v_0 \in \mathbb{N}^k$ and $|\tau_0| = -n$ and $|v_0| = n$ and $v_0 + \tau_0 = 0$.

From $\tau_0 \in V$ and $v_0 \in \mathbb{N}^k$, we obtain $(\tau_0, v_0) \in V \times \mathbb{N}^k$. Also, $|v_0| = -|\tau_0|$ (since $|v_0| + |\tau_0| = n + (-n) = 0$) and $v_0 + \tau_0 = 0 = \gamma$. Thus, we have shown that $(\tau_0, v_0) \in V \times \mathbb{N}^k$ and $|v_0| = -|\tau_0|$ and $v_0 + \tau_0 = \gamma$. In other words, $(\tau_0, v_0) \in Q$ (by the definition of $Q$). In other words, $(\tau_0, v_0) \subseteq Q$.

On the other hand, $Q \subseteq \{(\tau_0, v_0)\}$. Combining this with $(\tau_0, v_0) \subseteq Q$, we obtain $Q = \{(\tau_0, v_0)\}$.

---

**Proof.** Let $(\tau, \nu) \in Q$. We shall prove that $(\tau, \nu) = (\tau_0, v_0)$.

Most of the following argument is copypasted from the previous footnote.

We have $(\tau, \nu) \in Q$. According to the definition of $Q$, this means that $(\tau, \nu) \in V \times \mathbb{N}^k$ and $|\nu| = -|\tau|$ and $\nu + \tau = \gamma$.

From $(\tau, \nu) \in V \times \mathbb{N}^k$, we obtain $\tau \in V$ and $\nu \in \mathbb{N}^k$.

We have $\tau \in V$ and $\nu \in \mathbb{N}^k$. According to the definition of $V$, this means that $\tau$ is a $k$-tuple $(-n, \tau_2, \ldots, \tau_k) \in \mathbb{Z}^k$ satisfying (120). In other words, $\tau \in \mathbb{Z}^k$ and $\tau = -n$ and the condition (120) holds.

Now, fix $j \in \{2,3,\ldots,k\}$. Then, $\tau_j \in \{0,1\}$ (by (120), applied to $i = j$). Hence, $\tau_j \geq 0$.

Also, $\nu_j \in \mathbb{N}$ (since $\nu \in \mathbb{N}^k$), so that $\nu_j \geq 0$. But $(\gamma_2, \ldots, \gamma_k) = (0,0,\ldots,0)$, and thus $\gamma_j = 0$ (since $j \in \{2,3,\ldots,k\}$). But $\gamma = \nu + \tau$, and thus $\gamma_j = (\nu + \tau)_j = \nu_j + \tau_j$. Hence, $\nu_j + \tau_j = \gamma_j = 0$, so that $\nu_j = -\tau_j \leq 0$. Combining this with $\nu_j \geq 0$, we obtain $\nu_j = 0$.

Hence, $\nu_j = -\tau_j$ rewrites as $0 = -\tau_j$, so that $\tau_j = 0$.

Now, forget that we fixed $j$. Thus, we have shown that each $j \in \{2,3,\ldots,k\}$ satisfies $\tau_j = 0$. In other words, $(\tau_2, \ldots, \tau_k) = (0,0,\ldots,0)$. Combining this with $\tau_1 = -n$, we obtain $\tau = (-n,0,0,\ldots,0) = \tau_0$.

From $\nu + \tau = \gamma$, we obtain $\nu = \gamma - \tau = \gamma - \tau_0 = v_0$ (since $v_0 + \tau_0 = \gamma$). Combining this with $\tau = \tau_0$, we obtain $(\tau, \nu) = (\tau_0, v_0) \subseteq \{(\tau_0, v_0)\}$.

Now, forget that we fixed $(\tau, \nu)$. We thus have proven that $(\tau, \nu) \in \{(\tau_0, v_0)\}$ for each $(\tau, \nu) \in Q$. In other words, $Q \subseteq \{(\tau_0, v_0)\}$. 

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But (125) yields
\[
\sum_{\tau \in V} \sum_{\nu \in \mathbb{N}^k; \nu + \tau = \gamma} (-1)^{n + |\tau|} = \sum_{(\tau, \nu) \in Q} (-1)^{n + |\tau|} = (-1)^{n + |\tau_0|} \quad \text{(since } Q = \{(\tau_0, v_0)\})
\]
\[
= (-1)^0 \quad \text{(since } n + |\tau_0| = n + (-n) = 0)
\]
\[
= 1 = \begin{cases} 1, & \text{if } \gamma = 0; \\ 0, & \text{if } \gamma \neq 0 \end{cases}
\]
(since \(1, \text{ if } \gamma = 0; 0, \text{ if } \gamma \neq 0 = 1\)). Thus, Lemma 11.16 is proven in Case 3.

We have now proven Lemma 11.16 in each of the three Cases 1, 2 and 3. Hence, Lemma 11.16 always holds. \(\square\)

**Proof of Theorem 11.14.** Each \(\tau \in V\) satisfies \(|\tau| \in \{n - k + 1, n - k + 2, \ldots, n\}\) (by Proposition 11.11). Thus, we have the following equality of summation signs:
\[
\sum_{\tau \in V} = \sum_{i=n-k+1}^{n} \sum_{|\tau|=i}^{\tau \in V} = \sum_{j=1}^{k} \sum_{|\tau|=n-k+j}^{\tau \in V} \quad \text{(131)}
\]
(here, we have substituted \(n - k + j\) for \(i\) in the outer sum). Now,
\[
\sum_{j=1}^{k} (-1)^{k-j} h_{n-k+j} \sum_{|\tau|=n-k+j}^{\tau \in V} s_{\mu + \tau}
\]
\[
= \sum_{j=1}^{k} \sum_{|\tau|=n-k+j}^{\tau \in V} \underbrace{(-1)^{k-j}}_{= (-1)^{n+|\tau|}} \underbrace{h_{n-k+j}}_{= h_{-|\tau|}} \sum_{|\tau|=n-k+j}^{\tau \in V} s_{\mu + \tau} \quad \text{(since } k-j=n+|\tau|\text{)} \quad \text{(since } n-k+j=-|\tau|\text{)} \quad \text{(because } -|\tau|=n-k+j\text{)} \quad \text{(because } -|\tau|=n-k+j\text{)}
\]
\[
= \sum_{\tau \in V} (-1)^{n+|\tau|} h_{-|\tau|} s_{\mu + \tau}. \quad \text{(132)}
\]

But each \(\tau \in V\) satisfies
\[
h_{-|\tau|} s_{\mu + \tau} = \sum_{\nu \in \mathbb{N}^k; |\nu|=-|\tau|} s_{\mu + (\nu + \tau)}. \quad \text{(133)}
\]

**Proof of (133):** Let \(\tau \in V\). According to the definition of \(V\), this means that \(\tau\) is a \(k\)-tuple \((-n, \tau_2, \tau_3, \ldots, \tau_k) \in \mathbb{Z}^k\) satisfying (120). In other words, \(\tau \in \mathbb{Z}^k\) and \(\tau_1 = -n\) and the relation (120) holds.
Proposition 11.11 yields \(-|\tau| \in \{n - k + 1, n - k + 2, \ldots, n\} \subseteq \mathbb{N}\).

Also, \(\mu \in P_k \subseteq \mathbb{N}^k\); hence,

\[
\mu_i \geq 0 \quad \text{for each } i \in \{1, 2, \ldots, k\}.
\] (134)

Also, \(\rho_1 = k - 1\) (by the definition of \(\rho\)) and \(\rho \in \mathbb{N}^k\) (likewise). Now,

\[
(\mu + \tau + \rho)_1 = \mu_1 + \tau_1 + \rho_1 > (n - k) + (-n) + (k - 1) = -1.
\]

Thus, \((\mu + \tau + \rho)_1 \geq 0\) (since \((\mu + \tau + \rho)_1\) is an integer). In other words, \((\mu + \tau + \rho)_1 \in \mathbb{N}\). Furthermore, for each \(i \in \{2, 3, \ldots, k\}\), we have

\[
(\mu + \tau + \rho)_i = \mu_i + \tau_i + \rho_i \in \mathbb{N}.
\]

This also holds for \(i = 1\) (since \((\mu + \tau + \rho)_1 \in \mathbb{N}\)). Thus, we have \((\mu + \tau + \rho)_i \in \mathbb{N}\) for each \(i \in \{1, 2, \ldots, k\}\). In other words, \(\mu + \tau + \rho \in \mathbb{N}^k\). Hence, Theorem 11.7 (applied to \(\lambda = \mu + \tau\) and \(m = -|\tau|\)) yields

\[
\sum_{\nu \in \mathbb{N}^k; \ |\nu| = -|\tau|} s_{\mu + \tau + \nu} = \sum_{\nu \in \mathbb{N}^k; \ |\nu| = -|\tau|} s_{\mu + (\nu + |\tau|)}.
\]

Thus,

\[
\sum_{\nu \in \mathbb{N}^k; \ |\nu| = -|\tau|} s_{\mu + \tau} = \sum_{\nu \in \mathbb{N}^k; \ |\nu| = -|\tau|} s_{\mu + (\nu + |\tau|)}.
\]

This proves (133).
Now, (132) becomes
\[
\sum_{j=1}^{k} (-1)^{k-j} h_{n-k+j} \sum_{\tau \in V; -|\tau| = n-k+j} s_{\mu+\tau}
\]
\[
= \sum_{\tau \in V} (-1)^{n+|\tau|} \sum_{\substack{v \in \mathbb{N}^k; \\
|v| = -|\tau|}} \frac{h_{-|\tau|} s_{\mu+\tau}}{s_{\mu+(v+\tau)}}
\]
\[
= \sum_{\tau \in V} (-1)^{n+|\tau|} s_{\mu+(v+\tau)} = \sum_{\tau \in V} (-1)^{n+|\tau|} \sum_{\gamma \in \mathbb{Z}^k} \sum_{\substack{v \in \mathbb{N}^k; \\
|v| = -|\tau|; v+\tau = \gamma}} s_{\mu+\gamma}
\]
\[
= \sum_{\tau \in V} (-1)^{n+|\tau|} \sum_{\gamma \in \mathbb{Z}^k} \sum_{\substack{v \in \mathbb{N}^k; \\
|v| = -|\tau|; v+\tau = \gamma}} (-1)^{n+|\tau|} s_{\mu+\gamma}
\]
\[
= \sum_{\gamma \in \mathbb{Z}^k} \begin{cases}
1, & \text{if } \gamma = 0; \\
0, & \text{if } \gamma \neq 0
\end{cases} s_{\mu+\gamma}
\]
\[
= \sum_{\gamma \in \mathbb{Z}^k} \begin{cases}
1, & \text{if } \gamma = 0; \\
0, & \text{if } \gamma \neq 0
\end{cases} s_{\mu+\gamma} = \sum_{\gamma \neq 0} \begin{cases}
1, & \text{if } \gamma = 0; \\
0, & \text{if } \gamma \neq 0
\end{cases} s_{\mu+\gamma}
\]
\[
= s_{\mu+0} = s_{\mu}.
\]
This proves Theorem 11.14.

Proof of Theorem 11.12. Theorem 11.14 yields
\[
s_{\mu} = \sum_{j=1}^{k} (-1)^{k-j} h_{n-k+j} \sum_{\tau \in V; -|\tau| = n-k+j} s_{\mu+\tau} \equiv \sum_{j=1}^{k} \sum_{\tau \in V; -|\tau| = n-k+j} a_j \sum_{\tau \in V; -|\tau| = n-k+j} s_{\mu+\tau} \mod I.
\]
Thus, in $S/I$, we have

$$\sum_{\tau \in V; -|\tau|=n-k+j} s_{\mu+\tau} = \sum_{j=1}^{k} (-1)^{k-j} a_j \sum_{\tau \in V; -|\tau|=n-k+j} s_{\mu+\tau}. \tag{11.12}$$

This proves Theorem 11.12.

12. Deforming symmetric functions

We have so far studied a quotient $S/I$ of the ring $S$ of symmetric polynomials in $k$ variables $x_1, x_2, \ldots, x_k$. But $S$ itself is a quotient of a larger ring – the ring $\Lambda$ of symmetric functions in infinitely many variables. More precisely,

$$S \cong \Lambda/\langle e_{k+1}, e_{k+2}, e_{k+3}, \ldots \rangle \tag{12.1}$$

(where $\langle \cdot \rangle$ means the ideal generated by whatever is inside the brackets). Hence,

$$S/I \cong \Lambda/\left( \langle h_{n-k+1} - a_1, h_{n-k+2} - a_2, \ldots, h_n - a_k \rangle \right).$$

This suggests a further generalization: What if we replace $e_{k+1}, e_{k+2}, e_{k+3}, \ldots$ by $e_{k+1} - b_1, e_{k+2} - b_2, e_{k+3} - b_3, \ldots$ for some constants $b_1, b_2, b_3, \ldots$? In the next corollary, we take a quick look at this generalization, at least in the case when $a_1, a_2, \ldots, a_k \in k$:

**Theorem 12.1.** Assume that $a_1, a_2, \ldots, a_k$ as well as $b_1, b_2, b_3, \ldots$ are elements of $k$. Let $\Lambda$ be the ring of symmetric functions in infinitely many indeterminates $x_1, x_2, x_3, \ldots$ over $k$. (See [GriRei18, Chapter 2] for more about this ring $\Lambda$.) Let $e_m$ and $h_m$ be the elementary symmetric functions and the complete homogeneous symmetric functions in $\Lambda$. For each partition $\lambda$, let $s_\lambda$ be the Schur function in $\Lambda$ corresponding to $\lambda$.

Let $K$ be the ideal

$$\langle h_{n-k+1} - a_1, h_{n-k+2} - a_2, \ldots, h_n - a_k \rangle + \langle e_{k+1} - b_1, e_{k+2} - b_2, e_{k+3} - b_3, \ldots \rangle$$

of $\Lambda$ (where $\langle \cdot \rangle$ means the ideal generated by whatever is inside the brackets). Then, $\Lambda/K$ is a free $k$-module with basis $(s_\lambda)_{\lambda \in P_{k,n}}$.

**Proof of Theorem 12.1.** Again, the Jacobi-Trudi identities ([GriRei18, (2.4.9) and (2.4.10)]) easily yield that the family $(s_\lambda)_{\lambda \in P_{k,n}}$ spans the $k$-module $\Lambda/K$. We need to prove that it is a basis of $\Lambda/K$.

Let us first recall that $\Lambda/\langle e_{k+1}, e_{k+2}, e_{k+3}, \ldots \rangle \cong S$. More precisely, there is a canonical surjective $k$-algebra homomorphism $\Lambda \to S$ which is given by substituting 0 for each of the variables $x_{k+1}, x_{k+2}, x_{k+3}, \ldots$; the kernel of this homomorphism is precisely the ideal $\langle e_{k+1}, e_{k+2}, e_{k+3}, \ldots \rangle$ of $\Lambda$. This homomorphism sends each $h_m \in \Lambda$ to the polynomial $h_m \in S$ defined in \[1\].
It is well-known that the commutative \( k \)-algebra \( \Lambda \) is freely generated by its elements \( e_1, e_2, e_3, \ldots \). Hence, we can define an endomorphism \( \varphi \) of the \( k \)-algebra \( \Lambda \) by letting
\[
\varphi(e_i) = e_i \quad \text{for each } i \in \{1, 2, \ldots, k\}; \\
\varphi(e_i) = e_i + b_{i-k} \quad \text{for each } i \in \{k + 1, k + 2, k + 3, \ldots\}.
\]

Consider this \( \varphi \). Then, clearly, \( \varphi \) is an automorphism of \( \Lambda \) (indeed, the inverse of \( \varphi \) can easily be constructed: it sends \( e_i \) to \( e_i \) for each \( i \in \{1, 2, \ldots, k\} \), and sends \( e_i \) to \( e_i - b_{i-k} \) for each \( i \in \{k + 1, k + 2, k + 3, \ldots\} \)). Thus,
\[
\varphi((e_{k+1} - b_1, e_{k+2} - b_2, e_{k+3} - b_3, \ldots)) = (e_{k+1}, e_{k+2}, e_{k+3}, \ldots). \tag{135}
\]

For each \( i \in \{1, 2, \ldots, k\} \), define \( c_i \in \Lambda \) by \( c_i = h_{n-k+i} - \varphi(h_{n-k+i}) + a_i \). Then,
\[
\varphi(h_{n-k+i} - a_i) = h_{n-k+i} - c_i \quad \text{for each } i \in \{1, 2, \ldots, k\}. \]

Adding this together with (135), we obtain
\[
\varphi(K) = \langle h_{n-k+1} - c_1, h_{n-k+2} - c_2, \ldots, h_n - c_k \rangle + \langle e_{k+1}, e_{k+2}, e_{k+3}, \ldots \rangle. \tag{136}
\]

The definition of \( \varphi \) shows that the endomorphism \( \varphi \) respects the filtration of \( \Lambda \); moreover, it shows that the endomorphism of the associated graded ring \( \text{gr} \Lambda \) induced by \( \varphi \) is the identity endomorphism. In other words, if \( f \in \Lambda \) is a symmetric function of degree \( \leq g \), then \( f - \varphi(f) \) is a symmetric function of degree \( < g \). Hence, for each \( i \in \{1, 2, \ldots, k\} \), the symmetric function \( h_{n-k+i} - \varphi(h_{n-k+i}) \) is a symmetric function of degree \( < n - k + i \). Hence, for each \( i \in \{1, 2, \ldots, k\} \), the symmetric function \( c_i \in \Lambda \) is a symmetric function of degree \( < n - k + i \) (since \( a_i \in k \)).

But \( \varphi \) is an automorphism of \( \Lambda \). Thus,
\[
\Lambda/K \cong \Lambda/\varphi(K) \\
\cong \Lambda/\langle h_{n-k+1} - c_1, h_{n-k+2} - c_2, \ldots, h_n - c_k \rangle + \langle e_{k+1}, e_{k+2}, e_{k+3}, \ldots \rangle \tag{by (136)}
\]
\[
\cong (\Lambda/\langle e_{k+1}, e_{k+2}, e_{k+3}, \ldots \rangle)/\langle h_{n-k+1} - c_1, h_{n-k+2} - c_2, \ldots, h_n - c_k \rangle \\
\cong S/\langle h_{n-k+1} - c_1, h_{n-k+2} - c_2, \ldots, h_n - c_k \rangle. \tag{137}
\]

But recall that for each \( i \in \{1, 2, \ldots, k\} \), the symmetric function \( c_i \in \Lambda \) is a symmetric function of degree \( < n - k + i \). Hence, for each \( i \in \{1, 2, \ldots, k\} \), the symmetric polynomial \( \overline{c_i} \in S \) is a symmetric polynomial of degree \( < n -
Thus, Theorem 2.7 (applied to $c_i$ instead of $a_i$) yields that the $k$-module $S/ \langle h_{n-k+1} - c_1, h_{n-k+2} - c_2, \ldots, h_n - c_k \rangle$ is free with basis $(\bar{s}_\lambda)_{\lambda \in P_{k,n}}$. Hence, this $k$-module is free and has a basis of size $|P_{k,n}|$. Thus, (137) shows that the $k$-module $\Lambda/K$ also is free and has a basis of size $|P_{k,n}|$.

Now, recall that the family $(\bar{s}_\lambda)_{\lambda \in P_{k,n}}$ spans the $k$-module $\Lambda/K$. Hence, this family must be a basis of $\Lambda/K$ (since it has the same size as a basis of $\Lambda/K$). This proves Theorem 12.1.

TODO: Explain why Theorem 12.1 remains true if we replace the requirements $a_i \in k$ and $b_i \in k$ by the weaker requirements $\deg(a_i) < n - k + i$ and $\deg(b_i) < k + i$.

References


