Arithmetic and Anti-Arithmetic Products of Symmetric Functions: A Representation-Theoretic Proof of Integrality

GPT-6, partly edited by Darij Grinberg, not yet proofread

unproofread draft, September 18, 2026

Abstract

The power sums form a basis of the algebra of symmetric functions over $\mathbb{Q}$, but not over $\mathbb{Z}$. Thus an operation defined by a simple rule on power sums need not preserve integral symmetric functions. Yet, many of them do, each time requiring proof. We study two such operations. The *arithmetic product* $\boxdot$ is given by

$$p_\lambda \boxdot p_\mu = \prod_{i,j} p_{\mathrm{lcm}(\lambda_i,\mu_j)}^{\gcd(\lambda_i,\mu_j)},$$

and the *anti-arithmetic product* by the same formula with gcd and lcm interchanged. The arithmetic product is known to preserve integral symmetric functions, and is in fact Schur-positive on pairs of Schur functions. The analogous integrality of the anti-arithmetic product was posed as an open question on MathOverflow in 2014.

We prove both integrality statements using representation theory. For the arithmetic product, the representation-theoretic interpretation is well-known: The Frobenius characteristic identifies the n-th graded components of $\mathbf{Symm}_\mathbb{Z}$ and of $\mathbf{Symm}_\mathbb{Q}$ with the representation ring and the class function algebra of the symmetric group S_n; thus, $\boxdot$ becomes a bilinear product on the class functions, and in this guise it turns out to be adjoint to the pullback of a natural group homomorphism $S_m \times S_n \to S_{mn}$. This yields both integrality and Schur positivity.

For the anti-arithmetic product, there is no Schur positivity, and thus $\diamondsuit$ cannot be adjoint to the pullback of a group homomorphism $S_m \times S_n \to S_{mn}$. However, using the notions of λ-rings and Adams operations, we can define a stand-in for this missing homomorphism, producing virtual representations out of actual ones. The key insight is that the adjoint of $\diamondsuit$ is a λ-ring homomorphism, but – as Boorman proved in 1975 – the representation ring of S_n is generated by the natural permutation representation $\mathbb{Q}^n$ as a λ-ring. Thus, in order to show that this adjoint preserves the integral structure, it suffices to compute its value on $\mathbb{Q}^n$. This value (a virtual representation) is constructed using Adams operations and Möbius inversion.

This note is aimed at readers familiar with representation rings and the representation theory of symmetric groups. No prior knowledge of λ-rings is presumed. Boorman's theorem is proved in two different ways, one of which lifts it to the descent algebra of S_n.

Manifest. This note was written by GPT-6 based on a proof found by GPT-5.5. I have then edited it.

This note is in the public domain. –DG

Contents

1 Definitions and statements

1.1 Integral and rational symmetric functions

Let $\mathbf{Symm}_{\mathbb{Z}}$ denote the ring of symmetric functions over $\mathbb{Z}$, and let $\mathbf{Symm}_{\mathbb{Q}}$ be the analogous ring over $\mathbb{Q}$. Thus,

$$\mathbf{Symm}_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{Symm}_{\mathbb{Z}}.$$

We write $\mathbf{Symm}_{\mathbb{Z},n}$ and $\mathbf{Symm}_{\mathbb{Q},n}$ for the homogeneous components of degree n of $\mathbf{Symm}_{\mathbb{Z}}$ and $\mathbf{Symm}_{\mathbb{Q}}$. Our conventions for symmetric functions are the standard ones; see, for example, Sagan [27], Stanley [32, Chapter 7], or Grinberg–Reiner [12, Section 2].

For $r \geq 1$, let p_r be the r-th power-sum symmetric function, and for a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ let

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}.$$

Each p_λ belongs to $\mathbf{Symm}_{\mathbb{Z}}$, but the family $(p_\lambda)_\lambda$ is only a $\mathbb{Q}$-basis of $\mathbf{Symm}_{\mathbb{Q}}$, not a $\mathbb{Z}$-basis of $\mathbf{Symm}_{\mathbb{Z}}$. For instance,

$$h_2 = \frac{p_1^2 + p_2}{2}.$$

Equivalently,

$$\mathbf{Symm}_{\mathbb{Q}} = \mathbb{Q}[p_1, p_2, p_3, \ldots],$$

whereas the integral ring is

$$\mathbf{Symm}_{\mathbb{Z}} = \mathbb{Z}[h_1, h_2, h_3, \ldots] = \mathbb{Z}[e_1, e_2, e_3, \ldots].$$

Consequently, it is very easy to define multilinear operations on $\mathbf{Symm}_{\mathbb{Q}}$ by prescribing their values on power sums, but it is a separate question whether such operations preserve $\mathbf{Symm}_{\mathbb{Z}}$. Many familiar operations do: the Kronecker product and plethysm are standard examples[1]. This phenomenon, and several operations conveniently defined on the power-sum basis, are discussed for example in [12, Exercise 2.9.4]; see also the symmetric-function chapters of Stanley [32, Chapter 7] and Macdonald [19, Chapter I]. The two operations considered below provide another illustration of precisely this integrality issue.

1.2 The arithmetic and anti-arithmetic products

Since the p_λ form a $\mathbb{Q}$-basis of $\mathbf{Symm}_{\mathbb{Q}}$, the following definitions make sense over $\mathbb{Q}$.

Definition 1.1. *The* arithmetic product $\boxdot$ *is the* $\mathbb{Q}$*-bilinear operation on* $\mathbf{Symm}_{\mathbb{Q}}$ *defined by*

$$p_\lambda \boxdot p_\mu = \prod_{i=1}^{r} \prod_{j=1}^{s} p_{\operatorname{lcm}(\lambda_i, \mu_j)}^{\gcd(\lambda_i, \mu_j)} \tag{1}$$

for all pairs of partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ *and* $\mu = (\mu_1, \ldots, \mu_s)$ *(written without trailing zeroes).*

[1] Admittedly, plethysm is not multilinear, but the same construction principle applies with some complications.

The anti-arithmetic product $\diamondsuit$ *is the* $\mathbb{Q}$*-bilinear operation on* $\mathbf{Symm}_{\mathbb{Q}}$ *defined by*

$$p_\lambda \diamondsuit p_\mu = \prod_{i=1}^{r} \prod_{j=1}^{s} p_{\gcd(\lambda_i,\mu_j)}^{\operatorname{lcm}(\lambda_i,\mu_j)} \tag{2}$$

for all pairs of partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ *and* $\mu = (\mu_1, \ldots, \mu_s)$ *(written without trailing zeroes).*

If $|\lambda| = m$ and $|\mu| = n$, then both right-hand sides have degree mn, because each pair (λ_i, μ_j) contributes total degree $\lambda_i \mu_j$. Thus, for $m, n \geq 0$, we have

$$\boxdot, \diamondsuit : \mathbf{Symm}_{\mathbb{Q},m} \times \mathbf{Symm}_{\mathbb{Q},n} \longrightarrow \mathbf{Symm}_{\mathbb{Q},mn}.$$

The arithmetic product is the symmetric-function shadow of the arithmetic product of combinatorial species of Maia and Méndez [20]; its integrality is discussed in [12, Exercise 4.4.9] and in the MathOverflow question [11]. For a further development of the species-theoretic side, Li [18] introduced an exponential composition based on the arithmetic product and used it to study prime graphs under Cartesian product. The anti-arithmetic product was introduced in the MathOverflow question [10], where its integrality was asked for. Unlike the arithmetic product, it is not Schur-positive on pairs of Schur functions; for example, $s_{(2,1,1)} \diamondsuit s_{(2,1,1)}$ has Schur coefficients of both signs [10].

Our main result is the following.

Theorem 1.2 (Integrality)**.** *For all* $m, n \geq 0$*, the following statements hold.*

(a) We have

$$\mathbf{Symm}_{\mathbb{Z},m} \boxdot \mathbf{Symm}_{\mathbb{Z},n} \subseteq \mathbf{Symm}_{\mathbb{Z},mn}.$$

(b) If $f \in \mathbf{Symm}_{\mathbb{Z},m}$ *and* $g \in \mathbf{Symm}_{\mathbb{Z},n}$ *are Schur-positive, then* $f \boxdot g$ *is Schur-positive.*

(c) We have

$$\mathbf{Symm}_{\mathbb{Z},m} \diamondsuit \mathbf{Symm}_{\mathbb{Z},n} \subseteq \mathbf{Symm}_{\mathbb{Z},mn}.$$

The statement about $\boxdot$ is classical. We include it because its proof is almost the same as the proof of the $\diamondsuit$ statement, except at exactly one point. Both proofs proceed using the classical Frobenius correspondence between symmetric functions and representations/class functions of symmetric groups. For the arithmetic product $\boxdot$, a genuine group homomorphism[2]

$$S_m \times S_n \longrightarrow S_{mn}$$

provides the required pullback on representations. For the anti-arithmetic product no such homomorphism is available; instead, we construct a certain *virtual* representation by Adams operations and Möbius inversion.

Before we present these proofs, we shall develop the machinery needed for them. Very little of it is new, but not all of it is found in textbooks.

[2]We let S_n denote the n-th symmetric group; this group consists of the permutations of the set $[n] := \{1, 2, \ldots, n\}$.

Remark 1.3 (Degree zero)**.** *Theorem 1.2 is easy when one of m and n is 0. Indeed, by the empty product convention, each partition λ satisfies*

$$1 \boxdot p_\lambda = p_\lambda \boxdot 1 = 1 \diamondsuit p_\lambda = p_\lambda \diamondsuit 1 = 1 = p_\lambda (1, 0, 0, 0, \ldots).$$

Thus, by linearity, for any $f \in \mathbf{Symm}_{\mathbb{Z}}$, we have

$$1 \boxdot f = f \boxdot 1 = 1 \diamondsuit f = f \diamondsuit 1 = f (1, 0, 0, 0, \ldots) \in \mathbb{Z} \subseteq \mathbf{Symm}_{\mathbb{Z}}.$$

In particular, if λ is a partition, then $1 \boxdot s_\lambda = s_\lambda \boxdot 1$ is 1 if λ is a one-row partition and is 0 otherwise.

Thus, in proving Theorem 1.2, we can focus on the case when $m, n \geq 1$. We thus WLOG assume that $m, n \geq 1$ from now on.

Remark 1.4. *Both operations $\boxdot$ and $\diamondsuit$ are easily seen to be commutative.*

2 Exterior powers, Adams operations, and class functions

We will use the basic language of λ-rings and their Adams operations. The words "λ-ring" and "Adams operation" can suggest a good deal more machinery than we need. In this section we develop the small part of the theory used later, starting from exterior powers and the standard relation between elementary symmetric functions and power sums. For systematic treatments, see Knutson [16], Yau [34], or Hazewinkel [15, Section 16]. None of the general structure theorems from these references will be used. For the symmetric-group representation rings specifically, Thibon [33] and Scharf–Thibon [29] develop Adams operations and inner plethysm directly in symmetric-function language. A modern structural perspective, emphasizing Adams operations as natural transformations of the representation-ring functor, is given by Meir–Szymik [22].

2.1 Representation rings and class functions

For simplicity, whenever a general finite group G occurs below, we assume that the representations under discussion are realizable over $\mathbb{Q}$. This includes all symmetric groups and their finite direct products, which are the only groups needed in the proof. Alternatively, one may replace $\mathbb{Q}$ throughout this section by any characteristic-zero splitting field of G.

Let $\mathrm{Cl}_{\mathbb{Q}}(G)$ be the $\mathbb{Q}$-algebra of $\mathbb{Q}$-valued class functions on G, with pointwise addition and multiplication. In particular, the character χ_V of any finite-dimensional representation V of G over $\mathbb{Q}$ belongs to $\mathrm{Cl}_{\mathbb{Q}}(G)$. Let $R(G)$ be the Grothendieck ring of finite-dimensional $\mathbb{Q}G$-modules (see, e.g., [31, Chapter 9]). Thus $R(G)$ is a $\mathbb{Z}$-algebra: addition comes from direct sum and multiplication from tensor product. We write $[V]$ for the class of a representation V. Maschke's theorem and ordinary character theory give an injective ring homomorphism

$$\chi : R(G) \hookrightarrow \mathrm{Cl}_{\mathbb{Q}}(G), \qquad [V] \longmapsto \chi_V, \tag{3}$$

which we shall call the *character embedding*. We shall usually identify $R(G)$ with its image under this map. Standard background on these facts may be found, for example, in Etingof–Golberg–Hensel–Liu–Schwendner–Vaintrob–Yudovina [6, §4.2], Serre [31, Chapter 2], or the lecture notes of Lassueur [17, §9].

2.2 A weak notion of λ-ring

Our notion of a λ-ring is a weak one, imposing axioms for $\lambda^0(x)$ and $\lambda^1(x)$ and $\lambda^n(x+y)$. Many authors know such λ-rings under the name of *pre-λ-rings*, and only deem them worthy of the name "λ-ring" if they satisfy additional axioms for $\lambda^n(xy)$ and $\lambda^n(\lambda^m(y))$. To us, these extra axioms are unnecessary.

Definition 2.1. *A* λ-ring *in this note (often called a* pre-λ ring*) is a commutative unital ring A equipped with (usually non-linear) maps*

$$\lambda^r : A \longrightarrow A \qquad \text{for all } r \geq 0$$

such that all $x, y \in A$ satisfy the axioms

$$\lambda^0(x) = 1, \tag{4}$$

$$\lambda^1(x) = x, \tag{5}$$

$$\lambda^n(x+y) = \sum_{i=0}^{n} \lambda^i(x)\lambda^{n-i}(y). \tag{6}$$

We can encode these maps λ^r into a single generating function

$$\lambda_t(x) = \sum_{r \geq 0} \lambda^r(x) t^r \in A[[t]] \tag{7}$$

defined for all $x \in A$ (that is, into a map $\lambda_t : A \to A[[t]]$); then the axioms (4) *and* (5) *say that*

$$\lambda_t(x) = 1 + xt + (\text{higher powers of } t)\,,$$

whereas the axiom (6) *can be equivalently rewritten as*

$$\lambda_t(x+y) = \lambda_t(x)\lambda_t(y). \tag{8}$$

In other words, λ_t must be a group homomorphism from the additive group of A to the multiplicative group of formal power series with constant term 1 over A, and it must have the property that $\dfrac{d}{dt}\lambda_t(x)\,|_{t=0} = x$ for each $x \in A$.

A λ-ring morphism *is a unital ring homomorphism $f : A \to B$ between two λ-rings that commutes with every λ^r (that is, satisfies $f \circ \lambda^r = \lambda^r \circ f$ for each $r \geq 0$). A* λ-subring *of a λ-ring A is a subring closed under every λ^r. The λ-subring of A* generated *by a subset $X \subseteq A$ is the smallest λ-subring of A containing X.*

Note that the operations λ^r on a λ-ring A, taken in combination, carry the same information as their generating series λ_t. In particular, a λ-ring can be defined by providing λ_t instead of the λ^r's.

Example 2.2 (The binomial λ-ring)**.** *The ring $\mathbb{Z}$ is a λ-ring under*

$$\lambda^r(a) = \binom{a}{r}, \qquad \lambda_t(a) = (1+t)^a.$$

The binomial theorem shows that these two equalities fit together with (7)*; the equality* (6) *is the Chu–Vandermonde convolution.*

Example 2.3 (Representation rings)**.** *Let G be a group, and consider its representation ring $R(G)$. For an actual G-representation V, put*

$$\lambda_t([V]) = \sum_{r\geq 0} [\textstyle\bigwedge^r V] t^r \in R(G)[[t]]. \tag{9}$$

The canonical decomposition

$$\bigwedge^r (V \oplus W) \cong \bigoplus_{i+j=r} \bigwedge^i V \otimes \bigwedge^j W$$

gives

$$\lambda_t([V \oplus W]) = \lambda_t([V])\lambda_t([W]).$$

Thus, (9) *defines a monoid homomorphism $x \mapsto \lambda_t(x)$ from the additive monoid of actual representations of G to the multiplicative group $1 + tR(G)[[t]]$ of power series with constant term 1. Since $R(G)$ is the Grothendieck completion of the former monoid, this homomorphism extends uniquely to a group homomorphism*

$$\lambda_t : R(G) \longrightarrow 1 + tR(G)[[t]]$$

by the rule

$$\lambda_t(x - y) = \frac{\lambda_t(x)}{\lambda_t(y)}.$$

Thus $R(G)$ is a λ-ring (the proof of (5) *is easy).*

2.3 ψ-rings

Adams operations are even easier to axiomatize.

Definition 2.4. *A* ψ-ring *is a commutative unital ring A equipped with unital ring endomorphisms*

$$\psi^r : A \longrightarrow A \qquad \text{for all } r \geq 1$$

such that

$$\psi^1 = \mathrm{id} \qquad \text{and} \qquad \psi^{rs} = \psi^r \circ \psi^s \quad \text{for all } r, s \geq 1.$$

The endomorphisms ψ^r are known as the Adams operations *of A. A* morphism of ψ-rings *is a unital ring homomorphism commuting with all ψ^r.*

Proposition 2.5. *For every finite group G, the class-function algebra $\mathrm{Cl}_{\mathbb{Q}}(G)$ is a ψ-ring under the ψ-operations ψ^r defined by*

$$(\psi^r f)(g) = f(g^r) \qquad \text{for all } r \geq 1 \text{ and } f \in \mathrm{Cl}_{\mathbb{Q}}(G) \text{ and } g \in G. \tag{10}$$

Proof. The function $\psi^r f$ defined in (10) is a class function, because if g and h are two conjugate elements of G, then their r-th powers g^r and h^r are also conjugate. The map ψ^r preserves sums, products, and the constant function 1, because all operations on class functions are pointwise. Moreover, for all $r, s \geq 1$ and all $f \in \mathrm{Cl}_{\mathbb{Q}}(G)$ and all $g \in G$, we have

$$\psi^r(\psi^s f)(g) = (\psi^s f)(g^r) = f((g^r)^s) = f(g^{rs}) = (\psi^{rs} f)(g).$$

Thus, $\psi^{rs} = \psi^r \circ \psi^s$. □

2.4 From ψ to λ over $\mathbb{Q}$

Here is the only general construction we need from the theory of λ-rings. The qualification "over $\mathbb{Q}$" matters: over an arbitrary ring the formula below can introduce denominators, and their cancellation is a genuine integrality question.

Proposition 2.6. *Let A be a ψ-ring over $\mathbb{Q}$ (that is, a $\mathbb{Q}$-algebra equipped with a ψ-ring structure). For any $x \in A$, define $\lambda_t(x) \in A[[t]]$ by*

$$\lambda_t(x) = \exp\left(\sum_{k \geq 1} (-1)^{k-1} \psi^k(x) \frac{t^k}{k}\right). \tag{11}$$

Then the following statements hold.

(a) The operations λ^r defined by (7) in terms of this λ_t make A into a λ-ring.

(b) Every morphism of ψ-rings over $\mathbb{Q}$ is a morphism of the resulting λ-rings.

The familiar symmetric-function identity

$$\sum_{r \geq 0} e_r t^r = \exp\left(\sum_{k \geq 1} (-1)^{k-1} p_k \frac{t^k}{k}\right) \tag{12}$$

is one way to remember the formula (11); comparing the two identities shows that the $\mathbb{Q}$-algebra homomorphism $\mathbf{Symm}_{\mathbb{Q}} \to A$ that sends all power-sums p_k to $\psi^k(x)$ (for a given $x \in A$) will send all e_r to $\lambda^r(x)$.

Proof of Proposition 2.6. (a) Since $\psi^1 = \mathrm{id}$, the coefficient of t in (11) is x, while the constant coefficient is 1. Since every ψ^k is additive,

$$\begin{aligned}
\lambda_t(x+y) &= \exp\left(\sum_{k \geq 1} (-1)^{k-1} (\psi^k(x) + \psi^k(y)) \frac{t^k}{k}\right) \\
&= \exp\left(\sum_{k \geq 1} (-1)^{k-1} \psi^k(x) \frac{t^k}{k}\right) \cdot \exp\left(\sum_{k \geq 1} (-1)^{k-1} \psi^k(y) \frac{t^k}{k}\right) \\
&= \lambda_t(x) \lambda_t(y).
\end{aligned}$$

This proves the λ-ring axioms (4), (5) and (8), thus showing that A is indeed a λ-ring.

(b) If a ring homomorphism $\varphi : A \to B$ commutes with all ψ^k, then applying φ to (11) shows that it commutes with every λ^r. $\square$

Remark 2.7. *The proof of Proposition 2.6 uses much less than a ψ-ring structure. To construct the weak λ-ring structure* (11)*, it is enough that A be a commutative $\mathbb{Q}$-algebra and that we be given additive maps*

$$\psi^r : A \longrightarrow A \qquad \text{for all } r \geq 1$$

with $\psi^1 = \mathrm{id}$. Neither multiplicativity of the ψ^k nor the relations $\psi^{rs} = \psi^r \circ \psi^s$ enter the proof. (Additivity already implies $\mathbb{Q}$-linearity here.) These stronger properties are part of the definition of a ψ-ring because they hold for the Adams operations that interest us and are what lead to the usual special λ-ring structure; they are not needed for the weak λ-ring axioms used in Proposition 2.6.

Differentiating the logarithm of (11) gives

$$\left(\frac{d}{dt}\lambda_t(x)\right)/\lambda_t(x) = \sum_{k\geq 1}(-1)^{k-1}\psi^k(x)t^{k-1}, \tag{13}$$

that is,

$$\frac{d}{dt}\lambda_t(x) = \lambda_t(x)\cdot\sum_{k\geq 1}(-1)^{k-1}\psi^k(x)t^{k-1}. \tag{14}$$

Comparing t^{r-1}-coefficients, we obtain the Newton recurrence

$$r\lambda^r(x) = \sum_{k=1}^{r}(-1)^{k-1}\lambda^{r-k}(x)\psi^k(x) \tag{15}$$

for each $r \geq 1$ and $x \in A$. This equation can be solved for $\psi^r(x)$, yielding

$$\psi^r(x) - \lambda^1(x)\psi^{r-1}(x) + \lambda^2(x)\psi^{r-2}(x) - \cdots + (-1)^r r\lambda^r(x) = 0. \tag{16}$$

Thus, $\psi^r(x)$ can be expressed as a polynomial in the inputs $\psi^1(x), \psi^2(x), \ldots, \psi^{r-1}(x)$ and $\lambda^1(x), \lambda^2(x), \ldots, \lambda^{r-1}(x)$. This shows (by induction) that each $\psi^r(x)$ can be expressed as an *integral* polynomial in the elements

$$\lambda^1(x), \lambda^2(x), \ldots, \lambda^r(x)$$

(a version of the usual Newton identities for power-sum symmetric functions). For example,

$$\psi^1(x) = \lambda^1(x), \qquad \psi^2(x) = \lambda^1(x)^2 - 2\lambda^2(x).$$

The absence of denominators in this direction will be useful below, as it shows that the λ-ring structure on A in Proposition 2.6 uniquely determines the ψ-ring structure it originates from.

2.5 Characters respect exterior powers

Consider again a finite group G. The class function algebra $\mathrm{Cl}_{\mathbb{Q}}(G)$ of G is a ψ-ring over $\mathbb{Q}$, and thus (by Proposition 2.6) becomes a λ-ring. We shall now show that $R(G)$ is a λ-subring of this λ-ring $\mathrm{Cl}_{\mathbb{Q}}(G)$; that is, the λ-ring structure on $R(G)$ is a restriction of that induced by the ψ-ring $\mathrm{Cl}_{\mathbb{Q}}(G)$.

Proposition 2.8. *(a) The character embedding*

$$R(G) \hookrightarrow \mathrm{Cl}_{\mathbb{Q}}(G)$$

is a morphism of λ-rings.

(b) The operations ψ^r of (10) preserve $R(G)$.

(c) For every virtual representation $x \in R(G)$ and every $g \in G$, we have

$$\chi_{\psi^r(x)}(g) = \chi_x(g^r), \tag{17}$$

where χ_y denotes the image of any $y \in R(G)$ under the character embedding.

Proof. (a) Let V be an actual representation of G, and let A be the matrix by which a given element $g \in G$ acts on V. On one hand, by the classical formula for the characteristic polynomial of a matrix in terms of its principal minors[3], we have

$$\det(I + tA) = \sum_{j \geq 0} \mathrm{tr}(\textstyle\bigwedge^j A) t^j. \tag{18}$$

On the other hand, if the eigenvalues of A in an algebraic closure are $\alpha_1, \ldots, \alpha_d$, then

$$\begin{aligned} \det(I + tA) &= \prod_{i=1}^{d} (1 + \alpha_i t) = \sum_{r \geq 0} e_r(\alpha_1, \alpha_2, \ldots, \alpha_d) t^r \\ &= \exp\left(\sum_{k \geq 1} (-1)^{k-1} \left(\sum_i \alpha_i^k \right) \frac{t^k}{k} \right) \qquad \text{(by (12))} \\ &= \exp\left(\sum_{k \geq 1} (-1)^{k-1} (\psi^k(\chi_V))(g) \frac{t^k}{k} \right), \end{aligned} \tag{19}$$

since each $k \geq 1$ satisfies $\sum_i \alpha_i^k = \mathrm{tr}(A^k) = \chi_V(g^k) = (\psi^k(\chi_V))(g)$. Comparing this with (18), we find

$$\sum_{j \geq 0} \mathrm{tr}(\textstyle\bigwedge^j A) t^j = \exp\left(\sum_{k \geq 1} (-1)^{k-1} (\psi^k(\chi_V))(g) \frac{t^k}{k} \right).$$

On the other hand, applying (11) to $x = \chi_V$, and evaluating at g, we obtain

$$\sum_{j \geq 0} (\lambda^j(\chi_V))(g) t^j = \exp\left(\sum_{k \geq 1} (-1)^{k-1} (\psi^k(\chi_V))(g) \frac{t^k}{k} \right)$$

[3]To be fully precise, the characteristic polynomial of A is $\det(tI - A)$ rather than $\det(I + tA)$. But these two polynomials have the same coefficients up to sign and order. Alternatively, Proposition 11 in [3, §III.8.5], applied with $u = A$, $\xi = 1$, and $\eta = t$, gives (18) directly.

(since the algebra structure on $\mathrm{Cl}_{\mathbb{Q}}(G)$ is pointwise). Comparing these two equalities, we find

$$\sum_{j\geq 0}(\lambda^j(\chi_V))(g)t^j = \sum_{j\geq 0}\mathrm{tr}(\textstyle\bigwedge^j A)t^j.$$

Thus, for each $j \geq 0$, we obtain $\lambda^j(\chi_V)(g) = \mathrm{tr}(\bigwedge^j A) = \chi_{\bigwedge^j V}(g) = \chi_{\lambda^j([V])}(g)$. Since g was arbitrary, this proves that the class-function $\lambda^j(\chi_V)$ equals $\chi_{\lambda^j([V])}$.

This shows that the character embedding $R(G) \to \mathrm{Cl}_{\mathbb{Q}}(G)$ commutes with λ^j (and thus with λ_t) at least on actual representations. The fact that λ_t is a group homomorphism extends this to all virtual representations (since $R(G)$ is generated as an abelian group by the actual representations). Thus the character embedding is a λ-ring morphism.

(b) Let $x \in R(G)$. Then, by (16), we can write $\psi^r(x)$ as an integral polynomial in the exterior-power operations on x. Hence $\psi^r(x) \in R(G)$.

(c) The character embedding is a λ-ring morphism by part (a), and thus commutes with the ψ^r operations. Thus, for any $x \in R(G)$ and $r \geq 1$, we have $\chi_{\psi^r(x)} = \psi^r(\chi_x)$. Evaluating this at a $g \in G$ yields (17). □

Thus we may regard $R(G)$ simultaneously as a λ-subring and a ψ-subring of $\mathrm{Cl}_{\mathbb{Q}}(G)$.

2.6 A word about special λ-rings

In the usual terminology, one often reserves "λ-ring" for a structure satisfying additional universal identities governing $\lambda^r(xy)$ and $\lambda^r(\lambda^s(x))$; such rings are often called *special λ-rings*. Representation rings are special, and a ψ-ring over $\mathbb{Q}$ as above yields a special λ-ring because its Adams operations are commuting ring endomorphisms. These stronger axioms play no role in our proof, so we do not state them. See [16, 34, 15] for the full theory.

2.7 Tensor products of ψ-rings

If A and B are ψ-rings over $\mathbb{Q}$, then $A \otimes_{\mathbb{Q}} B$ becomes a ψ-ring over $\mathbb{Q}$ by

$$\psi^r(a \otimes b) = \psi^r(a) \otimes \psi^r(b). \tag{20}$$

This is well-defined because the ψ^r are $\mathbb{Q}$-linear ring endomorphisms, and the identities $\psi^{rs} = \psi^r\psi^s$ are inherited factorwise. Proposition 2.6 then supplies the corresponding λ-operations.

For finite groups G and H, the map

$$\mathrm{Cl}_{\mathbb{Q}}(G) \otimes_{\mathbb{Q}} \mathrm{Cl}_{\mathbb{Q}}(H) \longrightarrow \mathrm{Cl}_{\mathbb{Q}}(G \times H), \qquad f \otimes h \longmapsto \Big((g,k) \mapsto f(g)h(k)\Big) \tag{21}$$

is an isomorphism of $\mathbb{Q}$-algebras, since conjugacy classes in $G \times H$ are pairs of conjugacy classes (and since $\mathbb{Q}^X \otimes_{\mathbb{Q}} \mathbb{Q}^Y \cong \mathbb{Q}^{X\times Y}$ for any two finite sets X and Y). It is visibly an isomorphism of ψ-rings, because

$$(g,k)^r = (g^r, k^r).$$

Hence it is also an isomorphism of the induced λ-rings.

There is also a (subtler) notion of tensor products of λ-rings, but we will not need it. The only tensor products of λ-rings that we will use are $\mathrm{Cl}_{\mathbb{Q}}(S_m) \otimes_{\mathbb{Q}} \mathrm{Cl}_{\mathbb{Q}}(S_n)$ and $R(S_m) \otimes_{\mathbb{Z}}$

$R(S_n)$; the former is obtained by tensoring two ψ-rings (as above), while the latter will be identified with the representation ring $R(S_m \times S_n)$ by external tensor product. The required isomorphism is proved in Subsection 3.5, once the irreducible representations of the symmetric groups have been recalled.

3 The representation ring of the symmetric group

3.1 Frobenius characteristic

For any partition $\lambda = (1^{m_1} 2^{m_2} \cdots) \vdash n$, set

$$z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!.$$

For a class function $f \in \mathrm{Cl}_{\mathbb{Q}}(S_n)$, write $f(\lambda)$ for its value on the conjugacy class of cycle type λ. The Frobenius characteristic is the $\mathbb{Q}$-linear map

$$\mathrm{ch}_n : \mathrm{Cl}_{\mathbb{Q}}(S_n) \longrightarrow \mathbf{Symm}_{\mathbb{Q},n}, \qquad \mathrm{ch}_n(f) = \sum_{\lambda \vdash n} f(\lambda) \frac{p_\lambda}{z_\lambda}. \tag{22}$$

It is an isomorphism of $\mathbb{Q}$-vector spaces. The basic theorem of Frobenius says that

$$\mathrm{ch}_n(\chi^\lambda) = s_\lambda, \tag{23}$$

where χ^λ is the irreducible character of the Specht module S^λ. Since the Specht modules are defined over $\mathbb{Q}$ and form a complete set of absolutely irreducible $\mathbb{Q}S_n$-modules, restriction of (22) gives an isomorphism of abelian groups

$$R(S_n) \xrightarrow{\sim} \mathbf{Symm}_{\mathbb{Z},n}. \tag{24}$$

Thus we have a commutative square

$$\begin{array}{ccc} R(S_n) & \hookrightarrow & \mathrm{Cl}_{\mathbb{Q}}(S_n) \\ \downarrow{\scriptstyle \mathrm{ch}_n} & & \downarrow{\scriptstyle \mathrm{ch}_n} \\ \mathbf{Symm}_{\mathbb{Z},n} & \hookrightarrow & \mathbf{Symm}_{\mathbb{Q},n}. \end{array}$$

This is the bridge between the integrality problem for symmetric functions in $\mathbf{Symm}_{\mathbb{Q},n}$ and the integral lattice of virtual characters in $\mathrm{Cl}_{\mathbb{Q}}(S_n)$. Note that the product on $R(S_n)$ and $\mathrm{Cl}_{\mathbb{Q}}(S_n)$ corresponds to the so-called *Kronecker product* (also known as the *internal product*) on the symmetric functions; but we will not gain anything from this fact.

3.2 The natural representation generates as a λ-ring

Let

$$M_n = \mathbb{Q}^n$$

be the natural permutation representation of S_n, with basis $e_1, \ldots, e_n$. Its action is given by $\sigma(e_i) = e_{\sigma(i)}$ for any $\sigma \in S_n$ and $i \in [n]$. Let

$$V_n = S^{(n-1,1)}$$

be the reflection representation. Then

$$M_n \cong \mathbf{1} \oplus V_n. \tag{25}$$

We will need a 1975 result of Evelyn Boorman [1]: the representation M_n generates $R(S_n)$ as a λ-ring. Marin later proved the equivalent statement that the exterior powers of V_n generate $R(S_n)$ as a ring [21]; his proof uses a formula of Dvir and points out earlier equivalent symmetric-function results of Butler [4] and Boorman. We record the theorem in the form needed later and give a self-contained proof using triangularity (leading term analysis).

Theorem 3.1 (Boorman; Marin)**.** *For every $n \geq 1$, the smallest λ-subring of $R(S_n)$ containing M_n is all of $R(S_n)$.*

The proof below is neither Boorman's nor Marin's original proof; it is a triangular argument using the number of boxes below the first row. It is included so that Theorem 1.2 does not depend on any nonstandard representation-theoretic black box. For generalizations of hook-type generating sets to wreath products, see Harman [14]. Appendix A gives a second proof, obtained from a stronger integral generation theorem for Solomon's descent algebra (or, equivalently, for noncommutative symmetric functions). For a readable introduction to Solomon's descent algebra, including its realization through the face semigroup algebra of the braid arrangement, see Saliola [28, §2, especially §2.1].

3.3 A triangular proof of Theorem 3.1

For a partition $\lambda \vdash n$, define its *depth* by

$$\operatorname{depth}(\lambda) = n - \lambda_1. \tag{26}$$

Thus, if $d = \operatorname{depth}(\lambda)$, we can write uniquely

$$\lambda = (n-d, \alpha), \qquad \text{where } \alpha \vdash d \text{ with } \alpha_1 \leq n-d.$$

Here $(n-d, \alpha)$ means the partition consisting of $n-d$ followed by the entries of α. We call α the *tail* of λ. Let

$$\alpha' = (c_1, c_2, \ldots, c_r)$$

be the conjugate partition of α, and define

$$T_\lambda = \bigotimes_{j=1}^{r} \bigwedge^{c_j} M_n. \tag{27}$$

Clearly $[T_\lambda]$ belongs to the λ-subring of $R(S_n)$ generated by M_n.

We shall prove that T_λ contains S^λ once, and that all other constituents S^μ of T_λ are triangularly smaller: either they have smaller depth, or they have the same depth and a strictly smaller tail in dominance order.

For $\beta \vdash d$, let $\mathbb{S}^\beta(M_n)$ denote the Schur functor corresponding to β, applied to M_n. One convenient definition in characteristic zero is

$$\mathbb{S}^\beta(M_n) = \operatorname{Hom}_{S_d}(S^\beta, M_n^{\otimes d}), \tag{28}$$

where the tensor power $M_n^{\otimes d}$ has two commuting actions of S_d and S_n (in the obvious ways: S_d permutes tensor positions and S_n acts diagonally on $M_n^{\otimes d}$).

Lemma 3.2. *Let $\beta \vdash d$, and assume $\beta_1 \leq n-d$. Then, in $R(S_n)$, we have*

$$[\mathbb{S}^\beta(M_n)] = [S^{(n-d,\beta)}] + \sum_{\substack{\mu \vdash n; \\ \operatorname{depth}(\mu) < d}} a_{\beta\mu}[S^\mu] \tag{29}$$

for some nonnegative integers $a_{\beta\mu}$.

Proof. Step 1: the lower part of the filtration. Filter $M_n^{\otimes d}$ by the number of distinct basis vectors that occur in a pure tensor. More precisely, let F_r be the span of the basis tensors

$$e_{i_1} \otimes \cdots \otimes e_{i_d} \qquad \text{for which } |\{i_1, \ldots, i_d\}| \leq r.$$

This span F_r is stable under the commuting actions of S_n and S_d. Thus, we obtain a filtration

$$0 = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_d = M_n^{\otimes d}$$

of $M_n^{\otimes d}$ by S_n-subrepresentations.

For a tuple $\mathbf{i} = (i_1, \ldots, i_d) \in [n]^d$, let $\pi(\mathbf{i})$ be the set partition of $[d]$ whose blocks are the fibers of the map $j \mapsto i_j$; equivalently, j and k lie in the same block of $\pi(\mathbf{i})$ if and only if $i_j = i_k$. We call $\pi(\mathbf{i})$ the *kernel partition* of $\mathbf{i}$. For example, the tuple $(3, 7, 3, 3, 7)$ has kernel partition $\{\{1, 3, 4\}, \{2, 5\}\}$.

Now fix a set partition π of $[d]$ with exactly r blocks. The basis tensors $e_{i_1} \otimes \cdots \otimes e_{i_d}$ with $\pi(\mathbf{i}) = \pi$ are naturally in S_n-equivariant bijection with the injections from the r-element set of blocks of π into $[n]$: an injection records the common value i_j on each block. This S_n-set is transitive, and the stabilizer of one such injection is isomorphic to S_{n-r} (since a permutation in the stabilizer must fix all the r elements of the image of the injection, but can permute the remaining $n-r$ elements of $[n]$ arbitrarily). Hence its permutation representation is

$$\operatorname{Ind}_{S_{n-r}}^{S_n} \mathbf{1}.$$

By the branching rule[4], every Specht module S^μ occurring in this representation satisfies

$$\mu_1 \geq n - r,$$

and hence $\operatorname{depth}(\mu) \leq r$.

So we have broken up F_r into a direct sum of S_n-representations of the form $\operatorname{Ind}_{S_{n-r}}^{S_n} \mathbf{1}$ (one for each possible kernel partition), and showed that every Specht module S^μ in each of these representations satisfies $\operatorname{depth}(\mu) \leq r$. In other words, every S_n-constituent of F_r has depth at most r (where "depth" means the depth of the corresponding partition). It follows that every S_n-constituent of F_{d-1} has depth at most $d-1$.

Step 2: the top quotient. The top quotient F_d/F_{d-1} of our filtration has a basis consisting of tensors with pairwise distinct indices. Thus, our above reasoning shows that

$$F_d/F_{d-1} \cong \mathbb{Q}[\operatorname{Inj}([d], [n])] \tag{30}$$

[4]In terms of symmetric functions, this is just the Pieri rule. Under the Frobenius correspondence ch, the representation $\operatorname{Ind}_{S_{n-r}}^{S_n} \mathbf{1}$ corresponds to the symmetric function $h_{n-r}h_1^r = s_{(n-r)}h_1^r$, which (by repeated application of the Pieri rule) is obtained from $s_{(n-r)}$ by adding r cells to the Young diagram. Obviously, adding cells cannot make the first row any shorter, so the resulting Schur functions s_μ all satisfy $\mu_1 \geq n-r$.

as S_n-representations, where $\mathrm{Inj}(X,Y)$ denotes the set of all injections from X to Y. As an $S_n \times S_d$-bimodule, its S^β-multiplicity space is

$$\mathrm{Hom}_{S_d}\Big(S^\beta, \mathbb{Q}[\mathrm{Inj}([d],[n])]\Big) \cong \mathrm{Ind}_{S_d \times S_{n-d}}^{S_n}(S^\beta \boxtimes \mathbf{1}). \tag{31}$$

By Pieri's rule, the right-hand side is the multiplicity-free sum of S^ν over partitions $\nu \vdash n$ such that ν/β is a horizontal strip of size $n-d$.

Every such ν satisfies $\nu_1 \geq n-d$, hence $\mathrm{depth}(\nu) \leq d$. Suppose equality holds. Then $\nu_1 = n-d$. The horizontal-strip condition is equivalent to the interlacing inequalities

$$\nu_{i+1} \leq \beta_i \qquad (i \geq 1).$$

Since

$$\sum_{i \geq 2} \nu_i = d = |\beta| = \sum_{i \geq 1} \beta_i,$$

all these inequalities must be equalities. Hence

$$\nu = (n-d, \beta).$$

It occurs with multiplicity one. All other constituents of the top quotient have depth $< d$.

Step 3: take the S^β-multiplicity space. Applying the exact functor $\mathrm{Hom}_{S_d}(S^\beta, -)$ to the filtration, and combining Steps 1 and 2, proves (29). □

We now use the standard Kostka-triangular decomposition of a tensor product of exterior powers. The symmetric-function identity

$$e_{\alpha'} = \prod_{j=1}^{r} e_{c_j} = \sum_{\beta \vdash d} K_{\beta',\alpha'} s_\beta \tag{32}$$

translates, by Schur–Weyl theory, to

$$T_\lambda \cong \bigoplus_{\beta \vdash d} K_{\beta',\alpha'}\, \mathbb{S}^\beta(M_n). \tag{33}$$

The Kostka number $K_{\beta',\alpha'}$ is nonzero only if $\beta' \trianglerighteq \alpha'$, equivalently $\beta \trianglelefteq \alpha$, and

$$K_{\alpha',\alpha'} = 1.$$

In particular, whenever a term occurs, $\beta_1 \leq \alpha_1 \leq n-d$, so Lemma 3.2 applies. We obtain the promised triangularity.

This lemma is a small triangular piece of the classical restriction problem from GL_n to S_n. Indeed, regarding S_n as the group of permutation matrices in GL_n, the S_n-representation $\mathbb{S}^\beta(M_n)$ is the restriction of the polynomial GL_n-representation $\mathbb{S}^\beta(\mathbb{Q}^n)$. Orellana–Zabrocki [25, Introduction] discuss this restriction problem and encode such restrictions by evaluating symmetric functions at the eigenvalues of permutation matrices. We only need the top-depth triangular statement above.

Proposition 3.3. *Let $\lambda = (n-d, \alpha) \vdash n$. Then*

$$[T_\lambda] = [S^\lambda] + \sum_{\beta \lhd \alpha} K_{\beta',\alpha'}[S^{(n-d,\beta)}] + \sum_{\substack{\mu \vdash n; \\ \mathrm{depth}(\mu) < d}} b_{\lambda\mu}[S^\mu], \tag{34}$$

where the $b_{\lambda\mu}$ are nonnegative integers, and $\beta \lhd \alpha$ means strict dominance.

Proof. Combine (33), Kostka triangularity, and Lemma 3.2. The term $\beta = \alpha$ contributes $S^{(n-d,\alpha)} = S^\lambda$ exactly once. Every other depth-d term has tail $\beta \lhd \alpha$. □

Proof of Theorem 3.1. Let A_n be the λ-subring of $R(S_n)$ generated by M_n. We prove $S^\lambda \in A_n$ for every $\lambda \vdash n$, by induction first on $d = \mathrm{depth}(\lambda)$ and then, for fixed d, upward along dominance of the tail α in $\lambda = (n-d, \alpha)$.

The element $[T_\lambda]$ belongs to A_n. In (34), every constituent in the last sum has smaller depth, so belongs to A_n by the outer induction. Every constituent in the middle sum has the same depth but strictly smaller tail, so belongs to A_n by the inner induction. Subtracting these already-known classes from $[T_\lambda]$ gives $[S^\lambda] \in A_n$.

Since the Specht classes form a $\mathbb{Z}$-basis of $R(S_n)$, this proves $A_n = R(S_n)$. □

Remark 3.4. *The two parts of the triangular order can be seen concretely already for $n = 6$ and $d = 3$. First consider the Schur functor corresponding to $\beta = (3)$. Sorting the basis tensors of $\mathrm{Sym}^3(M_6)$ according to whether they involve three, two, or one distinct basis vectors (as in the proof of Lemma 3.2) gives*

$$\mathbb{S}^{(3)}(M_6) = \mathrm{Sym}^3(M_6) \cong S^{(3,3)} \oplus \underbrace{S^{(4,1,1)} \oplus 2S^{(4,2)} \oplus 4S^{(5,1)} \oplus 3S^{(6)}}_{\text{constituents of depth} < 3}.$$

Thus the lower-depth terms need not be lower in dominance order. For instance, $(3,3)$ and $(4,1,1)$ are incomparable: the first partial sum favors $(4,1,1)$, whereas the first two partial sums favor $(3,3)$. This is why dominance order alone is not sufficient.

The second part of the induction—dominance among tails of the same depth—is visible if we take $\lambda = (3,2,1)$, so that $d = 3$, $\alpha = (2,1)$, and

$$T_\lambda = \bigwedge^2 M_6 \otimes M_6.$$

Since $e_2 e_1 = s_{(2,1)} + s_{(1,1,1)}$, we have

$$T_\lambda \cong \mathbb{S}^{(2,1)}(M_6) \oplus \mathbb{S}^{(1,1,1)}(M_6),$$

and the same calculation gives

$$T_{(3,2,1)} \cong S^{(3,2,1)} \oplus \underbrace{S^{(3,1,1,1)}}_{\substack{\text{same depth } 3, \\ (1,1,1) \lhd (2,1)}} \oplus \underbrace{3S^{(4,1,1)} \oplus 2S^{(4,2)} \oplus 3S^{(5,1)} \oplus S^{(6)}}_{\text{constituents of depth} < 3}.$$

Thus this one example displays exactly the two kinds of already-known terms that occur in (34): smaller-depth constituents, and same-depth constituents whose tails are strictly smaller in dominance order.

Remark 3.5 (Relation with Marin's formulation)**.** *Marin's theorem is usually stated by saying that the exterior powers of the standard representation V_n generate $R(S_n)$ as a ring. This is equivalent to Theorem 3.1. Indeed, $M_n = \mathbf{1} \oplus V_n$, and therefore*

$$\bigwedge^k M_n \cong \bigwedge^k V_n \oplus \bigwedge^{k-1} V_n.$$

Thus the λ-subring generated by M_n contains all the exterior powers of V_n, and conversely Marin's generators contain V_n and hence M_n.

3.4 The scalar product and integral self-duality

For class functions on a finite group, define the bilinear scalar product

$$\langle f, g \rangle_G = \frac{1}{|G|} \sum_{x \in G} f(x) g(x^{-1}). \tag{35}$$

For symmetric groups, every conjugacy class is invariant under inversion, so this is simply the usual character scalar product without any need for complex conjugation. The irreducible characters χ^λ are an orthonormal basis.

The Hall scalar product on $\mathbf{Symm}_{\mathbb{Q},n}$ is characterized by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda. \tag{36}$$

From (22) one checks immediately that Frobenius characteristic is an isometry:

$$\langle \mathrm{ch}_n(f), \mathrm{ch}_n(g) \rangle = \langle f, g \rangle_{S_n}. \tag{37}$$

Equivalently, the Schur functions form an orthonormal basis:

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}.$$

In particular, the lattice $\mathbf{Symm}_{\mathbb{Z},n}$ is self-dual:

Lemma 3.6. *If $F \in \mathbf{Symm}_{\mathbb{Q},n}$ satisfies*

$$\langle F, H \rangle \in \mathbb{Z} \qquad \text{for every } H \in \mathbf{Symm}_{\mathbb{Z},n},$$

then $F \in \mathbf{Symm}_{\mathbb{Z},n}$.

Proof. Write $F = \sum_{\lambda \vdash n} c_\lambda s_\lambda$. Pairing with s_λ gives $c_\lambda \in \mathbb{Z}$ for every λ. □

We note one simple fact connecting the scalar product with the Frobenius characteristic ch_n: If $h \in \mathrm{Cl}_{\mathbb{Q}}(S_n)$ is a class function, then

$$\langle p_\nu, \mathrm{ch}_n(h) \rangle = h(\nu) \tag{38}$$

for every partition ν of n. This follows from (22) and (36).

3.5 Products of symmetric groups

The irreducible $\mathbb{Q}[S_m \times S_n]$-modules are exactly the external tensor products

$$S^\lambda \boxtimes S^\mu \qquad (\lambda \vdash m,\ \mu \vdash n).$$

Let us give a quick proof that stays over $\mathbb{Q}$. The character of $S^\lambda \boxtimes S^\mu$ is

$$(\sigma, \tau) \longmapsto \chi^\lambda(\sigma)\chi^\mu(\tau).$$

Consequently, a direct double-sum calculation gives

$$\begin{aligned}\left\langle \chi^\lambda \boxtimes \chi^\mu, \chi^\nu \boxtimes \chi^\rho \right\rangle_{S_m \times S_n} \\ = \left\langle \chi^\lambda, \chi^\nu \right\rangle_{S_m} \langle \chi^\mu, \chi^\rho \rangle_{S_n} = \delta_{\lambda\nu}\delta_{\mu\rho}.\end{aligned}$$

In particular, each $S^\lambda \boxtimes S^\mu$ remains irreducible after extension of scalars to $\mathbb{C}$, since its complex character has norm 1. Thus these are pairwise nonisomorphic absolutely irreducible $\mathbb{Q}[S_m \times S_n]$-modules. There are $p(m)p(n)$ of them, which is exactly the number of conjugacy classes of $S_m \times S_n$. Since the number of complex irreducible characters equals the number of conjugacy classes, these external tensor products exhaust all irreducibles. This is the product-group theorem for irreducibles in the present rational setting; compare the usual algebraically closed version [6, Theorem 3.10.2].

It follows that external tensor product gives an isomorphism of rings

$$R(S_m) \otimes_{\mathbb{Z}} R(S_n) \xrightarrow{\sim} R(S_m \times S_n). \tag{39}$$

Likewise, (21) gives

$$\mathrm{Cl}_{\mathbb{Q}}(S_m) \otimes_{\mathbb{Q}} \mathrm{Cl}_{\mathbb{Q}}(S_n) \cong \mathrm{Cl}_{\mathbb{Q}}(S_m \times S_n).$$

These two isomorphisms fit into the commutative diagram

$$\begin{array}{ccc} R(S_m) \otimes_{\mathbb{Z}} R(S_n) & \xrightarrow{\sim} & R(S_m \times S_n) \\ \downarrow & & \downarrow \\ \mathrm{Cl}_{\mathbb{Q}}(S_m) \otimes_{\mathbb{Q}} \mathrm{Cl}_{\mathbb{Q}}(S_n) & \xrightarrow{\sim} & \mathrm{Cl}_{\mathbb{Q}}(S_m \times S_n), \end{array}$$

where the vertical arrows are the character embeddings. Indeed, $\chi_{V \boxtimes W}(g,h) = \chi_V(g)\chi_W(h)$ for all S_m-representations V, all S_n-representations W, and all $g \in S_m$ and $h \in S_n$. Under these identifications, the scalar product factors:

$$\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle = \langle f_1, f_2 \rangle \langle g_1, g_2 \rangle. \tag{40}$$

We will therefore move freely between class functions on $S_m \times S_n$ and tensor products of class functions on the two factors.

4 Arithmetic and anti-arithmetic products

We now prove Theorem 1.2. The argument is most transparent after passing to the adjoints of the two products.

4.1 Two maps on conjugacy classes

We now define two maps from $S_m \times S_n$ to S_{mn}: the *arithmetic rule* $\mathcal{B}$ and the *anti-arithmetic rule* $\mathcal{A}$.

To define them, we let $\sigma \in S_m$ and $\tau \in S_n$.

For the arithmetic rule, let $\mathcal{B}(\sigma, \tau)$ be the permutation of $[m] \times [n]$ defined by

$$\mathcal{B}(\sigma, \tau)(i, j) = (\sigma(i), \tau(j)). \tag{41}$$

The cycle type of this permutation $\mathcal{B}(\sigma, \tau)$ can be easily described: Each pair consisting of an a-cycle $(x_1, x_2, \ldots, x_a)$ of σ and a b-cycle $(y_1, y_2, \ldots, y_b)$ of τ induces

$$\gcd(a, b) \text{ cycles of length } \operatorname{lcm}(a, b)$$

in the cycle decomposition of $\mathcal{B}(\sigma, \tau)$ (their union is the whole Cartesian product of the two chosen cycles). Thus the map

$$\mathcal{B} : S_m \times S_n \longrightarrow S_{mn} \tag{42}$$

is a genuine group homomorphism (after choosing an identification $[m] \times [n] \cong [mn]$), and its cycle rule is exactly (1).

For the anti-arithmetic rule, we define $\mathcal{A}(\sigma, \tau) \in S_{mn}$ only up to conjugacy, by specifying the cycle type of $\mathcal{A}(\sigma, \tau)$ rather than the permutation itself: Namely, for every pair consisting of an a-cycle of σ and a b-cycle of τ, the permutation $\mathcal{A}(\sigma, \tau)$ shall get

$$\operatorname{lcm}(a, b) \text{ cycles of length } \gcd(a, b). \tag{43}$$

The total number of letters contributed by this pair is again ab, so these data define a partition of mn, hence a conjugacy class of S_{mn}. The specific value of $\mathcal{A}(\sigma, \tau)$ in this conjugacy class can be chosen arbitrarily; thus, $\mathcal{A}$ will not (usually) be a group homomorphism.

The two class maps define pullbacks

$$\Delta^{\boxdot}_{m,n} : \mathrm{Cl}_{\mathbb{Q}}(S_{mn}) \longrightarrow \mathrm{Cl}_{\mathbb{Q}}(S_m \times S_n), \tag{44}$$

$$\Delta^{\diamondsuit}_{m,n} : \mathrm{Cl}_{\mathbb{Q}}(S_{mn}) \longrightarrow \mathrm{Cl}_{\mathbb{Q}}(S_m \times S_n) \tag{45}$$

by

$$(\Delta^{\boxdot}_{m,n} f)(\sigma, \tau) = f(\mathcal{B}(\sigma, \tau)),$$
$$(\Delta^{\diamondsuit}_{m,n} f)(\sigma, \tau) = f(\mathcal{A}(\sigma, \tau)).$$

Both are unital algebra homomorphisms because multiplication of class functions is pointwise.

4.2 These pullbacks are the adjoints of the two products

Transport the maps (44) and (45) through Frobenius characteristic. We use the same symbols for the resulting maps

$$\mathbf{Symm}_{\mathbb{Q},mn} \longrightarrow \mathbf{Symm}_{\mathbb{Q},m} \otimes_{\mathbb{Q}} \mathbf{Symm}_{\mathbb{Q},n}.$$

Proposition 4.1. *For $F \in \mathbf{Symm}_{\mathbb{Q},m}$, $G \in \mathbf{Symm}_{\mathbb{Q},n}$, and $H \in \mathbf{Symm}_{\mathbb{Q},mn}$, the following statements hold.*

(a) We have

$$\langle F \boxdot G, H\rangle = \langle F \otimes G, \Delta^{\boxdot}_{m,n} H\rangle. \tag{46}$$

(b) We have

$$\langle F \diamondsuit G, H\rangle = \langle F \otimes G, \Delta^{\diamondsuit}_{m,n} H\rangle. \tag{47}$$

Proof. (a) By bilinearity, it suffices to take $F = p_\lambda$ and $G = p_\mu$. If h is the class function with $\mathrm{ch}_{nm}(h) = H$, then from (38) we have

$$\langle p_\nu, H\rangle = h(\nu) \tag{48}$$

for every partition ν of mn. The left-hand side of (46) is therefore h evaluated on the arithmetic cycle type associated with (λ, μ). By definition of the pullback, this is also

$$\langle p_\lambda \otimes p_\mu, \Delta^{\boxdot}_{m,n} H\rangle.$$

(b) The anti-arithmetic case is identical, with the anti-arithmetic cycle type in place of the arithmetic one. □

Thus integrality of the products follows from integrality of their adjoints.

Proposition 4.2. *Let $\circ$ denote either $\boxdot$ or $\diamondsuit$. If*

$$\Delta^{\circ}_{m,n}\Big(\mathbf{Symm}_{\mathbb{Z},mn}\Big) \subseteq \mathbf{Symm}_{\mathbb{Z},m} \otimes_{\mathbb{Z}} \mathbf{Symm}_{\mathbb{Z},n}, \tag{49}$$

then

$$\mathbf{Symm}_{\mathbb{Z},m} \circ \mathbf{Symm}_{\mathbb{Z},n} \subseteq \mathbf{Symm}_{\mathbb{Z},mn}.$$

Proof. Take $F \in \mathbf{Symm}_{\mathbb{Z},m}$ and $G \in \mathbf{Symm}_{\mathbb{Z},n}$. For every $H \in \mathbf{Symm}_{\mathbb{Z},mn}$, Proposition 4.1 and (49) give

$$\langle F \circ G, H\rangle = \langle F \otimes G, \Delta^{\circ}_{m,n} H\rangle \in \mathbb{Z},$$

because the Schur bases on the two tensor factors are orthonormal. Lemma 3.6 now implies $F \circ G \in \mathbf{Symm}_{\mathbb{Z},mn}$. □

Under Frobenius characteristic, condition (49) is exactly

$$\Delta^{\circ}_{m,n}\Big(R(S_{mn})\Big) \subseteq R(S_m \times S_n). \tag{50}$$

We now prove this for both pullbacks.

4.3 Power compatibility

The first decisive observation is that both class maps commute with raising permutations to powers.

For the arithmetic map this is immediate from the group homomorphism (42):

$$\mathcal{B}(\sigma^r, \tau^r) = \mathcal{B}(\sigma, \tau)^r.$$

For the anti-arithmetic map, it is not automatic, but it is still true at the level of conjugacy classes. We first record the elementary number-theoretic identity behind the verification.

Lemma 4.3 (A gcd identity)**.** *For all positive integers a, b, r, one has*

$$\gcd\left(\frac{a}{\gcd(a,r)}, \frac{b}{\gcd(b,r)}\right) = \frac{\gcd(a,b)}{\gcd(\gcd(a,b),r)}. \tag{51}$$

Proof. A straightforward proof can be done using p-valuations: Fix a prime p, and put

$$\alpha = v_p(a), \qquad \beta = v_p(b), \qquad \rho = v_p(r).$$

It is easy to see that every nonzero integer m satisfies $v_p\left(m/\gcd(m,r)\right) = \left(v_p(m) - \rho\right)_+$, where $x_+ = \max(x, 0)$. Thus, the p-adic valuation of the left-hand side of (51) is

$$\min\Big((\alpha-\rho)_+, (\beta-\rho)_+\Big) = (\min(\alpha,\beta) - \rho)_+.$$

The valuation of the right-hand side is

$$\min(\alpha,\beta) - \min\Big(\min(\alpha,\beta), \rho\Big) = (\min(\alpha,\beta) - \rho)_+.$$

Thus the two sides have the same p-adic valuation for every prime p. □

Lemma 4.4. *For all $r \geq 1$,*

$$\mathcal{A}(\sigma^r, \tau^r) \sim \mathcal{A}(\sigma, \tau)^r, \tag{52}$$

where $\sim$ denotes conjugacy in S_{mn}.

Proof. Consider one a-cycle of σ and one b-cycle of τ. We shall show that they contribute the same amount of cycles to $\mathcal{A}(\sigma^r, \tau^r)$ as they do to $\mathcal{A}(\sigma, \tau)^r$, and that these cycles all have the same length.

Set

$$g = \gcd(a,b), \qquad c = \gcd(g,r).$$

The definition of $\mathcal{A}(\sigma, \tau)$ shows that our two chosen cycles of σ and τ produce $\operatorname{lcm}(a,b)$ cycles of length g in $\mathcal{A}(\sigma, \tau)$. Taking the r-th power splits each such g-cycle into c cycles of length g/c. Thus the right-hand side of (52) gains

$$\operatorname{lcm}(a,b)c \quad \text{cycles of length } g/c \tag{53}$$

from our two cycles.

Now take the r-th powers of σ and τ first. The a-cycle splits into $\gcd(a,r)$ cycles, each of length $a/\gcd(a,r)$. The same holds for the b-cycle. Lemma 4.3 gives

$$\gcd\left(\frac{a}{\gcd(a,r)}, \frac{b}{\gcd(b,r)}\right) = \frac{g}{c},$$

so every cycle of $\mathcal{A}(\sigma^r, \tau^r)$ obtained from our two cycles has length g/c. Since the block has ab letters in total, the number of such cycles is

$$\frac{ab}{g/c} = \frac{ab}{g}c = \operatorname{lcm}(a,b)c,$$

which agrees with (53). □

Corollary 4.5. *On rational class functions, the following statements hold.*

(a) The map $\Delta^{\boxdot}_{m,n}$ is a morphism of ψ-rings, and hence of λ-rings.

(b) The map $\Delta^{\diamondsuit}_{m,n}$ is a morphism of ψ-rings, and hence of λ-rings.

Proof. For either part, let $\mathcal{C}$ denote the relevant class map ($\mathcal{B}$ or $\mathcal{A}$). For any class function f and any $r \geq 1$ and any $\sigma \in S_m$ and $\tau \in S_n$, we have

$$\begin{aligned}(\Delta\psi^r f)(\sigma,\tau) &= f(\mathcal{C}(\sigma,\tau)^r) \\ &= f(\mathcal{C}(\sigma^r,\tau^r)) \\ &= (\psi^r \Delta f)(\sigma,\tau).\end{aligned}$$

For part (a), the middle equality follows from the homomorphism property of $\mathcal{B}$; for part (b), it follows from Lemma 4.4. Thus each Δ is a ψ-ring morphism. Proposition 2.6(b) makes each a λ-ring morphism. □

At this point, Theorem 3.1 reduces the entire integrality problem to one calculation: we need only check that the pullback of the natural representation M_{mn} is an integral virtual representation of $S_m \times S_n$.

4.4 The arithmetic product: an actual pullback

For the arithmetic map there is nothing to construct. Restricting the natural permutation representation of S_{mn} along (42) gives the permutation representation on $[m] \times [n]$. On the basis vector $e_i \otimes e_j$ of $M_m \boxtimes M_n$, the element (σ, τ) acts by

$$e_i \otimes e_j \longmapsto e_{\sigma(i)} \otimes e_{\tau(j)},$$

which is exactly the product action on $[m] \times [n]$. Therefore

$$\Delta^{\boxdot}_{m,n}(M_{mn}) = M_m \boxtimes M_n \in R(S_m \times S_n). \tag{54}$$

Corollary 4.5 says that $\Delta^{\boxdot}_{m,n}$ is a λ-ring morphism on rational class functions. Since $R(S_m \times S_n)$ is a λ-subring of $\mathrm{Cl}_{\mathbb{Q}}(S_m \times S_n)$, equation (54), together with the fact that M_{mn} generates $R(S_{mn})$ as a λ-ring, gives

$$\Delta^{\boxdot}_{m,n}(R(S_{mn})) \subseteq R(S_m \times S_n).$$

Proposition 4.2 proves the arithmetic integrality assertion

$$\mathbf{Symm}_{\mathbb{Z},m} \boxdot \mathbf{Symm}_{\mathbb{Z},n} \subseteq \mathbf{Symm}_{\mathbb{Z},mn}.$$

There is also a stronger conclusion. Since $\mathcal{B} : S_m \times S_n \hookrightarrow S_{mn}$ is a genuine group embedding, $\Delta^{\boxdot}_{m,n}$ on representation rings is ordinary restriction. Its adjoint is induction. Thus for $\lambda \vdash m$ and $\mu \vdash n$,

$$s_\lambda \boxdot s_\mu = \mathrm{ch}_{mn}\left(\mathrm{Ind}_{S_m \times S_n}^{S_{mn}}(S^\lambda \boxtimes S^\mu)\right), \tag{55}$$

where $S_m \times S_n$ is embedded by the product action. This proves Schur positivity. The same product-action restriction $S_{mn} \downarrow S_m \times S_n$ has recently been studied by Ryba [26] from the viewpoint of stable symmetric-group characters; his Kronecker-comultiplication formulas and stability results give a close representation-theoretic companion to the arithmetic side of the present note.

4.5 Detecting exact cycle lengths by Adams operations

For the anti-arithmetic map, the pullback of M_{mn} is not supplied by an actual restriction functor. We now construct it as a virtual representation.

For $1 \le a \le m$, define

$$C_{m,a} = \sum_{d \mid a} \mu(a/d)\, \psi^d(M_m) \in R(S_m), \tag{56}$$

where μ is the number-theoretic Möbius function. This is an integral virtual representation because the Adams operations preserve $R(S_m)$ by Proposition 2.8(b).

If $\sigma \in S_m$, write $m_a(\sigma)$ for its number of a-cycles.

Lemma 4.6. *For every $\sigma \in S_m$ and $1 \le a \le m$, we have*

$$\chi_{C_{m,a}}(\sigma) = a\, m_a(\sigma). \tag{57}$$

Proof. The character of the natural permutation representation is the number of fixed points. Hence, by (17), each $d \ge 1$ satisfies

$$\chi_{\psi^d(M_m)}(\sigma) = \chi_{M_m}(\sigma^d) = |\operatorname{Fix}(\sigma^d)| = \sum_{b \mid d} b\, m_b(\sigma). \tag{58}$$

Here, the middle equality holds because M_m is a permutation representation, whose character counts fixed basis vectors. For the last equality, a b-cycle of σ is fixed pointwise by σ^d exactly when $b \mid d$; in that case it contributes all of its b letters, and otherwise it contributes none. Substituting (58) into (56) gives

$$\begin{aligned}\chi_{C_{m,a}}(\sigma) &= \sum_{d \mid a} \mu(a/d) \sum_{b \mid d} b\, m_b(\sigma) \\ &= \sum_{b \mid a} b\, m_b(\sigma) \sum_{d:\ b \mid d \mid a} \mu(a/d).\end{aligned}$$

The inner sum is 1 for $b = a$ and 0 otherwise, by Möbius inversion. This yields (57). □

Thus $C_{m,a}$ is a virtual representation whose character counts the letters lying in cycles of *exactly* length a.

Example 4.7. *For the first few values of a, formula* (56) *gives*

$$C_{m,1} = M_m, \qquad C_{m,2} = \psi^2(M_m) - M_m,$$

and

$$C_{m,6} = \psi^6(M_m) - \psi^3(M_m) - \psi^2(M_m) + M_m.$$

Accordingly, their characters are respectively $m_1(\sigma)$, $2m_2(\sigma)$, and $6m_6(\sigma)$.

Remark 4.8 (A direct precedent)**.** *The Frobenius characteristic ch_m sends the virtual character $C_{m,a}$ to the symmetric function $h_{m-a}p_a$. This is not hard to prove using the Murnaghan–Nakayama rule. It also follows easily from Giannelli–Law–Long–Vallejo [9, Definition 3.7 and Theorem 3.9]. For a partition λ and a positive integer e, they define a virtual character $V^\lambda[e]$ by a signed sum over all ways of adding an e-hook to λ. The sign is exactly the usual Murnaghan–Nakayama sign $(-1)^\ell$, where ℓ is the leg length of the added hook; indeed, their proof starts from the power-sum Murnaghan–Nakayama identity*

$$s_\lambda p_e = \sum_\alpha (-1)^{\ell(\alpha/\lambda)} s_\alpha.$$

Their Theorem 3.9 says that, if a permutation has exactly k cycles of length e, then

$$V^\lambda[e](\sigma) = ke\,\chi^\lambda(\tau),$$

where τ is obtained by deleting one such e-cycle (and the value is 0 when $k = 0$). Taking $\lambda = (m-a)$ and $e = a$ (with λ the empty partition when $a = m$) gives

$$V^{(m-a)}[a](\sigma) = a\,m_a(\sigma).$$

Hence Lemma 4.6 shows that

$$C_{m,a} = V^{(m-a)}[a] \qquad \text{in } R(S_m).$$

Thus the virtual character $C_{m,a}$ has appeared before; formula (56) *gives a different Adams–Möbius expression for this special case.*

Remark 4.9 (Cycle-counting functions and character polynomials)**.** *The functions*

$$X_a(\sigma) = m_a(\sigma)$$

are the classical cycle-counting variables used in character polynomials; see, for example, Garsia–Goupil [7] for a combinatorial account and references to the earlier literature. A particularly close symmetric-function precedent for the calculation above is the permutation-eigenvalue viewpoint of Orellana–Zabrocki [25]. If Ξ_μ denotes the multiset of eigenvalues of a permutation matrix of cycle type μ, then

$$p_d[\Xi_\mu] = \sum_{b \mid d} b\,m_b(\mu).$$

This is exactly equation (58)*. Thus Lemma 4.6 is Möbius inversion of this familiar cycle-counting identity, with Proposition 2.8 providing the additional fact needed here: the resulting function aX_a is an* integral virtual character*.*

4.6 The anti-arithmetic substitute for the natural representation

Define

$$W_{m,n} = \sum_{\substack{1\le a\le m,\ 1\le b\le n;\\ \gcd(a,b)=1}} C_{m,a} \boxtimes C_{n,b} \in R(S_m \times S_n). \tag{59}$$

Proposition 4.10. *The following statements hold.*

(a) As class functions on $S_m \times S_n$,

$$\Delta^{\diamondsuit}_{m,n}(M_{mn}) = W_{m,n}. \tag{60}$$

(b) We have $\Delta^{\diamondsuit}_{m,n}(M_{mn}) \in R(S_m \times S_n)$.

Proof. For $(\sigma,\tau) \in S_m \times S_n$, Lemma 4.6 gives

$$\chi_{W_{m,n}}(\sigma,\tau) = \sum_{\gcd(a,b)=1} a\, m_a(\sigma)\, b\, m_b(\tau). \tag{61}$$

Indeed, characters multiply under external tensor products:

$$\chi_{C_{m,a}\boxtimes C_{n,b}}(\sigma,\tau) = \chi_{C_{m,a}}(\sigma)\chi_{C_{n,b}}(\tau),$$

and Lemma 4.6 evaluates these two factors as $a\, m_a(\sigma)$ and $b\, m_b(\tau)$, respectively.

On the other hand, the character of M_{mn} at a permutation is its number of fixed points. A pair consisting of an a-cycle of σ and a b-cycle of τ contributes, under the anti-arithmetic rule, $\operatorname{lcm}(a,b)$ cycles of length $\gcd(a,b)$ to $\mathcal{A}(\sigma,\tau)$. These are fixed points exactly when $\gcd(a,b)=1$; in that case there are

$$\operatorname{lcm}(a,b) = ab$$

of them. Summing over all pairs of cycles gives exactly (61). Hence

$$\chi_{W_{m,n}}(\sigma,\tau) = \chi_{M_{mn}}(\mathcal{A}(\sigma,\tau)),$$

which proves part (a). Part (b) follows from part (a) and the fact that $W_{m,n} \in R(S_m \times S_n)$ by its definition (59). □

We can now finish the proof in one line of λ-ring reasoning.

Proposition 4.11. *For all $m, n \ge 1$, we have*

$$\Delta^{\diamondsuit}_{m,n}(R(S_{mn})) \subseteq R(S_m \times S_n). \tag{62}$$

Proof. By Corollary 4.5, $\Delta^{\diamondsuit}_{m,n}$ is a λ-ring morphism

$$\mathrm{Cl}_{\mathbb{Q}}(S_{mn}) \longrightarrow \mathrm{Cl}_{\mathbb{Q}}(S_m \times S_n).$$

By Proposition 4.10, it sends the natural representation M_{mn} into the λ-subring $R(S_m \times S_n) \subseteq \mathrm{Cl}_{\mathbb{Q}}(S_m \times S_n)$. By Theorem 3.1, M_{mn} generates $R(S_{mn})$ as a λ-ring. Therefore the whole of $R(S_{mn})$ is sent into $R(S_m \times S_n)$. □

Proof of Theorem 1.2. (a) The arithmetic integrality statement follows from (54), Theorem 3.1, and Proposition 4.2.

(b) The Schur-positivity statement is exactly (55).

(c) The anti-arithmetic integrality statement follows from Proposition 4.11 and Proposition 4.2. □

4.7 What the proof is really using

It may be useful to isolate the short core of the anti-arithmetic argument. There are four ingredients.

(i) The anti-arithmetic map on conjugacy classes is compatible with powers:

$$\mathcal{A}(\sigma^r, \tau^r) \sim \mathcal{A}(\sigma, \tau)^r.$$

Therefore its pullback is a morphism for Adams operations and hence for exterior-power operations on rational class functions.

(ii) The Adams operations of the natural S_m-representation detect fixed points of powers:

$$\chi_{\psi^d(M_m)}(\sigma) = |\operatorname{Fix}(\sigma^d)|.$$

Möbius inversion therefore produces the exact-cycle virtual representations $C_{m,a}$.

(iii) Coprime pairs of cycle lengths are exactly the pairs that produce fixed points under the anti-arithmetic rule. This gives the virtual representation $W_{m,n}$ in (59), which is the anti-arithmetic pullback of M_{mn}.

(iv) The single representation M_{mn} generates $R(S_{mn})$ as a λ-ring.

No G-sets or Burnside rings are needed for the proof. They can be a useful way to discover the cycle formulas, but once the natural permutation representations have been introduced, all integrality takes place inside ordinary linear representation rings.

4.8 A general class-map criterion

The proof also isolates a general mechanism that may be useful elsewhere. Let G and H be finite groups. We shall call any map

$$a : \{\text{conjugacy classes of } G\} \longrightarrow \{\text{conjugacy classes of } H\}$$

a *class map*. Pullback gives an algebra homomorphism

$$a^* : \mathrm{Cl}_{\mathbb{Q}}(H) \longrightarrow \mathrm{Cl}_{\mathbb{Q}}(G).$$

For a conjugacy class C of H and $r \geq 1$, write $C^{[r]}$ for the conjugacy class containing h^r, where h is any element of C. If

$$a([g^r]) = a([g])^{[r]} \qquad (g \in G,\ r \geq 1), \tag{63}$$

then a^* commutes with all Adams operations, hence is a λ-ring morphism on rational class functions. To restrict this morphism to the integral representation rings, one needs the additional arithmetic condition

$$a^*(R(H)) \subseteq R(G).$$

The anti-arithmetic proof establishes this condition by checking the image of one λ-generator.

This viewpoint explains both the similarity and the difference between the two products. The arithmetic class map comes from a group homomorphism, so integrality of pullback is automatic. The anti-arithmetic class map only has the weaker power-compatibility property; integrality has to be manufactured separately, and Proposition 4.10 is exactly the missing step.

Example 4.12. *A basic non-homomorphic example is the power class map*

$$[g] \longmapsto [g^q] \qquad (q \geq 1)$$

from the conjugacy classes of a finite group G to themselves. It satisfies (63)*, and its pullback on class functions is exactly the Adams operation ψ^q. Proposition 2.8(b) says that this pullback preserves $R(G)$.*

Remark 4.13. *Single λ-generation is not peculiar to symmetric groups, but the integral statement is stronger than it may first appear. To avoid field-of-definition issues in this remark, let $R_{\mathbb{C}}(G)$ denote the complex representation ring of a finite group G. If $G = C_m$ is cyclic, then a faithful one-dimensional character generates $R_{\mathbb{C}}(G)$ already as a ring.*

More generally, let V be a finite-dimensional complex representation of G, and let A_V be the λ-subring of $R_{\mathbb{C}}(G)$ generated by V. Then

$$A_V \otimes_{\mathbb{Z}} \mathbb{C} = R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{C}$$

if and only if the characteristic polynomials

$$\det(t - V(g)) \qquad (g \in G)$$

separate the conjugacy classes of G. Indeed, their coefficients are, up to signs, the characters of the exterior powers $\bigwedge^j V$, while $R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{C}$ is the algebra of all complex-valued functions on the finite set of conjugacy classes. Thus the algebra generated by these coefficients is the whole function algebra exactly when they separate its points. Integral λ-generation asks in addition that this full-rank subring have index 1 in $R_{\mathbb{C}}(G)$.

For comparison, Adams and Conway found a related phenomenon for compact, simply connected Lie groups: along each arm of the Dynkin diagram, the fundamental representations can be recovered successively from exterior powers of the representation at the end of the arm. Guillot [13] gives an elementary proof. Thus his construction replaces the usual set of fundamental representations by a smaller set indexed by the arms of the Dynkin diagram; for example, his discussion of E_6 uses three λ-generators. What is particularly convenient for S_n is that the very elementary permutation representation M_n already works integrally.

4.9 Further precedents and nearby literature

Several parts of the proof have close relatives in the literature, although we do not know a previous occurrence of the anti-arithmetic integrality argument itself.

First, the use of Adams operations on $R(S_n)$ is classical. In symmetric-function language it is precisely inner plethysm by a power sum: under the Frobenius characteristic, the r-th Adams operation corresponds to

$$f \longmapsto p_r\{f\},$$

where $\{\,\}$ denotes inner plethysm (with the power sum in the outer slot). Thibon [33] and Scharf–Thibon [29] use the Hopf algebra of symmetric functions to study precisely these Adams operators and to recover Littlewood's formulas for inner plethysm. The present proof uses only the easiest part of that theory, namely the power-trace identity $\chi_{\psi^r V}(g) = \chi_V(g^r)$ and Newton's formulas. Meir–Szymik [22] give a useful modern account of Adams operations on finite group representation rings and characterize them as natural operations on the representation-ring functor.

Second, the cycle-length arithmetic behind the ordinary arithmetic product belongs to a broader gcd/lcm circle of ideas. The necklace ring of Metropolis–Rota [23] has multiplication whose structure constants involve gcd and lcm, and Dress–Siebeneicher [5] identify closely related necklace and Burnside-ring constructions with the big Witt vectors and λ-rings. These works are not needed for the proof above, but they provide a conceptual home for the same arithmetic on cycle lengths. On the species side, Maia–Méndez [20] and Li [18] show how the ordinary arithmetic product interacts with Dirichlet series, Cartesian products, and prime decompositions of combinatorial structures. Ryba [26] studies the corresponding restriction $S_{mn} \downarrow S_m \times S_n$ in the stable-character basis and proves stability results for its multiplicities.

Finally, the generation result for $R(S_n)$ has a substantial history. Murnaghan [24] studied generation of irreducible representations under Kronecker products already in 1955. Butler [4] and especially Boorman [1] obtained forms of the one-generator result in the language of S-operations and λ-rings; Marin [21] later gave a short proof based on Dvir's formula. Harman [14] extends this circle of results to representation rings of certain wreath products.

A A descent-algebra proof of the Boorman–Marin theorem

This appendix records another proof of Theorem 3.1. Its main ingredient is a slightly stronger integral statement: the degree-n component of the algebra of noncommutative symmetric functions is generated, for its internal product, by the complete functions with two parts. Equivalently, Solomon's descent algebra is generated by the sums of permutations that are increasing on two consecutive blocks. After abelianization this says that $R(S_n)$ is generated, as an ordinary ring, by the permutation representations on k-subsets. A short argument with symmetric powers then shows that all of these permutation representations already lie in the λ-subring generated by M_n.

Schocker [30] proves a related generation theorem. His main theorem generates Solomon's descent algebra by sums of descent classes associated with hook shapes. Those generators are different from the two-block complete elements used below, but the two results belong to the same circle of descent-algebra generation phenomena.

A.1 Two-block generators for noncommutative symmetric functions

Let $\mathbf{NSym}_{\mathbb{Z}}$ be the free associative $\mathbb{Z}$-algebra on generators $H_1, H_2, \ldots$, with $\deg H_i = i$, and put $H_0 = 1$. If $I = (i_1, \ldots, i_r)$ is a composition, write

$$H_I = H_{i_1} \cdots H_{i_r}.$$

The elements H_I, for $I \models n$, form a $\mathbb{Z}$-basis of the homogeneous component $\mathbf{NSym}_{\mathbb{Z},n}$.

Besides its usual product, $\mathbf{NSym}_{\mathbb{Z},n}$ carries an *internal product* $*$, corresponding to multiplication in Solomon's descent algebra. We shall only need the standard matrix rule of Gelfand–Krob–Lascoux–Leclerc–Retakh–Thibon [8, Proposition 5.1]. There is also a useful set-theoretic interpretation of this rule. Blessenohl–Schocker [2, Appendix B, especially B.1 and B.5; see also §§12.11–12.12] realize the relevant Young characters using ordered set partitions. If two ordered set partitions have block sizes I and J, then the matrix entry m_{uv} records the cardinality of the intersection of the u-th block of the first partition with the v-th block of the second. Thus the row sums and column sums are I and J, while reading the nonzero intersection sizes row by row gives precisely the composition occurring in Solomon's multiplication rule. In particular, the matrix formula below is the noncommutative-symmetric-function form of the same Mackey-type calculation.

If $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ are compositions of n, then

$$H_I * H_J = \sum_M H_{\mathrm{comp}(M)}, \tag{64}$$

where $M = (m_{uv})$ ranges over all $p \times q$ matrices with nonnegative integer entries whose row-sum vector is I and whose column-sum vector is J; explicitly,

$$\sum_{v=1}^{q} m_{uv} = i_u \quad (1 \le u \le p), \qquad \sum_{u=1}^{p} m_{uv} = j_v \quad (1 \le v \le q).$$

The composition $\mathrm{comp}(M)$ is obtained by reading the entries of M row by row and deleting the zero entries.

Example A.1. *Take $I = (3,1)$ and $J = (2,2)$. There are exactly two nonnegative 2×2 matrices with row-sum vector I and column-sum vector J:*

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

Their row-reading compositions are $(1,2,1)$ and $(2,1,1)$. Thus (64) gives

$$H_{(3,1)} * H_{(2,2)} = H_{(1,2,1)} + H_{(2,1,1)}.$$

Theorem A.2. *Fix $n \ge 1$. Under the internal product, the $\mathbb{Z}$-algebra $\mathbf{NSym}_{\mathbb{Z},n}$ is generated by*

$$H_{(k,n-k)} = H_k H_{n-k} \qquad (1 \le k \le n), \tag{65}$$

where a zero part is omitted.

Proof. Let A_n be the $\mathbb{Z}$-subalgebra of $\mathbf{NSym}_{\mathbb{Z},n}$ (with the internal product) generated by the elements in (65). We prove that $H_I \in A_n$ for every composition $I \models n$, by downward induction on the first part of I. The initial case is $I = (n)$, for which $H_I = H_{(n,0)}$ is one of the generators.

Now let

$$I = (m, a_2, a_3, \ldots, a_r), \qquad r \geq 2,$$

and assume that $H_K \in A_n$ for every composition $K \models n$ whose first part is strictly larger than m. Set

$$I' = (m + a_2, a_3, \ldots, a_r) \qquad \text{and} \qquad J = (n - a_2, a_2).$$

The first part of I' is $m+a_2 > m$, so $H_{I'} \in A_n$ by induction, while H_J is one of the generators and thus belongs to A_n as well.

Apply (64) to $H_{I'} * H_J$. Every matrix that occurs has two columns. Write its first row as

$$(m + a_2 - d, d).$$

Since the second column has total sum a_2, we have $0 \leq d \leq a_2$. If $d < a_2$, then the first part of $\mathrm{comp}(M)$ is $m + a_2 - d > m$. If $d = a_2$, all later entries in the second column must vanish, and there is exactly one possible matrix, namely

$$\begin{pmatrix} m & a_2 \\ a_3 & 0 \\ a_4 & 0 \\ \vdots & \vdots \\ a_r & 0 \end{pmatrix}.$$

Its row-reading composition is I. Consequently

$$H_{I'} * H_J = H_I + \sum_{\substack{K \models n; \\ K_1 > m}} c_K H_K \tag{66}$$

for some $c_K \in \mathbb{N}$, where K_1 denotes the first entry of K. Every term in the sum belongs to A_n by induction, so (66) yields $H_I \in A_n$. This completes the induction. □

Remark A.3. *The proof is triangular in a precise elementary sense: when* (66) *is solved for H_I, every other basis element that occurs has first part strictly larger than I_1. Thus the downward induction eliminates the standard basis elements one first-part level at a time. No stronger "unitriangular basis" statement is needed here.*

Under the usual identification of $\mathbf{NSym}_{\mathbb{Z},n}$ with Solomon's descent algebra (up to the harmless opposite-algebra convention for the internal product), H_I corresponds to the sum of all permutations whose descent set is contained in the set of partial sums of I. Hence $H_{(k,n-k)}$ corresponds to

$$a_k = \sum_{\substack{w \in S_n; \\ w(1) < \cdots < w(k); \\ w(k+1) < \cdots < w(n)}} w,$$

with empty chains omitted. We therefore obtain the following equivalent form.

Corollary A.4. *For every $n \geq 1$, the integral Solomon descent algebra of S_n is generated by $a_1, a_2, \ldots, a_n$.*

A.2 Abelianization and the Boorman–Marin theorem

The abelianization map

$$\pi : \mathbf{NSym}_{\mathbb{Z}} \longrightarrow \mathbf{Symm}_{\mathbb{Z}}, \qquad H_r \longmapsto h_r, \tag{67}$$

is surjective. On each homogeneous component it also respects the internal products: the image of (64) is the corresponding matrix rule for the Kronecker product of complete symmetric functions. Thus Theorem A.2 immediately gives:

Corollary A.5. *For every $n \geq 1$, the ring $(\mathbf{Symm}_{\mathbb{Z},n}, *)$ is generated by*

$$h_k h_{n-k} \qquad (0 \leq k \leq n).$$

Equivalently, $R(S_n)$ is generated under tensor product by the permutation representations

$$P_{n,k} := \mathbb{Q}\left[\binom{[n]}{k}\right] \qquad (0 \leq k \leq n) \tag{68}$$

on the k-subsets of $[n]$.

Proof. Under the Frobenius characteristic,

$$\mathrm{ch}_n(P_{n,k}) = h_k h_{n-k},$$

since $P_{n,k} \cong \mathrm{Ind}_{S_k \times S_{n-k}}^{S_n} \mathbf{1}$. Equivalently, its character is the two-part Young character of type $(k, n-k)$. In the notation of Blessenohl–Schocker [2, Chapter 12 and Appendix B], this is $\xi^{(k,n-k)}$: their ordered set partitions of type $(k, n-k)$ are simply pairs (A, A^c), hence are naturally identified with the k-subsets A of $[n]$. □

It remains to connect these generators with the single λ-generator $M_n = P_{n,1}$ used in Theorem 3.1. We do this without invoking that theorem.

For positive integers $c_1, \ldots, c_r$ with $d = c_1 + \cdots + c_r \leq n$, define the permutation module

$$Y_{c_1,\ldots,c_r} = \mathbb{Q}\Big[\{(A_1, \ldots, A_r) : |A_i| = c_i \text{ and the } A_i \text{ are pairwise disjoint}\}\Big]. \tag{69}$$

Thus $Y_k = P_{n,k}$. We call d the *support size* of this permutation module.

Lemma A.6. *Let L_n be the λ-subring of $R(S_n)$ generated by M_n. Suppose that $P_{n,j} \in L_n$ for every $j < d$. Then every $Y_{c_1,\ldots,c_r}$ of support size d with $r \geq 2$ belongs to L_n.*

Proof. We prove the assertion simultaneously for all support sizes d by strong induction on d. For $d = 1$ there is nothing to prove, since $r \geq 2$. Fix $d \geq 2$, assume the assertion known for all smaller support sizes, and assume, as in the statement, that $P_{n,j} \in L_n$ for every $j < d$.

Since $r \geq 2$, each $c_i < d$, so each P_{n,c_i} belongs to L_n. Their tensor product is the permutation representation on all tuples $(A_1, \ldots, A_r)$ with $|A_i| = c_i$, with no disjointness condition. For each nonempty $B \subseteq [r]$, define the *membership region*

$$R_B = \left(\bigcap_{i \in B} A_i\right) \setminus \left(\bigcup_{i \notin B} A_i\right).$$

Decompose this S_n-set into orbits according to the cardinalities $|R_B|$ of all these membership regions. Exactly one orbit is the pairwise-disjoint locus (69). Every other orbit has union of size strictly smaller than d, and is itself a permutation module of the form $Y_{b_1,\dots,b_s}$ with support size $< d$, after the nonempty membership regions are listed as labeled blocks.

For every such smaller-support orbit module, the induction hypothesis applies: if its support size is $e < d$, then all $P_{n,j}$ with $j < e$ belong to L_n because $j < e < d$. Hence all these other orbit modules belong to L_n. Subtracting them from $P_{n,c_1} \otimes \cdots \otimes P_{n,c_r}$ leaves $Y_{c_1,\dots,c_r}$. □

Proposition A.7. *For every $0 \le k \le n$, one has $P_{n,k} \in L_n$.*

Proof. We use strong induction on k. The cases $k = 0$ and $k = 1$ are clear. Assume $k \ge 2$ and that $P_{n,j} \in L_n$ for all $j < k$.

Step 1: decompose the symmetric power into permutation orbits. The symmetric power $\mathrm{Sym}^k(M_n)$ has a basis consisting of the monomials of total degree k in the basis vectors $e_1, \dots, e_n$, and S_n permutes this basis. Its orbits are indexed by partitions $\lambda \vdash k$: the partition records the positive multiplicities of the basis vectors occurring in a monomial. If $m_r(\lambda)$ denotes the number of parts of λ equal to r, then the orbit of type λ affords

$$Y_{m_1(\lambda),m_2(\lambda),\dots},$$

with zero entries omitted. Its support size is $\ell(\lambda) = \sum_r m_r(\lambda)$.

For $\lambda = (1^k)$ this orbit module is $P_{n,k}$. For every other partition λ we have $\ell(\lambda) < k$, so Lemma A.6, together with the induction hypothesis, puts its orbit module in L_n. Hence

$$[\mathrm{Sym}^k(M_n)] = [P_{n,k}] + \sum_{\substack{\lambda \vdash k;\\ \lambda \neq (1^k)}} [Y_{m_1(\lambda),m_2(\lambda),\dots}]. \tag{70}$$

Step 2: isolate the k-subset orbit. The left-hand side belongs to L_n. Indeed, in the ring $R(S_n)[[t]]$, we have

$$\sum_{j \ge 0} [\mathrm{Sym}^j(M_n)] t^j = \frac{1}{\lambda_{-t}([M_n])} = \lambda_{-t}(-[M_n]),$$

so each symmetric-power class belongs to the λ-subring L_n even with the weak notion of λ-ring used in this paper. All terms in the sum in (70) belong to L_n, so $P_{n,k} \in L_n$ as well. □

Second proof of Theorem 3.1. By Corollary A.5, the classes $[P_{n,k}]$ generate $R(S_n)$ as an ordinary ring. By Proposition A.7, every one of these classes belongs to the λ-subring L_n generated by M_n. Hence $R(S_n) \subseteq L_n$. The reverse inclusion is tautological, so $L_n = R(S_n)$. □

Remark A.8. *The argument proves a little more than what is needed for Theorem 3.1: independently of λ-operations, Corollary A.5 says that the permutation modules on subsets of fixed cardinalities generate the entire integral representation ring under tensor product. The role of the last two results is only to show that these many ring generators can all be manufactured from the single λ-generator M_n.*

References

[1] E. H. Boorman, *S-operations in representation theory*, Trans. Amer. Math. Soc. **205** (1975), 127–149, https://doi.org/10.1090/S0002-9947-1975-0364424-3.

[2] D. Blessenohl and M. Schocker, *Noncommutative Character Theory of the Symmetric Group*, Imperial College Press, London, 2005, https://doi.org/10.1142/P369.

[3] N. Bourbaki, *Algebra I: Chapters 1–3*, English translation, Addison–Wesley, 1974; reprinted by Springer, https://link.springer.com/book/9783540642435.

[4] P. H. Butler, *S-functions and symmetry in physics*, J. Physique Colloq. **31** (1970), C4-47–C4-50, https://doi.org/10.1051/jphyscol:1970408.

[5] A. W. M. Dress and C. Siebeneicher, *The Burnside ring of the infinite cyclic group and its relations to the necklace algebra, λ-rings, and the universal ring of Witt vectors*, Adv. Math. **78** (1989), no. 1, 1–41, https://doi.org/10.1016/0001-8708(89)90027-3.

[6] P. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, D. Vaintrob, and E. Yudovina, *Introduction to Representation Theory*, Student Mathematical Library 59, American Mathematical Society, 2011; also arXiv:0901.0827, https://doi.org/10.1090/stml/059.

[7] A. M. Garsia and A. Goupil, *Character polynomials, their q-analogs and the Kronecker product*, Electron. J. Combin. **16** (2009), no. 2, Research Paper R19, 40 pp., https://doi.org/10.37236/85.

[8] I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh, and J.-Y. Thibon, *Noncommutative symmetric functions*, Adv. Math. **112** (1995), no. 2, 218–348, https://doi.org/10.1006/aima.1995.1032.

[9] E. Giannelli, S. Law, J. Long, and C. Vallejo, *Sylow branching coefficients and a conjecture of Malle and Navarro*, Bull. Lond. Math. Soc. **54** (2022), no. 2, 552–567, https://doi.org/10.1112/blms.12584. See https://eugeniomaths.wordpress.com/wp-content/uploads/2022/01/5.-gllv22.pdf for a preprint.

[10] D. Grinberg, *Anti-arithmetic product of symmetric functions: (why) is it integral?*, MathOverflow question 182083 (2014), https://mathoverflow.net/questions/182083/anti-arithmetic-product-of-symmetric-functions-why-is-it-integral.

[11] D. Grinberg, *Arithmetic product of symmetric functions: why is it integral?*, MathOverflow question 138148 (2013), https://mathoverflow.net/questions/138148/arithmetic-product-of-symmetric-functions-why-is-it-integral.

[12] D. Grinberg and V. Reiner, *Hopf algebras in combinatorics*, arXiv:1409.8356v7 (2020), https://arxiv.org/abs/1409.8356.

[13] P. Guillot, *The representation ring of a simply connected Lie group as a λ-ring*, Comm. Algebra **35** (2007), no. 3, 875–883; arXiv:math/0510283, https://doi.org/10.1080/00927870601115799.

[14] N. Harman, *Generators for the representation rings of certain wreath products*, J. Algebra **445** (2016), 125–135; arXiv:1410.1786, https://doi.org/10.1016/j.jalgebra.2015.09.003.

[15] M. Hazewinkel, *Witt vectors. Part 1*, in *Handbook of Algebra*, Vol. 6, Elsevier, 2009, pp. 319–472; arXiv:0804.3888, https://doi.org/10.1016/S1570-7954(08)00207-6.

[16] D. Knutson, *λ-Rings and the Representation Theory of the Symmetric Group*, Lecture Notes in Mathematics 308, Springer, 1973, https://doi.org/10.1007/BFb0069217.

[17] C. Lassueur, *Character Theory of Finite Groups*, lecture notes, RPTU Kaiserslautern–Landau, Summer Semesters 2022 and 2023, version of June 30, 2023, https://classueur.github.io/maths/teaching/skripte/CharaktertheorieSS23.pdf.

[18] J. Li, *Prime graphs and exponential composition of species*, J. Combin. Theory Ser. A **115** (2008), no. 8, 1374–1401; arXiv:0705.0038, https://doi.org/10.1016/j.jcta.2008.02.008.

[19] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed., Oxford Mathematical Monographs, Oxford University Press, 1995, https://doi.org/10.1093/oso/9780198534891.001.0001.

[20] M. Maia and M. Méndez, *On the arithmetic product of combinatorial species*, Discrete Math. **308** (2008), no. 23, 5407–5427; arXiv:math/0503436, https://doi.org/10.1016/j.disc.2007.09.062.

[21] I. Marin, *Hooks generate the representation ring of the symmetric group*, Expositiones Math. **30** (2012), no. 3, 268–276; arXiv:1112.3127, https://doi.org/10.1016/j.exmath.2012.03.005.

[22] E. Meir and M. Szymik, *Adams operations and symmetries of representation categories*, Indiana Univ. Math. J. **70** (2021), no. 2, 501–523; arXiv:1704.03389, https://doi.org/10.1512/iumj.2021.70.8377.

[23] N. Metropolis and G.-C. Rota, *Witt vectors and the algebra of necklaces*, Adv. Math. **50** (1983), no. 2, 95–125, https://doi.org/10.1016/0001-8708(83)90035-X.

[24] F. D. Murnaghan, *On the generation of the irreducible representations of the symmetric group*, Proc. Natl. Acad. Sci. USA **41** (1955), no. 7, 514–515, https://doi.org/10.1073/pnas.41.7.514.

[25] R. Orellana and M. Zabrocki, *Symmetric group characters as symmetric functions*, Adv. Math. **390** (2021), Paper No. 107943, 34 pp.; arXiv:1605.06672, https://doi.org/10.1016/j.aim.2021.107943.

[26] C. Ryba, *Kronecker comultiplication of stable characters and restriction from S_{mn} to $S_m \times S_n$*, J. Algebra **638** (2024), 720–738, https://doi.org/10.1016/j.jalgebra.2023.07.043.

[27] B. E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, 2nd ed., Graduate Texts in Mathematics 203, Springer, 2001, https://doi.org/10.1007/978-1-4757-6804-6.

[28] F. V. Saliola, *Hyperplane arrangements and descent algebras*, lecture notes, version of January 10, 2006, https://saliola.github.io/maths/publications/LectureNotes/DesAlgLectureNotes.pdf; corrections at https://www.cip.ifi.lmu.de/~grinberg/algebra/saliola-errata.pdf.

[29] T. Scharf and J.-Y. Thibon, *A Hopf-algebra approach to inner plethysm*, Adv. Math. **104** (1994), no. 1, 30–58, https://doi.org/10.1006/aima.1994.1019.

[30] M. Schocker, *A generating set of Solomon's descent algebra*, J. Algebra **263** (2003), no. 1, 151–158, https://doi.org/10.1016/S0021-8693(03)00069-3.

[31] J.-P. Serre, *Linear Representations of Finite Groups*, Graduate Texts in Mathematics 42, Springer, 1977, https://doi.org/10.1007/978-1-4684-9458-7.

[32] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, 2nd ed., Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2023, https://doi.org/10.1017/9781009262538.

[33] J.-Y. Thibon, *Fonctions symétriques et opérations d'Adams de l'anneau de Grothendieck d'un groupe symétrique*, C. R. Acad. Sci. Paris Sér. I Math. **313** (1991), no. 12, 829–832; listed on the author's publication page, https://igm.univ-mlv.fr/~jyt/articles.html.

[34] D. Yau, *Lambda-Rings*, World Scientific, 2010, https://doi.org/10.1142/7664.