4th QEDMO, Problem 13[1]

Let n and k be two nonnegative integers. Prove that

$$\sum_{u=0}^{k} \binom{n+u-1}{u} \binom{n}{k-2u} = \binom{n+k-1}{k}.$$

Remark. Note that we use the following conventions:

$$\binom{r}{0} = 1 \qquad \text{for every integer } r;$$

$$\binom{u}{v} = 0 \qquad \text{if } u \text{ is a nonnegative integer and } v \text{ is an integer satisfying } v < 0 \text{ or } v > u.$$

Solution by Darij Grinberg

Let $\mathbb{N}$ denote the set of all nonnegative integers (in other words, let $\mathbb{N}$ denote the set $\{0, 1, 2, ...\}$).

Define a mapping $\operatorname{sum} : \mathbb{N}^n \to \mathbb{N}$ by $\operatorname{sum}(x_1;\ x_2;\ ...;\ x_n) = x_1 + x_2 + ... + x_n$ for every $(x_1;\ x_2;\ ...;\ x_n) \in \mathbb{N}^n$.

We need two lemmata:

Lemma 1. For every integer v, we have

$$\binom{n}{v} = \left| \left\{ \mathbf{x} \in \{0;\ 1\}^n \mid \operatorname{sum} \mathbf{x} = v \right\} \right|.$$

Proof of Lemma 1. For every set S, let $\mathcal{P}(S)$ denote the power set of S (that is, the set of all subsets of S).

We know that $\binom{n}{v}$ is the number of all subsets of $\{1;\ 2;\ ...;\ n\}$ which have v elements; in other words,

$$\binom{n}{v} = \left| \left\{ S \in \mathcal{P}(\{1;\ 2;\ ...;\ n\}) \mid |S| = v \right\} \right|.$$

Now, define a mapping $K_{\text{set}} : \mathcal{P}(\{1;\ 2;\ ...;\ n\}) \to \{0;\ 1\}^n$ as follows: For every $S \in \mathcal{P}(\{1;\ 2;\ ...;\ n\})$, let $K_{\text{set}}(S)$ be the n-tuple $(x_1;\ x_2;\ ...;\ x_n) \in \{0;\ 1\}^n$ defined by $x_i = \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{if } i \notin S \end{cases}$ for every $i \in \{1;\ 2;\ ...;\ n\}$. Then, this mapping K_{set} is a bijection,

[1] corrected version (28 May 2016)

and $|S| = \operatorname{sum}(K_{\text{set}}(S))$ for every $S \in \mathcal{P}(\{1;\ 2;\ ...;\ n\})$ [2]. Hence,

$$\begin{aligned}
&|\{S \in \mathcal{P}(\{1;\ 2;\ ...;\ n\}) \mid |S| = v\}| \\
&= \left| \underbrace{\{S \in \mathcal{P}(\{1;\ 2;\ ...;\ n\}) \mid \operatorname{sum}(K_{\text{set}}(S)) = v\}}_{\substack{=\{S\in\mathcal{P}(\{1;\ 2;\ ...;\ n\}) \mid K_{\text{set}}(S)\in\{\mathbf{x}\in\{0;\ 1\}^n \mid \operatorname{sum}\mathbf{x}=v\}\} \\ =(K_{\text{set}})^{-1}(\{\mathbf{x}\in\{0;\ 1\}^n \mid \operatorname{sum}\mathbf{x}=v\})}} \right| \\
&= \left|(K_{\text{set}})^{-1}(\{\mathbf{x} \in \{0;\ 1\}^n \mid \operatorname{sum}\mathbf{x} = v\})\right| \\
&= |\{\mathbf{x} \in \{0;\ 1\}^n \mid \operatorname{sum}\mathbf{x} = v\}| \qquad (\text{since } K_{\text{set}} \text{ is a bijection}).
\end{aligned}$$

Thus,

$$\binom{n}{v} = |\{S \in \mathcal{P}(\{1;\ 2;\ ...;\ n\}) \mid |S| = v\}| = |\{\mathbf{x} \in \{0;\ 1\}^n \mid \operatorname{sum}\mathbf{x} = v\}|,$$

and Lemma 1 is proven.

Lemma 2. For every nonnegative integer v, we have

$$\binom{n+v-1}{v} = |\{\mathbf{x} \in \mathbb{N}^n \mid \operatorname{sum}\mathbf{x} = v\}|.$$

Proof of Lemma 2. We know that $\binom{n+v-1}{v}$ is the number of all multisets consisting of elements from $\{1;\ 2;\ ...;\ n\}$ and having exactly v (not necessarily distinct) elements; in other words,

$$\binom{n+v-1}{v} = |\{S \text{ is a multiset consisting of elements from } \{1;\ 2;\ ...;\ n\} \mid |S| = v\}|.$$

Now, define a mapping

$$K_{\text{multiset}} : \{S \text{ is a multiset consisting of elements from } \{1;\ 2;\ ...;\ n\}\} \to \mathbb{N}^n$$

[2] In fact, $K_{\text{set}}(S)$ is the n-tuple $(x_1;\ x_2;\ ...;\ x_n) \in \{0;\ 1\}^n$ defined by $x_i = \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{if } i \notin S \end{cases}$ for every $i \in \{1;\ 2;\ ...;\ n\}$, and thus

$$\begin{aligned}
\operatorname{sum}(K_{\text{set}}(S)) &= \operatorname{sum}(x_1;\ x_2;\ ...;\ x_n) = x_1 + x_2 + ... + x_n = \sum_{i\in\{1;\ 2;\ ...;\ n\}} x_i \\
&= \sum_{i\in\{1;\ 2;\ ...;\ n\}} \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{if } i \notin S \end{cases} = \sum_{i\in S} \underbrace{\begin{cases} 1, & \text{if } i \in S \\ 0, & \text{if } i \notin S \end{cases}}_{=1,\text{ since } i\in S} + \sum_{i\in\{1;\ 2;\ ...;\ n\}\setminus S} \underbrace{\begin{cases} 1, & \text{if } i \in S \\ 0, & \text{if } i \notin S \end{cases}}_{\substack{=0,\text{ since} \\ i\in\{1;\ 2;\ ...;\ n\}\setminus S \text{ yields } i\notin S}} \\
&\qquad (\text{since the set } \{1;\ 2;\ ...;\ n\} \text{ is the union of its disjoint subsets } S \text{ and } \{1;\ 2;\ ...;\ n\} \setminus S) \\
&= \underbrace{\sum_{i\in S} 1}_{=|S|\cdot 1=|S|} + \underbrace{\sum_{i\in\{1;\ 2;\ ...;\ n\}\setminus S} 0}_{=0} = |S|.
\end{aligned}$$

as follows: For every multiset S consisting of elements from $\{1;\ 2;\ ...;\ n\}$, let $K_{\text{multiset}}(S)$ be the n-tuple $(x_1;\ x_2;\ ...;\ x_n) \in \mathbb{N}^n$, where x_i is defined as the number of times the element i occurs in our multiset S (particularly, 0 if i doesn't occur in our multiset S at all) for every $i \in \{1;\ 2;\ ...;\ n\}$. Then, this mapping K_{multiset} is a bijection, and $|S| = \operatorname{sum}(K_{\text{multiset}}(S))$ for every multiset S consisting of elements from $\{1;\ 2;\ ...;\ n\}$ [3]. Hence,

$$\begin{aligned}
&\left|\{S \text{ is a multiset consisting of elements from } \{1;\ 2;\ ...;\ n\} \mid |S| = v\}\right| \\
&= \left|\underbrace{\{S \text{ is a multiset consisting of elements from } \{1;\ 2;\ ...;\ n\} \mid \operatorname{sum}(K_{\text{multiset}}(S)) = v\}}_{\substack{=\{S \text{ is a multiset consisting of elements from } \{1;\ 2;\ ...;\ n\} \mid K_{\text{multiset}}(S) \in \{\mathbf{x} \in \mathbb{N}^n \mid \operatorname{sum}\mathbf{x} = v\}\} \\ =(K_{\text{multiset}})^{-1}(\{\mathbf{x} \in \mathbb{N}^n \mid \operatorname{sum}\mathbf{x} = v\})}}\right| \\
&= \left|(K_{\text{multiset}})^{-1}(\{\mathbf{x} \in \mathbb{N}^n \mid \operatorname{sum}\mathbf{x} = v\})\right| \\
&= \left|\{\mathbf{x} \in \mathbb{N}^n \mid \operatorname{sum}\mathbf{x} = v\}\right| \qquad (\text{since } K_{\text{multiset}} \text{ is a bijection}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\binom{n+v-1}{v} &= \left|\{S \text{ is a multiset consisting of elements from } \{1;\ 2;\ ...;\ n\} \mid |S| = v\}\right| \\
&= \left|\{\mathbf{x} \in \mathbb{N}^n \mid \operatorname{sum}\mathbf{x} = v\}\right|,
\end{aligned}$$

and Lemma 2 is proven.

Now we will solve the problem.

Define a mapping $F : \mathbb{N} \times \{0;\ 1\} \to \mathbb{N}$ by $F(q;\ r) = 2q + r$ for every $(q;\ r) \in \mathbb{N} \times \{0;\ 1\}$. Then, F is a bijection (since every $p \in \mathbb{N}$ can be uniquely written in the form $p = 2q + r$ with $(q;\ r) \in \mathbb{N} \times \{0;\ 1\}$ (in fact, q is the quotient and r is the remainder of p upon division by 2)). Thus, the mapping

$$\Phi : \mathbb{N}^n \times \{0;\ 1\}^n \to \mathbb{N}^n$$

defined by

$$\Phi((q_1;\ q_2;\ ...;\ q_n);\ (r_1;\ r_2;\ ...;\ r_n)) = (F(q_1;\ r_1);\ F(q_2;\ r_2);\ ...;\ F(q_n;\ r_n))$$
$$\text{for every } ((q_1;\ q_2;\ ...;\ q_n);\ (r_1;\ r_2;\ ...;\ r_n)) \in \mathbb{N}^n \times \{0;\ 1\}^n$$

[3] In fact, $K_{\text{multiset}}(S)$ is the n-tuple $(x_1;\ x_2;\ ...;\ x_n) \in \mathbb{N}^n$, where

$$x_i = (\text{the number of times the element } i \text{ occurs in our multiset } S)$$

for every $i \in \{1;\ 2;\ ...;\ n\}$, and thus

$$\begin{aligned}
\operatorname{sum}(K_{\text{multiset}}(S)) &= \operatorname{sum}(x_1;\ x_2;\ ...;\ x_n) = x_1 + x_2 + ... + x_n = \sum_{i \in \{1;\ 2;\ ...;\ n\}} x_i \\
&= \sum_{i \in \{1;\ 2;\ ...;\ n\}} (\text{the number of times the element } i \text{ occurs in our multiset } S) \\
&= |S|.
\end{aligned}$$

is a bijection as well.[4] Besides, for every $(\mathbf{q};\ \mathbf{r}) \in \mathbb{N}^n \times \{0;\ 1\}^n$, we have $\operatorname{sum}(\Phi(\mathbf{q};\ \mathbf{r})) = 2\operatorname{sum}\mathbf{q} + \operatorname{sum}\mathbf{r}$. [5]

---

[4] In fact, $\Phi = F^n \circ \Psi$, where the mapping

$$\Psi : \mathbb{N}^n \times \{0;\ 1\}^n \to (\mathbb{N} \times \{0;\ 1\})^n$$

is defined by

$$\Psi((q_1;\ q_2;\ ...;\ q_n);\ (r_1;\ r_2;\ ...;\ r_n)) = ((q_1;\ r_1);\ (q_2;\ r_2);\ ...;\ (q_n;\ r_n))$$
$$\text{for every } ((q_1;\ q_2;\ ...;\ q_n);\ (r_1;\ r_2;\ ...;\ r_n)) \in \mathbb{N}^n \times \{0;\ 1\}^n,$$

and the mapping

$$F^n : (\mathbb{N} \times \{0;\ 1\})^n \to \mathbb{N}^n$$

is defined by

$$F^n((q_1;\ r_1);\ (q_2;\ r_2);\ ...;\ (q_n;\ r_n)) = (F(q_1;\ r_1);\ F(q_2;\ r_2);\ ...;\ F(q_n;\ r_n))$$
$$\text{for every } ((q_1;\ r_1);\ (q_2;\ r_2);\ ...;\ (q_n;\ r_n)) \in (\mathbb{N} \times \{0;\ 1\})^n.$$

Since both F^n and Ψ are bijections (in fact, F^n is a bijection, since F is a bijection), it follows that $\Phi = F^n \circ \Psi$ is a bijection as well (since the composition of two bijections is a bijection).

[5] In fact, we can write $\mathbf{q}$ in the form $\mathbf{q} = (q_1;\ q_2;\ ...;\ q_n)$ (since $\mathbf{q} \in \mathbb{N}^n$), and we can write $\mathbf{r}$ in the form $\mathbf{r} = (r_1;\ r_2;\ ...;\ r_n)$ (since $\mathbf{r} \in \{0;\ 1\}^n$). Then,

$$\begin{aligned}\Phi(\mathbf{q};\ \mathbf{r}) &= \Phi((q_1;\ q_2;\ ...;\ q_n);\ (r_1;\ r_2;\ ...;\ r_n)) = (F(q_1;\ r_1);\ F(q_2;\ r_2);\ ...;\ F(q_n;\ r_n)) \\ &= (2q_1 + r_1;\ 2q_2 + r_2;\ ...;\ 2q_n + r_n)\end{aligned}$$

and thus

$$\begin{aligned}\operatorname{sum}(\Phi(\mathbf{q};\ \mathbf{r})) &= \operatorname{sum}(2q_1 + r_1;\ 2q_2 + r_2;\ ...;\ 2q_n + r_n) \\ &= (2q_1 + r_1) + (2q_2 + r_2) + ... + (2q_n + r_n) \\ &= 2\underbrace{(q_1 + q_2 + ... + q_n)}_{=\operatorname{sum}(q_1;\ q_2;\ ...;\ q_n) = \operatorname{sum}\mathbf{q}} + \underbrace{(r_1 + r_2 + ... + r_n)}_{=\operatorname{sum}(r_1;\ r_2;\ ...;\ r_n) = \operatorname{sum}\mathbf{r}} \\ &= 2\operatorname{sum}\mathbf{q} + \operatorname{sum}\mathbf{r}.\end{aligned}$$

Now,

$$\begin{aligned}
&\binom{n+k-1}{k} \\
&= \left|\left\{\mathbf{x} \in \mathbb{N}^n \mid \operatorname{sum}\mathbf{x} = k\right\}\right| \qquad \text{(by Lemma 2, applied to } v = k\text{)} \\
&= \left| \underbrace{\Phi^{-1}\left(\left\{\mathbf{x} \in \mathbb{N}^n \mid \operatorname{sum}\mathbf{x} = k\right\}\right)}_{\substack{=\{(\mathbf{q};\ \mathbf{r}) \in \mathbb{N}^n \times \{0;\ 1\}^n \mid \Phi(\mathbf{q};\ \mathbf{r}) \in \{\mathbf{x} \in \mathbb{N}^n \mid \operatorname{sum}\mathbf{x} = k\}\} \\ =\{(\mathbf{q};\ \mathbf{r}) \in \mathbb{N}^n \times \{0;\ 1\}^n \mid \operatorname{sum}(\Phi(\mathbf{q};\ \mathbf{r})) = k\}}} \right| \qquad \text{(since } \Phi \text{ is a bijection)} \\
&= \left| \left\{ (\mathbf{q};\ \mathbf{r}) \in \mathbb{N}^n \times \{0;\ 1\}^n \mid \underbrace{\operatorname{sum}\left(\Phi\left(\mathbf{q};\ \mathbf{r}\right)\right)}_{=2\operatorname{sum}\mathbf{q} + \operatorname{sum}\mathbf{r}} = k \right\} \right| \\
&= \left|\left\{(\mathbf{q};\ \mathbf{r}) \in \mathbb{N}^n \times \{0;\ 1\}^n \mid 2\operatorname{sum}\mathbf{q} + \operatorname{sum}\mathbf{r} = k\right\}\right| \\
&= \sum_{u\in\mathbb{N}} \left|\left\{(\mathbf{q};\ \mathbf{r}) \in \mathbb{N}^n \times \{0;\ 1\}^n \mid 2\operatorname{sum}\mathbf{q} + \operatorname{sum}\mathbf{r} = k \text{ and } \operatorname{sum}\mathbf{q} = u\right\}\right| \\
&= \sum_{u\in\mathbb{N}} \left|\left\{(\mathbf{q};\ \mathbf{r}) \in \mathbb{N}^n \times \{0;\ 1\}^n \mid 2u + \operatorname{sum}\mathbf{r} = k \text{ and } \operatorname{sum}\mathbf{q} = u\right\}\right| \\
&\qquad \left(\begin{array}{c} \text{because the assertions } (2\operatorname{sum}\mathbf{q} + \operatorname{sum}\mathbf{r} = k \text{ and } \operatorname{sum}\mathbf{q} = u) \text{ and} \\ (2u + \operatorname{sum}\mathbf{r} = k \text{ and } \operatorname{sum}\mathbf{q} = u) \text{ are equivalent, since if } \operatorname{sum}\mathbf{q} = u, \\ \text{then the assertions } (2\operatorname{sum}\mathbf{q} + \operatorname{sum}\mathbf{r} = k) \text{ and } (2u + \operatorname{sum}\mathbf{r} = k) \text{ are equivalent} \end{array}\right) \\
&= \sum_{u\in\mathbb{N}} \left| \underbrace{\left\{(\mathbf{q};\ \mathbf{r}) \in \mathbb{N}^n \times \{0;\ 1\}^n \mid \operatorname{sum}\mathbf{r} = k - 2u \text{ and } \operatorname{sum}\mathbf{q} = u\right\}}_{=\{\mathbf{q} \in \mathbb{N}^n \mid \operatorname{sum}\mathbf{q} = u\} \times \{\mathbf{r} \in \{0;\ 1\}^n \mid \operatorname{sum}\mathbf{r} = k-2u\}} \right| \\
&\qquad \text{(since } 2u + \operatorname{sum}\mathbf{r} = k \text{ is equivalent to } \operatorname{sum}\mathbf{r} = k - 2u\text{)} \\
&= \sum_{u\in\mathbb{N}} \underbrace{\left|\left\{\mathbf{q} \in \mathbb{N}^n \mid \operatorname{sum}\mathbf{q} = u\right\} \times \left\{\mathbf{r} \in \{0;\ 1\}^n \mid \operatorname{sum}\mathbf{r} = k - 2u\right\}\right|}_{=|\{\mathbf{q} \in \mathbb{N}^n \mid \operatorname{sum}\mathbf{q} = u\}| \cdot |\{\mathbf{r} \in \{0;\ 1\}^n \mid \operatorname{sum}\mathbf{r} = k-2u\}|} \\
&= \sum_{u\in\mathbb{N}} \underbrace{\left|\left\{\mathbf{q} \in \mathbb{N}^n \mid \operatorname{sum}\mathbf{q} = u\right\}\right|}_{\substack{=|\{\mathbf{x} \in \mathbb{N}^n \mid \operatorname{sum}\mathbf{x} = u\}| = \binom{n+u-1}{u} \\ \text{by Lemma 2 (applied to } v=u\text{)}}} \cdot \underbrace{\left|\left\{\mathbf{r} \in \{0;\ 1\}^n \mid \operatorname{sum}\mathbf{r} = k - 2u\right\}\right|}_{\substack{=|\{\mathbf{x} \in \{0;\ 1\}^n \mid \operatorname{sum}\mathbf{x} = k-2u\}| = \binom{n}{k-2u} \\ \text{by Lemma 1 (applied to } v=k-2u\text{)}}} \\
&= \sum_{u\in\mathbb{N}} \binom{n+u-1}{u} \cdot \binom{n}{k-2u} = \underbrace{\sum_{u\in\{0,1,\ldots,k\}}}_{=\sum_{u=0}^{k}} \binom{n+u-1}{u} \cdot \binom{n}{k-2u} \\
&\qquad \left(\begin{array}{c} \text{here we replaced the } \sum\limits_{u\in\mathbb{N}} \text{ sign by a } \sum\limits_{u\in\{0,1,\ldots,k\}} \text{ sign, since all addends for} \\ u \in \mathbb{N} \setminus \{0,1,\ldots,k\} \text{ are zero (because if } u \in \mathbb{N} \setminus \{0,1,\ldots,k\}\text{, then } u > k\text{, so that} \\ 2u > 2k \geq k\text{, thus } k - 2u < 0 \text{ and thus } \binom{n}{k-2u} = 0\text{)} \end{array}\right) \\
&= \sum_{u=0}^{k} \binom{n+u-1}{u} \cdot \binom{n}{k-2u}.
\end{aligned}$$

Thus, we see that the problem is solved.