An equality for balanced digraphs

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Consider a directed multigraph D that is balanced (i.e., at each vertex, the indegree equals the outdegree). Let A be its set of arcs. Fix an integer k. Let s be a vertex of D. We show that the number of k-element subsets B of A that contain no cycles but contain a path from each vertex to s (we call them "s-convergences") is independent on s. This generalizes known facts about spanning arborescences, acyclic orientations and maximal acyclic subdigraphs (or, equivalently, minimum feedback arc sets).

1. The theorem

In this note, we shall discuss *balanced multidigraphs* – i.e., directed multigraphs (allowing loops and multiple arcs) in which each vertex satisfies "outdegree = indegree". We recall the relevant definitions in more detail:

A *multidigraph* (henceforth just *digraph*) means a triple (V, A, ψ), where V and A are two finite sets and $\psi : A \to V \times V$ is a map. The elements of V are called the *vertices* of this digraph, and the elements of A are called the *arcs* of this digraph. The *source* and *target* of an arc $a \in A$ are, respectively, the first and second entries of the pair $\psi(a)$. The *indegree* $\deg^- v$ of a vertex $v \in V$ means the number of arcs $a \in A$ whose target is v. The *outdegree* $\deg^+ v$ of a vertex $v \in V$ means the number of arcs $a \in A$ whose source is v. We say that a digraph (V, A, ψ) is *balanced* if and only if each vertex $v \in V$ satisfies $\deg^+ v = \deg^- v$. For further terminology on digraphs, we refer to [22s].[1]

A *to-root* of a digraph D means a vertex s of D such that for each vertex v of D, the digraph D has a path from v to s (equivalently, a walk from v to s).

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[1]The famous directed Euler–Hierholzer theorem (which will not be used in this note) says that a weakly connected digraph has an Eulerian circuit if and only if it is balanced. Thus, weakly connected balanced digraphs are also known as *Eulerian digraphs*.

From now on, we **fix a balanced digraph** $D = (V, A, \psi)$. If B is any subset of A, then $D\langle B\rangle$ will denote the induced subdigraph $(V, B, \psi\mid_B)$. A subset B of A will be called *acyclic* if the subdigraph $D\langle B\rangle$ has no cycles. ("Cycle" always means "directed cycle", as in [22s, Definition 4.5.1].)

Given a vertex s of D, we define an *s-convergence* to be an acyclic subset B of A such that s is a to-root of the subdigraph $D\langle B\rangle$.

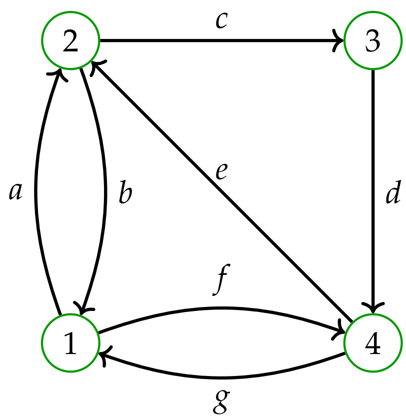
We can think of an s-convergence as a set B of arcs of D such that by following the B-arcs (i.e. the arcs in B) from any vertex $v \in V$, we will always arrive at s (no matter which B-arcs we take), and we will be stuck at s. (This intuition is formalized in Proposition 4.1.)

For any $k \in \mathbb{N}$ and $s \in V$, we let $\gamma_k(s)$ denote the number of s-convergences of size k (that is, with k arcs).[2]

In this note, we shall prove the following result:

Theorem 1.1. Let $k \in \mathbb{N}$. Then, the number $\gamma_k(s)$ does not depend on s. That is, $\gamma_k(s) = \gamma_k(t)$ for any $s, t \in V$.

Example 1.2. Let D be the following balanced multidigraph:



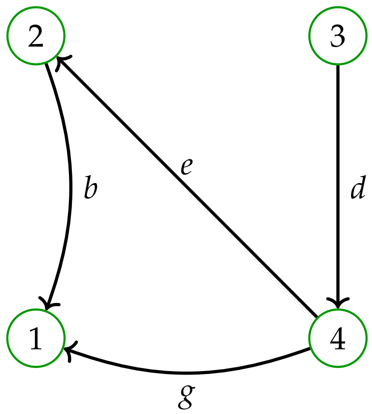
Then, the 1-convergences are the subsets

$$\{b, d, g\}, \qquad \{b, d, e\}, \qquad \{c, d, g\}, \qquad \{b, d, e, g\}, \qquad \{b, c, d, g\}.$$

Hence, $\gamma_3(1) = 3$, $\gamma_4(1) = 2$, and $\gamma_k(1) = 0$ for all $k \notin \{3, 4\}$. As a visual

[2] The symbol $\mathbb{N}$ denotes the set $\{0, 1, 2, \ldots\}$.

aid, below is the spanning subdigraph $D\langle B\rangle$ for $B = \{b, d, e, g\}$:



Theorem 1.1 says that for all $v \in \{1, 2, 3, 4\}$, we have $\gamma_3(v) = 3$, $\gamma_4(v) = 2$, and $\gamma_k(v) = 0$ for all $k \notin \{3, 4\}$. For example, the 2-convergences are the subsets

$$\{a, d, e\}, \qquad \{a, d, g\}, \qquad \{d, e, f\}, \qquad \{a, d, e, f\}, \qquad \{a, d, e, g\}.$$

Hence, $\gamma_3(2) = 3 = \gamma_3(1)$, $\gamma_4(2) = 2 = \gamma_4(1)$, and $\gamma_k(2) = 0 = \gamma_k(1)$ for all $k \notin \{3, 4\}$. The same holds for $v \in \{3, 4\}$.

2. Particular cases

Theorem 1.1 was inspired by a talk of Karla Leipold (NORCOM 2025), which made the first author aware of [LeiVal24, Lemma 4.1] and [PerPha15, §3.1]. The talk discussed no enumerative questions, yet was haunted by a perceptible aura of identities. The present note follows this aura to its source.

Some particular cases of Theorem 1.1 are known:

1. When $k = |V| - 1$, the s-convergences B of size k (or, more precisely, the respective subgraphs $D\langle B\rangle$ of D) are precisely the spanning arborescences of D rooted to s (see [22s, Definition 5.10.1 **(b)**] for the definition of this). Indeed, the condition $|B| = |V| - 1$, combined with the to-rootness of s, forces $D\langle B\rangle$ to be an arborescence rooted to s (by [22s, Theorem 5.10.5]), and conversely, if $D\langle B\rangle$ is a spanning arborescence rooted to s, then [22s, Theorem 5.10.5] shows that B is acyclic and $|B| = |V| - 1$. Thus, in the case $k = |V| - 1$, Theorem 1.1 is just [22s, Corollary 5.12.1].

   Likewise, if $k < |V| - 1$, then Theorem 1.1 is just saying that $0 = 0$, since a spanning subdigraph $D\langle B\rangle$ with fewer than $|V| - 1$ arcs cannot have a to-root.

2. If $D = G^{\text{bidir}}$ for some undirected multigraph $G = (V, E, \varphi)$ (this means that D is obtained from G by replacing each edge e with two arcs $e^{\rightarrow}$ and $e^{\leftarrow}$, going in opposite directions), and if $k = |E| = |A|/2$, then the s-convergences B of size k are just the acyclic orientations of G with unique sink s (because the acyclicity condition forbids B from containing both $e^{\rightarrow}$ and $e^{\leftarrow}$ for any given edge $e \in E$, but the size condition $|B| = k = |E|$ forces B to contain at least one of these two arcs; see Proposition 4.1 for the uniqueness of the sink). Thus, in this case, Theorem 1.1 is saying that the number of acyclic orientations of a given multigraph G with unique sink s does not depend on s. This is part of a result by Greene and Zaslavsky [GreZas83, Theorem 7.3], proved using hyperplane arrangements, and has recently been reproved combinatorially by Foissy [Foissy22, Proposition 4.6].

3. Up to reversing the directions of the arcs, [PerPha15, Proposition 3.7] is Theorem 1.1 for a specific value of k – namely, for the maximum possible that makes $\gamma_k(s)$ nonzero.

   We note that when D is weakly connected, then this maximum k is also the maximum size of an acyclic subset of A (not just of an s-convergence). This is a consequence of [PerPha15, Theorem 3.4]. It thus follows that finding this maximum k is equivalent to the *maximum acyclic subdigraph problem* for Eulerian (= weakly connected balanced) digraphs, which is known to be NP-hard by [PerPha15, Theorem 3.10] (see also [BanGut18, §3.7.1 and Lemma 4.4.3]). In the terminology of algorithmic combinatorics, this problem is often stated in terms of the complement of the acyclic subset; this complement is known as a *feedback arc set*. In these terms, our $\gamma_k(s)$ counts the feedback arc sets of size $|A| - k$; this counting problem is #P-complete [Perrot19, Theorem 5].

3. The proof

We will prove Theorem 1.1 through a sequence of lemmas, which are self-contained and might be of independent interest.

If P, Q and S are three sets, then the notation "$S = P \sqcup Q$" shall mean that $S = P \cup Q$ and $P \cap Q = \varnothing$. In other words, it shall mean that S is the union of the two disjoint sets P and Q. Of course, if $S = P \sqcup Q$, then P and Q are subsets of S and we have $|S| = |P| + |Q|$.

For any subsets P and Q of V, we let $A(P, Q)$ denote the set of arcs in A whose source belongs to P and whose target belongs to Q. The following fact is a simple but crucial property of balanced digraphs:

Proposition 3.1. Let P and Q be two subsets of V such that $V = P \sqcup Q$. Then,

$$|A(P, Q)| = |A(Q, P)|.$$

Proof. This is a known fact (see, e.g., [22s, Exercise 9.1]). The easiest way to prove it is as follows: By the definition of an outdegree, we have[3]

$$\begin{aligned}
&\sum_{p\in P} \deg^+ p \\
&= \sum_{p\in P} (\text{\# of arcs } a \in A \text{ with source } p) \\
&= (\text{\# of arcs } a \in A \text{ with source in } P) \\
&= \underbrace{(\text{\# of arcs } a \in A \text{ with source in } P \text{ and target in } P)}_{=|A(P,P)|} \\
&\qquad + \underbrace{(\text{\# of arcs } a \in A \text{ with source in } P \text{ and target in } Q)}_{=|A(P,Q)|} \\
&\qquad\qquad \left(\begin{array}{c}\text{since the target of any arc } a \in A \text{ belongs to either } P \text{ or } Q, \\ \text{but not to both (because } V = P \sqcup Q)\end{array}\right) \\
&= |A(P,P)| + |A(P,Q)|. \qquad (1)
\end{aligned}$$

An analogous argument (but with the roles of sources and targets switched) shows that

$$\sum_{p\in P} \deg^- p = |A(P,P)| + |A(Q,P)|. \qquad (2)$$

However, each $p \in P$ satisfies $\deg^+ p = \deg^- p$ (since D is balanced). Thus, the left hand sides of the equalities (1) and (2) are equal. Therefore, their right hand sides are equal as well. In other words, we have

$$|A(P,P)| + |A(P,Q)| = |A(P,P)| + |A(Q,P)|.$$

Subtracting $|A(P,P)|$ from this equality, we obtain $|A(P,Q)| = |A(Q,P)|$. Thus, Proposition 3.1 is proved. □

Now, given a subset B of A and two vertices $v, w \in V$, we say that "*v can B-reach w*" if the digraph $D\langle B\rangle$ has a path from v to w (or, equivalently, a walk from v to w).

Fix two vertices $s, t \in V$ and an integer $k \in \mathbb{N}$. We want to show that $\gamma_k(s) = \gamma_k(t)$.

For any subset B of A, we define the subsets

$$S(B) := \{v \in V \mid v \text{ can } B\text{-reach } s\} \qquad \text{and}$$
$$T(B) := \{v \in V \mid v \text{ can } B\text{-reach } t\} \qquad \text{of } V.$$

We call them the *attraction basins* of s and t with respect to B. Note that $s \in S(B)$ and $t \in T(B)$ always hold, since each vertex can B-reach itself (by a path of length 0).

The following property of attraction basins will be useful to our later arguments:

[3] The symbol "#" means "number".

Lemma 3.2. Let P and Q be two subsets of V such that $V = P \sqcup Q$. Let E be a subset of A. Let C be a subset of $A(P, Q)$. Then, $S(E) = P$ if and only if $S(E \cup C) = P$.

Proof. First, we observe that if B_1 and B_2 are two subsets of A satisfying $B_1 \subseteq B_2$, then any path of $D\langle B_1 \rangle$ is a path of $D\langle B_2 \rangle$, and thus we have $S(B_1) \subseteq S(B_2)$. Hence, $S(E) \subseteq S(E \cup C)$ (since $E \subseteq E \cup C$).

Note furthermore that P is disjoint from Q (since $V = P \sqcup Q$).

We must prove the equivalence $(S(E) = P) \iff (S(E \cup C) = P)$. We shall verify the $\Longrightarrow$ and $\Longleftarrow$ directions separately:

$\Longrightarrow$: Assume that $S(E) = P$. We must show that $S(E \cup C) = P$.

By assumption, we have $P = S(E) \subseteq S(E \cup C)$. It remains to prove the converse inclusion.

Let $v \in S(E \cup C)$. Thus, v can $E \cup C$-reach s. In other words, the digraph $D\langle E \cup C \rangle$ has a path $\mathbf{p}$ from v to s. Consider this path $\mathbf{p}$.

We claim that all arcs of $\mathbf{p}$ belong to E. Indeed, let us assume the contrary. Then, at least one of the arcs of $\mathbf{p}$ does not belong to E. Let c be the **last** arc of $\mathbf{p}$ that does not belong to E, and let q be the target of this arc c.

The arc c belongs to $E \cup C$ (since it is part of the path $\mathbf{p}$, which is a path of $D\langle E \cup C \rangle$). Since it does not belong to E, it must thus belong to C. Thus, $c \in C \subseteq A(P, Q)$, so that the target of c belongs to Q. In other words, $q \in Q$ (since q is the target of c).

However, c is the **last** arc of $\mathbf{p}$ that does not belong to E. Thus, all the arcs of $\mathbf{p}$ that come after c must belong to E. Therefore, the arcs of $\mathbf{p}$ that come after c form a path of the digraph $D\langle E \rangle$. This path starts at q (the target of c) and ends at s. Hence, we have shown that $D\langle E \rangle$ has a path from q to s. In other words, q can E-reach s, meaning that $q \in S(E)$. Thus, $q \in S(E) = P$, so that $q \notin Q$ (since P is disjoint from Q). This contradicts $q \in Q$.

This contradiction shows that our assumption was false. Hence, all arcs of $\mathbf{p}$ belong to E. Therefore, $\mathbf{p}$ is a path of $D\langle E \rangle$. Hence, v can E-reach s (by the path $\mathbf{p}$). That is, $v \in S(E) = P$.

Since we have proved this for each $v \in S(E \cup C)$, we thus conclude that $S(E \cup C) \subseteq P$. Combining this with $P \subseteq S(E \cup C)$, we obtain $S(E \cup C) = P$. This proves the "$\Longrightarrow$" direction of Lemma 3.2.

$\Longleftarrow$: Assume that $S(E \cup C) = P$. We must show that $S(E) = P$.

We have $S(E) \subseteq S(E \cup C) = P$ (by assumption). It remains to prove the converse inclusion.

Let $p \in P$. Then, $p \in P = S(E \cup C)$ (by assumption). Hence, p can $E \cup C$-reach s. That is, the digraph $D\langle E \cup C \rangle$ has a path $\mathbf{p}$ from p to s. Any vertex v of this path $\mathbf{p}$ must itself belong to $S(E \cup C)$ (since it can $E \cup C$-reach s by walking along $\mathbf{p}$ from v to s), so it cannot belong to Q (since $S(E \cup C) = P$ is disjoint from Q). Therefore, no arc of this path $\mathbf{p}$ can belong to $A(P, Q)$ (since this would require its target to belong to Q). Thus, all arcs of this path

$\mathbf{p}$ belong to $(E \cup C) \setminus \underbrace{A(P,Q)}_{\supseteq C} \subseteq (E \cup C) \setminus C \subseteq E$. Hence, $\mathbf{p}$ is a path of $D\langle E \rangle$. Consequently, the vertex p can E-reach s (via the path $\mathbf{p}$). In other words, $p \in S(E)$. Since we have proved this for each $p \in P$, we conclude that $P \subseteq S(E)$. Therefore, $S(E) = P$ (since $S(E) \subseteq P$). This proves the "$\Longleftarrow$" direction of Lemma 3.2. □

For any $i \in \mathbb{Z}$, we let $\mathcal{P}_i(A)$ denote the set of all i-element subsets of A. Define the subset

$$\begin{aligned} U_k &:= \{B \in \mathcal{P}_k(A) \text{ is acyclic} \mid \text{each vertex can } B\text{-reach } s \text{ or } t\} \\ &= \{B \in \mathcal{P}_k(A) \text{ is acyclic} \mid S(B) \cup T(B) = V\} \end{aligned}$$

of $\mathcal{P}_k(A)$. ("Vertex" means "vertex of D", that is, "element of V".)

We observe the following:

Lemma 3.3. We have

$$|\{B \in U_k \mid S(B) = V\}| = \gamma_k(s).$$

Proof. Note that an acyclic subset B of A is an s-convergence
if and only if s is a to-root of $D\langle B \rangle$,
i.e., if and only if each vertex $v \in V$ has a path to s in $D\langle B \rangle$,
i.e., if and only if each vertex $v \in V$ can B-reach s,
i.e., if and only if $S(B) = V$
(since $S(B)$ is defined as the set of all vertices $v \in V$ that can B-reach s). Thus, an s-convergence is the same thing as an acyclic subset B of A that satisfies $S(B) = V$.

Hence, the number $\gamma_k(s)$ of all s-convergences of size k can also be described as the number of all acyclic subsets B of A of size k that satisfy $S(B) = V$. In other words,

$$\gamma_k(s) = |\{B \in \mathcal{P}_k(A) \text{ is acyclic} \mid S(B) = V\}|. \tag{3}$$

But any acyclic $B \in \mathcal{P}_k(A)$ satisfying $S(B) = V$ must also satisfy $B \in U_k$ (since $\underbrace{S(B)}_{=V} \cup T(B) = V \cup T(B) = V$). Conversely, the definition of U_k shows that any $B \in U_k$ is acyclic and belongs to $\mathcal{P}_k(A)$. These two facts show that

$$\{B \in \mathcal{P}_k(A) \text{ is acyclic} \mid S(B) = V\} = \{B \in U_k \mid S(B) = V\}.$$

Hence, we can rewrite (3) as

$$\gamma_k(s) = |\{B \in U_k \mid S(B) = V\}|.$$

This proves Lemma 3.3. □

Now we claim the following:

Lemma 3.4. We have

$$\gamma_k(s) = |U_k| - \sum_{\substack{P,Q \subseteq V \text{ are nonempty;} \\ V = P \sqcup Q}} |\{B \in U_k \mid S(B) = P\}|.$$

Proof. Each $B \in U_k$ satisfies $S(B) = P$ for some nonempty subset P of V (indeed, the set $S(B)$ is nonempty since it contains s). Hence, by the sum rule,

$$\begin{aligned}
|U_k| &= \sum_{P \subseteq V \text{ nonempty}} |\{B \in U_k \mid S(B) = P\}| \\
&= \sum_{\substack{P,Q \subseteq V; \\ P \text{ is nonempty;} \\ V = P \sqcup Q}} |\{B \in U_k \mid S(B) = P\}| \\
&\qquad \left(\begin{array}{c}\text{here, we have introduced an extra index } Q \\ \text{into the sum (to make it more symmetric);} \\ \text{the condition ``} V = P \sqcup Q \text{'' ensures that it} \\ \text{is uniquely determined by } P\end{array}\right) \\
&= \sum_{\substack{P,Q \subseteq V \text{ are nonempty;} \\ V = P \sqcup Q}} |\{B \in U_k \mid S(B) = P\}| + \underbrace{|\{B \in U_k \mid S(B) = V\}|}_{\substack{= \gamma_k(s) \\ \text{(by Lemma 3.3)}}} \\
&\qquad \left(\begin{array}{c}\text{here, we have split off the addend} \\ \text{for } (P,Q) = (V,\varnothing) \text{ from the sum}\end{array}\right) \\
&= \sum_{\substack{P,Q \subseteq V \text{ are nonempty;} \\ V = P \sqcup Q}} |\{B \in U_k \mid S(B) = P\}| + \gamma_k(s).
\end{aligned}$$

Solving this for $\gamma_k(s)$, we obtain the claim of the lemma. □

Similarly, we find:

Lemma 3.5. We have

$$\gamma_k(t) = |U_k| - \sum_{\substack{P,Q \subseteq V \text{ are nonempty;} \\ V = P \sqcup Q}} |\{B \in U_k \mid T(B) = Q\}|.$$

Proof. Analogous to the proof of Lemma 3.4. (Just switch the roles of s and t and also the roles of P and Q.) □

Our goal is to prove $\gamma_k(s) = \gamma_k(t)$. In light of Lemma 3.4 and Lemma 3.5, it will suffice to show the following:

Proposition 3.6. Let P and Q be two nonempty subsets of V such that $V = P \sqcup Q$. Then,

$$|\{B \in U_k \mid S(B) = P\}| = |\{B \in U_k \mid T(B) = Q\}|.$$

We will prove this by finding analogous expressions for both sides. First, we need some notation:

Definition 3.7. If P and Q are two subsets of V satisfying $V = P \sqcup Q$, and if $i \in \mathbb{Z}$ is arbitrary, then we set

$$X_i^{P,Q} := \{B \in \mathcal{P}_i(A) \text{ is acyclic} \mid S(B) = P \text{ and } T(B) = Q\}.$$

Lemma 3.8. Let P and Q be two nonempty subsets of V such that $V = P \sqcup Q$. Then,

$$|\{B \in U_k \mid S(B) = P\}| = \sum_{m \in \mathbb{N}} \binom{|A(P,Q)|}{m} \cdot \left| X_{k-m}^{P,Q} \right|.$$

We note that the infinite sum on the right hand side here is well-defined, since it has only finitely many nonzero addends.[4]

Proof of Lemma 3.8. By the sum rule,

$$|\{B \in U_k \mid S(B) = P\}| = \sum_{C \subseteq A(P,Q)} |\{B \in U_k \mid S(B) = P \text{ and } B \cap A(P,Q) = C\}|, \qquad (4)$$

since the intersection $B \cap A(P,Q)$ is always a subset of $A(P,Q)$.

Fix a subset $C \subseteq A(P,Q)$, and let

$$Y_k := \{B \in U_k \mid S(B) = P \text{ and } B \cap A(P,Q) = C\}.$$

We want to show that $|Y_k| = \left| X_{k-|C|}^{P,Q} \right|$. Define

$$\text{the map } \Phi : Y_k \to X_{k-|C|}^{P,Q}$$
$$\text{by } \Phi(B) = B \setminus C$$

[4]Indeed, all negative integers i satisfy $\mathcal{P}_i(A) = \varnothing$ and thus $X_i^{P,Q} = \varnothing$, so that $\left|X_i^{P,Q}\right| = 0$; thus, we conclude that $\left|X_{k-m}^{P,Q}\right| = 0$ whenever $m > k$. Alternatively, we can observe that $\binom{|A(P,Q)|}{m} = 0$ whenever $m > |A(P,Q)|$.

and

$$\text{the map } \Psi : X^{P,Q}_{k-|C|} \to Y_k \quad \text{by } \Psi(E) = E \cup C.$$

Let us first show that these maps are well-defined:

> *Claim 1.* The map Φ is well-defined. That is, $B \setminus C \in X^{P,Q}_{k-|C|}$ for all $B \in Y_k$.

Proof of Claim 1. Let $B \in Y_k$. Then $B \in U_k$ as well as $S(B) = P$ and $B \cap A(P,Q) = C$. From $B \in U_k$, we see that B is an acyclic k-element subset of A, and that each vertex can B-reach s or B-reach t. In particular, $|B| = k$.

We must show that $B \setminus C \in X^{P,Q}_{k-|C|}$. In other words, we must show that $B \setminus C$ is an acyclic set in $\mathcal{P}_{k-|C|}(A)$ and satisfies $S(B \setminus C) = P$ and $T(B \setminus C) = Q$.

It is clear that $B \setminus C$ is acyclic, since removing arcs from the acyclic set B cannot create any cycles.

Moreover, from $C = B \cap A(P,Q) \subseteq B$, we obtain

$$|B \setminus C| = |B| - |C| = k - |C| \qquad (\text{since } |B| = k).$$

Thus, $B \setminus C \in \mathcal{P}_{k-|C|}(A)$. Also, from $C \subseteq B$, we obtain $(B \setminus C) \cup C = B$.

Note also that

$$B \setminus A(P,Q) = B \setminus \underbrace{(B \cap A(P,Q))}_{=C} = B \setminus C.$$

Thus, the set $B \setminus C$ is disjoint from $A(P,Q)$ (since $B \setminus A(P,Q)$ clearly is).

Next we show that $B \cap A(Q,P) = \varnothing$. Indeed, assume the contrary. Then, there is an arc $b \in B \cap A(Q,P)$. Consider this b. Thus, $b \in B$ and $b \in A(Q,P)$. The latter shows that the arc b has a source $q \in Q$ and a target $p \in P$. The vertex p can B-reach s (since $p \in P = S(B)$); hence, the vertex q can B-reach s as well (by first stepping to p via the arc b). This means that $q \in S(B) = P$, which contradicts $q \in Q$ since $V = P \sqcup Q$. This contradiction shows that our assumption was false, so we know that $B \cap A(Q,P) = \varnothing$.

In other words, no arc in B belongs to $A(Q,P)$. Hence, no arc in $B \setminus C$ belongs to $A(Q,P)$ either (since $B \setminus C \subseteq B$). Furthermore, no arc in $B \setminus C$ belongs to $A(P,Q)$ (since the set $B \setminus C$ is disjoint from $A(P,Q)$).

Let us now prove that $S(B \setminus C) = P$. Indeed, Lemma 3.2 (applied to $E = B \setminus C$) yields that $S(B \setminus C) = P$ if and only if $S((B \setminus C) \cup C) = P$. Since $S((B \setminus C) \cup C) = P$ does hold (because $S\left(\underbrace{(B \setminus C) \cup C}_{=B}\right) = S(B) = P$), we thus conclude that $S(B \setminus C) = P$.

It remains to show that $T(B \setminus C) = Q$.

First, we claim that $t \in Q$. Indeed, there exists some vertex $v \in Q$ (since Q is nonempty). Consider this v. From $v \in Q$ and $V = P \sqcup Q$, we obtain $v \notin P = S(B)$, and thus v cannot B-reach s. Hence, v can B-reach t (since each vertex can B-reach s or B-reach t). If t could B-reach s, then this would entail that v can also B-reach s (by concatenating the path from v to t and the path from t to s), and this would contradict the fact that v cannot B-reach s. Hence, t cannot B-reach s. In other words, $t \notin S(B) = P$, so that $t \in Q$ (since $V = P \sqcup Q$).

Let us now prove that $T(B \setminus C) \subseteq Q$. Indeed, let $x \in T(B \setminus C)$ be any vertex. We must show that $x \in Q$. Assume the contrary. Then, $x \in P$ (since $V = P \sqcup Q$). But $x \in T(B \setminus C)$ shows that x can $B \setminus C$-reach t. So the digraph $D\langle B \setminus C\rangle$ has a path $\mathbf{x}$ from x to t. This path $\mathbf{x}$ starts at a vertex in P (namely, at $x \in P$) and ends at a vertex in Q (namely, at $t \in Q$). So this path $\mathbf{x}$ must cross from P to Q at some point (since $V = P \sqcup Q$). In other words, it must contain an arc that belongs to $A(P, Q)$. But this is impossible, since all arcs of $\mathbf{x}$ come from the set $B \setminus C$, which is disjoint from $A(P, Q)$. This contradiction shows that $x \in Q$. Since we have proved this for each $x \in T(B \setminus C)$, we thus conclude that $T(B \setminus C) \subseteq Q$.

Now, we claim that $Q \subseteq T(B \setminus C)$. Indeed, let $q \in Q$ be any vertex. Then, $q \notin P$ (since $V = P \sqcup Q$), so that $q \notin P = S(B)$. In other words, q cannot B-reach s. Hence, q can B-reach t (since each vertex can B-reach s or B-reach t). That is, the digraph $D\langle B\rangle$ has a path $\mathbf{q}$ from q to t. Any vertex w of this path $\mathbf{q}$ must satisfy $w \notin P$ (since otherwise, w would belong to $P = S(B)$ and thus could B-reach s, so that the digraph $D\langle B\rangle$ would have a path $\mathbf{w}$ from w to s; but then the vertex q could also B-reach s by first following the path $\mathbf{q}$ from q until w and then following the path $\mathbf{w}$ from w to s, contradicting the fact that q cannot B-reach s). Therefore, no arc of the path $\mathbf{q}$ can belong to $A(P, Q)$ (since the source of such an arc would be a vertex of $\mathbf{q}$ and belong to P). Thus, all arcs of $\mathbf{q}$ belong to $B \setminus A(P, Q) = B \setminus C$. Hence, $\mathbf{q}$ is a path of $D\langle B \setminus C\rangle$. This shows that q can $B \setminus C$-reach t (via the path $\mathbf{q}$). In other words, $q \in T(B \setminus C)$. Since we have proved this for each $q \in Q$, we thus conclude that $Q \subseteq T(B \setminus C)$.

Combining this with $T(B \setminus C) \subseteq Q$, we obtain $T(B \setminus C) = Q$.

Altogether, we now have shown that $B \setminus C$ is an acyclic set in $\mathcal{P}_{k-|C|}(A)$ and satisfies $S(B \setminus C) = P$ and $T(B \setminus C) = Q$. In other words, $B \setminus C \in X^{P,Q}_{k-|C|}$. Claim 1 is thus proved. □

Claim 2. The map Ψ is well-defined. That is, $E \cup C \in Y_k$ for all $E \in X^{P,Q}_{k-|C|}$.

Proof of Claim 2. Let $E \in X^{P,Q}_{k-|C|}$. Then E is an acyclic $(k - |C|)$-element subset of A satisfying $S(E) = P$ and $T(E) = Q$.

We must show that $E \cup C \in Y_k$. In other words, we must show that $E \cup C \in U_k$, $S(E \cup C) = P$, and $(E \cup C) \cap A(P, Q) = C$.

From $S(E) = P$, we immediately obtain $S(E \cup C) = P$ by Lemma 3.2.

Next, we show that $E \cap A(P, Q) = \varnothing$. Indeed, assume the contrary. Then, there exists some arc $e \in E \cap A(P, Q)$. Thus, $e \in E$ and $e \in A(P, Q)$. The latter shows that e has a source $p \in P$ and a target $q \in Q$. From $q \in Q = T(E)$, we see that q can E-reach t. Thus, p can also E-reach t (via the arc $e \in E$ followed by the path from q to t). Therefore, $p \in T(E) = Q$, which contradicts $p \in P$ because $V = P \sqcup Q$. This contradiction shows that our assumption was false. Hence, $E \cap A(P, Q) = \varnothing$ is proved.

An analogous argument shows that $E \cap A(Q, P) = \varnothing$.

Note that the set $A(P, Q)$ is disjoint from $A(Q, P)$, since the source of an arc cannot belong to P and Q at the same time (because of $V = P \sqcup Q$). Hence, the set C (being a subset of $A(P, Q)$) must be disjoint from $A(Q, P)$ as well. In other words, $C \cap A(Q, P) = \varnothing$.

Also, note that C is a subset of $A(P, Q)$, and thus we have

$$E \cap \underbrace{C}_{\subseteq A(P,Q)} \subseteq E \cap A(P, Q) = \varnothing.$$

Hence,

$$E \cap C = \varnothing, \tag{5}$$

so that

$$|E \cup C| = |E| + |C| = k \qquad (\text{since } |E| = k - |C|).$$

Therefore, $E \cup C \in \mathcal{P}_k(A)$.

Furthermore,

$$(E \cup C) \cap A(P, Q) = \underbrace{(E \cap A(P, Q))}_{=\varnothing} \cup \underbrace{(C \cap A(P, Q))}_{\substack{=C \\ (\text{since } C \subseteq A(P,Q))}} = \varnothing \cup C = C$$

and

$$(E \cup C) \cap A(Q, P) = \underbrace{(E \cap A(Q, P))}_{=\varnothing} \cup \underbrace{(C \cap A(Q, P))}_{=\varnothing} = \varnothing \cup \varnothing = \varnothing.$$

Next, we show that $E \cup C$ is acyclic. Indeed, assume the contrary. Then, the digraph $D\langle E \cup C \rangle$ has a cycle $\mathbf{c}$. Since E itself is acyclic, this cycle $\mathbf{c}$ must use at least one arc from C. This arc must have source in P and target in Q (because it belongs to $C \subseteq A(P, Q)$). Thus, the cycle $\mathbf{c}$ contains both a vertex in P and a vertex in Q. Consequently, it must cross from Q to P at some point (since $V = P \sqcup Q$). In other words, it contains an arc $a \in A(Q, P)$. But such an arc a cannot belong to $E \cup C$ (since $(E \cup C) \cap A(Q, P) = \varnothing$). This is a contradiction, since it is an arc of $\mathbf{c}$, which is a cycle of $D\langle E \cup C \rangle$. This contradiction shows that our assumption was wrong, so $E \cup C$ is indeed acyclic.

Each vertex in P can $E \cup C$-reach s (since it lies in $P = S(E \cup C)$). Each vertex in Q can E-reach t (since it lies in $Q = T(E)$) and thus can $E \cup C$-reach

t as well (since $E \subseteq E \cup C$). Since each vertex in V belongs to either P or Q (because $V = P \sqcup Q$), we thus conclude that each vertex in V can $E \cup C$-reach s (if it belongs to P) or can $E \cup C$-reach t (if it belongs to Q). This shows that $E \cup C \in U_k$ (since $E \cup C \in \mathcal{P}_k(A)$ is acyclic).

So, we have shown that $E \cup C \in U_k$, and $S(E \cup C) = P$, and $(E \cup C) \cap A(P,Q) = C$. All together, this yields $E \cup C \in Y_k$. Claim 2 is thus proved. □

Claim 3. The maps Φ and Ψ are mutually inverse.

Proof of Claim 3. For each $B \in Y_k$, we have $B \cap A(P,Q) = C$ and therefore $C = B \cap A(P,Q) \subseteq B$. Thus, for each $B \in Y_k$, we have

$$\Psi(\Phi(B)) = \Psi(B \setminus C) = (B \setminus C) \cup C = B \qquad (\text{since } C \subseteq B).$$

In other words, $\Psi \circ \Phi = \text{id}$.

On the other hand, for each $E \in X_{k-|C|}^{P,Q}$, we have $E \cap C = \varnothing$ (see the equality (5) in the proof of Claim 2) and thus

$$\Phi(\Psi(E)) = \Phi(E \cup C) = (E \cup C) \setminus C = E \qquad (\text{since } E \cap C = \varnothing).$$

This shows that $\Phi \circ \Psi = \text{id}$.

From $\Phi \circ \Psi = \text{id}$ and $\Psi \circ \Phi = \text{id}$, we conclude that the maps Φ and Ψ are mutually inverse. This proves Claim 3. □

Claim 3 shows that the map Ψ is a bijection from $X_{k-|C|}^{P,Q}$ to Y_k. Thus,

$$\left| X_{k-|C|}^{P,Q} \right| = |Y_k| = |\{B \in U_k \mid S(B) = P \text{ and } B \cap A(P,Q) = C\}|$$

(by the definition of Y_k).

We have proved this equality for each subset C of $A(P,Q)$. Summing it over all such subsets, we obtain

$$\sum_{C \subseteq A(P,Q)} \left| X_{k-|C|}^{P,Q} \right| = \sum_{C \subseteq A(P,Q)} |\{B \in U_k \mid S(B) = P \text{ and } B \cap A(P,Q) = C\}|$$
$$= |\{B \in U_k \mid S(B) = P\}| \qquad (\text{by (4)}).$$

Thus,

$$\begin{aligned}
\left|\left\{B \in U_k \ \mid \ S\left(B\right) = P\right\}\right| &= \sum_{C \subseteq A(P,Q)} \left| X_{k-|C|}^{P,Q} \right| = \sum_{m \in \mathbb{N}} \ \sum_{\substack{C \subseteq A(P,Q); \\ |C| = m}} \underbrace{\left| X_{k-|C|}^{P,Q} \right|}_{\substack{= \left| X_{k-m}^{P,Q} \right| \\ (\text{since } |C| = m)}} \\
&= \sum_{m \in \mathbb{N}} \underbrace{\sum_{\substack{C \subseteq A(P,Q); \\ |C| = m}} \left| X_{k-m}^{P,Q} \right|}_{\substack{= \binom{\left|A\left(P,Q\right)\right|}{m} \cdot \left| X_{k-m}^{P,Q} \right| \\ (\text{since this is a sum of } \binom{\left|A\left(P,Q\right)\right|}{m} \\ \text{many equal addends})}} \\
&= \sum_{m \in \mathbb{N}} \binom{\left|A\left(P,Q\right)\right|}{m} \cdot \left| X_{k-m}^{P,Q} \right| .
\end{aligned}$$

This proves Lemma 3.8. □

Lemma 3.9. Let P and Q be two nonempty subsets of V such that $V = P \sqcup Q$. Then,

$$\left|\left\{B \in U_k \ \mid \ T\left(B\right) = Q\right\}\right| = \sum_{m \in \mathbb{N}} \binom{\left|A\left(Q,P\right)\right|}{m} \cdot \left| X_{k-m}^{P,Q} \right| .$$

Proof. Analogous to the proof of Lemma 3.8. (Just switch the roles of s and t and also the roles of P and Q.) □

Proof of Proposition 3.6. Lemma 3.8 yields

$$\begin{aligned}
\left|\left\{B \in U_k \ \mid \ S\left(B\right) = P\right\}\right| &= \sum_{m \in \mathbb{N}} \binom{\left|A\left(P,Q\right)\right|}{m} \cdot \left| X_{k-m}^{P,Q} \right| \\
&= \sum_{m \in \mathbb{N}} \binom{\left|A\left(Q,P\right)\right|}{m} \cdot \left| X_{k-m}^{P,Q} \right| \\
&\qquad \left(\begin{array}{c} \text{since Proposition 3.1} \\ \text{yields } \left|A\left(P,Q\right)\right| = \left|A\left(Q,P\right)\right| \end{array} \right) \\
&= \left|\left\{B \in U_k \ \mid \ T\left(B\right) = Q\right\}\right| \qquad (\text{by Lemma 3.9}) .
\end{aligned}$$

This proves Proposition 3.6. □

Now, Lemma 3.4 yields

$$\gamma_k(s) = |U_k| - \sum_{\substack{P,Q \subseteq V \text{ are nonempty;} \\ V = P \sqcup Q}} \underbrace{|\{B \in U_k \mid S(B) = P\}|}_{\substack{= |\{B \in U_k \mid T(B) = Q\}| \\ \text{(by Proposition 3.6)}}}$$
$$= |U_k| - \sum_{\substack{P,Q \subseteq V \text{ are nonempty;} \\ V = P \sqcup Q}} |\{B \in U_k \mid T(B) = Q\}| = \gamma_k(t)$$

(by Lemma 3.5). This completes the proof of Theorem 1.1.

4. Further remarks

1. One might wonder whether our proof of Theorem 1.1 is, or can be made, bijective. In the given form, it is not, as it uses subtraction twice: once in proving Proposition 3.1 and once again in the "subtractive flip" that is involved in Lemma 3.4 (and Lemma 3.5). Both of these instances of subtraction can be made bijective using the Garsia–Milne involution principle [StaWhi86, §4.6].

In the case of Lemma 3.4, only the simplest case of the involution principle is needed: Let $\Gamma_k(s)$ be the set of all s-convergences of size k, and let $\Gamma_k(t)$ be the set of all t-convergences of size k. Our above proof of Proposition 3.6 gives a bijection

$$\{B \in U_k \mid S(B) = P\} \to \{B \in U_k \mid T(B) = Q\}$$

for every pair of nonempty subsets $P, Q \subseteq V$ satisfying $V = P \sqcup Q$. Combining these bijections for all such pairs (P, Q), we obtain a bijection $\phi : U_k \setminus \Gamma_k(s) \to U_k \setminus \Gamma_k(t)$. Now we need to construct a bijection $\psi : \Gamma_k(s) \to \Gamma_k(t)$ from it. The involution principle tells us that this ψ acts on a given element $B \in \Gamma_k(s)$ by repeatedly applying ϕ^{-1} to it until it no longer belongs to $U_k \setminus \Gamma_k(t)$ (which means that it belongs to $\Gamma_k(t)$). It is not hard to see that this does not require more than $|V|$ many iterations, since $|T(B)|$ increases with each application of ϕ^{-1}.

In the case of Proposition 3.1, however, we can also give a direct bijective proof: By the directed Euler–Hierholzer theorem, each weak component of the balanced digraph D has a Eulerian circuit. Pick such a circuit for each weak component of D. Note that on each of these circuits, arcs from $A(P, Q)$ and arcs from $A(Q, P)$ alternate (if we remove the arcs from $A(P, P)$ and $A(Q, Q)$). Thus, we can define a bijection $A(P, Q) \to A(Q, P)$ that sends each arc $a \in A(P, Q)$ to the next arc $b \in A(Q, P)$ following it on the chosen Eulerian circuit. At a second thought, this does not even require Eulerian circuits; it suffices to pick any decomposition of A into (arc-disjoint) circuits.

2. We can also characterize s-convergences in terms of sinks. Recall that a *sink* of a digraph means a vertex with no outgoing arcs (i.e., a vertex with outdegree 0). Now our alternative characterization of s-convergences is as follows:

Proposition 4.1. Let B be an acyclic subset of A. Let $s \in V$. Then, B is an s-convergence if and only if s is the only sink of the subdigraph $D\langle B\rangle$.

Proof. $\Longrightarrow$: Assume that B is an s-convergence. Thus, s is a to-root of $D\langle B\rangle$.

We shall first show that s is a sink of $D\langle B\rangle$. Indeed, assume the contrary; thus, there exists an arc a of $D\langle B\rangle$ with source s. Let v be the target of this arc a. Since s is a to-root of $D\langle B\rangle$, there exists a path from v to s in $D\langle B\rangle$. This path, together with the arc a, creates a cycle[5] in $D\langle B\rangle$, which contradicts the acyclicity of B. So, we see that s is a sink of $D\langle B\rangle$.

Furthermore, no vertex $v \neq s$ can also be a sink of $D\langle B\rangle$, for $D\langle B\rangle$ must have a path from v to s (because s is a to-root of $D\langle B\rangle$) and this path must begin with an arc with source v. So we conclude that s is the only sink of $D\langle B\rangle$.

$\Longleftarrow$: Assume that s is the only sink of $D\langle B\rangle$. Let $v \in V$ be any vertex.

The digraph $D\langle B\rangle$ has no cycles (since B is acyclic). Thus, any walk of $D\langle B\rangle$ is a path (since a walk that is not a path would contain a cycle). Consequently, the walks of $D\langle B\rangle$ are precisely the paths of $D\langle B\rangle$. Hence, the digraph $D\langle B\rangle$ has only finitely many walks that start at v (since it clearly has only finitely many **paths** that start at v). Moreover, there is at least one such walk (the length-0 walk (v)).

Thus, there is a longest such walk. Let $\mathbf{p}$ be a such. Then, $\mathbf{p}$ must end at a sink of $D\langle B\rangle$ (since otherwise, we could extend $\mathbf{p}$ by an additional arc at the end, obtaining an even longer walk that starts at v). Since the only sink of $D\langle B\rangle$ is s, this means that $\mathbf{p}$ must end at s. Thus, $\mathbf{p}$ is a walk from v to s, hence a path from v to s (since any walk of $D\langle B\rangle$ is a path). Hence, $D\langle B\rangle$ has a path from v to s (namely, $\mathbf{p}$). Since v was arbitrary, this shows that s is a to-root of $D\langle B\rangle$. Therefore, B is an s-convergence (since B is acyclic). □

3. In Lemma 3.4, Lemma 3.5, Lemma 3.8, Lemma 3.9 and Proposition 4.1, there is no need for the digraph D to be balanced.

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