

4th QEDMO, Problem 13

Let n and k be integers such that $0 \leq k \leq n$. Prove that

$$\sum_{u=0}^k \binom{n+u-1}{u} \binom{n}{k-2u} = \binom{n+k-1}{k}.$$

Remark. Note that we use the following conventions:

$$\begin{aligned} \binom{r}{0} &= 1 && \text{for every integer } r; \\ \binom{u}{v} &= 0 && \text{if } u \text{ is a nonnegative integer and } v \text{ is an integer satisfying } v < 0 \text{ or } v > u. \end{aligned}$$

Solution by Darij Grinberg

Let \mathbb{N} denote the set of all nonnegative integers (in other words, let \mathbb{N} denote the set $\{0, 1, 2, \dots\}$).

Define a mapping $\text{sum} : \mathbb{N}^n \rightarrow \mathbb{N}$ by $\text{sum}(x_1; x_2; \dots; x_n) = x_1 + x_2 + \dots + x_n$ for every $(x_1; x_2; \dots; x_n) \in \mathbb{N}^n$.

We need two lemmata:

Lemma 1. For every integer v , we have

$$\binom{n}{v} = |\{\mathbf{x} \in \{0; 1\}^n \mid \text{sum } \mathbf{x} = v\}|.$$

Proof of Lemma 1. We know that $\binom{n}{v}$ is the number of all subsets of $\{1; 2; \dots; n\}$ which have v elements; in other words,

$$\binom{n}{v} = |\{S \in \mathcal{P}(\{1; 2; \dots; n\}) \mid |S| = v\}|.$$

Now, define a mapping $K_{\text{set}} : \mathcal{P}(\{1; 2; \dots; n\}) \rightarrow \{0; 1\}^n$ as follows: For every $S \in \mathcal{P}(\{1; 2; \dots; n\})$, let $K_{\text{set}}(S)$ be the n -tuple $(x_1; x_2; \dots; x_n) \in \{0; 1\}^n$ defined by $x_i = \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{if } i \notin S \end{cases}$ for every $i \in \{1; 2; \dots; n\}$. Then, this mapping K_{set} is a bijection, and $|S| = \text{sum}(K_{\text{set}}(S))$ for every $S \in \mathcal{P}(\{1; 2; \dots; n\})$ ¹. Hence,

$$\begin{aligned} & |\{S \in \mathcal{P}(\{1; 2; \dots; n\}) \mid |S| = v\}| \\ &= |\{S \in \mathcal{P}(\{1; 2; \dots; n\}) \mid \text{sum}(K_{\text{set}}(S)) = v\}| \\ &= |K_{\text{set}}^{-1}(\{\mathbf{x} \in \{0; 1\}^n \mid \text{sum } \mathbf{x} = v\})| \\ &= |\{\mathbf{x} \in \{0; 1\}^n \mid \text{sum } \mathbf{x} = v\}| \quad (\text{since } K_{\text{set}} \text{ is a bijection}). \end{aligned}$$

¹In fact, $K_{\text{set}}(S)$ is the n -tuple $(x_1; x_2; \dots; x_n) \in \{0; 1\}^n$ defined by $x_i = \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{if } i \notin S \end{cases}$ for every

Thus,

$$\binom{n}{v} = |\{S \in \mathcal{P}(\{1; 2; \dots; n\}) \mid |S| = v\}| = |\{\mathbf{x} \in \{0; 1\}^n \mid \text{sum } \mathbf{x} = v\}|,$$

and Lemma 1 is proven.

Lemma 2. For every nonnegative integer v , we have

$$\binom{n+v-1}{v} = |\{\mathbf{x} \in \mathbb{N}^n \mid \text{sum } \mathbf{x} = v\}|.$$

Proof of Lemma 2. We know that $\binom{n+v-1}{v}$ is the number of all multisets consisting of elements from $\{1; 2; \dots; n\}$ and having exactly v (not necessarily distinct) elements; in other words,

$$\binom{n+v-1}{v} = |\{S \text{ is a multiset consisting of elements from } \{1; 2; \dots; n\} \mid |S| = v\}|.$$

Now, define a mapping

$$K_{\text{multiset}} : \{S \text{ is a multiset consisting of elements from } \{1; 2; \dots; n\}\} \rightarrow \mathbb{N}^n$$

as follows: For every multiset S consisting of elements from $\{1; 2; \dots; n\}$, let $K_{\text{multiset}}(S)$ be the n -tuple $(x_1; x_2; \dots; x_n) \in \mathbb{N}^n$, where x_i is defined as the number of times the element i occurs in our multiset S (particularly, 0 if i doesn't occur in our multiset S at all) for every $i \in \{1; 2; \dots; n\}$. Then, this mapping K_{multiset} is a bijection, and $|S| = \text{sum}(K_{\text{multiset}}(S))$ for every multiset S consisting of elements from $\{1; 2; \dots; n\}$ ². Hence,

$$\begin{aligned} & |\{S \text{ is a multiset consisting of elements from } \{1; 2; \dots; n\} \mid |S| = v\}| \\ &= |\{S \text{ is a multiset consisting of elements from } \{1; 2; \dots; n\} \mid \text{sum}(K_{\text{multiset}}(S)) = v\}| \\ &= |K_{\text{multiset}}^{-1}(\{\mathbf{x} \in \mathbb{N}^n \mid \text{sum } \mathbf{x} = v\})| \\ &= |\{\mathbf{x} \in \mathbb{N}^n \mid \text{sum } \mathbf{x} = v\}| \quad (\text{since } K_{\text{multiset}} \text{ is a bijection}). \end{aligned}$$

 $i \in \{1; 2; \dots; n\}$, and thus

$$\begin{aligned} \text{sum}(K_{\text{set}}(S)) &= \text{sum}(x_1; x_2; \dots; x_n) = x_1 + x_2 + \dots + x_n = \sum_{i \in \{1; 2; \dots; n\}} x_i \\ &= \sum_{i \in \{1; 2; \dots; n\}} \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{if } i \notin S \end{cases} = \underbrace{\sum_{i \in S} \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{if } i \notin S \end{cases}}_{=1, \text{ since } i \in S} + \sum_{i \in \{1; 2; \dots; n\} \setminus S} \underbrace{\begin{cases} 1, & \text{if } i \in S \\ 0, & \text{if } i \notin S \end{cases}}_{=0, \text{ since } i \in \{1; 2; \dots; n\} \setminus S \text{ yields } i \notin S} \\ & \quad (\text{since the set } \{1; 2; \dots; n\} \text{ is the union of its disjoint subsets } S \text{ and } \{1; 2; \dots; n\} \setminus S) \\ &= \underbrace{\sum_{i \in S} 1}_{=|S| \cdot 1 = |S|} + \underbrace{\sum_{i \in \{1; 2; \dots; n\} \setminus S} 0}_{=0} = |S|. \end{aligned}$$

²In fact, $K_{\text{multiset}}(S)$ is the n -tuple $(x_1; x_2; \dots; x_n) \in \mathbb{N}^n$, where

$$x_i = (\text{the number of times the element } i \text{ occurs in our multiset } S)$$

Thus,

$$\begin{aligned} \binom{n+v-1}{v} &= |\{S \text{ is a multiset consisting of elements from } \{1; 2; \dots; n\} \mid |S| = v\}| \\ &= |\{\mathbf{x} \in \mathbb{N}^n \mid \text{sum } \mathbf{x} = v\}|, \end{aligned}$$

and Lemma 2 is proven.

Now we will solve the problem.

Define a mapping $F : \mathbb{N} \times \{0; 1\} \rightarrow \mathbb{N}$ by $F(q; r) = 2q + r$ for every $(q; r) \in \mathbb{N} \times \{0; 1\}$. Then, F is a bijection (since every $p \in \mathbb{N}$ can be uniquely written in the form $p = 2q + r$ with $(q; r) \in \mathbb{N} \times \{0; 1\}$ (in fact, q is the quotient and r is the remainder of p upon division by 2)). Thus, the mapping

$$\Phi : \mathbb{N}^n \times \{0; 1\}^n \rightarrow \mathbb{N}^n$$

defined by

$$\begin{aligned} \Phi((q_1; q_2; \dots; q_n); (r_1; r_2; \dots; r_n)) &= (F(q_1; r_1); F(q_2; r_2); \dots; F(q_n; r_n)) \\ \text{for every } ((q_1; q_2; \dots; q_n); (r_1; r_2; \dots; r_n)) &\in \mathbb{N}^n \times \{0; 1\}^n \end{aligned}$$

is a bijection as well.³ Besides, for every $(\mathbf{q}; \mathbf{r}) \in \mathbb{N}^n \times \{0; 1\}^n$, we have $\text{sum } (\Phi(\mathbf{q}; \mathbf{r})) = 2 \text{sum } \mathbf{q} + \text{sum } \mathbf{r}$.⁴

for every $i \in \{1; 2; \dots; n\}$, and thus

$$\begin{aligned} \text{sum}(K_{\text{multiset}}(S)) &= \text{sum}(x_1; x_2; \dots; x_n) = x_1 + x_2 + \dots + x_n = \sum_{i \in \{1; 2; \dots; n\}} x_i \\ &= \sum_{i \in \{1; 2; \dots; n\}} (\text{the number of times the element } i \text{ occurs in our multiset } S) \\ &= |S|. \end{aligned}$$

³In fact, $\Phi = F^n \circ \Psi$, where the mapping

$$\Psi : \mathbb{N}^n \times \{0; 1\}^n \rightarrow (\mathbb{N} \times \{0; 1\})^n$$

is defined by

$$\begin{aligned} \Psi((q_1; q_2; \dots; q_n); (r_1; r_2; \dots; r_n)) &= ((q_1; r_1); (q_2; r_2); \dots; (q_n; r_n)) \\ \text{for every } ((q_1; q_2; \dots; q_n); (r_1; r_2; \dots; r_n)) &\in \mathbb{N}^n \times \{0; 1\}^n, \end{aligned}$$

and the mapping

$$F^n : (\mathbb{N} \times \{0; 1\})^n \rightarrow \mathbb{N}^n$$

is defined by

$$\begin{aligned} F^n((q_1; r_1); (q_2; r_2); \dots; (q_n; r_n)) &= (F(q_1; r_1); F(q_2; r_2); \dots; F(q_n; r_n)) \\ \text{for every } ((q_1; r_1); (q_2; r_2); \dots; (q_n; r_n)) &\in (\mathbb{N} \times \{0; 1\})^n. \end{aligned}$$

Since both F^n and Ψ are bijections (in fact, F^n is a bijection, since F is a bijection), it follows that $\Phi = F^n \circ \Psi$ is a bijection as well (since the composition of two bijections is a bijection).

⁴In fact, we can write \mathbf{q} in the form $\mathbf{q} = (q_1; q_2; \dots; q_n)$ (since $\mathbf{q} \in \mathbb{N}^n$), and we can write \mathbf{r} in the

form $\mathbf{r} = (r_1; r_2; \dots; r_n)$ (since $\mathbf{r} \in \{0; 1\}^n$). Then,

$$\begin{aligned}\Phi(\mathbf{q}; \mathbf{r}) &= \Phi((q_1; q_2; \dots; q_n); (r_1; r_2; \dots; r_n)) = (F(q_1; r_1); F(q_2; r_2); \dots; F(q_n; r_n)) \\ &= (2q_1 + r_1; 2q_2 + r_2; \dots; 2q_n + r_n)\end{aligned}$$

and thus

$$\begin{aligned}\text{sum}(\Phi(\mathbf{q}; \mathbf{r})) &= \text{sum}(2q_1 + r_1; 2q_2 + r_2; \dots; 2q_n + r_n) \\ &= (2q_1 + r_1) + (2q_2 + r_2) + \dots + (2q_n + r_n) \\ &= 2 \underbrace{(q_1 + q_2 + \dots + q_n)}_{=\text{sum}(q_1; q_2; \dots; q_n)=\text{sum } \mathbf{q}} + \underbrace{(r_1 + r_2 + \dots + r_n)}_{=\text{sum}(r_1; r_2; \dots; r_n)=\text{sum } \mathbf{r}} \\ &= 2 \text{sum } \mathbf{q} + \text{sum } \mathbf{r}.\end{aligned}$$

Now,

$$\begin{aligned}
& \binom{n+k-1}{k} \\
&= |\{\mathbf{x} \in \mathbb{N}^n \mid \text{sum } \mathbf{x} = k\}| \quad (\text{by Lemma 2, applied to } v = k) \\
&= |\Phi^{-1}(\{\mathbf{x} \in \mathbb{N}^n \mid \text{sum } \mathbf{x} = k\})| \quad (\text{since } \Phi \text{ is a bijection}) \\
&= \left| \left\{ (\mathbf{q}; \mathbf{r}) \in \mathbb{N}^n \times \{0; 1\}^n \mid \underbrace{\text{sum}(\Phi(\mathbf{q}; \mathbf{r}))}_{=2 \text{sum } \mathbf{q} + \text{sum } \mathbf{r}} = k \right\} \right| \\
&= |\{(\mathbf{q}; \mathbf{r}) \in \mathbb{N}^n \times \{0; 1\}^n \mid 2 \text{sum } \mathbf{q} + \text{sum } \mathbf{r} = k\}| \\
&= \sum_{u \in \mathbb{N}} |\{(\mathbf{q}; \mathbf{r}) \in \mathbb{N}^n \times \{0; 1\}^n \mid 2 \text{sum } \mathbf{q} + \text{sum } \mathbf{r} = k \text{ and } \text{sum } \mathbf{q} = u\}| \\
&= \sum_{u \in \mathbb{N}} |\{(\mathbf{q}; \mathbf{r}) \in \mathbb{N}^n \times \{0; 1\}^n \mid 2u + \text{sum } \mathbf{r} = k \text{ and } \text{sum } \mathbf{q} = u\}| \\
&\quad \left(\begin{array}{l} \text{because the assertions } (2 \text{sum } \mathbf{q} + \text{sum } \mathbf{r} = k \text{ and } \text{sum } \mathbf{q} = u) \text{ and} \\ (2u + \text{sum } \mathbf{r} = k \text{ and } \text{sum } \mathbf{q} = u) \text{ are equivalent, since if } \text{sum } \mathbf{q} = u, \\ \text{then the assertions } (2 \text{sum } \mathbf{q} + \text{sum } \mathbf{r} = k) \text{ and } (2u + \text{sum } \mathbf{r} = k) \text{ are equivalent} \end{array} \right) \\
&= \sum_{u \in \mathbb{N}} \left| \underbrace{\{(\mathbf{q}; \mathbf{r}) \in \mathbb{N}^n \times \{0; 1\}^n \mid \text{sum } \mathbf{r} = k - 2u \text{ and } \text{sum } \mathbf{q} = u\}}_{= \{ \mathbf{q} \in \mathbb{N}^n \mid \text{sum } \mathbf{q} = u \} \times \{ \mathbf{r} \in \{0; 1\}^n \mid \text{sum } \mathbf{r} = k - 2u \}} \right| \\
&\quad (\text{since } 2u + \text{sum } \mathbf{r} = k \text{ is equivalent to } \text{sum } \mathbf{r} = k - 2u) \\
&= \sum_{u \in \mathbb{N}} |\{ \mathbf{q} \in \mathbb{N}^n \mid \text{sum } \mathbf{q} = u \} \times \{ \mathbf{r} \in \{0; 1\}^n \mid \text{sum } \mathbf{r} = k - 2u \}| \\
&\quad = |\{ \mathbf{q} \in \mathbb{N}^n \mid \text{sum } \mathbf{q} = u \}| \cdot |\{ \mathbf{r} \in \{0; 1\}^n \mid \text{sum } \mathbf{r} = k - 2u \}| \\
&= \sum_{u \in \mathbb{N}} \underbrace{|\{ \mathbf{q} \in \mathbb{N}^n \mid \text{sum } \mathbf{q} = u \}|}_{= |\{ \mathbf{x} \in \mathbb{N}^n \mid \text{sum } \mathbf{x} = u \}|} \cdot \underbrace{|\{ \mathbf{r} \in \{0; 1\}^n \mid \text{sum } \mathbf{r} = k - 2u \}|}_{= |\{ \mathbf{x} \in \{0; 1\}^n \mid \text{sum } \mathbf{x} = k - 2u \}|} \\
&\quad \begin{array}{l} \text{by Lemma 2 (applied to } v = u) \end{array} \quad \begin{array}{l} \text{by Lemma 1 (applied to } v = k - 2u) \end{array} \\
&= \sum_{u \in \mathbb{N}} \binom{n+u-1}{u} \cdot \binom{n}{k-2u} = \underbrace{\sum_{u \in \{0, 1, \dots, k\}} \binom{n+u-1}{u} \cdot \binom{n}{k-2u}}_{= \sum_{u=0}^k} \\
&\quad \left(\begin{array}{l} \text{here we replaced the } \sum_{u \in \mathbb{N}} \text{ sign by a } \sum_{u \in \{0, 1, \dots, k\}} \text{ sign, since all addends for} \\ u \in \mathbb{N} \setminus \{0, 1, \dots, k\} \text{ are zero (because if } u \in \mathbb{N} \setminus \{0, 1, \dots, k\}, \text{ then } u > k, \text{ so that} \\ 2u > 2k \geq k, \text{ thus } k - 2u < 0 \text{ and thus } \binom{n}{k-2u} = 0) \end{array} \right) \\
&= \sum_{u=0}^k \binom{n+u-1}{u} \cdot \binom{n}{k-2u}.
\end{aligned}$$

Thus, we see that the problem is solved.