7th QEDMO 2010, Problem 8 (a variation on AMM problem #E2353 by J. G. Rau)

Let $(a_1, a_2, ..., a_n)$ be an n-tuple of reals. Let $(b_1, b_2, ..., b_n)$ be an n-tuple of positive reals. We are searching for a permutation π of the set $\{1, 2, ..., n\}$ that minimizes the sum

$$\sum_{k=1}^{n} a_{\pi(k)} \sum_{i=k}^{n} b_{\pi(i)} = a_{\pi(1)} \left(b_{\pi(1)} + b_{\pi(2)} + ... + b_{\pi(n)}\right) + a_{\pi(2)} \left(b_{\pi(2)} + b_{\pi(3)} + ... + b_{\pi(n)}\right) + ... + a_{\pi(n)} b_{\pi(n)}.$$

Prove that any permutation π that satisfies

$$\frac{a_{\pi(1)}}{b_{\pi(1)}} \leq \frac{a_{\pi(2)}}{b_{\pi(2)}} \leq ... \leq \frac{a_{\pi(n)}}{b_{\pi(n)}}$$

minimizes this sum.

Remark: This problem is slightly stronger than American Mathematical Monthly problem #E2353 by J. G. Rau. My solution below follows R. J. Dickson's solution in [1].

Solution (according to R. J. Dickson)

We will prove a stronger assertion:

Theorem 1. Let $(a_1, a_2, ..., a_n)$ be an n-tuple of reals. Let $(b_1, b_2, ..., b_n)$ be an n-tuple of positive reals. Let π be a permutation of the set $\{1, 2, ..., n\}$. Then,

$$\sum_{k=1}^{n} a_{\pi(k)} \sum_{i=k}^{n} b_{\pi(i)} \geq \sum_{k=1}^{n} a_k b_k + \frac{1}{2} \sum_{\substack{(i,k) \in \{1,2,...,n\}^2; \\ i \neq k}} \min \{a_k b_i, a_i b_k\}. \qquad (1)$$

This inequality (1) becomes an equality if and only if

$$\frac{a_{\pi(1)}}{b_{\pi(1)}} \leq \frac{a_{\pi(2)}}{b_{\pi(2)}} \leq ... \leq \frac{a_{\pi(n)}}{b_{\pi(n)}}.$$

Proof of Theorem 1. We have

$$\sum_{k=1}^{n} a_{\pi(k)} \underbrace{\sum_{i=k}^{n} b_{\pi(i)}}_{= b_{\pi(k)} + \sum_{i=k+1}^{n} b_{\pi(i)}} = \underbrace{\sum_{k=1}^{n} a_{\pi(k)} b_{\pi(k)}}_{= \sum_{k=1}^{n} a_k b_k, \text{ since } \pi \text{ is a permutation}} + \underbrace{\sum_{k=1}^{n} a_{\pi(k)} \sum_{i=k+1}^{n} b_{\pi(i)}}_{= \sum_{k=1}^{n} \sum_{i=k+1}^{n} a_{\pi(k)} b_{\pi(i)} = \sum_{\substack{(i,k) \in \{1,2,...,n\}^2; \\ i > k}} a_{\pi(k)} b_{\pi(i)}}$$

$$= \sum_{k \in \{1,2,...,n\}} a_k b_k + \sum_{\substack{(i,k) \in \{1,2,...,n\}^2; \\ i > k}} a_{\pi(k)} b_{\pi(i)} \geq \sum_{k \in \{1,2,...,n\}} a_k b_k + \sum_{\substack{(i,k) \in \{1,2,...,n\}^2; \\ i > k}} \min \left\{a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)}\right\} \qquad (2)$$

(since $a_{\pi(k)} b_{\pi(i)} \geq \min \left\{ a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)} \right\}$ for any pair $(i,k) \in \{1,2,...,n\}^2$ satisfying $i > k$). On the other hand,

$$\begin{aligned}
& 2 \cdot \sum_{\substack{(i,k) \in \{1,2,...,n\}^2; \\ i>k}} \min \left\{ a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)} \right\} \\
&= \sum_{\substack{(i,k) \in \{1,2,...,n\}^2; \\ i>k}} \min \left\{ a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)} \right\} + \underbrace{\sum_{\substack{(i,k) \in \{1,2,...,n\}^2; \\ i>k}} \min \left\{ a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)} \right\}}_{\substack{= \sum_{\substack{(i,k) \in \{1,2,...,n\}^2; \\ k>i}} \min \left\{ a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)} \right\}, \\ \text{since } \min \left\{ a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)} \right\} \text{ is invariant} \\ \text{under the substitution } (i,k) \mapsto (k,i)}} \\
&= \sum_{\substack{(i,k) \in \{1,2,...,n\}^2; \\ i>k}} \min \left\{ a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)} \right\} + \sum_{\substack{(i,k) \in \{1,2,...,n\}^2; \\ k>i}} \min \left\{ a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)} \right\} \\
&= \sum_{\substack{(i,k) \in \{1,2,...,n\}^2; \\ i \neq k}} \min \left\{ a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)} \right\} = \sum_{\substack{(i,k) \in \{1,2,...,n\}^2; \\ i \neq k}} \min \left\{ a_k b_i, a_i b_k \right\}
\end{aligned}$$

(here, we substituted $\pi(i)$ and $\pi(k)$ for i and k, since π is a permutation). Dividing this equation by 2, we obtain

$$\sum_{\substack{(i,k) \in \{1,2,...,n\}^2; \\ i>k}} \min \left\{ a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)} \right\} = \frac{1}{2} \sum_{\substack{(i,k) \in \{1,2,...,n\}^2; \\ i \neq k}} \min \left\{ a_k b_i, a_i b_k \right\}.$$

Hence, (2) becomes

$$\sum_{k=1}^{n} a_{\pi(k)} \sum_{i=k}^{n} b_{\pi(i)} \geq \sum_{k \in \{1,2,...,n\}} a_k b_k + \frac{1}{2} \sum_{\substack{(i,k) \in \{1,2,...,n\}^2; \\ i \neq k}} \min \left\{ a_k b_i, a_i b_k \right\}.$$

Thus, the inequality (1) is proven. It remains to see when it becomes an equality. But this is more or less obvious: In our above proof of (1), we added together the inequalities $a_{\pi(k)} b_{\pi(i)} \geq \min \left\{ a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)} \right\}$ for all pairs $(i,k) \in \{1,2,...,n\}^2$ satisfying $i > k$. Hence, the inequality (1) becomes an equality if and only if each of these inequalities $a_{\pi(k)} b_{\pi(i)} \geq \min \left\{ a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)} \right\}$ for $(i,k) \in \{1,2,...,n\}^2$ satisfying $i > k$ becomes an equality. Thus, we have the following equivalence of assertions:

$$\begin{aligned}
& (\text{the inequality (1) becomes an equality}) \\
\iff & \left(\begin{array}{c} \text{for each pair } (i,k) \in \{1,2,...,n\}^2 \text{ satisfying } i>k \text{, the inequality} \\ a_{\pi(k)} b_{\pi(i)} \geq \min \left\{ a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)} \right\} \text{ becomes an equality} \end{array} \right) \\
\iff & \left(\begin{array}{c} \text{for each pair } (i,k) \in \{1,2,...,n\}^2 \text{ satisfying } i>k \text{, we have} \\ a_{\pi(k)} b_{\pi(i)} = \min \left\{ a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)} \right\} \end{array} \right) \\
\iff & \left(\text{for each pair } (i,k) \in \{1,2,...,n\}^2 \text{ satisfying } i>k \text{, we have } a_{\pi(k)} b_{\pi(i)} \leq a_{\pi(i)} b_{\pi(k)} \right) \\
\iff & \left(\text{for each pair } (i,k) \in \{1,2,...,n\}^2 \text{ satisfying } i>k \text{, we have } \frac{a_{\pi(k)}}{b_{\pi(k)}} \leq \frac{a_{\pi(i)}}{b_{\pi(i)}} \right) \\
\iff & \left(\frac{a_{\pi(1)}}{b_{\pi(1)}} \leq \frac{a_{\pi(2)}}{b_{\pi(2)}} \leq ... \leq \frac{a_{\pi(n)}}{b_{\pi(n)}} \right).
\end{aligned}$$

Thus, Theorem 1 is proven.

References

[1] J. G. Rau, W. O. J. Moser, R. J. Dickson, L. P. Prostanstus, *Optimal Sequence of Products (problem E 2353 and solutions)*, American Mathematical Monthly vol. 80 (1973), pp. 437-438.