Synthetic proof of Paul Yiu's excircles theorem / Darij Grinberg

I will present a synthetic proof of one of Paul Yiu's excircle theorems from [2] and an extension from [4]. Here is the theorem:

Let ABC be a triangle. Denote by B_a the point of tangency of the a-excircle with CA, and define similarly the points C_a, C_b, A_b, A_c and B_c.

Now construct the following triangles (Fig. 1):

The triangle $A'B'C'$ with the vertices

$$A' = A_cB_c \cap C_bA_b; \qquad B' = B_aC_a \cap A_cB_c; \qquad C' = C_bA_b \cap B_aC_a.$$

The triangle $A''B''C''$ with the vertices

$$A'' = C_aA_c \cap A_bB_a; \qquad B'' = A_bB_a \cap B_cC_b; \qquad C'' = B_cC_b \cap C_aA_c.$$

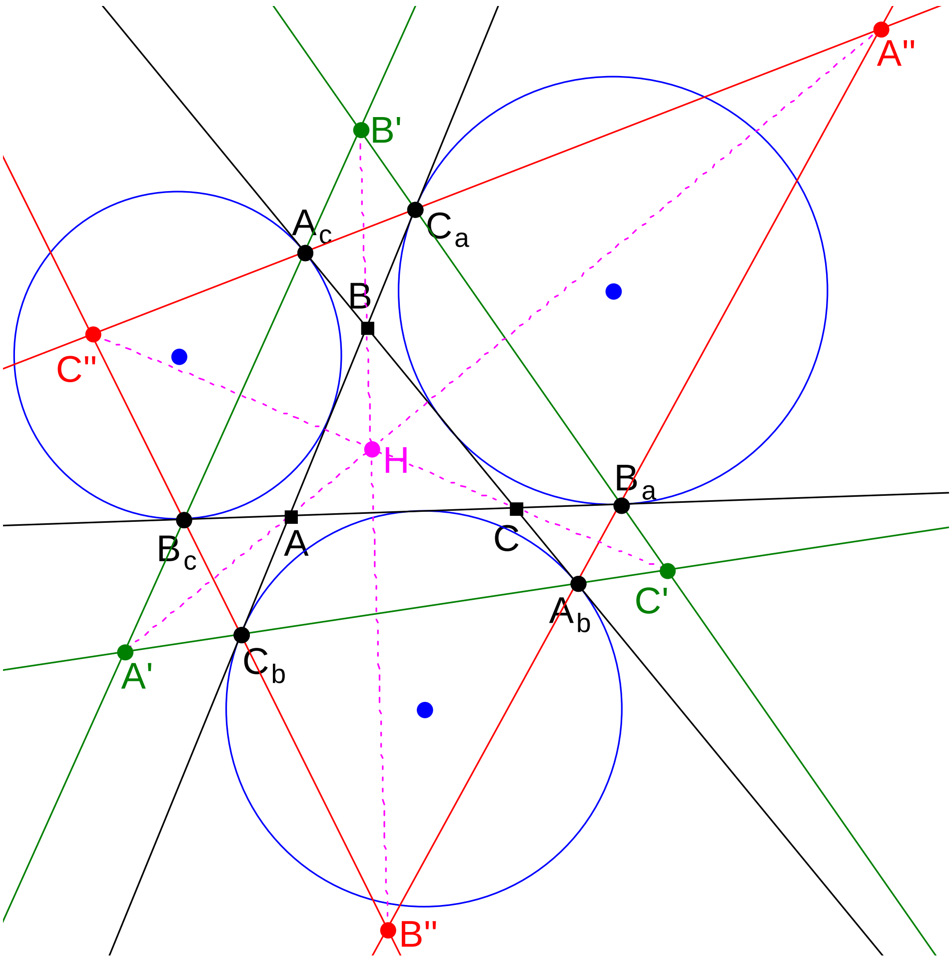

Fig. 1

Then, the theorem states:

a) The lines AA', BB' and CC' meet at one point, and this point is the orthocenter H of ΔABC (see also Fig. 2).

b) The lines AA'', BB'' and CC'' also meet at H.

Note: The theorem of part **a**), together with the fact that H is the circumcenter of triangle $A'B'C'$, were known to J. Hadamard ([4], Exercise 379). Analytic proofs were given in [2], 3.1 and [3]. Part **b**) was only mentioned in [4] and can be also proven with barycentric coordinates.

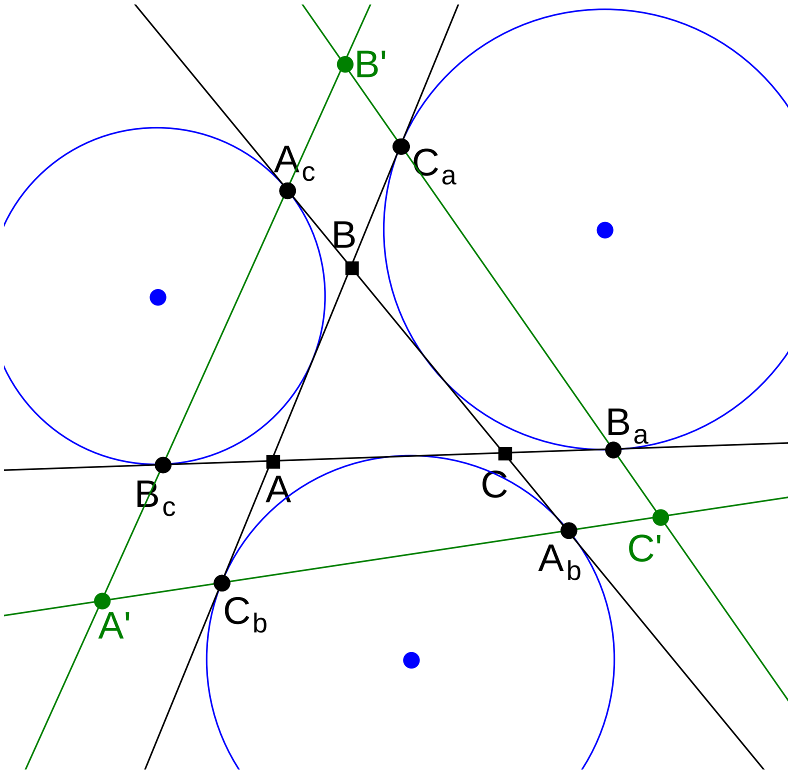

Fig. 2

Proof

a) We start with some heuristics: How should we attack the problem? The theorem consists of three symmetric parts: AA' goes through H; BB' goes through H; CC' goes through H. We will prove only one of these parts, and the other will follow by analogy.

We have to prove that the line AA' passes through the orthocenter H of triangle ABC, i. e. that A' lies on the a-altitude h_a of ΔABC. But we know that A' lies on C_bA_b and A_cB_c. Thus, we must prove that the lines C_bA_b, A_cB_c and h_a concur.

The well-known Steiner theorem gives a condition for the concurrence of three perpendiculars. We have to interpret the lines C_bA_b, A_cB_c and h_a as perpendiculars for applying the Steiner theorem.

The line A_cB_c joins the points of tangency of the c-excircle with BC and CA. By symmetry, A_cB_c is thus orthogonal to the angle bisector of C. Let O be the incenter of ΔABC; then, $A_cB_c \perp CO$. Similarly, we get $C_bA_b \perp BO$. Finally, we know $h_a \perp BC$. Therefore, A_cB_c, C_bA_b and h_a are the perpendiculars from B_c, C_b and A to the sidelines CO, BO and BC of the triangle BCO. After the Steiner theorem, they concur if and only if

$$CA^2 - BA^2 + BC_b^{\ 2} - OC_b^{\ 2} + OB_c^{\ 2} - CB_c^{\ 2} = 0. \qquad (1)$$

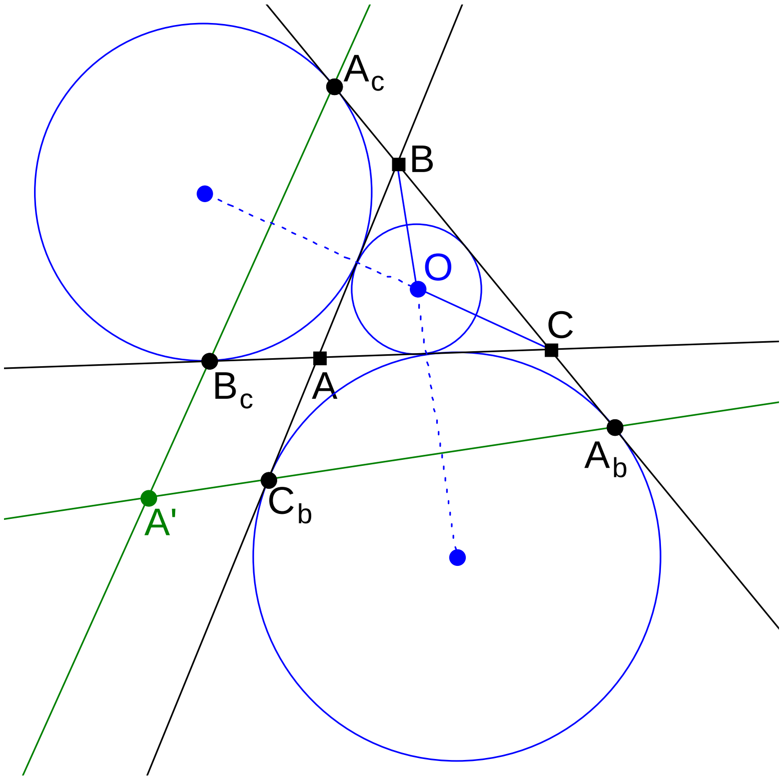

Fig. 3

All we have to do now is to prove (1). Obviously,

$$\begin{aligned}&CA^2 - BA^2 + BC_b{}^2 - OC_b{}^2 + OB_c{}^2 - CB_c{}^2\\ &= b^2 - c^2 + \left(BC_b{}^2 - CB_c{}^2\right) + \left(OB_c{}^2 - OC_b{}^2\right).\end{aligned}$$

After the well-known lengths $BC_b = s$ and $CB_c = s,$ where $s = \frac{1}{2}(a+b+c),$ we have

$$\begin{aligned}&CA^2 - BA^2 + BC_b{}^2 - OC_b{}^2 + OB_c{}^2 - CB_c{}^2\\ &= b^2 - c^2 + (s^2 - s^2) + \left(OB_c{}^2 - OC_b{}^2\right)\\ &= b^2 - c^2 + \left(OB_c{}^2 - OC_b{}^2\right).\end{aligned} \tag{2}$$

In order to compute the second brackets $\left(OB_c{}^2 - OC_b{}^2\right),$ we denote by B_1' the point of tangency of the incircle with CA. In the right-angled triangle $B_cB_1'O,$ we have

$$B_cB_1' = AB_c + AB_1' = (s-b) + (s-a) = c$$

and $\qquad OB_1' = \rho,$

where ρ is the inradius of ΔABC. After the Pythagoras theorem,

$$OB_c{}^2 = B_cB_1'{}^2 + OB_1'{}^2 = c^2 + \rho^2,$$

and analogously,

$$OC_b{}^2 = b^2 + \rho^2.$$

Inserted in (2), we have

$$\begin{aligned}&CA^2 - BA^2 + BC_b{}^2 - OC_b{}^2 + OB_c{}^2 - CB_c{}^2\\ &= b^2 - c^2 + ((c^2 + \rho^2) - (b^2 + \rho^2)) = b^2 - c^2 + (c^2 - b^2) = 0,\end{aligned}$$

what proves (1).

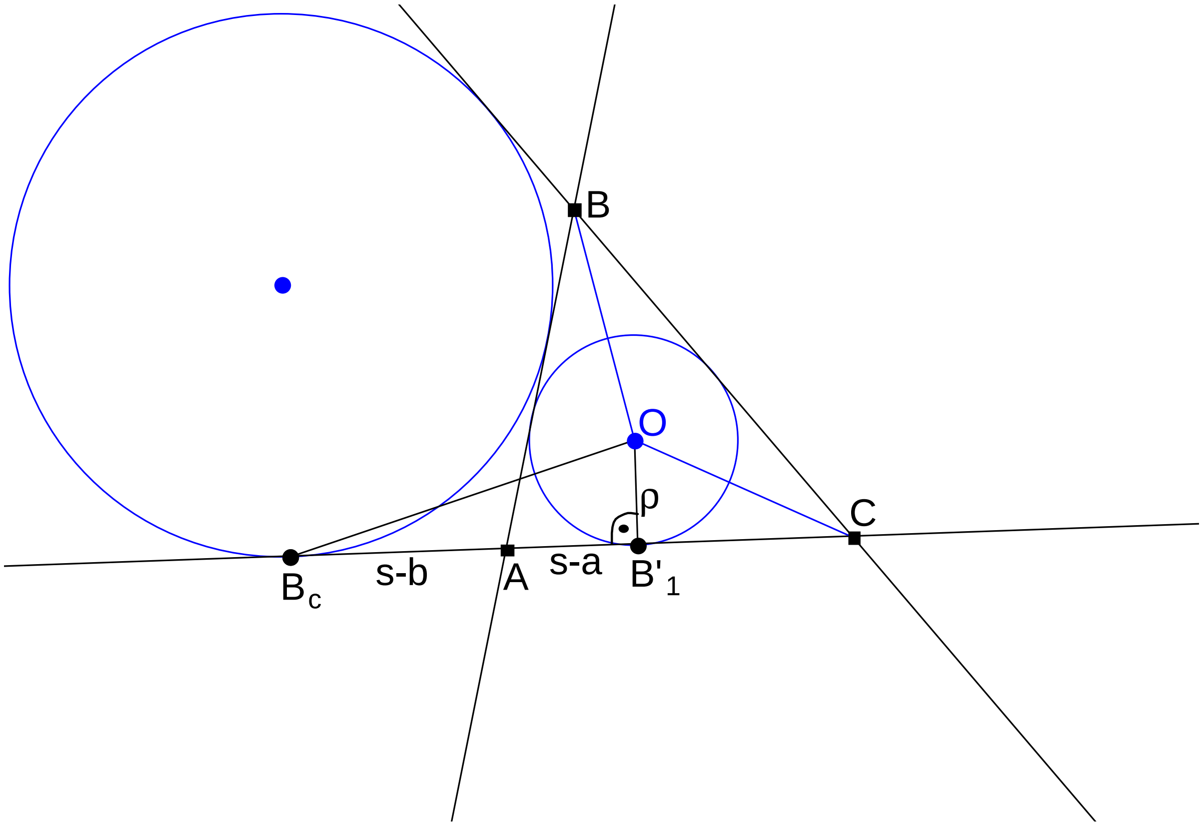

Fig. 4

Now, the Steiner theorem yields that the perpendiculars from B_c, C_b and A to CO, BO and BC concur, i. e. the lines A_cB_c, C_bA_b and h_a concur. Thus, A' lies on $h_a = AH$. Analogously, B' lies on BH and C' lies on CH, qed.

b) This part cannot be solved as easily as **a**), but it can be reduced to **a**). A projective theorem by Floor van Lamoen (projective dual of the desmic configuration) states that

If two triangles ABC and $A'B'C'$ are perspective with perspector $P = AA' \cap BB' \cap CC'$, and

$$A_b = C'A' \cap BC; \qquad A_c = A'B' \cap BC; \qquad C_a = B'C' \cap AB;$$

$$C_b = C'A' \cap AB; \qquad B_c = A'B' \cap CA; \qquad B_a = B'C' \cap CA,$$

then the triangle $A''B''C''$ enclosed by the lines B_cC_b, C_aA_c and A_bB_a, i. e. the triangle of

$$A'' = A_bB_a \cap C_aA_c; \qquad B'' = B_cC_b \cap A_bB_a; \qquad C'' = C_aA_c \cap B_cC_b,$$

is perspective with triangles ABC and $A'B'C'$ through one perspector, i. e. the lines AA'', BB'' and CC'' also meet at P.

Applied to our configuration, we see that AA'', BB'' and CC'' meet at H, qed.

Note: The projective theorem we have used is also the *generalized Desargues theorem* and can be deriven from the Desargues theorem.

References

[1] Jacques Hadamard: *Leçons de géométrie élémentaire I*: *Géométrie plane*, Paris 1911.

[2] Paul Yiu: *The Clawson Point and Excircles*.

[3] Paul Yiu, Niels Bejlegaard: *Crux Mathematicorum* #2579 *Problem and Solution*, Crux Mathematicorum 8/2001, S. 539-540.

[4] Steve Sigur: *H in the Gergonne-Nagel system*, Geometry-college Mathforum.org newsgroup, 23.8.1999.

`http://mathforum.org/epigone/geometry-college/glendgrahching`