Research statement

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I work in the field of *algebraic combinatorics*, centered on (but not limited to) *symmetric functions* and related concepts, such as *combinatorial Hopf algebras*, *Young tableaux* and *trees*. These objects live at the borderlands of algebra and combinatorics, often allowing for viewpoints from both sides and transfer of knowledge from one to the other. I also study adjacent subjects such as invariant theory, Lie algebras, representations, and constructive algebra.

Among my contributions to these disciplines are a new generalization of the dual stable Grothendieck polynomials (which themselves generalize the Schur functions), an antipode formula for quasisymmetric functions, a proof of a conjecture of K. Mészáros on the "subdivision algebra", and a periodicity result on "birational rowmotion" (originally conjectured by D. Einstein and J. Propp) that has seen several uses in dynamical algebraic combinatorics. Details on this and other work can be found below.

Background. The history of *symmetric functions* goes back at least as far as the 17th Century, when Newton and Girard explored the relations between elementary symmetric polynomials and power sums. The next major steps were Cauchy's 1815 introduction of what later came to be known as *Schur functions*, and Jacobi's and Trudi's determinantal formulas for them (1841 and 1864). Symmetric functions found various uses in the algebra of the 19th Century, particularly in *Galois* and *invariant theory*, as well as in *Schubert calculus*. However, their combinatorial meaning was not discovered until the 1930s, when Schur and others connected them to *Young tableaux* and the *representation theory of symmetric and general linear groups*. This connection opened the floodgates, and the research that followed since the mid-20th Century could fill bookshelves (see, e.g., [Stan99, Ch. 7], [Fulton97], [Sagan01]). Symmetric functions were found to form a *Hopf algebra*, which tended to appear in various guises in seemingly unrelated fields such as algebraic topology (as *cohomology* of some classifying spaces) and number theory (as coordinate ring of the *Witt vectors*, e.g., [Hazewi08]). The multiplication of Schur functions turned out to be governed by a combinatorial rule (the *Littlewood-Richardson rule*), formulated in 1934 and first proven in 1974, with applications in theoretical physics. The combinatorics of Young tableaux became a subject of its own, bordering on theoretical computer science (Knuth devoted a section in "The Art of Computer Programming" to it). Symmetric functions have been applied in fields as diverse as *random matrix theory*, *K-theory* (particularly of *Grassmannians*), *group theory* and *quantum groups*. The description of the representations of symmetric and general linear groups using Schur functions has become a mold in which many other representation theories have been shaped.

By now, even as various questions on Schur functions remain unanswered, the focus has broadened to include generalizations and analogues thereof, such as Hall-Littlewood and Macdonald polynomials, factorial Schur functions, Schubert and Grothendieck polynomials, P-partition enumerators, and others.

It is such generalizations that I have been dealing with in much of my research. From a bird's eye view, their theories follow a certain pattern: a family of power series is defined, and analogues of classical properties of the Schur functions (such as symmetry, determinantal formulas à la Jacobi-Trudi, Littlewood-Richardson rule(s), and antipode formulas) are proven for this family. However, this is rarely ever straightforward, as each generalization comes with its additional complications; consequently, these programs are at rather different stages of completion, and some of them (e.g., a full Littlewood-Richardson rule for Schubert polynomials) appear out of reach today. Additionally, each generalization has its own motivation, sometimes stemming from a totally different field.

Combinatorial Hopf algebras are one (although not the only) place where these generalizations live. They are interesting both in their intrinsic properties (e.g., some of them are free as algebras for non-obvious reasons) and for the special elements they contain (such as the above-mentioned generalizations of symmetric functions). They have found applications to algebraic groups, Lie groups, probability and renormalization theory. One of the most frequently seen among these Hopf algebras is the ring of *quasisymmetric functions*, which has been introduced by Gessel and Stanley for combinatorial purposes in the 1970s, but has recently appeared in topology ([BakRic08]) and K-theory ([Morava15], [Oesing18]); this ring has also been involved in much of my past research (e.g., [Grinbe15a], [Grinbe14], [Grinbe17c]).

Overview of selected past results. My results so far, as well as my research plans for the future, live in and around the algebro-combinatorial landscape surveyed above. Among my finished work on symmetric functions and related Hopf algebras, the following are the most relevant:

- In [Grinbe14], I prove a conjecture of Mike Zabrocki on a quasisymmetric analogue of *Bernstein's creation-operator approach* to the Schur functions. The proof relies on a *dendriform algebra* structure on the quasisymmetric functions – a structure I later apply to a combinatorial problem in [Grinbe17c].

- In [Grinbe15a], I reprove and generalize a formula of Malvenuto and Reutenauer for the *antipode of a P-partition enumerator* (which itself extends a classical formula for the antipode of a Schur function).

- In [GaGrLi15], Pavel Galashin, Gaku Liu and I refine the *dual stable Grothendieck polynomials* (a recent generalization of Schur functions motivated by K-theory) to include new parameters, and prove the symmetry of these new power series combinatorially.

- In [BorGri13], James Borger and I explore *positivity* properties of symmetric functions and apply them to *Witt vectors over semirings*.

- In [Grinbe15d], I leverage a universal property to obtain a new construction of the Bernstein homomorphism for commutative connected graded Hopf algebras (which generalizes the internal comultiplication of the quasisymmetric functions).

- The lecture notes [GriRei15] (joint with Victor Reiner) are an introduction to both symmetric functions and combinatorial Hopf algebras.

Some other works of mine are not directly concerned with symmetric functions:

- In the two papers [GriRob14], Tom Roby and I prove the periodicity of *birational rowmotion* on rectangles and some related results.
- In [Grinbe17a], I answer a question of Karola Mészáros on the *subdivision algebra* (a deformation of the *Orlik-Solomon algebra* of the type-A braid arrangement).
- In [GrHuRe17], Jia Huang, Victor Reiner and I study the combinatorics of the *Grothendieck groups of a finite-dimensional Hopf algebra*, revealing a parallel to the *chip-firing game* on directed graphs.
- In [GriPos17], Alexander Postnikov and I demonstrate a property of *reduced expressions* in *Coxeter groups*.
- In [GriOlv18], Peter Olver and I factorize the determinant of a matrix arising from the *n-body problem*. I have since generalized this factorization in [Grinbe19a].
- In [AaGrSc18], Erik Aas, Travis Scrimshaw and I prove two conjectures on *multiline queues* and the *totally asymmetric simple exclusion process* (one coming from physics, one from probability theory).

Current research

Cohomology-like quotients of symmetric polynomials.[1] It is well-known since the early 20th century that the symmetric functions can be used to describe the cohomology ring $\mathrm{H}^*(\mathrm{Gr}_{k,n})$ of the Grassmannian $\mathrm{Gr}_{k,n}$ (see [Fulton97, Part III] for an introduction). More precisely, this cohomology ring is a quotient of the ring of symmetric functions modulo an ideal generated by all "large" elementary symmetric functions $e_{k+1}, e_{k+2}, e_{k+3}, \ldots$ and all "large" complete homogeneous symmetric functions $h_{n-k+1}, h_{n-k+2}, h_{n-k+3}, \ldots$. More recently it has been revealed that a deformation of this ideal induces a quotient that is isomorphic to the *quantum* cohomology ring $\mathrm{QH}^*(\mathrm{Gr}_{k,n})$, whose structure constants are the Gromov-Witten invariants (see [Postni05] for a recent perspective). These facts motivate a more general scenario:

Fix integers $n \geq k \geq 0$, and let $\mathcal{S}$ be the ring of symmetric polynomials in k variables $x_1, x_2, \ldots, x_k$ over an arbitrary base ring $\mathbf{k}$. (We can view $\mathcal{S}$ itself as a quotient of the ring of symmetric functions by the elementary symmetric functions $e_{k+1}, e_{k+2}, \ldots$.) Fix k scalars $a_1, a_2, \ldots, a_k \in \mathbf{k}$, and let I be the ideal of $\mathcal{S}$ generated by $h_{n-k+i} - a_i$ for all $i \in \{1, 2, \ldots, k\}$. The quotient algebra $\mathcal{S}/I$ then generalizes both the classical and the quantum cohomology rings.

[1] See [Grinbe18b] for a survey (slightly out of date) and http://www.cip.ifi.lmu.de/~grinberg/algebra/drexel2019.pdf for a semi-introductory talk.

I have initiated the study of this quotient algebra $\mathcal{S}/I$ in [Grinbe18a] (work in progress). In particular, I have shown that as a $\mathbf{k}$-module, $\mathcal{S}/I$ is free with a basis consisting of the projections of the Schur polynomials s_λ, where λ ranges over all partitions that "fit inside" the $k \times (n-k)$-rectangle (that is, λ has at most k parts, and they are $\leq n-k$). Furthermore, I have shown that the structure coefficients of the algebra $\mathcal{S}/I$ (in this basis) satisfy an S_3-symmetry, generalizing those of the Littlewood-Richardson coefficients and of the Gromov-Witten invariants. I have found a Pieri rule for products of the form $s_\lambda h_i$ (with $i \leq n-k$) and a second basis of $\mathcal{S}/I$ consisting of complete homogeneous symmetric functions h_λ. A "rim hook algorithm" for reducing arbitrary Schur polynomials (that do not fit into the $k \times (n-k)$-rectangle) in the "projected Schur polynomial" basis has been found (significantly subtler than the original one of Bertram, Ciocan-Fontanine and Fulton [BeCiFu99]). Further work on this algebra is ongoing, the goal being to explore what other properties of the cohomology rings generalize to this situation:

- The structure coefficients of $\mathcal{S}/I$ in the s_λ-basis seem to satisfy a positivity property (more precisely, predictable signs) generalizing the positivity of Gromov-Witten invariants. This is so far a conjecture, which I expect to be highly difficult.
- What further properties of classical and quantum cohomology extend to $\mathcal{S}/I$? Computations have shown that neither the $\mathrm{Gr}_{k,n} \leftrightarrow \mathrm{Gr}_{n-k,n}$ duality nor Postnikov's "curious duality" [Postni05] do (at least not without further restrictions), but others remain to be tested.
- What other bases with combinatorial meaning does $\mathcal{S}/I$ have? For example, do the power-sum and monomial bases of $\mathcal{S}$ lead to bases of $\mathcal{S}/I$?
- How many of the properties survive when $a_1, a_2, \ldots, a_k$ are no longer scalars in $\mathbf{k}$, but symmetric polynomials with $\deg(a_i) < n-k+i$ for all i? For example, the s_λ-basis still holds in that generality, while the S_3-symmetry does not.

A connection with the splitting algebras of Laksov and Thorup [LakTho12] is also suspected.

Refined dual stable Grothendieck polynomials. Dual stable Grothendieck polynomials first appear in the work of Lam and Pylyavskyy [LamPyl07], after having been anticipated by Lenart and Buch. In joint work [GaGrLi15] with Pavel Galashin and Gaku Liu, I have extended their definition and some of their properties to a more general setup, involving an infinite family of new parameters.

A *weak composition* means a sequence $(\alpha_1, \alpha_2, \alpha_3, \ldots) \in \mathbb{N}^\infty$ (where $\mathbb{N} = \{0, 1, 2, \ldots\}$) such that all but finitely many i satisfy $\alpha_i = 0$.

Consider a skew partition λ/μ. A *reverse plane partition* (short: *rpp*) of shape λ/μ means a filling of the skew Young diagram of λ/μ with positive integers which increase weakly along rows and weakly along columns. (Requiring them to increase strictly along columns would instead yield the definition of a semistandard tableau.) For every rpp T, we let $\operatorname{ircont} T$ be the weak composition whose i-th entry is the number of columns of T which contain the entry i. Moreover, for every rpp T, we let $\operatorname{ceq} T$ be the weak composition whose i-th entry is the number of cells c in the i-th row of T such that the entry of T in cell c equals the entry of T in the cell directly below c (and, in particular, the latter entry exists). Notice that $\operatorname{ceq} T = (0, 0, 0, \ldots)$ if and only if T is a semistandard tableau.

The *refined dual stable Grothendieck polynomial* $\widetilde{g}_{\lambda/\mu}$ corresponding to the skew partition λ/μ is defined to be

$$\sum_{T \text{ is an rpp of shape } \lambda/\mu} \mathbf{t}^{\operatorname{ceq} T} \mathbf{x}^{\operatorname{ircont} T} \in (\mathbb{Z}[t_1, t_2, t_3, \ldots])[[x_1, x_2, x_3, \ldots]].$$

Here we are using the notation $\mathbf{x}^\alpha$ for the monomial $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$ whenever $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ is a weak composition, and similarly the notation $\mathbf{t}^\alpha$ stands for $t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3} \cdots$.

The dual stable Grothendieck polynomials are obtained from the $\widetilde{g}_{\lambda/\mu}$ by setting all t_i equal to 1, whereas the skew Schur functions are obtained by setting all t_i equal to 0. Other specializations of $\widetilde{g}_{\lambda/\mu}$ have not been explored so far, but the parameter space is obviously vast.

In [GaGrLi15], Galashin, Liu and I have shown that the power series $\widetilde{g}_{\lambda/\mu}$ is symmetric (in the $x_1, x_2, x_3, \ldots$).

Various questions suggest themselves now:

- Do the $\widetilde{g}_{\lambda/\mu}$ satisfy a determinantal formula generalizing (one of) the Jacobi-Trudi identities? The answer appears to be positive, but a proof has not been found so far. Damir Yeliussizov [Yelius16] has found a proof in the $\mu = \varnothing$ case; research on the general case is ongoing.

- Do the $\widetilde{g}_{\lambda/\mu}$ satisfy a Littlewood-Richardson rule? There are several ways to interpret the question, some of which (e.g., expanding the product $\widetilde{g}_{\lambda/\varnothing}\widetilde{g}_{\mu/\varnothing}$ as a linear combination of the $\widetilde{g}_{\nu/\varnothing}$) appear out of reach. I have proven one Littlewood-Richardson rule (expanding $s_\nu \widetilde{g}_{\lambda/\mu}$ in terms of the s_κ) using the results of [GaGrLi15] and an analogue of Stembridge's proof of the classical Littlewood-Richardson rule[2].

- It appears that a similar refinement can be done to the (non-dual) stable Grothendieck polynomials $G_{\lambda/\mu}$.

Critical groups in representation theory.[3] The concept of a *critical group* originated from the theory of *chip-firing* on digraphs (also known as the *sandpile model*). The group can be defined as the cokernel (over $\mathbb{Z}$) of the digraph's *reduced Laplacian* (the submatrix of its Laplacian obtained by removing a certain row and a certain column); from this point of view, the theory of chip-firing can be recast as a study of this Laplacian

[2]See `http://www.cip.ifi.lmu.de/~grinberg/algebra/chicago2015.pdf` for a statement of this (proof not yet written up) and also of the conjectural Jacobi-Trudi identity.

[3]See `http://www.cip.ifi.lmu.de/~grinberg/algebra/madison17.pdf` for slides of a talk on this subject, which might clarify the below.

acting on the integer lattice. In particular, it has been observed by Gabrielov, Benkart, Klivans, Reiner and others (e.g., [BeKlRe16]) that the reduced Laplacian of a strongly connected digraph is a "*nonsingular M-matrix*" (an integer matrix whose off-diagonal entries are nonpositive, and whose inverse has nonnegative entries; several equivalent definitions exist), and that most of the theory of chip-firing can be recovered from this property. Hence, wherever one encounters a nonsingular M-matrix, one can construct a "chip-firing theory" with similar properties to that arising from a digraph.

Benkart, Klivans, Reiner, and Gaetz ([BeKlRe16], [Gaetz16]) have recently used this strategy to build a "chip-firing theory" from a faithful representation V of a finite group G (over $\mathbb{C}$). The analogue of the Laplacian here is taken over by the matrix $L_V = nI - M_V$, where I is the identity matrix, n is the dimension of V, and M_V is the matrix ("*McKay matrix* of V") that represents tensoring by V on the Grothendieck ring of G. Explicitly, if the irreducible representations of G are $S_1, S_2, \ldots, S_{\ell+1}$, then M_V is an $(\ell+1) \times (\ell+1)$-matrix with (i, j)-th entry $[S_i \otimes V : S_j]$. The analogue of the reduced Laplacian is the submatrix of L_V obtained by removing the row and the column corresponding to the trivial representation. They have shown that this reduced Laplacian is a nonsingular M-matrix (it is here that the faithfulness of V shows its relevance), and computed the order of the critical group (the cokernel of the reduced Laplacian) in terms of the character of V.

In a paper [GrHuRe17] with Reiner and Huang, we extend this construction to an arbitrary representation V of a *finite-dimensional Hopf algebra* A over any algebraically closed field. Instead of faithfulness, we now need a slightly subtler property of V (which we call "*tensor-richness*": every simple A-module appears in a composition series of some $V^{\otimes k}$) to ensure that our "reduced Laplacian" is a nonsingular M-matrix. We fully describe the critical group of the regular representation and some further examples; moreover, we express the order of the critical group in the case when $A = \mathbb{F}_p[G]$ for some finite group G (in terms of Brauer characters of G). We have no formula for the order of the critical group of a general V over a general A. Some further questions left to explore are

- the extent to which theory generalizes even further to objects in tensor categories (the basic facts do);
- the behavior of the critical groups under restriction and induction.

Birational rowmotion. Given a finite poset P, we can form another poset $\widehat{P}$ by adjoining a global minimum (called 0) and a global maximum (called 1) to P. Given a field (or semifield) $\mathbb{K}$, we consider the set $\mathbb{K}^{\widehat{P}}$ of all labelings of the elements of $\widehat{P}$ by elements of $\mathbb{K}$. On this set, David Einstein and James Propp have defined a birational equivalence, which they call *birational rowmotion*, and which generalizes the notion of rowmotion on the order ideals of P. Einstein and Propp have experimentally observed that, for various special classes of posets P, this birational equivalence has finite order (i.e., a certain power of it is the identity). In [GriRob14], Tom Roby and I prove these observations and some others. The most prominent case is that when P is a "rectangle" (i.e., a product of two chains with p and q elements, respectively); in this case, the order of birational rowmotion is $p+q$. This generalizes Schützenberger's classical result that the *promotion* operator ∂ on the semistandard Young tableaux of a given rectangular shape with entries in $\{1, 2, \ldots, n\}$ satisfies $\partial^n = \mathrm{id}$.

The proof of this result on rectangles is inspired by Volkov's proof of the type-AA Zamolodchikov conjecture [Volkov06]; we also analyze some other posets like graded forests (by an inductive argument) and some triangle-shaped posets (via a "folding" reduction to the rectangular case). One case – that of "trapezoidal" posets, which can be seen as a type-B analogue of rectangles – is still unresolved, and a conjecture by Nathan Williams connects it to the rectangular case in a remarkable way [GriRob14, §19], which (if correct) generalizes a conjecture by Elizalde on noncrossing families of Dyck paths [Elizal14].

Further developments have happened in the last few years[4]. For one, Max Glick has found a more direct relation between birational rowmotion in the case of a rectangle and the type-AA Zamolodchikov Y-system, whereas Alexander Postnikov has suggested a connection to the octahedral recurrence which still remains to be fully understood. Richard Stanley suggested a generalization of the case of graded forests, which is currently being written up by others. James Propp has conjectured some "homomesies" (algebraic identities holding for each orbit under birational rowmotion), some of which I have proven. Tom Roby and Gregg Musiker have recently found explicit formulas for iterates of birational rowmotion on rectangles [MusRob17], which are inspired by [GriRob14, §19] but are independent of it (and provide new proofs for its main results). Their formulas could help prove the remaining homomesies.

Among the questions that I am planning to study are the following:

- The finite order of birational rowmotion in the above-mentioned "trapezoidal" case needs to be proven.
- The finite order of birational rowmotion proven for rectangles seems to hold even if $\mathbb{K}$ is replaced by a (noncommutative) semifield, up to a certain conjugation[5]; Tom Roby and I have proven this for the most part, but the correct theoretical underpinnings of the noncommutative setting still need to be found.

Research plans, work in progress, future directions

The Reiner-Saliola-Welker conjecture. Let $n \in \mathbb{N}$, and consider the group algebra $\mathbb{C}\mathfrak{S}_n$ of the n-th symmetric group $\mathfrak{S}_n$. For any $k \in \{1, 2, \ldots, n\}$, we define an element

[4] For a list of papers inspired by [GriRob14], see the citations listed at Google Scholar, e.g., for the second half of [GriRob14]: `https://scholar.google.com/scholar?cites=445460616416961588&as_sdt=5,24&sciodt=0,24&hl=en` .

[5] See the end of my 2014 talk in Vienna (`http://www.cip.ifi.lmu.de/~grinberg/algebra/vienna2014.pdf`) for an example of this noncommutative phenomenon.

RSW_k of $\mathbb{C}\mathfrak{S}_n$ as $\sum_{w\in\mathfrak{S}_n} (\operatorname{noninv}_k w) \cdot w$, where $\operatorname{noninv}_k w$ is the number of all k-element subsets I of $\{1, 2, \ldots, n\}$ such that $w\mid_I$ is strictly increasing.

A surprisingly difficult result by Reiner, Saliola and Welker ([ReSaWe11, Theorem 1.1]) states that the elements $\mathrm{RSW}_1, \mathrm{RSW}_2, \ldots, \mathrm{RSW}_n$ commute pairwise. They further conjecture that each of these elements (viewed as a $\mathbb{C}$-linear endomorphism of $\mathbb{C}\mathfrak{S}_n$, given by left multiplication) has integer spectrum (i.e., all its eigenvalues are integers). This suggests the existence of a combinatorially meaningful joint eigenbasis for these operators (similar to, e.g., the seminormal basis for $\mathbb{C}\mathfrak{S}_n$). Indeed, an eigenbasis for RSW_1 has been found recently by Dieker and Saliola [DieSal15], which led to a proof of the fact that the eigenvalues of RSW_1 are integers between 0 and n^2. The proof is a tour-de-force of tensor algebra, representation theory of symmetric groups and Young tableau theory, and naturally suggests further questions (for instance, it appears to contain homological arguments in disguise).

The element RSW_1 of $\mathbb{C}\mathfrak{S}_n$ is known as the "random-to-random operator" on $\mathfrak{S}_n$, due to the following probabilistic interpretation: Imagine a shelf with n books labelled by $1, 2, \ldots, n$. In one step, we take out a randomly chosen book from the shelf, and put it back at a randomly chosen position[6]. The transition matrix of this Markov chain is the representing matrix of RSW_1. Thus, the Dieker-Saliola result claims that this Markov chain has integer eigenvalues; from this viewpoint, it appears surprising that such a simple-looking result has not been proven until 2015, and not without such difficulties!

The Markov chain just described is reminiscent of a simpler and better-known Markov chain: the *Tsetlin library*. Here, one puts the book back at the beginning of the shelf rather than at a random point. This chain is, indeed, closely related, and has a number of similar properties. It, too, has integer eigenvalues, and also corresponds to the first element of a sequence $\mathrm{R2T}_1, \mathrm{R2T}_2, \ldots, \mathrm{R2T}_n$ of pairwise commuting elements of $\mathbb{C}\mathfrak{S}_n$ [7]. These elements $\mathrm{R2T}_k$ not only commute, but also (unlike the RSW_k) span a subalgebra of $\mathbb{C}\mathfrak{S}_n$, and their products can be explicitly expanded; this is a particular case of the famous "Solomon's Mackey formula" for the descent algebra of $\mathfrak{S}_n$.

The symmetric group algebra $\mathbb{C}\mathfrak{S}_n$ is a Hopf algebra (as any group algebra is) and thus has an antipode S. It is not hard to see that $\mathrm{RSW}_k = \dfrac{1}{(n-k)!} \mathrm{R2T}_k \, S(\mathrm{R2T}_k)$ for every k.

In [Grinbe15b], I describe the kernel of the action of the random-to-top operator $\mathrm{R2T}_1$ on the tensor algebra (or, more precisely, the kernels of two of its actions – one "unsigned" and one "signed") over fields of arbitrary characteristic (and, in the "signed" case, even over arbitrary commutative rings). While the methods used do not directly apply to diagonalizing $\mathrm{R2T}_k$ and RSW_k (which seems out of reach in positive characteristic), they might provide some valuable insights. I hope to explore the algebra of the RSW_k and $\mathrm{R2T}_k$ more thoroughly. Questions of interest are:

[6] I.e., we place it in one of the n gaps (either between two books or at the beginning or at the end of the shelf) with equal probability. We do not model gaps between different books as intervals of different sizes (although that, too, might lead to interesting questions).

[7] The notation R2T stands for "random-to-top", which relates to how these elements are defined.

- While the integrality of the eigenvalues of RSW_1 has been proved, the question still stands for $\mathrm{RSW}_2, \mathrm{RSW}_3, \ldots, \mathrm{RSW}_n$.
- The proof of the commutativity of the $\mathrm{RSW}_1, \mathrm{RSW}_2, \ldots, \mathrm{RSW}_n$ given in [ReSaWe11, Theorem 1.1] is unsatisfactory (neither slick nor intuitive – its author called it "horrendous"); a better one needs to be found.
- It also feels that the eigenvalues of RSW_1 (at least the fact that they are integers between 0 and n^2) should have a simpler proof than the one in [DieSal15].

Carlitz-Witt vectors and function-field symmetric functions. The "field with one element" ($\mathbb{F}_1$) stands for the meta-mathematical idea that the ring $\mathbb{Z}$ has deep similarities with the polynomial rings $\mathbb{F}_q[T]$ over finite fields; that the combinatorics of sets has analogies with the linear algebra of $\mathbb{F}_q$-vector spaces; that the symmetric group is, in some sense, the "$q = 1$" version of the general linear group $\mathrm{GL}_n(\mathbb{F}_q)$. No fully explicatory mathematical foundation for these analogies is known, but they have been highly useful as heuristics many times, and a vast number of objects have been translated from one world to the other. Some basic examples can be found in [Cohn04]; another is the theory of Carlitz polynomials [Conrad15].

The ring of symmetric functions is deeply connected with integer partitions (e.g., almost all of its well-known bases are indexed by partitions); these correspond to conjugacy classes of permutations in symmetric groups. This raises the question of finding an "$\mathbb{F}_q$-analogue" of this ring which is similarly connected to "$\mathbb{F}_q[T]$-partitions" (i.e., sequences $(p_1, p_2, p_3, \ldots)$ of monic polynomials in $\mathbb{F}_q[T]$ such that $\cdots \mid p_3 \mid p_2 \mid p_1$ and such that all but finitely many p_i are $= 1$), or, equivalently, to conjugacy classes of matrices in $\mathrm{GL}_n(\mathbb{F}_q)$. Partial results towards the construction of such an analogue can be found in my work-in-progress [Grinbe15c]. My approach to finding such an analogue takes a detour through the notion of *Witt vectors*, which are an affine group whose coordinate ring is the symmetric functions ([Hazewi08, §10]). An $\mathbb{F}_q$-analogue of the Witt vectors (the *Carlitz-Witt vectors*, as I call them due to their use of Carlitz polynomials) is not hard to construct, and its coordinate ring can then be regarded as an $\mathbb{F}_q$-analogue of the symmetric functions. However, the combinatorics of this $\mathbb{F}_q$-analogue still remains to be understood, as the theory of Witt vectors reflects but little of the combinatorics of symmetric functions.

Other projects. Further ongoing research of mine includes a sequel to [GriPos17] that generalizes the main result to non-reduced expressions (I have been able to prove the first half of that generalization).

Several questions have been left unresolved in [Grinbe17a], such as a connection between $\mathcal{X}/\mathcal{J}$ and the Whitehouse representations of symmetric group [Whiteh97].

My work on shuffle-compatible permutation statistics [Grinbe17c] has led to a further variant of shuffle-compatibility, related to the notion of quadri-algebras; a study of its behavior should lead to another note.

Another area that I am planning to explore is the use of proof assistants such as Coq/ssreflect and Lean to formalize algebraic combinatorics, building on the work done by the MSR-Inria joint centre and by Florent Hivert.

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