The diagonal derivative of a skew Schur polynomial

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version 1.0, 21 February 2024

Abstract. We prove a formula for the image of a skew Schur polynomial $s_{\lambda/\mu}(x_1, x_2, \ldots, x_N)$ under the differential operator $\nabla := \dfrac{\partial}{\partial x_1} + \dfrac{\partial}{\partial x_2} + \cdots + \dfrac{\partial}{\partial x_N}$. This generalizes a formula of Weigandt for $\nabla(s_\lambda)$.

1. Notations and definitions

Fix a nonnegative integer N. Let $R = \mathbb{Z}[x_1, x_2, \ldots, x_N]$ be the ring of polynomials in N indeterminates $x_1, x_2, \ldots, x_N$ with integer coefficients.

Let $\nabla : R \to R$ be the operator $\dfrac{\partial}{\partial x_1} + \dfrac{\partial}{\partial x_2} + \cdots + \dfrac{\partial}{\partial x_N}$. This map ∇ is a derivation (i.e., it is $\mathbb{Z}$-linear and satisfies $\nabla(fg) = (\nabla f)g + f(\nabla g)$ for all $f, g \in R$). We call it the *diagonal derivative* since (in the language of analysis) it is the directional derivative with respect to the vector $(1, 1, \ldots, 1)$. (The notation ∇ comes from the related operator in [Nenash20], but this is not the vector differential operator ∇ known from analysis.)

We let $[N]$ denote the set $\{1, 2, \ldots, N\}$.

For each $i \in [N]$, we let e_i be the N-tuple $(0, 0, \ldots, 0, 1, 0, 0, \ldots, 0) \in \mathbb{Z}^N$ where the 1 is at the i-th position. Addition and subtraction of N-tuples are defined entrywise (i.e., these N-tuples are viewed as vectors in the $\mathbb{Z}$-module $\mathbb{Z}^N$). Thus, if $\mu \in \mathbb{Z}^N$ is any N-tuple, then $\mu + e_i$ is the N-tuple obtained from μ by increasing the i-th entry by 1.

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We shall use the standard notations regarding symmetric polynomials in N variables $x_1, x_2, \ldots, x_N$ as introduced (e.g.) in [Stembr02] (but we write N for what was called n in [Stembr02]).

If $a \in \mathbb{Z}^N$ is any N-tuple, and if $i \in [N]$, then the notation a_i shall denote the i-th entry of a (so that $a = (a_1, a_2, \ldots, a_N)$).

We let $\mathcal{P}_N$ denote the set of all N-tuples $a \in \mathbb{Z}^N$ that satisfy

$$a_1 \geq a_2 \geq \cdots \geq a_N \geq 0.$$

For instance, if $N = 3$, then the N-tuples $(3, 1, 0)$ and $(2, 2, 1)$ belong to $\mathcal{P}_N$, while the N-tuples $(2, 1, -1)$ and $(1, 2, 1)$ don't.

The N-tuples in $\mathcal{P}_N$ are called *partitions of length* $\leq N$ (or *partitions with at most N nonzero terms*, following the terminology of [Stembr02]). In algebraic combinatorics, such an N-tuple $a \in \mathcal{P}_N$ usually gets identified with the infinite sequence $(a_1, a_2, \ldots, a_N, 0, 0, 0, \ldots)$, which is called an (integer) partition. Any integer partition has a so-called *Young diagram* assigned to it (see, e.g., [Stanle12, §1.7]). If $\mu \in \mathcal{P}_N$, and if $i \in [N]$ is such that the N-tuple $\mu + e_i$ again belongs to $\mathcal{P}_N$, then the Young diagram of $\mu + e_i$ is obtained from the Young diagram of μ by adding a single box.

For any integer n, we let h_n denote the complete homogeneous symmetric polynomial in the variables $x_1, x_2, \ldots, x_N$. It is defined by

$$h_n := \sum_{\substack{(i_1, i_2, \ldots, i_N) \in \mathbb{N}^N; \\ i_1 + i_2 + \cdots + i_N = n}} x_1^{i_1} x_2^{i_2} \cdots x_N^{i_N}, \tag{1}$$

where $\mathbb{N} := \{0, 1, 2, \ldots\}$ is the set of all nonnegative integers (so that the sum ranges over all N-tuples $(i_1, i_2, \ldots, i_N)$ of nonnegative integers satisfying $i_1 + i_2 + \cdots + i_N = n$). (Thus, $h_0 = 1$, and $h_n = 0$ for each negative n.) Clearly, $h_n \in R$ for each $n \in \mathbb{Z}$.

If λ and μ are two N-tuples in $\mathcal{P}_N$, then the notation $s_{\lambda/\mu}$ denotes the skew Schur polynomial $s_{\lambda/\mu}(x_1, x_2, \ldots, x_N)$ (see, e.g., [Stembr02] for a definition[1]). (Note that this is called a "skew Schur function" in [Stembr02], but more commonly the latter word is reserved for the analogous object in infinitely many variables.) The *Jacobi–Trudi formula* says that any $\lambda \in \mathcal{P}_N$ and $\mu \in \mathcal{P}_N$ satisfy

$$s_{\lambda/\mu} = \det \left(h_{\lambda_i - \mu_j - i + j} \right)_{i,j \in [N]} \tag{2}$$

(where the notation $\left(a_{i,j} \right)_{i,j \in [N]}$ means the $N \times N$-matrix with given entries $a_{i,j}$). Proofs of this formula can be found (e.g.) in [GriRei20, (2.4.16)] or [Stanle24,

[1] To be more precise, $s_{\lambda/\mu}$ is defined in [Stembr02] in the case when $\mu \leq \lambda$ (meaning that $\mu_i \leq \lambda_i$ for each $i \in [N]$). In all other cases, $s_{\lambda/\mu}$ is defined to be 0.

(7.69)][2]. We can use the formula (2) as a definition of $s_{\lambda/\mu}$ here, as we will need no other properties of $s_{\lambda/\mu}$. However, it is worth observing that

$$s_{\lambda/\mu} = 0 \tag{3}$$

unless $\mu \subseteq \lambda$ (that is, unless $\mu_i \leq \lambda_i$ for each $i \in [N]$).

2. The results

The main result of this note is the following formula, which generalizes Weigandt's recent result [Weigan23, The Symmetric Derivative Rule]:

Theorem 2.1. Let a and b be two integers such that $a + b = N - 1$. Let $\lambda \in \mathcal{P}_N$ and $\mu \in \mathcal{P}_N$. Define two further N-tuples $\ell \in \mathbb{Z}^N$ and $m \in \mathbb{Z}^N$ by setting

$$\ell_i := \lambda_i - i \qquad \text{and} \qquad m_i := \mu_i - i \qquad \text{for each } i \in [N].$$

Then,

$$\nabla\left(s_{\lambda/\mu}\right) = \sum_{\substack{i \in [N];\\ \lambda - e_i \in \mathcal{P}_N}} (\ell_i + a)\, s_{(\lambda - e_i)/\mu} + \sum_{\substack{i \in [N];\\ \mu + e_i \in \mathcal{P}_N}} (b - m_i)\, s_{\lambda/(\mu + e_i)}.$$

Example 2.2. Let $N = 3$ and $\lambda = (3,2,1)$ and $\mu = (1,1,0)$. Then, the N-tuples ℓ and m defined in Theorem 2.1 are $\ell = (2,0,-2)$ and $m = (0,-1,-3)$. Let a and b be two integers such that $a + b = N - 1 = 2$. Thus, Theorem 2.1 says that

$$\begin{aligned}
&\nabla\left(s_{(3,2,1)/(1,1,0)}\right)\\
&= \sum_{i \in \{1,2,3\}} (\ell_i + a)\, s_{((3,2,1)-e_i)/(1,1,0)} + \sum_{i \in \{1,3\}} (b - m_i)\, s_{(3,2,1)/((1,1,0)+e_i)}\\
&= (2 + a)\, s_{(2,1,1)/(1,1,0)} + (0 + a)\, s_{(3,1,1)/(1,1,0)} + (-2 + a)\, s_{(3,2,0)/(1,1,0)}\\
&\qquad + (b - 0)\, s_{(3,2,1)/(2,1,0)} + (b - (-3))\, s_{(3,2,1)/(1,1,1)}.
\end{aligned}$$

Note that the second sum has no $i = 2$ addend, since $\mu + e_2 = (1,2,0) \notin \mathcal{P}_N$.

[2] Both texts [GriRei20, (2.4.16)] and [Stanle24, (7.69)] state the Jacobi–Trudi formula for Schur functions in infinitely many variables $x_1, x_2, x_3, \ldots$ instead of Schur polynomials in finitely many variables $x_1, x_2, \ldots, x_N$. However, the latter version can be obtained from the former by setting $x_{N+1}, x_{N+2}, x_{N+3}, \ldots$ to 0.

Note also that [Stanle24, (7.69)] assumes that $\mu \subseteq \lambda$, but the proofs do not require this assumption.

Remark 2.3. Let us comment on the combinatorial meaning of Theorem 2.1. A pair (μ, λ) of N-tuples $\lambda, \mu \in \mathcal{P}_N$ is called a *skew partition* if $\mu \subseteq \lambda$ (that is, $\mu_i \leq \lambda_i$ for each $i \in [N]$). Such a skew partition (μ, λ) has a *skew Young diagram* assigned to it, which is defined as the set difference of the Young diagrams of λ and μ. This diagram is denoted by λ / μ.

Let λ / μ be a skew partition with $\lambda, \mu \in \mathcal{P}_N$. The sum $\sum\limits_{\substack{i \in [N]; \\ \lambda - e_i \in \mathcal{P}_N}} (\ell_i + a)\, s_{(\lambda - e_i)/\mu}$ in Theorem 2.1 can be rewritten as $\sum\limits_{\substack{i \in [N]; \\ \lambda - e_i \in \mathcal{P}_N; \\ \mu \subseteq \lambda - e_i}} (\ell_i + a)\, s_{(\lambda - e_i)/\mu}$, because any addend in which $\mu \subseteq \lambda - e_i$ does not hold is 0 by (3). This is a sum over all skew partitions obtained from λ / μ by removing an outer corner (i.e., a removable box on the southeastern boundary of λ / μ). Moreover, the number $\ell_i = \lambda_i - i$ tells us which diagonal this corner belongs to.

Likewise, the sum $\sum\limits_{\substack{i \in [N]; \\ \mu + e_i \in \mathcal{P}_N}} (b - m_i)\, s_{\lambda / (\mu + e_i)}$ can be rewritten as $\sum\limits_{\substack{i \in [N]; \\ \mu + e_i \in \mathcal{P}_N; \\ \mu + e_i \subseteq \lambda}} (b - m_i)\, s_{\lambda / (\mu + e_i)}$, which is a sum over all skew partitions obtained from λ / μ by removing an inner corner (i.e., a removable box on the northwestern boundary of λ / μ). Moreover, the number $m_i = \mu_i - i$ tells us which diagonal this corner belongs to.

Remark 2.4. Let $\lambda \in \mathcal{P}_N$. Let $\ell_i := \lambda_i - i$ for each $i \in [N]$. Set $\mu := (0, 0, \ldots, 0) \in \mathcal{P}_N$. Then, the skew Schur polynomial $s_{\lambda / \mu}$ is usually called s_λ. The only $i \in [N]$ satisfying $\mu + e_i \in \mathcal{P}_N$ is 1. Hence, Theorem 2.1 (applied to $a = N$ and $b = -1$) yields

$$\begin{aligned} \nabla (s_\lambda) &= \sum_{\substack{i \in [N]; \\ \lambda - e_i \in \mathcal{P}_N}} (\ell_i + N)\, s_{\lambda - e_i} + \underbrace{(1 - 0 + (-1))}_{=0} s_{\lambda / (\mu + e_1)} \\ &= \sum_{\substack{i \in [N]; \\ \lambda - e_i \in \mathcal{P}_N}} (\ell_i + N)\, s_{\lambda - e_i}, \end{aligned}$$

which recovers [Weigan23, The Symmetric Derivative Rule].

Once Theorem 2.1 is proved, we will derive the following curious corollary:

Corollary 2.5. Let $\lambda \in \mathcal{P}_N$ and $\mu \in \mathcal{P}_N$. Then,

$$\sum_{\substack{i \in [N]; \\ \lambda - e_i \in \mathcal{P}_N}} s_{(\lambda - e_i)/\mu} = \sum_{\substack{i \in [N]; \\ \mu + e_i \in \mathcal{P}_N}} s_{\lambda / (\mu + e_i)}.$$

We note that Corollary 2.5 follows easily from the theory of skewing operators (see [GriRei20, §2.8]) and the Pieri rule.[3] But we shall prove both Theorem 2.1 and Corollary 2.5 elementarily.

3. A lemma on $\nabla(h_n)$

First we need a formula for the image of the complete homogeneous symmetric polynomial h_n under the operator ∇:

Lemma 3.1. Let n be a integer. Then, $\nabla(h_n) = (n+N-1)\,h_{n-1}$.

Proof of Lemma 3.1. For each $k \in [N]$, we have

$$\begin{aligned}
\frac{\partial}{\partial x_k} h_n &= \frac{\partial}{\partial x_k} \sum_{\substack{(i_1,i_2,\ldots,i_N)\in\mathbb{N}^N;\\ i_1+i_2+\cdots+i_N=n}} x_1^{i_1}x_2^{i_2}\cdots x_N^{i_N} \qquad \text{(by (1))} \\
&= \sum_{\substack{(i_1,i_2,\ldots,i_N)\in\mathbb{N}^N;\\ i_1+i_2+\cdots+i_N=n}} \underbrace{\frac{\partial}{\partial x_k}\left(x_1^{i_1}x_2^{i_2}\cdots x_N^{i_N}\right)}_{\substack{=\begin{cases} i_k x_1^{i_1}x_2^{i_2}\cdots x_k^{i_k-1}\cdots x_N^{i_N}, & \text{if } i_k>0;\\ 0, & \text{if } i_k=0\end{cases}\\ \text{(where } x_1^{i_1}x_2^{i_2}\cdots x_k^{i_k-1}\cdots x_N^{i_N} \text{ means the monomial } x_1^{i_1}x_2^{i_2}\cdots x_N^{i_N}\\ \text{with the exponent on } x_k \text{ decremented by 1)}}} \\
&= \sum_{\substack{(i_1,i_2,\ldots,i_N)\in\mathbb{N}^N;\\ i_1+i_2+\cdots+i_N=n;\\ i_k>0}} i_k x_1^{i_1}x_2^{i_2}\cdots x_k^{i_k-1}\cdots x_N^{i_N} \\
&= \underbrace{\sum_{\substack{(i_1,i_2,\ldots,i_N)\in\mathbb{N}^N;\\ i_1+i_2+\cdots+(i_k+1)+\cdots+i_N=n}}}_{=\sum_{\substack{(i_1,i_2,\ldots,i_N)\in\mathbb{N}^N;\\ i_1+i_2+\cdots+i_N=n-1}}} (i_k+1)\underbrace{x_1^{i_1}x_2^{i_2}\cdots x_k^{(i_k+1)-1}\cdots x_N^{i_N}}_{=x_1^{i_1}x_2^{i_2}\cdots x_N^{i_N}} \\
&\qquad \text{(here, we substituted } i_k+1 \text{ for } i_k \text{ in the sum)} \\
&= \sum_{\substack{(i_1,i_2,\ldots,i_N)\in\mathbb{N}^N;\\ i_1+i_2+\cdots+i_N=n-1}} (i_k+1)\,x_1^{i_1}x_2^{i_2}\cdots x_N^{i_N}.
\end{aligned}$$

[3] In a nutshell: Corollary 2.5 is obtained by taking the equality $s_\mu^\perp\left(s_1^\perp s_\lambda\right) = \left(s_1 s_\mu\right)^\perp s_\lambda$, which holds on the level of symmetric functions in infinitely many indeterminates $x_1, x_2, x_3, \ldots$ (a consequence of [GriRei20, Proposition 2.8.2 (ii)]), and expanding both of its sides using the Pieri rules [GriRei20, (2.7.1) and (2.8.3)].

Summing this equality over all $k \in [N]$, we obtain

$$\begin{aligned}
\sum_{k\in[N]} \frac{\partial}{\partial x_k} h_n &= \sum_{k\in[N]} \sum_{\substack{(i_1,i_2,\ldots,i_N)\in\mathbb{N}^N;\\ i_1+i_2+\cdots+i_N=n-1}} (i_k+1)\, x_1^{i_1} x_2^{i_2} \cdots x_N^{i_N} \\
&= \sum_{\substack{(i_1,i_2,\ldots,i_N)\in\mathbb{N}^N;\\ i_1+i_2+\cdots+i_N=n-1}} \underbrace{\left(\sum_{k\in[N]} (i_k+1) \right)}_{\substack{=(i_1+1)+(i_2+1)+\cdots+(i_N+1)\\ =(i_1+i_2+\cdots+i_N)+N\\ =n-1+N\\ (\text{since } i_1+i_2+\cdots+i_N=n-1)}} x_1^{i_1} x_2^{i_2} \cdots x_N^{i_N} \\
&= \underbrace{(n-1+N)}_{=n+N-1} \underbrace{\sum_{\substack{(i_1,i_2,\ldots,i_N)\in\mathbb{N}^N;\\ i_1+i_2+\cdots+i_N=n-1}} x_1^{i_1} x_2^{i_2} \cdots x_N^{i_N}}_{\substack{=h_{n-1}\\ (\text{by the definition of } h_{n-1})}} = (n+N-1)\, h_{n-1}.
\end{aligned}$$

This can be rewritten as $\nabla (h_n) = (n+N-1)\, h_{n-1}$ (since $\nabla = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \cdots + \frac{\partial}{\partial x_N} = \sum_{k\in[N]} \frac{\partial}{\partial x_k}$). Thus, Lemma 3.1 is proved. □

4. Lemmas on determinants

We will next need a few simple lemmas about determinants:

Lemma 4.1. Let $\lambda, \mu \in \mathcal{P}_N$. Define $\ell, m \in \mathbb{Z}^N$ as in Theorem 2.1. Then,

$$\det \left(h_{\ell_i - m_j} \right)_{i,j\in[N]} = s_{\lambda/\mu}.$$

Proof of Lemma 4.1. For every $i, j \in [N]$, we have $\ell_i = \lambda_i - i$ (by the definition of ℓ) and $m_j = \mu_j - j$ (similarly). Subtracting these two equalities from each other, we obtain that

$$\ell_i - m_j = \lambda_i - i - (\mu_j - j) = \lambda_i - \mu_j - i + j \qquad \text{for every } i, j \in [N].$$

Hence, we can rewrite (2) as $s_{\lambda/\mu} = \det \left(h_{\ell_i - m_j} \right)_{i,j\in[N]}$. This proves Lemma 4.1. □

Lemma 4.2. Let $\lambda, \mu \in \mathcal{P}_N$ and $k \in [N]$ be such that $\lambda - e_k \in \mathcal{P}_N$. Define $\ell, m \in \mathbb{Z}^N$ as in Theorem 2.1. Then,

$$\det \left(h_{(\ell - e_k)_i - m_j} \right)_{i,j\in[N]} = s_{(\lambda - e_k)/\mu}.$$

Proof of Lemma 4.2. Recall that the N-tuple ℓ is defined from the N-tuple λ by subtracting 1 from its 1-st entry, subtracting 2 from its 2-nd entry, subtracting 3 from its 3-rd entry, etc.. Thus, the N-tuple $\ell - e_k$ is obtained from $\lambda - e_k$ in the same way. Hence, Lemma 4.1 (applied to $\lambda - e_k$ and $\ell - e_k$ instead of λ and ℓ) yields

$$\det \left(h_{(\ell - e_k)_i - m_j} \right)_{i,j \in [N]} = s_{(\lambda - e_k)/\mu}.$$

This proves Lemma 4.2. □

Lemma 4.3. Let $\lambda, \mu \in \mathcal{P}_N$ and $k \in [N]$ be such that $\mu + e_k \in \mathcal{P}_N$. Define $\ell, m \in \mathbb{Z}^N$ as in Theorem 2.1. Then,

$$\det \left(h_{\ell_i - (m + e_k)_j} \right)_{i,j \in [N]} = s_{\lambda/(\mu + e_k)}.$$

Proof of Lemma 4.3. Similarly to Lemma 4.2, we can show this by applying Lemma 4.1 to $\mu + e_k$ and $m + e_k$ instead of μ and m. □

Lemma 4.4. Let $\lambda, \mu \in \mathcal{P}_N$ and $k \in [N]$ be such that $\lambda - e_k \notin \mathcal{P}_N$. Define $\ell, m \in \mathbb{Z}^N$ as in Theorem 2.1. Then,

$$\det \left(h_{(\ell - e_k)_i - m_j} \right)_{i,j \in [N]} = 0.$$

Proof of Lemma 4.4. The definition of e_k yields that the only nonzero entry of e_k is $(e_k)_k = 1$. Hence, in particular, $(e_k)_{k+1} = 0$.

We have $\lambda \in \mathcal{P}_N$, so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$. Recall that $k \in [N]$, so that $k \leq N$. Hence, we are in one of the following two cases:

Case 1: We have $k < N$.

Case 2: We have $k = N$.

Let us first consider Case 1. In this case, we have $k < N$. The definition of e_k yields $\lambda - e_k = (\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k - 1, \lambda_{k+1}, \ldots, \lambda_N)$ (this is the N-tuple λ with its k-th entry decreased by 1). Thus, the chain of inequalities

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{k-1} \geq \lambda_k - 1 \geq \lambda_{k+1} \geq \cdots \geq \lambda_N \geq 0 \quad \textbf{does not} \text{ hold}$$

(since $\lambda - e_k \notin \mathcal{P}_N$). Therefore, the inequality $\lambda_k - 1 \geq \lambda_{k+1}$ must be violated (since all the other inequality signs in this chain follow from $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$). In other words, we have $\lambda_k - 1 < \lambda_{k+1}$. Since λ_k and λ_{k+1} are integers, this entails $\lambda_k - 1 \leq \lambda_{k+1} - 1$, so that $\lambda_k \leq \lambda_{k+1}$. Combining this with $\lambda_k \geq \lambda_{k+1}$ (which follows from $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$), we obtain $\lambda_k = \lambda_{k+1}$.

Now, the definition of ℓ yields $\ell_k = \lambda_k - k$. Hence,

$$(\ell - e_k)_k = \underbrace{\ell_k}_{=\lambda_k - k} - \underbrace{(e_k)_k}_{=1} = \underbrace{\lambda_k}_{=\lambda_{k+1}} - k - 1 = \lambda_{k+1} - k - 1.$$

Furthermore, the definition of ℓ yields $\ell_{k+1} = \lambda_{k+1} - (k+1)$. Hence,

$$(\ell - e_k)_{k+1} = \ell_{k+1} - \underbrace{(e_k)_{k+1}}_{=0} = \ell_{k+1} = \lambda_{k+1} - (k+1) = \lambda_{k+1} - k - 1.$$

Comparing this with $(\ell - e_k)_k = \lambda_{k+1} - k - 1$, we obtain $(\ell - e_k)_k = (\ell - e_k)_{k+1}$.

Hence, for each $j \in [N]$, we have

$$h_{(\ell - e_k)_k - m_j} = h_{(\ell - e_k)_{k+1} - m_j}.$$

In other words, each entry in the k-th row of the matrix $\left(h_{(\ell-e_k)_i - m_j}\right)_{i,j\in[N]}$ equals the corresponding entry in the $(k+1)$-st row of this matrix. Thus, the matrix $\left(h_{(\ell-e_k)_i - m_j}\right)_{i,j\in[N]}$ has two equal rows (namely, its k-th and $(k+1)$-st rows). Hence, its determinant is 0. This proves Lemma 4.4 in Case 1.

Let us now consider Case 2. In this case, we have $k = N$. Hence, $\lambda - e_k = \lambda - e_N = (\lambda_1, \lambda_2, \ldots, \lambda_{N-1}, \lambda_N - 1)$ (this is the N-tuple λ with its N-th entry decreased by 1). Thus, the chain of inequalities

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-1} \geq \lambda_N - 1 \geq 0 \quad \textbf{does not} \text{ hold}$$

(since $\lambda - e_k \notin \mathcal{P}_N$). Therefore, the inequality $\lambda_N - 1 \geq 0$ must be violated (since all the other inequality signs in this chain follow from $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$). In other words, we have $\lambda_N - 1 < 0$. In other words, $\lambda_k - 1 < 0$ (since $k = N$). Thus, $\lambda_k < 1$. Now,

$$(\ell - e_k)_k = \underbrace{\ell_k}_{\substack{=\lambda_k - k \\ \text{(by the definition of } \ell)}} - \underbrace{(e_k)_k}_{=1} = \underbrace{\lambda_k}_{<1} - \underbrace{k}_{=N} - 1 < 1 - N - 1 = -N$$

On the other hand, for each $j \in [N]$, we have $m_j = \mu_j - j$ (by the definition of m) and thus $m_j = \underbrace{\mu_j}_{\geq 0} - \underbrace{j}_{\leq N} \geq 0 - N = -N > (\ell - e_k)_k$ (since $(\ell - e_k)_k < -N$).

Thus, for each $j \in [N]$, we have $(\ell - e_k)_k - m_j < 0$ and therefore

$$h_{(\ell - e_k)_k - m_j} = 0 \qquad (\text{since } h_i = 0 \text{ for all } i < 0).$$

In other words, each entry in the k-th row of the matrix $\left(h_{(\ell-e_k)_i - m_j}\right)_{i,j\in[N]}$ is zero. Thus, the matrix $\left(h_{(\ell-e_k)_i - m_j}\right)_{i,j\in[N]}$ has a zero row (namely, its k-th row). Hence, its determinant is 0. This proves Lemma 4.4 in Case 2.

We have now proved Lemma 4.4 in both cases. $\square$

Lemma 4.5. Let $\lambda, \mu \in \mathcal{P}_N$ and $k \in [N]$ be such that $\mu + e_k \notin \mathcal{P}_N$. Define $\ell, m \in \mathbb{Z}^N$ as in Theorem 2.1. Then,

$$\det \left(h_{\ell_i - (m+e_k)_j} \right)_{i,j \in [N]} = 0.$$

Proof of Lemma 4.5. This is similar to Case 2 in the proof of Lemma 4.4. Here are the details:

The definition of e_k yields $\mu + e_k = (\mu_1, \mu_2, \ldots, \mu_{k-1}, \mu_k + 1, \mu_{k+1}, \ldots, \mu_N)$ (this is the N-tuple μ with its k-th entry increased by 1).

We have $\mu \in \mathcal{P}_N$, so that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N \geq 0$. However, we have $\mu + e_k \notin \mathcal{P}_N$, so that the chain of inequalities

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{k-1} \geq \mu_k + 1 \geq \mu_{k+1} \geq \cdots \geq \mu_N \geq 0 \quad \textbf{does not } \text{hold}$$

(since $\mu + e_k = (\mu_1, \mu_2, \ldots, \mu_{k-1}, \mu_k + 1, \mu_{k+1}, \ldots, \mu_N)$). Hence, the inequality $\mu_{k-1} \geq \mu_k + 1$ must be violated (since all the other inequalities in this chain follow from $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N \geq 0$). In other words, we must have $k > 1$ and $\mu_{k-1} < \mu_k + 1$.

From $\mu_{k-1} < \mu_k + 1$, we obtain $\mu_{k-1} \leq \mu_k$ (since μ_{k-1} and μ_k are integers). Combining this with $\mu_{k-1} \geq \mu_k$ (since $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N$), we obtain $\mu_{k-1} = \mu_k$.

The definition of e_k yields that the only nonzero entry of e_k is $(e_k)_k = 1$. Hence, in particular, $(e_k)_{k-1} = 0$.

The definition of m yields $m_k = \mu_k - k$ and $m_{k-1} = \mu_{k-1} - (k-1)$. Now,

$$(m + e_k)_k = \underbrace{m_k}_{= \mu_k - k} + \underbrace{(e_k)_k}_{=1} = \mu_k - k + 1 = \mu_k - (k-1).$$

Comparing this with

$$(m + e_k)_{k-1} = m_{k-1} + \underbrace{(e_k)_{k-1}}_{=0} = m_{k-1} = \underbrace{\mu_{k-1}}_{= \mu_k} - (k-1) = \mu_k - (k-1),$$

we obtain $(m + e_k)_k = (m + e_k)_{k-1}$.

Now, for each $i \in [N]$, we have

$$h_{\ell_i - (m+e_k)_k} = h_{\ell_i - (m+e_k)_{k-1}} \qquad \left(\text{since } (m+e_k)_k = (m+e_k)_{k-1} \right).$$

In other words, each entry in the k-th column of the matrix $\left(h_{\ell_i - (m+e_k)_j} \right)_{i,j \in [N]}$ equals the corresponding entry in the $(k-1)$-st column of this matrix. Thus, the matrix $\left(h_{\ell_i - (m+e_k)_j} \right)_{i,j \in [N]}$ has two equal columns (namely, its k-th and $(k-1)$-st columns). Hence, its determinant is 0. This proves Lemma 4.5. ☐

Finally, we will need the Leibniz rule for products of multiple factors:[4]

[4] Recall that $R = \mathbb{Z}[x_1, x_2, \ldots, x_n]$.

Lemma 4.6. For any $a_1, a_2, \ldots, a_n \in R$, we have

$$\nabla \left(a_1 a_2 \cdots a_n\right) = \sum_{k=1}^{n} a_1 a_2 \cdots a_{k-1} \nabla \left(a_k\right) a_{k+1} a_{k+2} \cdots a_n.$$

Proof of Lemma 4.6. This holds not just for ∇ but actually for any derivation of any ring, and can be easily proved by induction on n using the Leibniz rule. □

5. The last lemma

We now have everything in place for the proofs of Theorem 2.1 and Corollary 2.5. However, to keep our computation short, let us outsource a part of it to a lemma:

Lemma 5.1. Let σ be a permutation of the set $[N]$. Let $\ell, m \in \mathbb{Z}^N$ be two N-tuples of integers. Let a and b be two integers such that $a + b = N - 1$. Then,

$$\nabla \left(\prod_{i=1}^{N} h_{\ell_i - m_{\sigma(i)}} \right) = \sum_{k=1}^{N} \left(\ell_k + a\right) \prod_{i=1}^{N} h_{(\ell - e_k)_i - m_{\sigma(i)}} + \sum_{k=1}^{N} \left(b - m_k\right) \prod_{i=1}^{N} h_{\ell_i - (m+e_k)_{\sigma(i)}}.$$

Proof of Lemma 5.1. Fix $k \in [N]$. The N-tuple $\ell - e_k$ differs from the N-tuple ℓ only in its k-th entry, which is smaller by 1. Thus, the product $\prod_{i=1}^{N} h_{(\ell - e_k)_i - m_{\sigma(i)}}$ differs from the product $\prod_{i=1}^{N} h_{\ell_i - m_{\sigma(i)}}$ only in its k-th factor, which is $h_{(\ell_k - 1) - m_{\sigma(k)}}$ instead of $h_{\ell_k - m_{\sigma(k)}}$. Hence,

$$\begin{aligned} \prod_{i=1}^{N} h_{(\ell - e_k)_i - m_{\sigma(i)}} &= \underbrace{h_{(\ell_k - 1) - m_{\sigma(k)}}}_{= h_{\ell_k - m_{\sigma(k)} - 1}} \prod_{\substack{i \in [N]; \\ i \neq k}} h_{\ell_i - m_{\sigma(i)}} \\ &= h_{\ell_k - m_{\sigma(k)} - 1} \prod_{\substack{i \in [N]; \\ i \neq k}} h_{\ell_i - m_{\sigma(i)}}. \end{aligned} \tag{4}$$

The N-tuple $m + e_{\sigma(k)}$ differs from the N-tuple m only in its $\sigma(k)$-th entry, which is larger by 1. Thus, the product $\prod_{i=1}^{N} h_{\ell_i - \left(m + e_{\sigma(k)}\right)_{\sigma(i)}}$ differs from the product $\prod_{i=1}^{N} h_{\ell_i - m_{\sigma(i)}}$ only in its k-th factor, which is $h_{\ell_k - \left(m_{\sigma(k)} + 1\right)}$ instead of $h_{\ell_k - m_{\sigma(k)}}$ (since all the remaining factors satisfy $i \neq k$ and therefore $\sigma(i) \neq \sigma(k)$). Hence,

$$\prod_{i=1}^{N} h_{\ell_i - \left(m + e_{\sigma(k)}\right)_{\sigma(i)}} = \underbrace{h_{\ell_k - \left(m_{\sigma(k)}+1\right)}}_{= h_{\ell_k - m_{\sigma(k)} - 1}} \prod_{\substack{i \in [N]; \\ i \neq k}} h_{\ell_i - m_{\sigma(i)}} = h_{\ell_k - m_{\sigma(k)} - 1} \prod_{\substack{i \in [N]; \\ i \neq k}} h_{\ell_i - m_{\sigma(i)}}. \tag{5}$$

Forget that we fixed k. We thus have proved the equalities (4) and (5) for each $k \in [N]$.

Since the ring R is commutative, we can rewrite Lemma 4.6 as follows: For any $a_1, a_2, \ldots, a_n \in R$, we have

$$\nabla \left(\prod_{i=1}^{n} a_i \right) = \sum_{k=1}^{n} \nabla(a_k) \prod_{\substack{i \in [n]; \\ i \neq k}} a_i.$$

Applying this to $N = n$ and $a_i = h_{\ell_i - m_{\sigma(i)}}$, we obtain

$$\nabla\left(\prod_{i=1}^{N} h_{\ell_i - m_{\sigma(i)}}\right)$$
$$= \sum_{k=1}^{N} \underbrace{\nabla\left(h_{\ell_k - m_{\sigma(k)}}\right)}_{\substack{=\left(\ell_k - m_{\sigma(k)} + N - 1\right) h_{\ell_k - m_{\sigma(k)} - 1} \\ \text{(by Lemma 3.1)}}} \prod_{\substack{i\in[N];\\ i\neq k}} h_{\ell_i - m_{\sigma(i)}}$$
$$= \sum_{k=1}^{N} \underbrace{\left(\ell_k - m_{\sigma(k)} + N - 1\right)}_{\substack{=(\ell_k + a) + \left(b - m_{\sigma(k)}\right) \\ \text{(since } N-1=a+b\text{)}}} h_{\ell_k - m_{\sigma(k)} - 1} \prod_{\substack{i\in[N];\\ i\neq k}} h_{\ell_i - m_{\sigma(i)}}$$
$$= \sum_{k=1}^{N} \left((\ell_k + a) + \left(b - m_{\sigma(k)}\right)\right) h_{\ell_k - m_{\sigma(k)} - 1} \prod_{\substack{i\in[N];\\ i\neq k}} h_{\ell_i - m_{\sigma(i)}}$$
$$= \sum_{k=1}^{N} (\ell_k + a) \underbrace{h_{\ell_k - m_{\sigma(k)} - 1} \prod_{\substack{i\in[N];\\ i\neq k}} h_{\ell_i - m_{\sigma(i)}}}_{\substack{=\prod_{i=1}^{N} h_{(\ell - e_k)_i - m_{\sigma(i)}} \\ \text{(by (4))}}} + \sum_{k=1}^{N} \left(b - m_{\sigma(k)}\right) \underbrace{h_{\ell_k - m_{\sigma(k)} - 1} \prod_{\substack{i\in[N];\\ i\neq k}} h_{\ell_i - m_{\sigma(i)}}}_{\substack{=\prod_{i=1}^{N} h_{\ell_i - \left(m + e_{\sigma(k)}\right)_{\sigma(i)}} \\ \text{(by (5))}}}$$
$$= \sum_{k=1}^{N} (\ell_k + a) \prod_{i=1}^{N} h_{(\ell - e_k)_i - m_{\sigma(i)}} + \sum_{k=1}^{N} \left(b - m_{\sigma(k)}\right) \prod_{i=1}^{N} h_{\ell_i - \left(m + e_{\sigma(k)}\right)_{\sigma(i)}}$$
$$= \sum_{k=1}^{N} (\ell_k + a) \prod_{i=1}^{N} h_{(\ell - e_k)_i - m_{\sigma(i)}} + \sum_{k=1}^{N} (b - m_k) \prod_{i=1}^{N} h_{\ell_i - (m + e_k)_{\sigma(i)}}$$

(here, we have substituted k for $\sigma(k)$ in the second sum, since $\sigma : [N] \to [N]$ is a bijection). This proves Lemma 5.1. □

6. Proofs of the main results

Proof of Theorem 2.1. Let S_N denote the N-th symmetric group (i.e., the group of all permutations of $[N]$). Let $(-1)^{\sigma}$ denote the sign of any permutation σ. The definition of a determinant says that

$$\det\left(a_{i,j}\right)_{i,j\in[N]} = \sum_{\sigma\in S_N} (-1)^{\sigma} \prod_{i=1}^{N} a_{i,\sigma(i)} \tag{6}$$

for any matrix $\left(a_{i,j}\right)_{i,j\in[N]} \in R^{N\times N}$. Now, Lemma 4.1 yields

$$s_{\lambda/\mu} = \det\left(h_{\ell_i - m_j}\right)_{i,j\in[N]} = \sum_{\sigma\in S_N} (-1)^{\sigma} \prod_{i=1}^{N} h_{\ell_i - m_{\sigma(i)}}$$

(by (6)). Applying the $\mathbb{Z}$-linear map ∇ to both sides of this equality, we find

$$\begin{aligned}
&\nabla\left(s_{\lambda/\mu}\right) \\
&= \sum_{\sigma\in S_N} (-1)^{\sigma} \nabla\left(\prod_{i=1}^{N} h_{\ell_i - m_{\sigma(i)}}\right) \\
&= \sum_{\sigma\in S_N} (-1)^{\sigma} \left(\sum_{k=1}^{N} (\ell_k + a) \prod_{i=1}^{N} h_{(\ell - e_k)_i - m_{\sigma(i)}} + \sum_{k=1}^{N} (b - m_k) \prod_{i=1}^{N} h_{\ell_i - (m+e_k)_{\sigma(i)}}\right) \\
&\qquad \text{(by Lemma 5.1)} \\
&= \sum_{\sigma\in S_N} (-1)^{\sigma} \sum_{k=1}^{N} (\ell_k + a) \prod_{i=1}^{N} h_{(\ell - e_k)_i - m_{\sigma(i)}} + \sum_{\sigma\in S_N} (-1)^{\sigma} \sum_{k=1}^{N} (b - m_k) \prod_{i=1}^{N} h_{\ell_i - (m+e_k)_{\sigma(i)}}.
\end{aligned}$$

In view of

$$\begin{aligned}
&\sum_{\sigma\in S_N} (-1)^{\sigma} \sum_{k=1}^{N} (\ell_k + a) \prod_{i=1}^{N} h_{(\ell - e_k)_i - m_{\sigma(i)}} \\
&= \sum_{k=1}^{N} (\ell_k + a) \underbrace{\sum_{\sigma\in S_N} (-1)^{\sigma} \prod_{i=1}^{N} h_{(\ell - e_k)_i - m_{\sigma(i)}}}_{\substack{=\det\left(h_{(\ell-e_k)_i - m_j}\right)_{i,j\in[N]} \\ \text{(by (6))}}} \\
&= \sum_{k=1}^{N} (\ell_k + a) \underbrace{\det\left(h_{(\ell - e_k)_i - m_j}\right)_{i,j\in[N]}}_{\substack{=0 \text{ if } \lambda - e_k \notin \mathcal{P}_N \\ \text{(by Lemma 4.4)}}} \\
&= \sum_{\substack{k\in[N];\\ \lambda - e_k \in \mathcal{P}_N}} (\ell_k + a) \underbrace{\det\left(h_{(\ell - e_k)_i - m_j}\right)_{i,j\in[N]}}_{\substack{=s_{(\lambda - e_k)/\mu} \\ \text{(by Lemma 4.2)}}} \\
&= \sum_{\substack{k\in[N];\\ \lambda - e_k \in \mathcal{P}_N}} (\ell_k + a)\, s_{(\lambda - e_k)/\mu} = \sum_{\substack{i\in[N];\\ \lambda - e_i \in \mathcal{P}_N}} (\ell_i + a)\, s_{(\lambda - e_i)/\mu}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\sigma\in S_N}(-1)^\sigma \sum_{k=1}^N (b-m_k)\prod_{i=1}^N h_{\ell_i-(m+e_k)_{\sigma(i)}} \\
&= \sum_{k=1}^N (b-m_k) \underbrace{\sum_{\sigma\in S_N}(-1)^\sigma \prod_{i=1}^N h_{\ell_i-(m+e_k)_{\sigma(i)}}}_{\substack{=\det\left(h_{\ell_i-(m+e_k)_j}\right)_{i,j\in[N]} \\ \text{(by (6))}}} \\
&= \sum_{k=1}^N (b-m_k) \underbrace{\det\left(h_{\ell_i-(m+e_k)_j}\right)_{i,j\in[N]}}_{\substack{=0 \text{ if } \mu+e_k\notin\mathcal{P}_N \\ \text{(by Lemma 4.5)}}} \\
&= \sum_{\substack{k\in[N];\\ \mu+e_k\in\mathcal{P}_N}} (b-m_k) \underbrace{\det\left(h_{\ell_i-(m+e_k)_j}\right)_{i,j\in[N]}}_{\substack{=s_{\lambda/(\mu+e_k)} \\ \text{(by Lemma 4.3)}}} \\
&= \sum_{\substack{k\in[N];\\ \mu+e_k\in\mathcal{P}_N}} (b-m_k)\, s_{\lambda/(\mu+e_k)} = \sum_{\substack{i\in[N];\\ \mu+e_i\in\mathcal{P}_N}} (b-m_i)\, s_{\lambda/(\mu+e_i)},
\end{aligned}$$

this can be rewritten as

$$\nabla\left(s_{\lambda/\mu}\right) = \sum_{\substack{i\in[N];\\ \lambda-e_i\in\mathcal{P}_N}} (\ell_i+a)\, s_{(\lambda-e_i)/\mu} + \sum_{\substack{i\in[N];\\ \mu+e_i\in\mathcal{P}_N}} (b-m_i)\, s_{\lambda/(\mu+e_i)}.$$

Thus, Theorem 2.1 is proved. □

Proof of Corollary 2.5. Define $\ell, m \in \mathbb{Z}^N$ as in Theorem 2.1. Theorem 2.1 (applied to $a = N-1$ and $b = 0$) yields

$$\nabla\left(s_{\lambda/\mu}\right) = \sum_{\substack{i\in[N];\\ \lambda-e_i\in\mathcal{P}_N}} (\ell_i+N-1)\, s_{(\lambda-e_i)/\mu} + \sum_{\substack{i\in[N];\\ \mu+e_i\in\mathcal{P}_N}} (0-m_i)\, s_{\lambda/(\mu+e_i)}.$$

Theorem 2.1 (applied to $a = N$ and $b = -1$) yields

$$\nabla\left(s_{\lambda/\mu}\right) = \sum_{\substack{i\in[N];\\ \lambda-e_i\in\mathcal{P}_N}} (\ell_i+N)\, s_{(\lambda-e_i)/\mu} + \sum_{\substack{i\in[N];\\ \mu+e_i\in\mathcal{P}_N}} (-1-m_i)\, s_{\lambda/(\mu+e_i)}.$$

Subtracting the latter equality from the former, we find

$$\begin{aligned}
0 &= \sum_{\substack{i\in[N];\\ \lambda-e_i\in\mathcal{P}_N}} (-1)\, s_{(\lambda-e_i)/\mu} + \sum_{\substack{i\in[N];\\ \mu+e_i\in\mathcal{P}_N}} s_{\lambda/(\mu+e_i)} \\
&= -\sum_{\substack{i\in[N];\\ \lambda-e_i\in\mathcal{P}_N}} s_{(\lambda-e_i)/\mu} + \sum_{\substack{i\in[N];\\ \mu+e_i\in\mathcal{P}_N}} s_{\lambda/(\mu+e_i)}.
\end{aligned}$$

In other words,

$$\sum_{\substack{i\in[N];\\ \lambda-e_i\in\mathcal{P}_N}} s_{(\lambda-e_i)/\mu} = \sum_{\substack{i\in[N];\\ \mu+e_i\in\mathcal{P}_N}} s_{\lambda/(\mu+e_i)}.$$

This proves Corollary 2.5. □

7. Final remarks

1. Clearly, Theorem 2.1 can be generalized by replacing $\mathbb{Z}$ with any commutative ring $\mathbf{k}$. In this generality, a and b can be any two elements of $\mathbf{k}$ (rather than just integers) satisfying $a+b=(N-1)\cdot 1_{\mathbf{k}}$. However, not much generality is gained in this way, since Corollary 2.5 easily shows that all choices of a and b lead to the same sum.

2. Theorem 2.1 can also be lifted to the "infinite setting", i.e., to the ring of symmetric functions in infinitely many variables (see, e.g., [Stanle24, Chapter 7] or [GriRei20, Chapter 2] for introductions to this ring). This is not completely straightforward, since the diagonal derivative ∇ is defined only for finitely many indeterminates and depends on their number N (for instance, $\nabla\left(s_{(1)}\right)=N$). In the infinite setting, it has to be replaced by a derivation ∇_q depending on a scalar q:

Let $\mathbf{k}$ be a commutative ring, and let $q\in\mathbf{k}$ be an element. (For example, we can have $\mathbf{k}=\mathbb{Z}[q]$ and $q=q$.) Let Λ be the ring of symmetric functions in infinitely many indeterminates $x_1,x_2,x_3,\ldots$ over $\mathbf{k}$. For each $n\in\mathbb{Z}$, we let h_n denote the n-th complete homogeneous symmetric function in Λ (so that $h_0=1$ and $h_i=0$ for all $i<0$). Let $\nabla_q:\Lambda\to\Lambda$ be the unique derivation that satisfies

$$\nabla_q(h_n)=(n+q-1)\,h_{n-1} \qquad \text{for each } n>0.$$

[5] For $q=N\in\mathbb{N}$, this derivation is a lift of the directional derivative operator $\nabla:R\to R$ to Λ (meaning that $\nabla\circ\pi=\pi\circ\nabla_N$, where $\pi:\Lambda\to R$ is the evaluation homomorphism at $x_1,x_2,\ldots,x_N,0,0,0,\ldots$). With these definitions, we can extend Theorem 2.1 to a general property of ∇_q:

Theorem 7.1. Let a and b be two elements of $\mathbf{k}$ such that $a+b=q-1$. Let $\lambda=(\lambda_1,\lambda_2,\lambda_3,\ldots)$ and $\mu=(\mu_1,\mu_2,\mu_3,\ldots,)$ be two partitions. Set

$$\ell_i:=\lambda_i-i \qquad \text{and} \qquad m_i:=\mu_i-i \qquad \text{for each } i\geq 1.$$

Then,

$$\nabla_q\left(s_{\lambda/\mu}\right)=\sum_{\substack{i\geq 1;\\ \lambda-e_i \text{ is a partition}}}(\ell_i+a)\,s_{(\lambda-e_i)/\mu}+\sum_{\substack{i\geq 1;\\ \mu+e_i \text{ is a partition}}}(b-m_i)\,s_{\lambda/(\mu+e_i)}.$$

[5] This condition uniquely determines a derivation of Λ, since the elements $h_1,h_2,h_3,\ldots$ freely generate Λ as a commutative $\mathbf{k}$-algebra. It is easy to see that this derivation ∇_q satisfies the equality $\nabla_q(h_n)=(n+q-1)\,h_{n-1}$ for all $n\in\mathbb{Z}$.

Here, e_i means the infinite sequence $(0, 0, \ldots, 0, 1, 0, 0, 0, \ldots)$ with the 1 in its i-th position.

The proof of this theorem is similar to our above proof of Theorem 2.1 (with the minor complication that we have to fix an $N \in \mathbb{N}$ that is strictly larger than the lengths of λ and μ, in order to apply the Jacobi–Trudi formula), and is left to the reader.

3. It is natural to attempt generalizing Theorem 2.1 to higher-order differential operators, such as $\nabla' := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_N^2}$. However, finding similar formulas for $\nabla' \left(s_{\lambda/\mu}\right)$ appears significantly harder, as the "locality" (the fact that λ and μ change only a very little) disappears: For instance, for $N = 3$, the expansion of $\nabla' \left(s_{(5,3,0)}\right)$ in the Schur basis contains a $2s_{(2,2,2)}$ term.

Acknowledgments. The present note is a side product of the Yulia's Dream program 2023, in which three of its authors participated as student researchers and the remaining one as a mentor.

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