7th QEDMO 2010, Problem 8 (a variation on AMM problem #E2353 by J. G. Rau)

Let $(a_1, a_2, ..., a_n)$ be an n-tuple of reals. Let $(b_1, b_2, ..., b_n)$ be an n-tuple of positive reals. We are searching for a permutation π of the set $\{1, 2, ..., n\}$ that minimizes the sum

$$\begin{aligned}\sum_{k=1}^{n} a_{\pi(k)} \sum_{i=k}^{n} b_{\pi(i)} &= a_{\pi(1)} \left(b_{\pi(1)} + b_{\pi(2)} + ... + b_{\pi(n)}\right) \\ &\quad + a_{\pi(2)} \left(b_{\pi(2)} + b_{\pi(3)} + ... + b_{\pi(n)}\right) \\ &\quad + ... \\ &\quad + a_{\pi(n)} b_{\pi(n)}.\end{aligned}$$

Prove that any permutation π that satisfies

$$\frac{a_{\pi(1)}}{b_{\pi(1)}} \leq \frac{a_{\pi(2)}}{b_{\pi(2)}} \leq ... \leq \frac{a_{\pi(n)}}{b_{\pi(n)}}$$

minimizes this sum.

Remark: This problem is slightly stronger than American Mathematical Monthly problem #E2353 by J. G. Rau. My solution below follows R. J. Dickson's solution in [1].

Solution (according to R. J. Dickson)

We will prove a stronger assertion:

Theorem 1. Let $(a_1, a_2, ..., a_n)$ be an n-tuple of reals. Let $(b_1, b_2, ..., b_n)$ be an n-tuple of positive reals. Let π be a permutation of the set $\{1, 2, ..., n\}$. Then,

$$\sum_{k=1}^{n} a_{\pi(k)} \sum_{i=k}^{n} b_{\pi(i)} \geq \sum_{k \in \{1,2,...,n\}} a_k b_k + \frac{1}{2} \sum_{\substack{(i,k) \in \{1,2,...,n\}^2; \\ i \neq k}} \min\{a_k b_i, a_i b_k\}. \quad (1)$$

This inequality (1) becomes an equality if and only if

$$\frac{a_{\pi(1)}}{b_{\pi(1)}} \leq \frac{a_{\pi(2)}}{b_{\pi(2)}} \leq ... \leq \frac{a_{\pi(n)}}{b_{\pi(n)}}.$$

Proof of Theorem 1. We have

$$\sum_{k=1}^{n} a_{\pi(k)} \underbrace{\sum_{i=k}^{n} b_{\pi(i)}}_{= b_{\pi(k)} + \sum_{i=k+1}^{n} b_{\pi(i)}} = \sum_{k=1}^{n} a_{\pi(k)} \left(b_{\pi(k)} + \sum_{i=k+1}^{n} b_{\pi(i)} \right) = \sum_{k=1}^{n} a_{\pi(k)} b_{\pi(k)} + \sum_{k=1}^{n} a_{\pi(k)} \sum_{i=k+1}^{n} b_{\pi(i)}$$

$$= \sum_{k \in \{1,2,...,n\}} a_k b_k + \sum_{\substack{(i,k) \in \{1,2,...,n\}^2; \\ i > k}} a_{\pi(k)} b_{\pi(i)} \quad (2)$$

(since

$$\sum_{k=1}^{n} a_{\pi(k)} b_{\pi(k)} = \sum_{k\in\{1,2,...,n\}} a_{\pi(k)} b_{\pi(k)} = \sum_{k\in\{1,2,...,n\}} a_k b_k$$

(here, we substituted k for $\pi(k)$ in the sum, since π is a permutation of the set $\{1, 2, ..., n\}$) and

$$\underbrace{\sum_{k=1}^{n}}_{=\sum_{k\in\{1,2,...,n\}}} a_{\pi(k)} \underbrace{\sum_{i=k+1}^{n}}_{=\sum_{\substack{i\in\{1,2,...,n\};\\ i>k}}} b_{\pi(i)} = \sum_{k\in\{1,2,...,n\}} a_{\pi(k)} \sum_{\substack{i\in\{1,2,...,n\};\\ i>k}} b_{\pi(i)} = \underbrace{\sum_{k\in\{1,2,...,n\}} \sum_{\substack{i\in\{1,2,...,n\};\\ i>k}}}_{=\sum_{\substack{(i,k)\in\{1,2,...,n\}^2;\\ i>k}}} a_{\pi(k)} b_{\pi(i)}$$

$$= \sum_{\substack{(i,k)\in\{1,2,...,n\}^2;\\ i>k}} a_{\pi(k)} b_{\pi(i)}$$

). On the other hand,

$$\sum_{\substack{(i,k)\in\{1,2,...,n\}^2;\\ i>k}} \min\left\{a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)}\right\}$$

$$= \underbrace{\sum_{\substack{(k,i)\in\{1,2,...,n\}^2;\\ k>i}}}_{\substack{=\sum_{i\in\{1,2,...,n\}} \sum_{\substack{k\in\{1,2,...,n\};\\ k>i}} \\ =\sum_{\substack{(i,k)\in\{1,2,...,n\}^2;\\ k>i}}}} \min \underbrace{\left\{a_{\pi(i)} b_{\pi(k)}, a_{\pi(k)} b_{\pi(i)}\right\}}_{=\left\{a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)}\right\}} \qquad \text{(here we renamed } (i,k) \text{ as } (k,i) \text{ in the sum)}$$

$$= \sum_{\substack{(i,k)\in\{1,2,...,n\}^2;\\ k>i}} \min\left\{a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)}\right\}, \qquad (3)$$

and

$$2\cdot\sum_{\substack{(i,k)\in\{1,2,...,n\}^2;\\ i>k}}\min\left\{a_{\pi(k)}b_{\pi(i)},a_{\pi(i)}b_{\pi(k)}\right\}$$

$$=\sum_{\substack{(i,k)\in\{1,2,...,n\}^2;\\ i>k}}\min\left\{a_{\pi(k)}b_{\pi(i)},a_{\pi(i)}b_{\pi(k)}\right\}+\sum_{\substack{(i,k)\in\{1,2,...,n\}^2;\\ i>k}}\min\left\{a_{\pi(k)}b_{\pi(i)},a_{\pi(i)}b_{\pi(k)}\right\}$$

$$=\sum_{\substack{(i,k)\in\{1,2,...,n\}^2;\\ i>k}}\min\left\{a_{\pi(k)}b_{\pi(i)},a_{\pi(i)}b_{\pi(k)}\right\}+\sum_{\substack{(i,k)\in\{1,2,...,n\}^2;\\ k>i}}\min\left\{a_{\pi(k)}b_{\pi(i)},a_{\pi(i)}b_{\pi(k)}\right\}\qquad\text{(by (3))}$$

$$=\sum_{\substack{(i,k)\in\{1,2,...,n\}^2;\\ i\neq k}}\min\left\{a_{\pi(k)}b_{\pi(i)},a_{\pi(i)}b_{\pi(k)}\right\}$$

$$\left(\begin{array}{c}\text{since the set }\left\{(i,k)\in\{1,2,...,n\}^2\ \mid\ i\neq k\right\}\text{ is the union of the two disjoint sets}\\ \left\{(i,k)\in\{1,2,...,n\}^2\ \mid\ i>k\right\}\text{ and }\left\{(i,k)\in\{1,2,...,n\}^2\ \mid\ k>i\right\}\end{array}\right)$$

$$=\underbrace{\sum_{\substack{(i,k)\in\{1,2,...,n\}^2;\\ \pi(i)\neq\pi(k)}}}_{=\sum\limits_{i\in\{1,2,...,n\}}\sum\limits_{\substack{k\in\{1,2,...,n\};\\ \pi(i)\neq\pi(k)}}}\min\left\{a_{\pi(k)}b_{\pi(i)},a_{\pi(i)}b_{\pi(k)}\right\}\qquad\left(\begin{array}{c}\text{since }i\neq k\text{ is equivalent to }\pi(i)\neq\pi(k)\text{, because}\\ \pi\text{ is a permutation}\end{array}\right)$$

$$=\sum_{i\in\{1,2,...,n\}}\sum_{\substack{k\in\{1,2,...,n\};\\ \pi(i)\neq\pi(k)}}\min\left\{a_{\pi(k)}b_{\pi(i)},a_{\pi(i)}b_{\pi(k)}\right\}$$

$$=\sum_{i\in\{1,2,...,n\}}\sum_{\substack{k\in\{1,2,...,n\};\\ i\neq\pi(k)}}\min\left\{a_{\pi(k)}b_{i},a_{i}b_{\pi(k)}\right\}$$

(here, we substituted i for $\pi(i)$ in the sum, since π is a permutation of the set $\{1,2,...,n\}$))

$$=\underbrace{\sum_{i\in\{1,2,...,n\}}\sum_{\substack{k\in\{1,2,...,n\};\\ i\neq k}}}_{=\sum\limits_{\substack{(i,k)\in\{1,2,...,n\}^2;\\ i\neq k}}}\min\{a_kb_i,a_ib_k\}$$

(here, we substituted k for $\pi(k)$ in the sum, since π is a permutation of the set $\{1,2,...,n\}$))

$$=\sum_{\substack{(i,k)\in\{1,2,...,n\}^2;\\ i\neq k}}\min\{a_kb_i,a_ib_k\}$$

Dividing this equation by 2, we obtain

$$\sum_{\substack{(i,k)\in\{1,2,...,n\}^2;\\ i>k}}\min\left\{a_{\pi(k)}b_{\pi(i)},a_{\pi(i)}b_{\pi(k)}\right\}=\frac{1}{2}\sum_{\substack{(i,k)\in\{1,2,...,n\}^2;\\ i\neq k}}\min\{a_kb_i,a_ib_k\}.\qquad(4)$$

Now, for every pair $(i,k)\in\{1,2,...,n\}^2$ satisfying $i>k$, we have the inequality

$a_{\pi(k)}b_{\pi(i)}\geq\min\left\{a_{\pi(k)}b_{\pi(i)},a_{\pi(i)}b_{\pi(k)}\right\}$, with equality if and only if $a_{\pi(k)}b_{\pi(i)}\leq a_{\pi(i)}b_{\pi(k)}$.

In other words,

$$a_{\pi(k)} b_{\pi(i)} \geq \min \left\{ a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)} \right\}, \tag{5}$$
$$\text{with equality if and only if } \frac{a_{\pi(k)}}{b_{\pi(k)}} \leq \frac{a_{\pi(i)}}{b_{\pi(i)}}$$

(since $a_{\pi(k)} b_{\pi(i)} \leq a_{\pi(i)} b_{\pi(k)}$ is equivalent to $\frac{a_{\pi(k)}}{b_{\pi(k)}} \leq \frac{a_{\pi(i)}}{b_{\pi(i)}}$). Summing up the inequality (5) over all pairs $(i,k) \in \{1,2,...,n\}^2$ satisfying $i > k$, we obtain

$$\sum_{\substack{(i,k)\in\{1,2,\ldots,n\}^2; \\ i>k}} a_{\pi(k)} b_{\pi(i)} \geq \sum_{\substack{(i,k)\in\{1,2,\ldots,n\}^2; \\ i>k}} \min \left\{ a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)} \right\},$$

with equality if and only if $\left(\frac{a_{\pi(k)}}{b_{\pi(k)}} \leq \frac{a_{\pi(i)}}{b_{\pi(i)}} \text{ for every pair } (i,k) \in \{1,2,...,n\}^2 \text{ satisfying } i > k \right)$.

In other words,

$$\sum_{\substack{(i,k)\in\{1,2,\ldots,n\}^2; \\ i>k}} a_{\pi(k)} b_{\pi(i)} \geq \sum_{\substack{(i,k)\in\{1,2,\ldots,n\}^2; \\ i>k}} \min \left\{ a_{\pi(k)} b_{\pi(i)}, a_{\pi(i)} b_{\pi(k)} \right\}, \tag{6}$$
$$\text{with equality if and only if } \frac{a_{\pi(1)}}{b_{\pi(1)}} \leq \frac{a_{\pi(2)}}{b_{\pi(2)}} \leq ... \leq \frac{a_{\pi(n)}}{b_{\pi(n)}}$$

(because $\left(\frac{a_{\pi(k)}}{b_{\pi(k)}} \leq \frac{a_{\pi(i)}}{b_{\pi(i)}} \text{ for every pair } (i,k) \in \{1,2,...,n\}^2 \text{ satisfying } i > k \right)$ is equivalent to $\frac{a_{\pi(1)}}{b_{\pi(1)}} \leq \frac{a_{\pi(2)}}{b_{\pi(2)}} \leq ... \leq \frac{a_{\pi(n)}}{b_{\pi(n)}}$). Due to (4), we can rewrite the inequality (6) as

$$\sum_{\substack{(i,k)\in\{1,2,\ldots,n\}^2; \\ i>k}} a_{\pi(k)} b_{\pi(i)} \geq \frac{1}{2} \sum_{\substack{(i,k)\in\{1,2,\ldots,n\}^2; \\ i\neq k}} \min \{ a_k b_i, a_i b_k \}, \tag{7}$$
$$\text{with equality if and only if } \frac{a_{\pi(1)}}{b_{\pi(1)}} \leq \frac{a_{\pi(2)}}{b_{\pi(2)}} \leq ... \leq \frac{a_{\pi(n)}}{b_{\pi(n)}}$$

Upon addition of $\sum_{k\in\{1,2,\ldots,n\}} a_k b_k$, the inequality (7) becomes

$$\sum_{k\in\{1,2,\ldots,n\}} a_k b_k + \sum_{\substack{(i,k)\in\{1,2,\ldots,n\}^2; \\ i>k}} a_{\pi(k)} b_{\pi(i)} \geq \sum_{k\in\{1,2,\ldots,n\}} a_k b_k + \frac{1}{2} \sum_{\substack{(i,k)\in\{1,2,\ldots,n\}^2; \\ i\neq k}} \min \{ a_k b_i, a_i b_k \}, \tag{8}$$

with equality if and only if $\frac{a_{\pi(1)}}{b_{\pi(1)}} \leq \frac{a_{\pi(2)}}{b_{\pi(2)}} \leq ... \leq \frac{a_{\pi(n)}}{b_{\pi(n)}}$.

Due to (2), we can rewrite the inequality (8) as

$$\sum_{k=1}^{n} a_{\pi(k)} \sum_{i=k}^{n} b_{\pi(i)} \geq \sum_{k\in\{1,2,\ldots,n\}} a_k b_k + \frac{1}{2} \sum_{\substack{(i,k)\in\{1,2,\ldots,n\}^2; \\ i\neq k}} \min \{ a_k b_i, a_i b_k \},$$
$$\text{with equality if and only if } \frac{a_{\pi(1)}}{b_{\pi(1)}} \leq \frac{a_{\pi(2)}}{b_{\pi(2)}} \leq ... \leq \frac{a_{\pi(n)}}{b_{\pi(n)}}.$$

Thus, Theorem 1 is proven.

References

[1] J. G. Rau, W. O. J. Moser, R. J. Dickson, L. P. Prostanstus, *Optimal Sequence of Products (problem E 2353 and solutions)*, American Mathematical Monthly vol. 80 (1973), pp. 437-438.