New proof of the Symmedian Point to be the centroid of its pedal triangle, and the Converse

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The following theorem is an important property of the symmedian point of a triangle (Fig. 1):

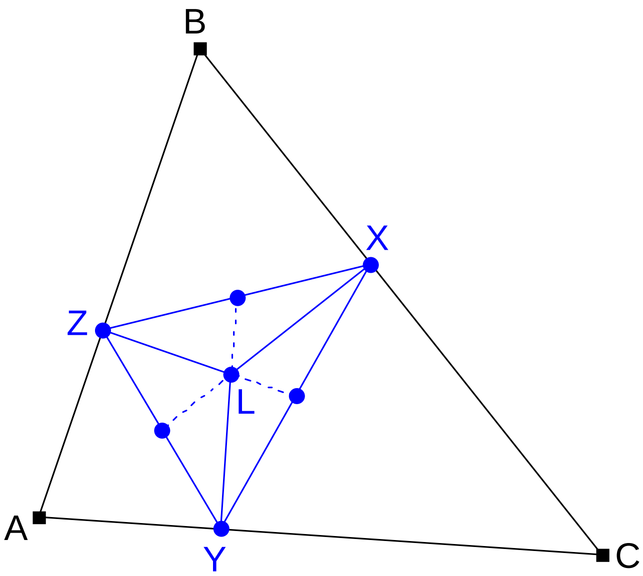

Fig. 1

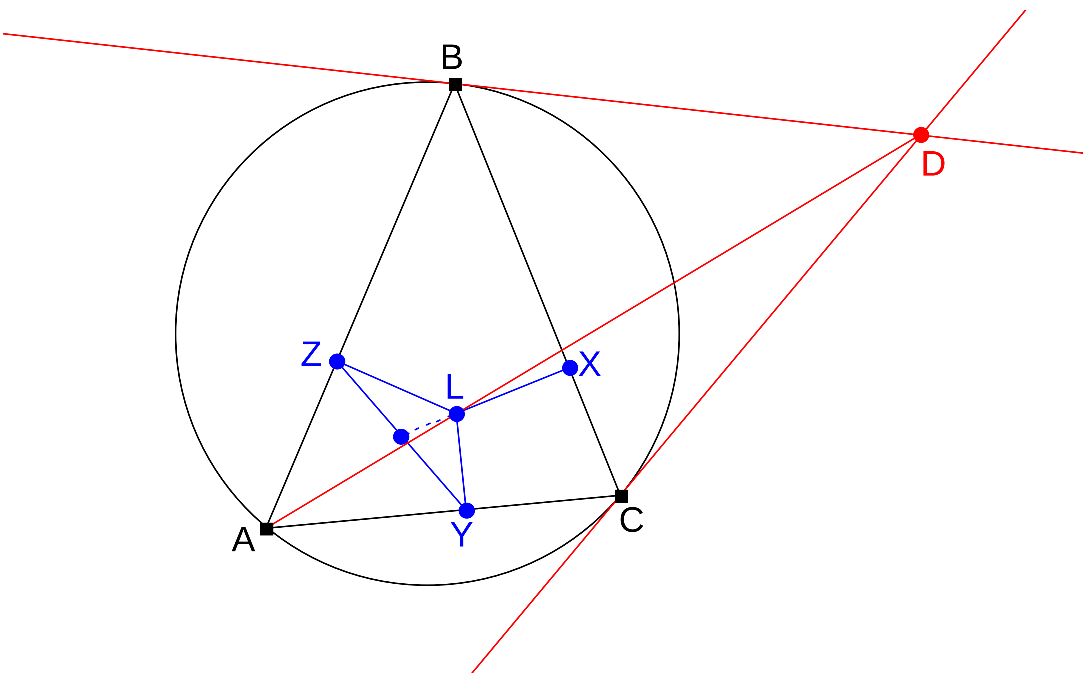

Fig. 2

Theorem 1: Let L be the symmedian point of a triangle ABC. From L, drop the perpendiculars LX, LY, LZ on the sides BC, CA, AB, where X, Y, Z are the respective feet of the perpendicular. Then L is the centroid of the triangle XYZ.

We mention that in the customary terminology, the triangle XYZ is called the pedal triangle of L with respect to the triangle ABC; but the triangle XYZ is also called **Lemoine**

pedal triangle of triangle ABC.

Theorem 1 can be paraphrased as follows: The symmedian point of a triangle is the centroid of the Lemoine pedal triangle.

Several proofs of Theorem 1 are known. In [2], p. 72-74, two proofs are presented. The proof given in [1] is a standard synthetic proof by constructing an auxiliary triangle.

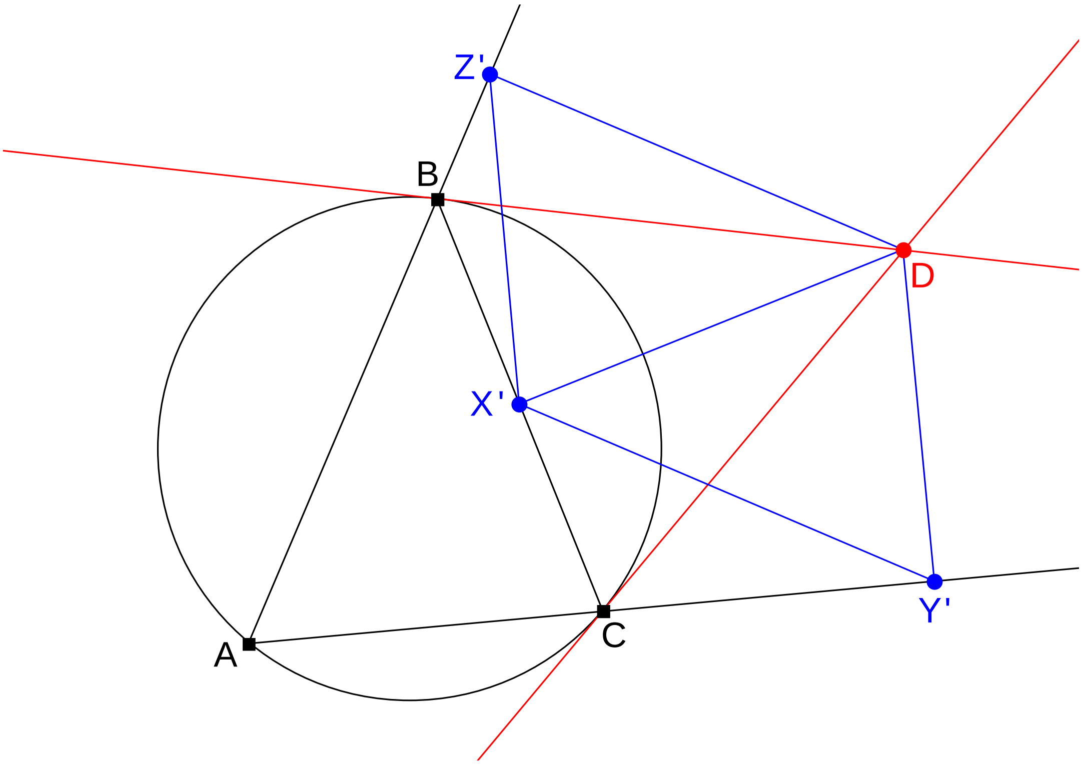

Fig. 3

We will prove Theorem 1 with the help of another construction; in fact, we regard the intersection D of the tangents to the circumcircle of ΔABC through the vertices B and C (Fig. 2). After [2], p. 60, the point D lies on the symedian from the vertex A, i. e. on the symmedian AL. Thus, we have:

Lemma 2: The symmedian point L lies on the line AD.

Now we will prove the following lemma (Fig. 3):

Lemma 3: Drop perpendiculars DX', DY', DZ' from D to the sides BC, CA, AB. Then $DY'X'Z'$ is a parallelogram. [This theorem is interesting to have another signification: It means that an ex-symmedian point D is an ex-centroid (exmedian point) of its pedal triangle $X'Y'Z'$.]

Proof (Fig. 4): We denote the angles of triangle ABC by $\measuredangle CAB = \alpha$, $\measuredangle ABC = \beta$ and $\measuredangle BCA = \gamma$. As chord-tangent angles, the angles $\measuredangle CBD$ and $\measuredangle BCD$ are both equal to the chordal angle of the chord BC, i. e. the angle α. From this, we have

$$\measuredangle DBZ' = 180° - \measuredangle ABC - \measuredangle CBD = 180° - \beta - \alpha = \gamma.$$

Since $\measuredangle DX'B = 90°$ and $\measuredangle DZ'B = 90°$, the points X' and Z' lie on the circle having the segment DB as diameter, and consequently, $BX'DZ'$ is a cyclic quadrilateral, and as chordal angles $\measuredangle DX'Z' = \measuredangle DBZ'$. Thus,

$$\measuredangle DX'Z' = \gamma. \qquad (1)$$

Since the points X' and Y' lie on the circle having the segment DC as diameter (because $\measuredangle DX'C = 90°$ and $\measuredangle DY'C = 90°$), $CX'DY'$ is a cyclic quadrilateral, and this yields

$$\measuredangle X'DY' = 180° - \measuredangle X'CY' = 180° - (180° - \measuredangle BCA) = 180° - (180° - \gamma) = \gamma.$$

By comparison with (1), we get $\measuredangle DX'Z' = \measuredangle X'DY'$, and from this, $X'Z' \parallel DY'$. Analogously, we can prove $X'Y' \parallel DZ'$, and thus, $DY'X'Z'$ is a parallelogram, qed.

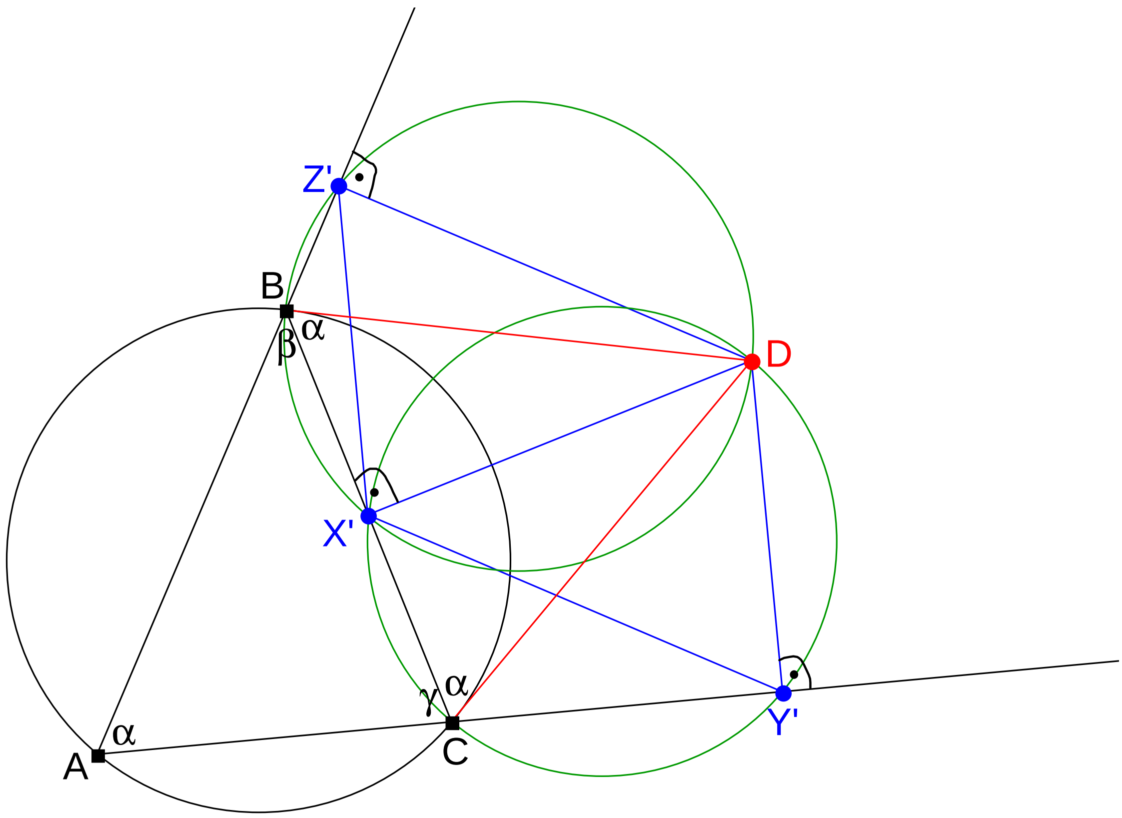

Fig. 4

Note that from the parallelogram $DY'X'Z'$, we get: The diagonal DX' bisects the diagonal $Y'Z'$. That means that the D-median of triangle $DY'Z'$ is DX'. Since DX' is orthogonal to BC, we have:

Lemma 4: The D-median of triangle $DY'Z'$ is orthogonal to BC.

Now we will connect this with Theorem 1 (Fig. 5). Since L lies on AD and Y lies on AY', and $LY \parallel DY'$ (because $LY \perp CA$ and $DY' \perp CA$), we have $AY : AY' = AL : AD$. Similarly, $AZ : AZ' = AL : AD$, and we conclude $AY : AY' = AZ : AZ'$. This yields $YZ \parallel Y'Z'$. Thus, the corresponding sides of triangles LYZ and $DY'Z'$ are parallel ($LY \parallel DY'$, $YZ \parallel Y'Z'$ und $ZL \parallel Z'L'$); therefore, also the L-median of triangle LYZ is parallel to the D-median of triangle $DY'Z'$. After Lemma 4, the latter one is orthogonal to BC; thus, also the L-median of triangle LYZ is orthogonal to BC, i. e. the perpendicular from L to BC bisects the segment YZ. But this perpendicular is the line LX. Thus, LX bisects the segment YZ, i. e. in the triangle XYZ, LX is a median. Similarly, LY and LZ are the two other medians in triangle XYZ, and therefore, L is the centroid of triangle XYZ. This proves Theorem 1.

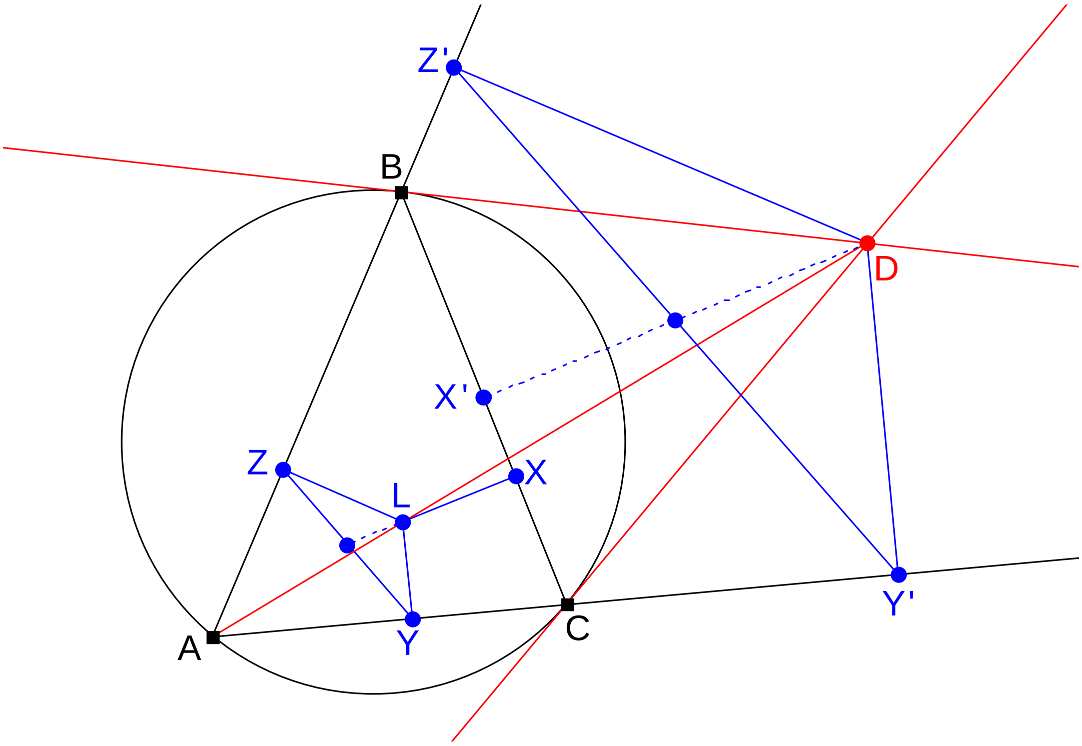

Fig. 5

This was my new proof. Also interesting is the observation that Theorem 1 possesses a valid converse theorem. We will now present a proof of *Theorem* 1 *together with the converse*:

Theorem 5 (**Theorem 1 together with the converse**): Let P be an arbitrary point in the plane of a triangle ABC, and let ΔXYZ be the pedal triangle of P with respect to triangle ABC. Then P is the centroid of triangle XYZ if and only if P is the symmedian point of triangle ABC.

In other words: There exists only one point which is the centroid of its pedal triangle, and this is the symmedian point.

The *proof* is similar to the trigonometric proof of Theorem 1 in [2], p. 72-73.

For first, we need the following lemma (proven in [2], p. 59):

Lemma 6: The distances of a point to the sides of a triangle ABC are in the ratios of these sides if and only if the point is the symmedian point of the triangle. In other words: For the distances $x = PX$, $y = PY$, $z = PZ$ of P to BC, CA, AB, the equation

$$x : y : z = a : b : c$$

holds if and only if P coincides with the symmedian point L of ΔABC.

Now we want to see when the point P is the centroid of its pedal triangle XYZ.

When does PY bisect the segment ZX ? Let D be the intersection of PY and ZX. Then, after the sine law in triangles PDZ and PDX, we have

$$\frac{ZD}{DX} = \frac{\sin\measuredangle ZPD \bullet PZ : \sin\measuredangle PDZ}{\sin\measuredangle XPD \bullet PX : \sin\measuredangle PDX} = \frac{\sin\measuredangle ZPD}{\sin\measuredangle XPD} \bullet \frac{PZ}{PX} : \frac{\sin\measuredangle PDZ}{\sin\measuredangle PDX}.$$

The angles $\measuredangle PDZ$ and $\measuredangle PDX$ sum up to $180°$; thus, their sines are equal: $\sin\measuredangle PDZ = \sin\measuredangle PDX$, and we get

$$\frac{ZD}{DX} = \frac{\sin\measuredangle ZPD}{\sin\measuredangle XPD} \bullet \frac{PZ}{PX} = \frac{\sin\measuredangle ZPD}{\sin\measuredangle XPD} \bullet \frac{z}{x}. \qquad (2)$$

For the angle $\measuredangle ZPD$, we have $\measuredangle ZPD = 180° - \measuredangle ZPY$; but we also have $\measuredangle ZAY = 180° - \measuredangle ZPY$, since $AZPY$ is a cyclic quadrangle (as for $\measuredangle AZP = 90°$ and

$\measuredangle AYP = 90°$, the points Z and Y lie on the circle having segment AP as diameter). This yields $\measuredangle ZPD = \measuredangle ZAY$, or $\measuredangle ZPD = \alpha$. Analogously, one finds $\measuredangle XPD = \gamma$; with this, the equation (2) is simplified to

$$\frac{ZD}{DX} = \frac{\sin\alpha}{\sin\gamma} \bullet \frac{z}{x} = \frac{a}{c} \bullet \frac{z}{x} = \frac{z}{x} : \frac{c}{a}.$$

Therefore, $ZD = DX$ holds if and only if $z : x = c : a$. This means that P lies on the Y-median of triangle XYZ if and only if $z : x = c : a$. Analogously, P lies on the X-median of triangle XYZ if and only if $y : z = b : c$. Thus, P is the centroid of triangle XYZ (lies on two medians) if and only if $x : y : z = a : b : c$. After Lemma 6, the condition $x : y : z = a : b : c$ holds if and only if P is the symmedian point of ΔABC. Therefore, P is the centroid of the pedal triangle XYZ if and only if P is the symmedian point of ΔABC. This proves Theorem 5.

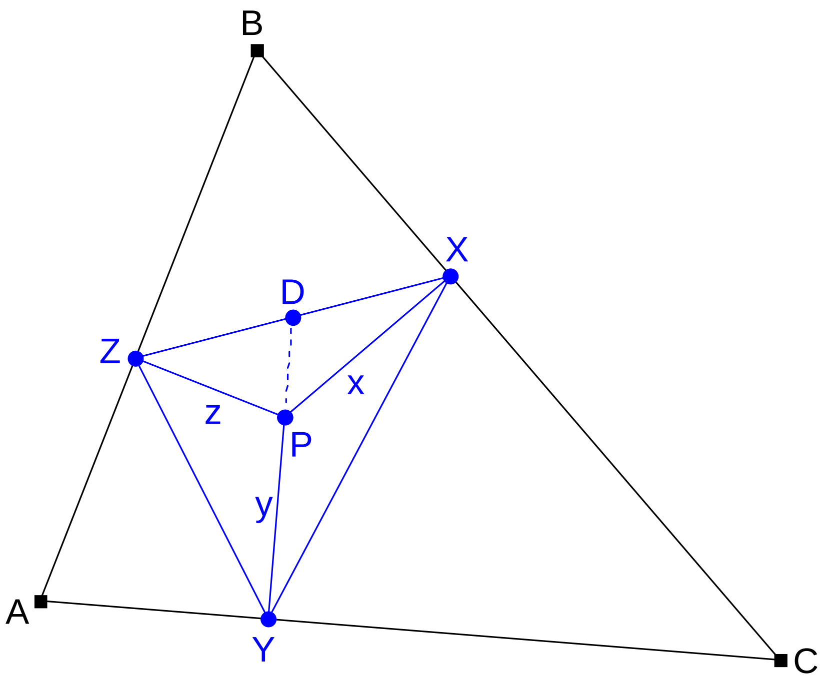

Fig. 6

References

[1] Emil Donath: *Die merkwürdigen Punkte und Linien des ebenen Dreiecks*, Berlin 1976.

[2] Ross Honsberger: *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, USA 1995.