Abelian networks IV. Dynamics of nonhalting networks
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Errata and addenda by Darij Grinberg (version of April 20, 2018)

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I will refer to the results appearing in the preprint "Abelian networks IV. Dynamics of nonhalting networks" by the numbers under which they appear in this preprint (specifically, in its version of 10 April 2018, posted on the arXiv as arXiv:1804.03322v1).

1. Errata

The errata below are relatively haphazard – I have skipped various proofs in my readthrough. A lot of them are pedantic suggestions rather than serious errors.

- **page 3:** "increasing functions the form" $\rightarrow$ "increasing functions of the form".
- **page 7:** In "the spectral radius of P", the "P" should be in mathmode.
- **page 9, §2.1:** In the definition of the Grothendieck group, I'd replace "$\mathcal{M} \times \mathcal{M} / \sim$" by "$(\mathcal{M} \times \mathcal{M}) / \sim$"; otherwise, it may be misread as "$\mathcal{M} \times (\mathcal{M} / \sim)$".
- **page 9, §2.1:** Before "Grothendieck group satisfies the *universal enveloping property*", add "The".
- **page 11, §2.1:** You say that "$\tau(\mathcal{K})$ is a finite group if $\mathcal{M}$ is finitely generated". I would suggest justifying this claim (it follows from the Fundamental Theorem of finitely generated abelian groups, but this should probably be said).
- **page 11, §2.2:** Your definition of "irreducible" (for an action of $\mathcal{F}$ on Y) differs from the one given in [BL16b]. Are the two definitions equivalent? A few extra sentences are probably in order, since you refer to [BL16b] for the proof of Lemma 2.7.
- **page 13, §3.1:** You begin this section with "Let $G = (V(G), E(G))$ be a directed graph". But if you want multiple edges, then your directed graph cannot just be a pair of two sets; it must also store the sources and the targets of the edges somewhere. I would encode a directed graph with multiple edges either as a triple (V, E, φ) (where φ is the map sending any edge to the pair consisting of its source and its target) or as a quadruple

(V, E, s, t) (where s is the map sending each edge to its source, and t is the map sending each edge to its target).

- **page 13, §3.1:** In the definition of "message-passing function", replace "for each edge $e = (v, u)$" by "for each edge e with source v and target u". This is, again, because you want to allow multiple edges, so you cannot just identify an edge with a pair of two vertices.

  This problem keeps coming up over and over in your paper, if the parts I have read so far are representative. I personally prefer to abbreviate "edge e with source v and target u" by "edge $e : v \to u$" for brevity (more so because it reminds me of the directedness of the graph). I don't know of a good way to quickly search/replace all the appearances of this, though.

- **page 13, §3.1:** After "and sends the message $T_e(q, a)$ to $\mathcal{P}_u$", add "for each edge e from v to u".

- **page 13, §3.1:** In the commutativity conditions, you should add "For each edge e with source v and target u" at the beginning of condition (ii).

- **page 14, §3.1:** In the first equality on this page, you seem to use the notation $\operatorname{Out}(v)$ for the set of all edges with source v. Please say so.

- **page 14, §3.1:** Also, in the first equality on this page, you want v to be the vertex of G satisfying $a \in A_v$.

- **page 14, §3.1:** In "Lemma 3.1(ii) implies that the function", replace "function" by "functions".

- **page 14, §3.2:** Do you really mean "the letters in $\mathcal{N}$", or perhaps "the letters in w"?

- **page 14, §3.2:** On the second-to-last line of this page, "for all $i \in \{1, \ldots, l\}$" $\to$ "for all $i \in \{1, \ldots, \ell\}$" (\ell, not just l).

- **page 15, Definition 3.2:** Explain, perhaps, what "sends" means (namely, w "sends" $\mathbf{x}.\mathbf{q}$ to $\pi_w(\mathbf{x}.\mathbf{q})$).

- **page 15, §3.2:** "The *support* of a vector $\mathbf{z}$" $\to$ "The *support* of a vector $\mathbf{u}$".

- **page 15, Lemma 3.3:** In part (ii), replace "$\operatorname{supp}(\mathbf{w})$" by "$\operatorname{supp}(|w|)$".

- **page 15, §3.3:** Your definition of a finite network seems to entail that the state space Q can be infinite, but only the Q_v and A_v for each given $v \in V$ are required to be finite (whereas V may be infinite). But I don't think this is what you want, because you next claim that M is finite. I don't know why you want to define finiteness for a single processor to begin with; do you ever use it? Why not just define a network $\mathcal{N}$ to be finite if its (total) state space Q and its (total) alphabet A are finite?

- **page 15, proof of Lemma 3.4:** In "Since $\mathbf{n} \geq 1$", the "1" should be boldfaced.
- **page 15, §3.3:** I don't understand the definition of "idempotent vector". Seeing that you're only using this word one time in the whole paper, maybe ditch it altogether?
- **page 16, §3.4:** In "*negative part* $\mathbf{z}^-$ of $\mathbf{x}$", replace "$\mathbf{x}$" by "$\mathbf{z}$".
- **page 16, §3.4:** After "such that $\mathbf{z} = \mathbf{z}^+ - \mathbf{z}^-$", add "and $\operatorname{supp}(\mathbf{z}^+) \cap \operatorname{supp}(\mathbf{z}^-) = \varnothing$".
- **page 16, §3.4:** When defining "locally irreducible", you probably also want to require that $Q \neq \varnothing$. Or, at least, you should require it in Definition 3.8, since otherwise $P(\mathbf{k})$ is not defined (you need a $\mathbf{q} \in \operatorname{Loc}(\mathcal{N})$ to ensure that $P_{\mathbf{q}}(\mathbf{k})$ makes sense).
- **page 17, §3.5:** You may want to point out what $\lambda(P)$ means when the matrix P is empty. (This happens, e.g., when $A = \varnothing$ in your network.)
- **page 17, Lemma 3.10:** I assume you require eigenvectors to be $\neq 0$, or otherwise part (i) wouldn't be as interesting as you may want it to be. Best to say so.
- **page 19, §3.6:** Somewhere here it helps to say that G denotes the underlying digraph of the network; V its set of vertices; and E its set of edges.
- **page 19, §3.6:** I suggest adding "Eulerian" after "into a" in "Any undirected graph can be changed into a directed graph".
- **page 20, §3.6:** In "For $n \geq 3$, the *bidirected cycle* C_n $(n \geq 3)$", remove the "$(n \geq 3)$" part (you already said it half a sentence ago).
- **page 20, Example 3.11:** What is a cyclic list? (I assume it is a family indexed by $\mathbb{Z}_m$ for some $m > 0$. Please define it.) Same for the concept of a "cyclic total order" later (in Example 3.14). The two things mean the same, right?
- **page 20, Example 3.11:** "by a *rotor configurations*" $\rightarrow$ "by a *rotor configuration*".
- **page 20, Example 3.11:** Not just in this example, but also in others, I'd recommend adding a few sentences about what $\operatorname{Loc}(\mathcal{N})$, e and K are. It is not hard to figure out (at least in Examples 3.11–3.19, the transition functions T_a are bijective, so that the total transition functions t_a are bijective as well, and thus M is a sub-unital submonoid of $\operatorname{End}(Q)$, so that $e = 1$, and therefore $\operatorname{Loc}(\mathcal{N}) = Q$; finding K is a bit trickier), but these things are among the trickiest to understand (being non-issues in classical chip-firing theory), so some handholding would help readers a lot here.

- **page 20, Example 3.12:** In "$Q_v := \{0, 1 \ldots, \operatorname{outdeg}(v) - 1\}$", add a comma after "$0, 1$".
- **page 21, Remark:** "has a subtle difference" → "have a subtle difference".
- **page 21, Example 3.13:** You don't want to fix a cyclic list "of the target vertices of v". You want to fix a cyclic list of the outgoing edges e_j^v of v, and then define u_j^v as the target of e_j^v. Otherwise, it is not clear how to order parallel edges.
- **page 22, Example 3.14:** In "is identical to the sinkless height-arrow networks", replace "networks" by "network".
- **page 22, Example 3.14:** "is equal the matrix" → "is equal to the matrix".
- **page 22, Example 3.14:** I suspect that the claim "$\lambda(P) < 1$" (and, hence, subcriticality) requires P to be strongly connected.
- **page 23, Example 3.15:** "with positive diagonal entries" → "with positive integer diagonal entries", I guess?
- **page 23, Example 3.15:** "For each $v \in V$ of" → "For each vertex $v \in V$ of".
- **page 23, Example 3.15:** In "$Q_v := \{0, 1 \ldots, d_v - 1\}$", add a comma after "$0, 1$".
- **page 24, Example 3.16:** In the definition of the message-passing function, replace "$(\operatorname{mod} \mathbb{Z}_{\operatorname{outdeg}(v)})$" by "$(\operatorname{mod} \operatorname{outdeg}(v))$" (or by "in $\mathbb{Z}_{\operatorname{outdeg}(v)}$").
- **page 24, Example 3.16:** "the states of this network can" → "a state of this network can".
- **page 25, Example 3.17:** In "$Q_v := \{0, 1 \ldots, t_v - 1\}$", add a comma after "$0, 1$".
- **page 25, Example 3.18:** In "where $M := (m_{i,j})_{0 \leq i,j \leq \ell}$", replace "$\ell$" by "$k$".
- **page 26, Example 3.19:** "For each $v \in V$ of" → "For each vertex $v \in V$ of".
- **page 26, Example 3.19:** "for all $i \in \mathbb{Z}_m$" → "for all $i \in \mathbb{Z}_{m_v}$".
- **page 27, and several times further:** "occurences" → "occurrences".
- **page 32, Lemma 4.11:** The first comma on the second line of this lemma should be a period.
- **page 33, proof of Lemma 4.13:** "$u = a_{j+1} \cdots a_k$" → "$u = a_{j+1} \cdots a_{j+k}$".
- **page 34, proof of Proposition 4.9:** Shouldn't "The conclusion of the lemma" be "The conclusion of the proposition"?

- **page 35, proof of Lemma 4.14:** "The Perron-Frobenius theorem (Lemma 3.10(iii)) then implies that $\lambda(P) = 1$" is overkill: You can just as well argue that $P|w| = |w|$ reveals that $|w|$ is an eigenvector of P for eigenvalue 1, and thus $\lambda(P) \geq 1$, which is equally sufficient for your aims.

- **page 44, proof of Lemma 5.3, (i) implies (ii):** "$\xrightarrow{w'w'''}$" $\to$ "$\xrightarrow{w'w''}$".

- **page 51, Example 5.15:** Why do you say that "the capacity of a sinkless rotor network is equal to zero" and not $-\infty$? Is the $\mathbb{Z}^A$ in Definition 5.14 meant to be an $\mathbb{N}^A$?

- **page 52, proof of Lemma 5.16:** Does this really prove that the capacity is well-defined? From what I see, it only shows that the maximum defining the capacity exists as a supremum; why does it exist as a maximum?

- **page 52, Definition 5.17:** "finitely" $\to$ "finite"?

- **page 82, §8.3:** In the definition of the weight function, replace "mod $\mathbb{Z}_n$" by "in $\mathbb{Z}_n$".

- **page 82, §8.3:** What do you mean by "level" here? I thought all levels in the sinkless rotor network would be 0; do you mean "level in the sandpile sense"? (Perhaps better to define this independently.)