**Isogonal conjugation with respect to a triangle**
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## 1. Trivialities on isogonal lines

The aim of this note is to provide a rather detailled and general introduction into isogonal conjugation with respect to a triangle. No new results of the author will be presented, but some results published elsewhere will be proven in a (presumably) new way.

Throughout this work, we will make use of directed angles modulo 180°. This is the kind of angles referred to as "directed angles" in [1], 1.7, and we refer the reader to [1] for their basic properties. (Also, [2], [3] and [4] provide introductions to this type of angles.)

Two preliminary conventions are to be made at first:

- "Wrt" is an abbreviation for "with respect to".
- The $A$**-altitude** of a triangle $ABC$ will mean the altitude of triangle $ABC$ issuing from its vertex $A$. Similarly, the $A$**-median** of triangle $ABC$ will mean the median of triangle $ABC$ issuing from its vertex $A$.

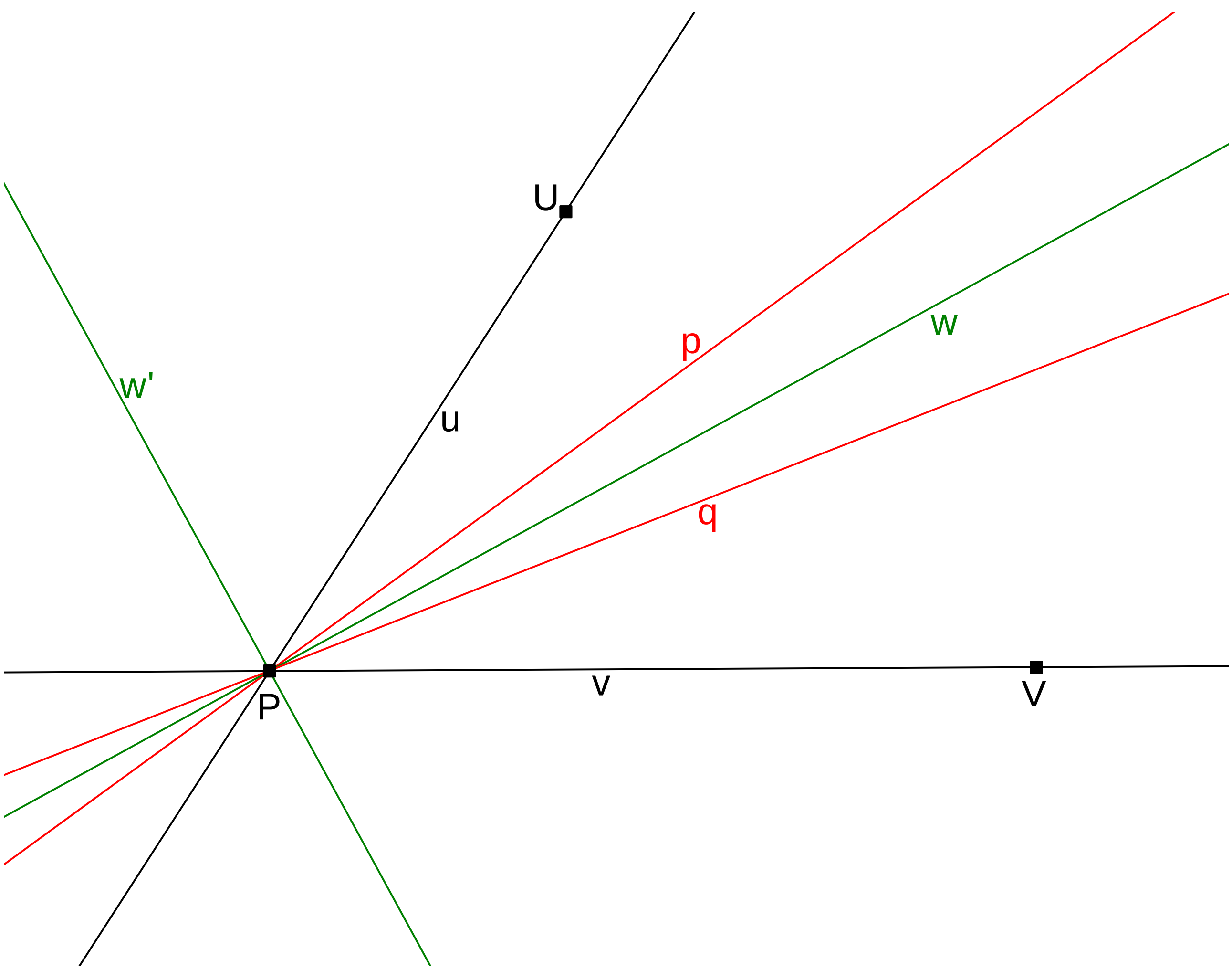

Fig. 1

We start with a simple property of lines (Fig. 1):

**Theorem 1.** Let $u$ and $v$ be two lines which intersect at an Euclidean (i. e. not infinite) point $P$. Let $p$ and $q$ be two more lines through the point $P$. Let $w$ and $w'$ be the two angle bisectors of the angles formed by the lines $u$ and $v$. Then, the following four assertions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ are pairwisely equivalent:

**Assertion $\mathcal{A}_1$:** We have $\measuredangle(u;\ q) = -\measuredangle(v;\ p)$.

**Assertion $\mathcal{A}_2$:** We have $\measuredangle(v;\ q) = -\measuredangle(u;\ p)$.

**Assertion $\mathcal{A}_3$:** The lines $p$ and $q$ are symmetric to each other wrt the line $w$.

**Assertion $\mathcal{A}_4$:** The lines $p$ and $q$ are symmetric to each other wrt the line $w'$.

*Proof of Theorem 1.* This proof is almost trivial; we are giving it here only for the sake of completeness.

First, if assertion $\mathcal{A}_1$ holds, i. e. if $\measuredangle(u;\ q) = -\measuredangle(v;\ p)$, then

$$\measuredangle(v;\ q) = \measuredangle(v;\ u)+\measuredangle(u;\ q) = \measuredangle(v;\ u)+(-\measuredangle(v;\ p)) = -(\measuredangle(v;\ p) - \measuredangle(v;\ u)) = -\measuredangle(u;\ p),$$

so assertion $\mathcal{A}_2$ also holds. Similarly, we show the converse: if assertion $\mathcal{A}_2$ holds, then assertion $\mathcal{A}_1$ also holds. Thus, the assertions $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent.

Now we will prove the equivalence of assertions $\mathcal{A}_1$ and $\mathcal{A}_3$: Since the line $w$ is an angle bisector of one of the angles between the lines $u$ and $v$, we have $\measuredangle(u;\ w) = -\measuredangle(v;\ w)$. Now, assertion $\mathcal{A}_1$ states that $\measuredangle(u;\ q) = -\measuredangle(v;\ p)$; this rewrites as $\measuredangle(u;\ w)+\measuredangle(w;\ q) = -(\measuredangle(v;\ w) + \measuredangle(w;\ p))$, what, in view of $\measuredangle(u;\ w) = -\measuredangle(v;\ w)$, simplifies to $\measuredangle(w;\ q) = -\measuredangle(w;\ p)$. But since the lines $w$, $p$, $q$ all pass through the point $P$, this equation is equivalent to stating that the lines $p$ and $q$ are symmetric to each other wrt the line $w$. This, however, is assertion $\mathcal{A}_3$. Thus we have shown that the assertions $\mathcal{A}_1$ and $\mathcal{A}_3$ are equivalent; similarly we can prove the assertions $\mathcal{A}_1$ and $\mathcal{A}_4$ to be equivalent. Hence, the equivalence of all four assertions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ is established, and Theorem 1 is proven.

Based on Theorem 1 we make a definition: If one of the four assertions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ holds - and therefore, according to Theorem 1, the three others hold as well -, then we say that the line $q$ is **isogonal** to the line $p$ wrt the lines $u$ and $v$.

For two lines $p$ and $q$ through the point $P$, we can easily see that the line $q$ is isogonal to the line $p$ wrt the lines $u$ and $v$ if and only if the line $p$ is isogonal to the line $q$ wrt the lines $u$ and $v$. (In fact, the line $q$ is isogonal to the line $p$ wrt the lines $u$ and $v$ if and only if $\measuredangle(u;\ q) = -\measuredangle(v;\ p)$; this is equivalent to $\measuredangle(v;\ p) = -\measuredangle(u;\ q)$, and this holds if and only if the line $p$ is isogonal to the line $q$ wrt the lines $u$ and $v$.) Hence, the assertions "the line $q$ is isogonal to the line $p$ wrt the lines $u$ and $v$ " and "the line $p$ is isogonal to the line $q$ wrt the lines $u$ and $v$ " are equivalent; thus, instead of any of these assertions, we can simply say that "the lines $p$ and $q$ are **isogonal to each other** wrt the lines $u$ and $v$ ".

Instead of saying "isogonal wrt the lines $u$ and $v$ ", we will often say "isogonal wrt the angle $UPV$ ", where $U$ is a point on the line $u$ (distinct from $P$) and $V$ is a point on the line $v$ (distinct from $P$).

For each line $p$ through the point $P$, there exists one and only one line $q$ through the point $P$ which is isogonal to the line $p$ wrt the lines $u$ and $v$; in fact, this line $q$ must satisfy the equation $\measuredangle(u;\ q) = -\measuredangle(v;\ p)$, and this holds for one and only one line $q$ through the point $P$ (this line can be constructed as a line through a given point which forms a given angle with another given line). This line $q$ which is isogonal to the line $p$ wrt the lines $u$ and $v$ is called the **isogonal** (or **isogonal line**) of the line $p$ wrt the lines $u$ and $v$, or, equivalently, the **isogonal** (or **isogonal line**) of the line $p$ wrt the angle $UPV$.

## 2. Isogonals and perpendicular bisectors

The next properties of isogonals we are going to show are not much harder to prove, but turn out to be of remarkable usefulness:

**Theorem 2.** Let $u$ and $v$ be two lines intersecting at an Euclidean point $P$. Let $T$ be a point in the plane.

**a)** Let $X$ and $Y$ be the orthogonal projections of the point $T$ on the lines $u$ and $v$. Then, the line $XY$ is perpendicular to the isogonal of the line $PT$ wrt the lines $u$ and $v$. (See Fig. 2.)

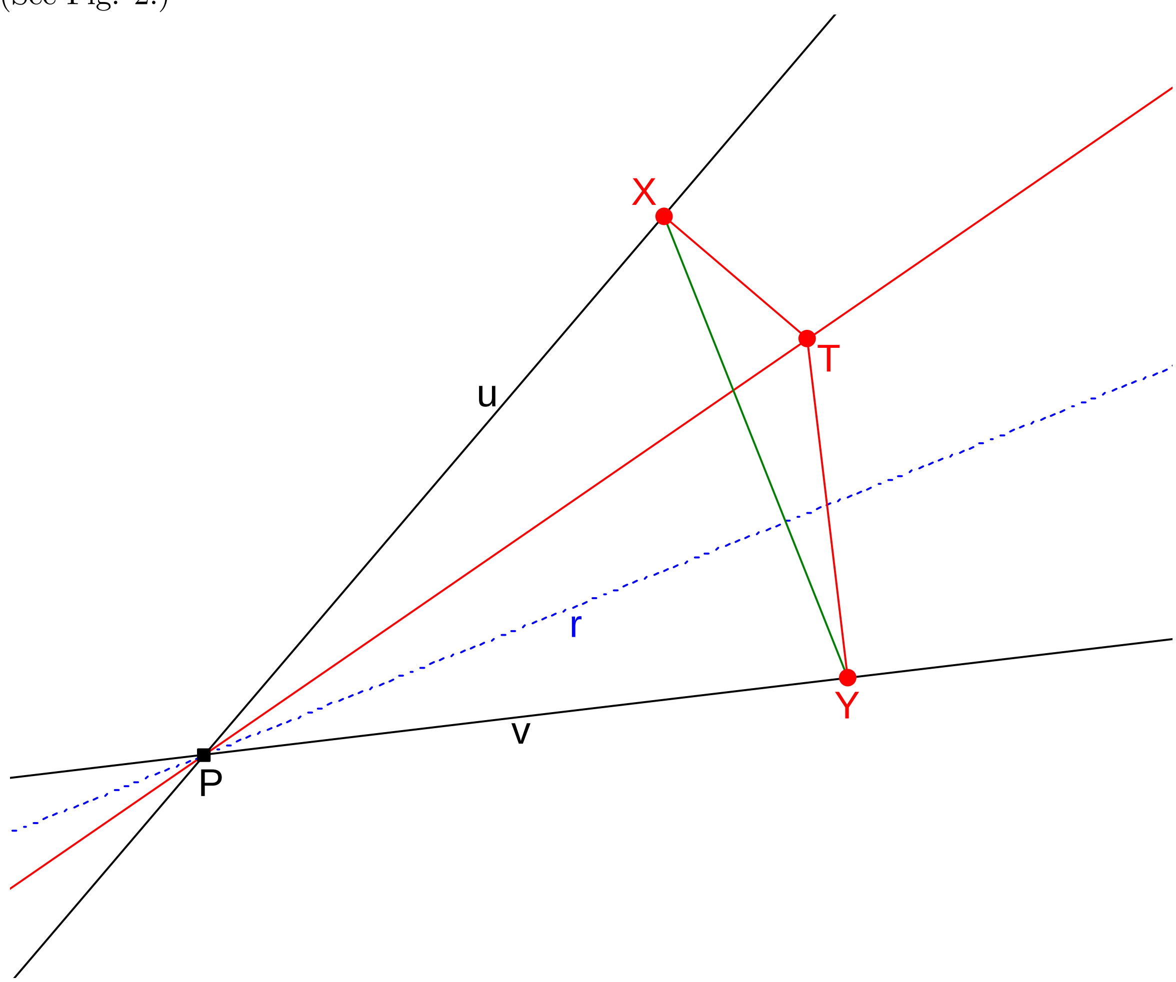


Fig. 2

**b)** Let $X'$ and $Y'$ be the reflections of the point $T$ in the lines $u$ and $v$. Then, the perpendicular bisector of the segment $X'Y'$ is the isogonal of the line $PT$ wrt the lines $u$ and $v$. (See Fig. 3.)

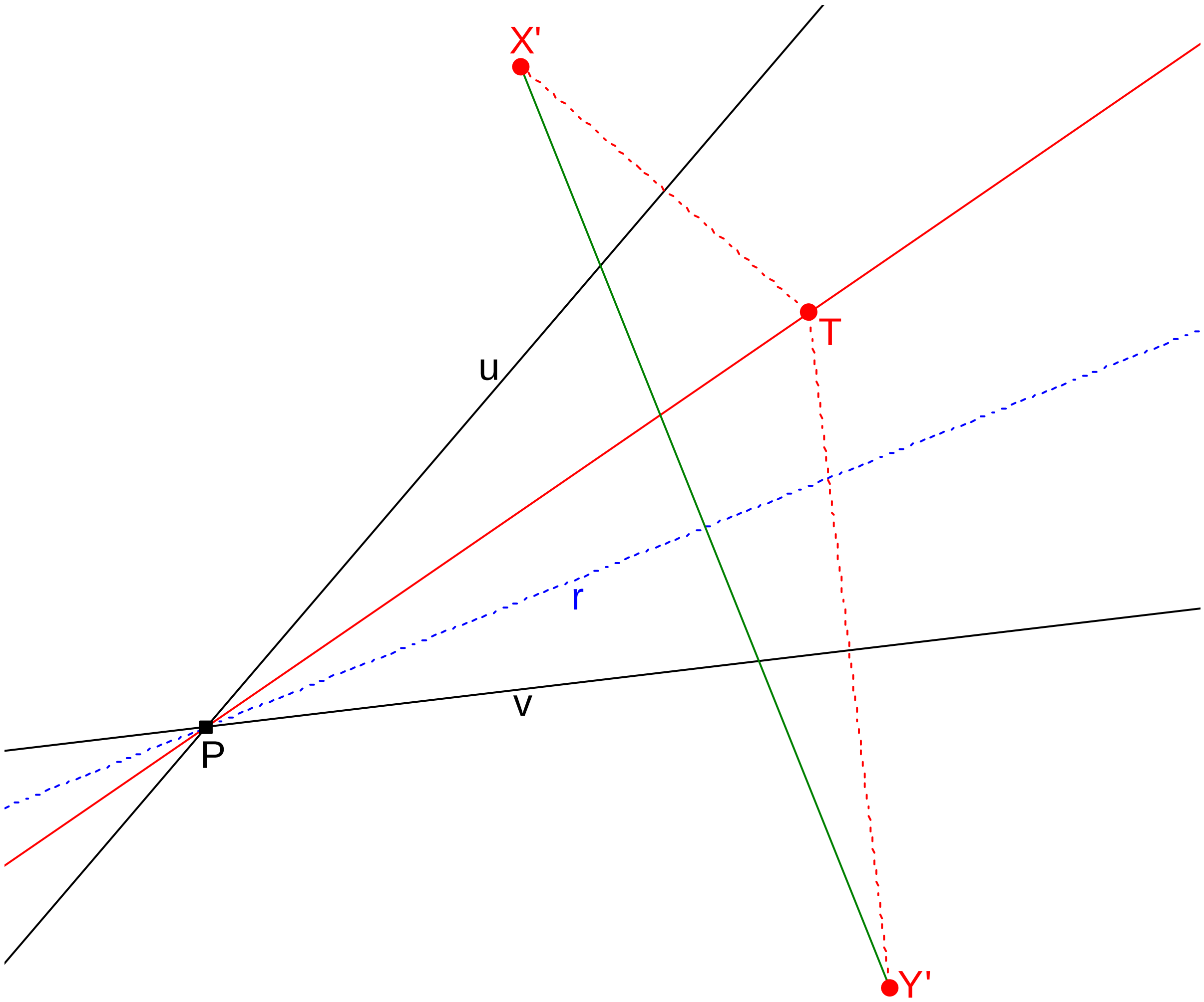


Fig. 3

*Proof of Theorem 2.* Let $r$ be the isogonal of the line $PT$ wrt the lines $u$ and $v$. Then, $\measuredangle(u;\ r) = -\measuredangle(v;\ PT)$ and $\measuredangle(v;\ r) = -\measuredangle(u;\ PT)$.

**a)** (See Fig. 4.) Since $\measuredangle PXT = 90^\circ$ and $\measuredangle PYT = 90^\circ$, the points $X$ and $Y$ lie on the circle with diameter $PT$. Thus, $\measuredangle YXP = \measuredangle YTP$, so that $\measuredangle(XY;\ u) = \measuredangle(TY;\ PT)$. But $TY \perp v$ yields $\measuredangle(TY;\ v) = 90^\circ$, so that $\measuredangle(XY;\ u) = \measuredangle(TY;\ PT) = \measuredangle(TY;\ v) + \measuredangle(v;\ PT) = 90^\circ + \measuredangle(v;\ PT)$.

Hence, $\measuredangle(XY;\ r) = \measuredangle(XY;\ u) + \measuredangle(u;\ r) = (90^\circ + \measuredangle(v;\ PT)) + (-\measuredangle(v;\ PT)) = 90^\circ$. Thus, the line $XY$ is perpendicular to the line $r$, that is, to the isogonal of the line $PT$ wrt the lines $u$ and $v$. This proves Theorem 2 **a)**.

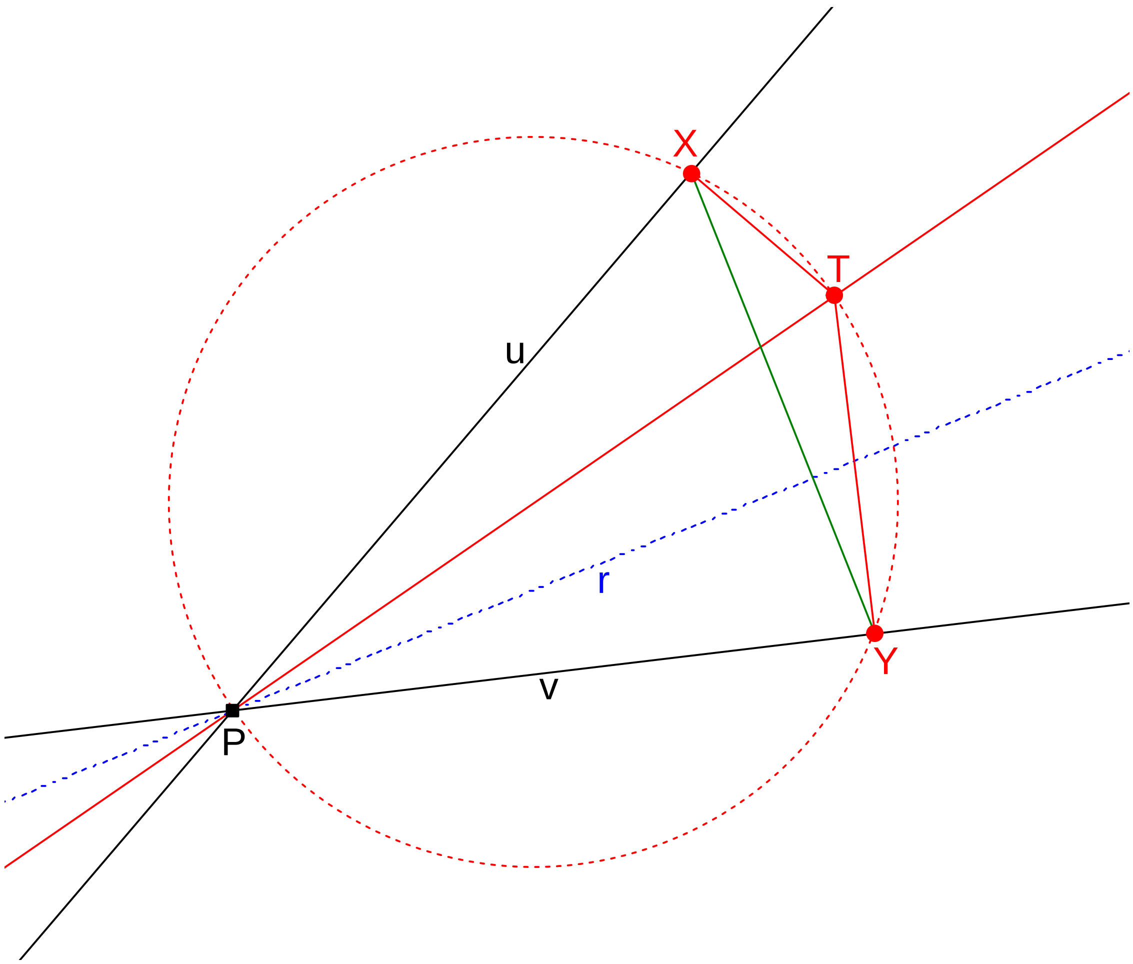


Fig. 4

**b)** (See Fig. 5.) Since $X'$ is the reflection of the point $T$ in the line $u$, we have $PX' = PT$; similarly, $PY' = PT$. Thus, $PX' = PT = PY'$, what entails that the point $P$ is the center of the circle through the points $X'$, $T$, $Y'$. Thus, on the one hand, the central angle theorem for directed angles modulo $180^\circ$ yields $\measuredangle PY'X' = 90^\circ - \measuredangle X'TY'$. On the other hand, $PX' = PY'$ implies that $P$ lies on the perpendicular bisector of the segment $X'Y'$.

Since $Y'$ is the reflection of the point $T$ in the line $v$, we get $\measuredangle(PY';\ v) = -\measuredangle(PT;\ v)$ and $TY' \perp v$; the latter yields $\measuredangle(v;\ TY') = 90^\circ$. Similarly, $\measuredangle(u;\ TX') = 90^\circ$. Therefore,

$$\begin{aligned}\measuredangle(PY';\ X'Y') &= \measuredangle PY'X' = 90^\circ - \measuredangle X'TY' = 90^\circ - \measuredangle(TX';\ TY') = \measuredangle(v;\ TY') - \measuredangle(TX';\ TY') \\ &= \measuredangle(v;\ TX') = \measuredangle(v;\ u) + \measuredangle(u;\ TX') = \measuredangle(v;\ u) + 90^\circ.\end{aligned}$$

Consequently,

$$\begin{aligned}\measuredangle(v;\ X'Y') &= \measuredangle(PY';\ X'Y') - \measuredangle(PY';\ v) = (\measuredangle(v;\ u) + 90^\circ) - (-\measuredangle(PT;\ v)) \\ &= (\measuredangle(PT;\ v) + \measuredangle(v;\ u)) + 90^\circ = \measuredangle(PT;\ u) + 90^\circ,\end{aligned}$$

so that

$$\begin{aligned}\measuredangle(r;\ X'Y') &= \measuredangle(v;\ X'Y') - \measuredangle(v;\ r) = (\measuredangle(PT;\ u) + 90^\circ) - (-\measuredangle(u;\ PT)) \\ &= (\measuredangle(PT;\ u) + 90^\circ) - \measuredangle(PT;\ u) = 90^\circ.\end{aligned}$$

This means that the line $r$ is perpendicular to the line $X'Y'$. Now, the perpendicular bisector of the segment $X'Y'$ is also perpendicular to the line $X'Y'$. Hence, the perpendicular bisector of the segment $X'Y'$ is parallel to the line $r$. But since the perpendicular bisector of the segment $X'Y'$ and the line $r$ have a common point (namely $P$), they can only be parallel if they coincide. Hence we see that the perpendicular bisector of the segment $X'Y'$ coincides with the line $r$, that is, with the isogonal of the line $PT$ wrt the lines $u$ and $v$. This proves Theorem 2 **b)** and thus completes the proof of Theorem 2.

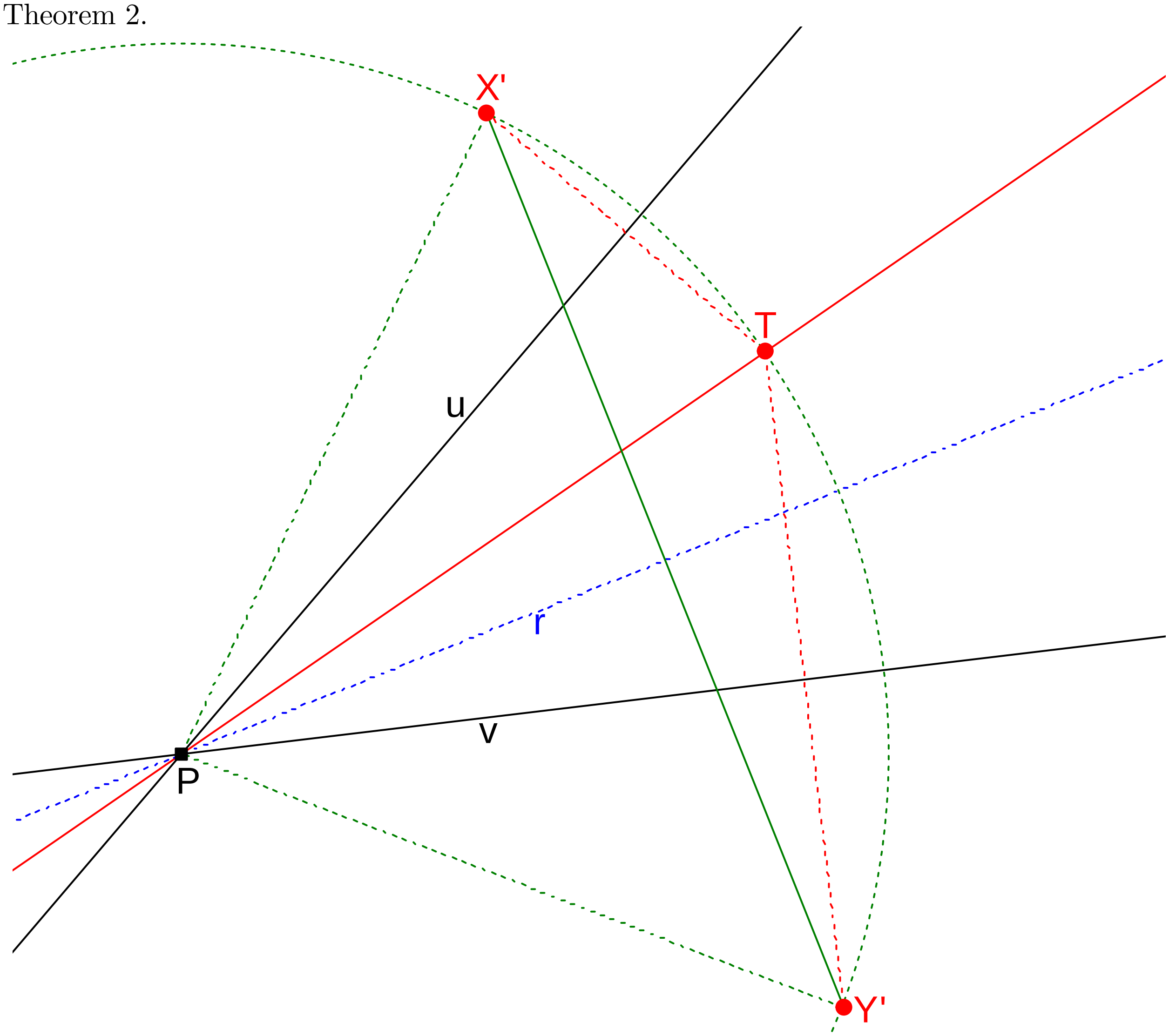


Fig. 5

*Remark.* (See Fig. 6.) As $X$ is the orthogonal projection of the point $T$ on the line $u$, whereas $X'$ is the reflection of $T$ in $u$, we see that $X$ is the midpoint of the segment $TX'$. Similarly, $Y$ is the midpoint of the segment $TY'$. Thus, the line $XY$ is a midparallel in triangle $X'TY'$ and thus parallel to its side $X'Y'$. Hence, the assertion that $X'Y' \perp r$ (the crucial assertion in the proof of Theorem 2 **b)**) is equivalent to the assertion that $XY \perp r$ (this is the assertion of Theorem 2 **a)**). This shows that Theorems 2 **a)** and 2 **b)** can be derived from each other.

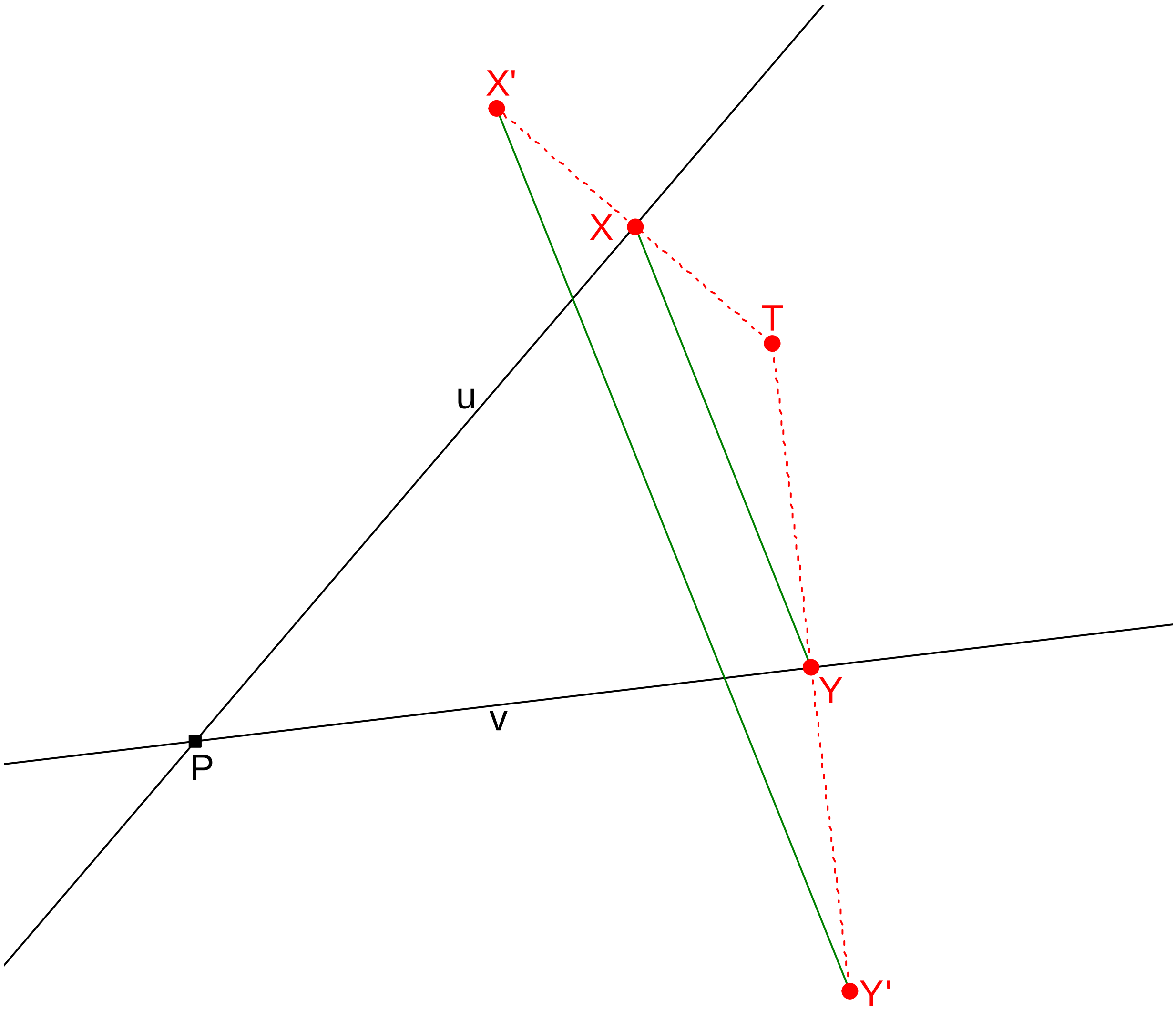


Fig. 6

Note that Theorem 2 will not only turn out useful to us in our study of isogonal conjugates, but it can also be applied to olympiad problems like the IMO 2004 problem 5 ([8], post #2).

## 3. Isogonal conjugation wrt triangles

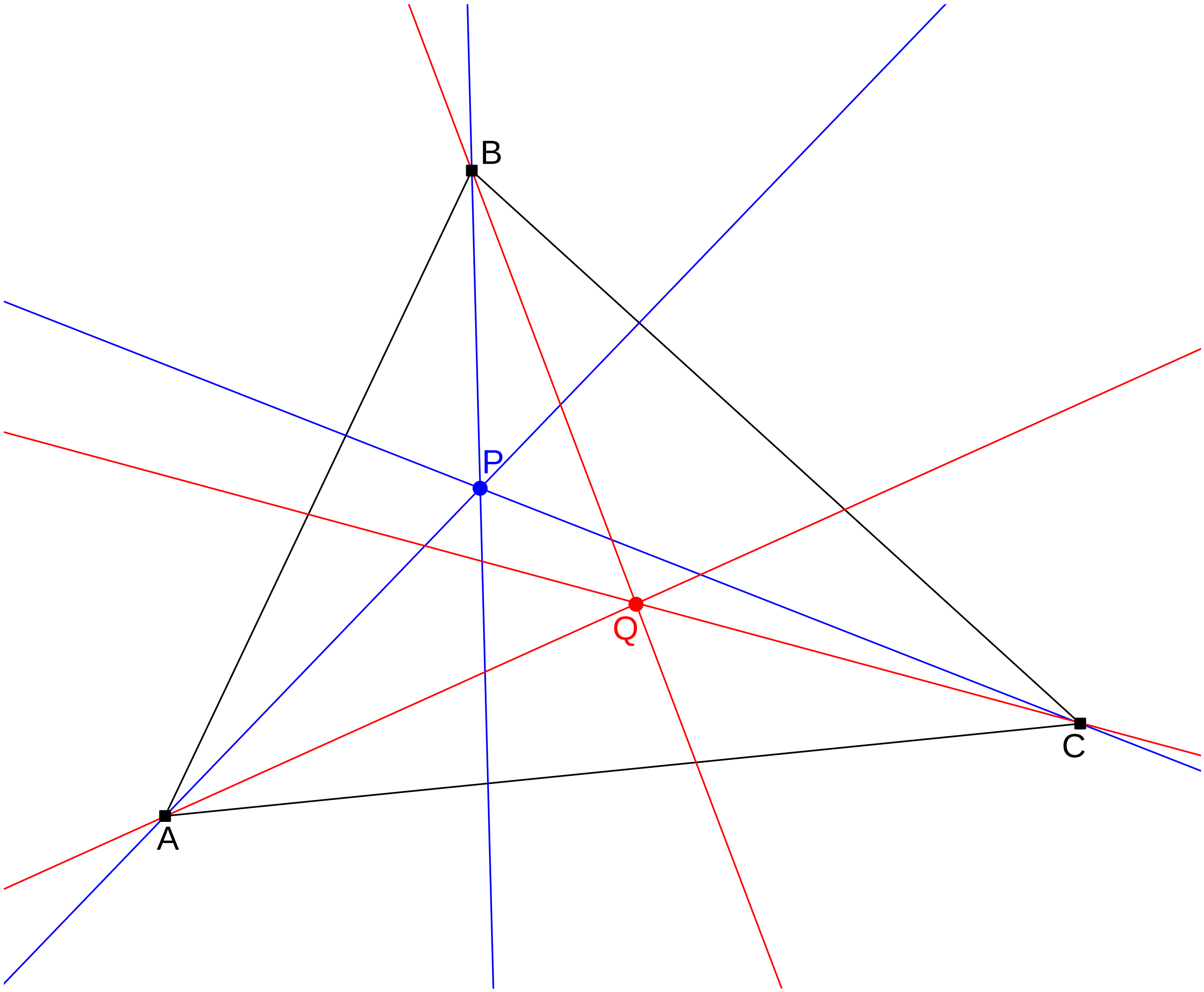


Fig. 7

Now we show the first serious result on isogonals, the **isogonal conjugate theorem**:

**Theorem 3.** Let $ABC$ be a triangle and $P$ a point (distinct from the vertices $A$, $B$, $C$). Then, the isogonals of the lines $AP$, $BP$, $CP$ wrt the angles $CAB$, $ABC$, $BCA$ concur at one point $Q$. (See Fig. 7.)

It has to be noticed that this theorem is formulated for the projective plane - this means that the point $P$ can be an Euclidean or an infinite point, and that the isogonals of the lines $AP$, $BP$, $CP$ wrt the angles $CAB$, $ABC$, $BCA$ can concur at an Euclidean or at an infinite point as well.

*Proof of Theorem 3.*[1] We distinguish between two cases:

**Case 1:** The point $P$ is an Euclidean (i. e. not infinite) point.

(See Fig. 8.) Let $X'$, $Y'$, $Z'$ be the reflections of the point $P$ in the lines $BC$, $CA$, $AB$.

[1] The following proof is basically the proof given in [9], apart from the difference that [9] doesn't use directed angles.

Since $Y'$ and $Z'$ are the reflections of the point $P$ in the lines $CA$ and $AB$, according to Theorem 2 **b)** it follows that the perpendicular bisector of the segment $Y'Z'$ is the isogonal of the line $AP$ wrt the lines $CA$ and $AB$, that is, the isogonal of the line $AP$ wrt the angle $CAB$. Similarly, the perpendicular bisectors of the segments $Z'X'$ and $X'Y'$ are the isogonals of the lines $BP$ and $CP$ wrt the angles $ABC$ and $BCA$. Thus, the isogonals of the lines $AP$, $BP$, $CP$ wrt the angles $CAB$, $ABC$, $BCA$ are the perpendicular bisectors of the segments $Y'Z'$, $Z'X'$, $X'Y'$ and thus concur at one point - at the circumcenter of triangle $X'Y'Z'$. This proves Theorem 3 in Case 1.

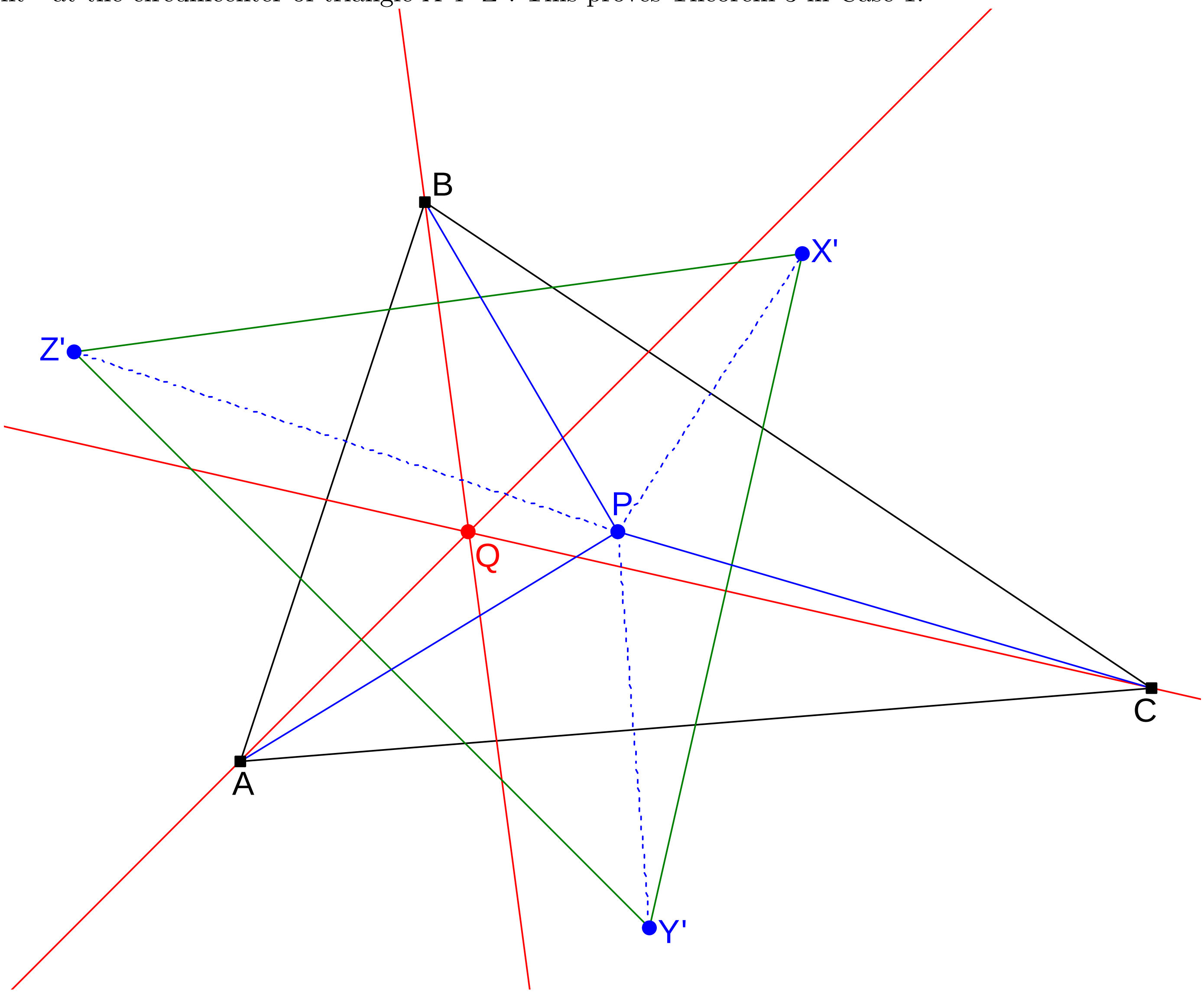


Fig. 8

**Case 2:** The point $P$ is an infinite point. Then, the lines $AP$, $BP$, $CP$ are parallel. (See Fig. 9.) Let the isogonal of the line $AP$ wrt the angle $CAB$ meet the circumcircle of triangle $ABC$ at a point $Q_1$ (apart from $A$). Since the lines $AP$ and $AQ_1$ are isogonal to each other wrt the angle $CAB$, we have $\measuredangle (CA;\ AQ_1) = -\measuredangle (AB;\ AP)$. Since $Q_1$ lies on the circumcircle of triangle $ABC$, we have $\measuredangle AQ_1C = \measuredangle ABC$, thus

$\measuredangle (AQ_1;\ CQ_1) = \measuredangle (AB;\ BC)$. Hence,

$$\begin{aligned}\measuredangle (CA;\ CQ_1) &= \measuredangle (CA;\ AQ_1) + \measuredangle (AQ_1;\ CQ_1) = -\measuredangle (AB;\ AP) + \measuredangle (AB;\ BC) \\ &= -(\measuredangle (AB;\ AP) - \measuredangle (AB;\ BC)) = -\measuredangle (BC;\ AP).\end{aligned}$$

But $AP \parallel CP$ yields $\measuredangle (BC;\ AP) = \measuredangle (BC;\ CP)$, and thus we obtain $\measuredangle (CA;\ CQ_1) = -\measuredangle (BC;\ CP)$. This means that the line $CQ_1$ is the isogonal of the line $CP$ wrt the angle $BCA$. In other words, the point $Q_1$ lies on the isogonal of the line $CP$ wrt the angle $BCA$. Similarly, the point $Q_1$ lies on the isogonal of the line $BP$ wrt the angle $ABC$. We already know that the point $Q_1$ lies on the isogonal of the line $AP$ wrt the angle $CAB$. Thus, the isogonals of the lines $AP$, $BP$, $CP$ wrt the angles $CAB$, $ABC$, $BCA$ concur at one point - namely, at the point $Q_1$. Herewith we have not only proved Theorem 3 in Case 2, but we have also shown that in this case - i. e. in the case when the point $P$ is infinite -, the point of intersection of the isogonals of the lines $AP$, $BP$, $CP$ wrt the angles $CAB$, $ABC$, $BCA$ lies on the circumcircle of triangle $ABC$ (in fact, this point of intersection is our point $Q_1$ and lies, by its definition, on the circumcircle of triangle $ABC$).

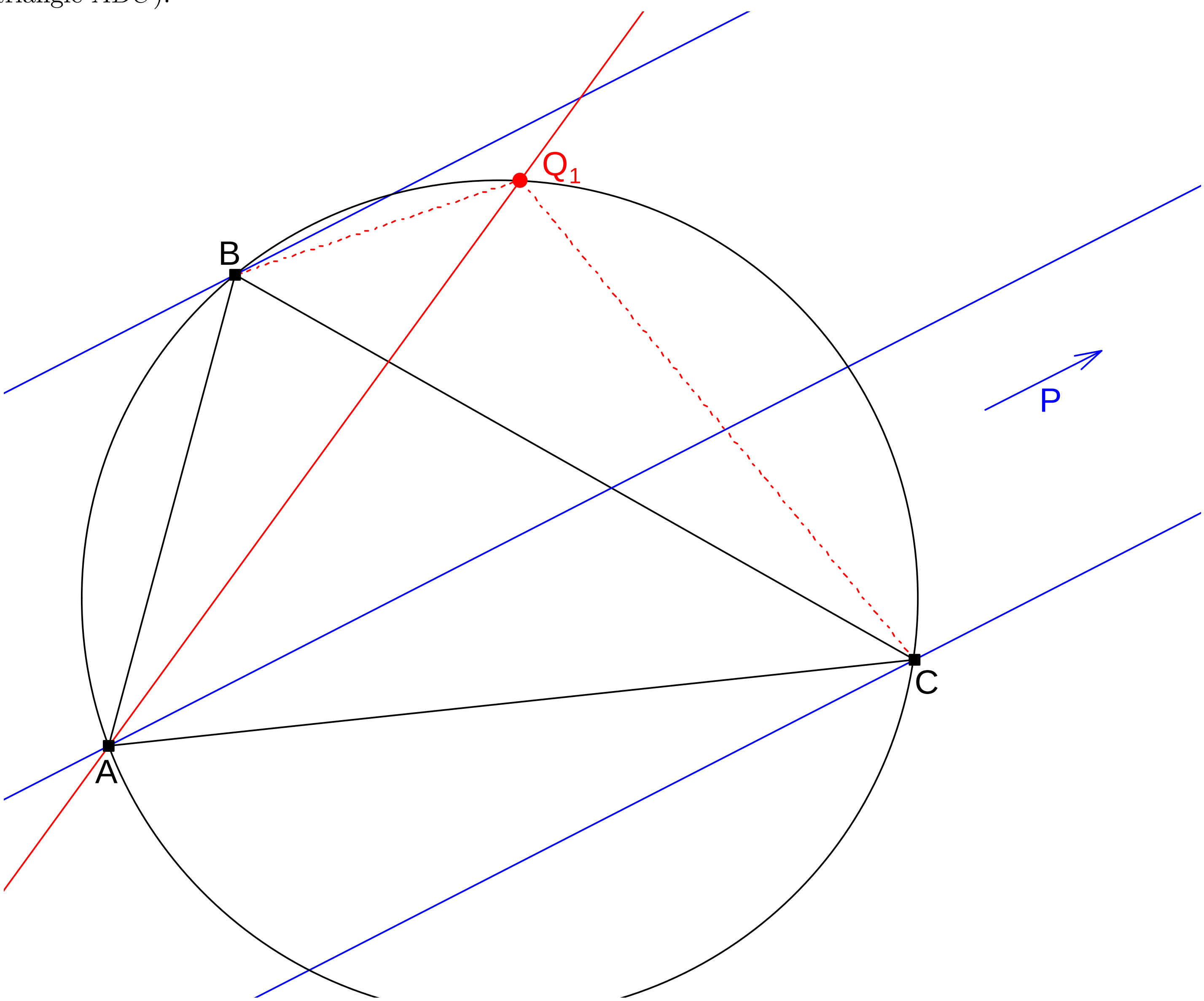

Fig. 9

Now Theorem 3 is completely proven. Based on Theorem 3, we introduce a notion:

The point of intersection of the isogonals of the lines $AP$, $BP$, $CP$ wrt the angles $CAB$, $ABC$, $BCA$ is called the **isogonal conjugate** of the point $P$ wrt the triangle $ABC$.

If $Q$ is the isogonal conjugate of a point $P$ wrt the triangle $ABC$, then the line $AQ$ is the isogonal of the line $AP$ wrt the angle $CAB$. This means that the lines $AP$ and $AQ$ are isogonal to each other wrt the angle $CAB$. This, in turn, shows that the line $AP$ is the isogonal of the line $AQ$ wrt the angle $CAB$. Similarly, the lines $BP$ and $CP$ are the isogonals of the lines $BQ$ and $CQ$ wrt the angles $ABC$ and $BCA$. Now, the point $P$ is the point of intersection of the lines $AP$, $BP$, $CP$, so it therefore must be the point of intersection of the isogonals of the lines $AQ$, $BQ$, $CQ$ wrt the angles $CAB$, $ABC$, $BCA$. Hence, the point $P$ is the isogonal conjugate of the point $Q$ wrt the triangle $ABC$ (as long as the point $P$ doesn't lie on any of the lines $BC$, $CA$, $AB$, since otherwise, the point $Q$ is one of the vertices $A$, $B$, $C$ of triangle $ABC$, and thus the isogonal conjugate of $Q$ is not defined).

Thus we have shown that if $Q$ is the isogonal conjugate of a point $P$ wrt a triangle $ABC$, then, in turn, the point $P$ is the isogonal conjugate of the point $Q$ wrt the triangle $ABC$ (as long as the point $P$ doesn't lie on any of the lines $BC$, $CA$, $AB$). Thus, instead of saying that "the point $Q$ is the isogonal conjugate of the point $P$ wrt the triangle $ABC$ " or that "the point $P$ is the isogonal conjugate of the point $Q$ wrt the triangle $ABC$ ", we can say that "the points $P$ and $Q$ are **isogonally conjugate points** wrt the triangle $ABC$ ".

Some first properties of isogonal conjugates can be obtained by harvesting our proof of Theorem 3. We start with the following fact:

**Theorem 4.** Let $P$ and $Q$ be two isogonally conjugate points wrt a triangle $ABC$. Then, the point $P$ is an infinite point if and only if the point $Q$ lies on the circumcircle of triangle $ABC$. (See Fig. 10.)

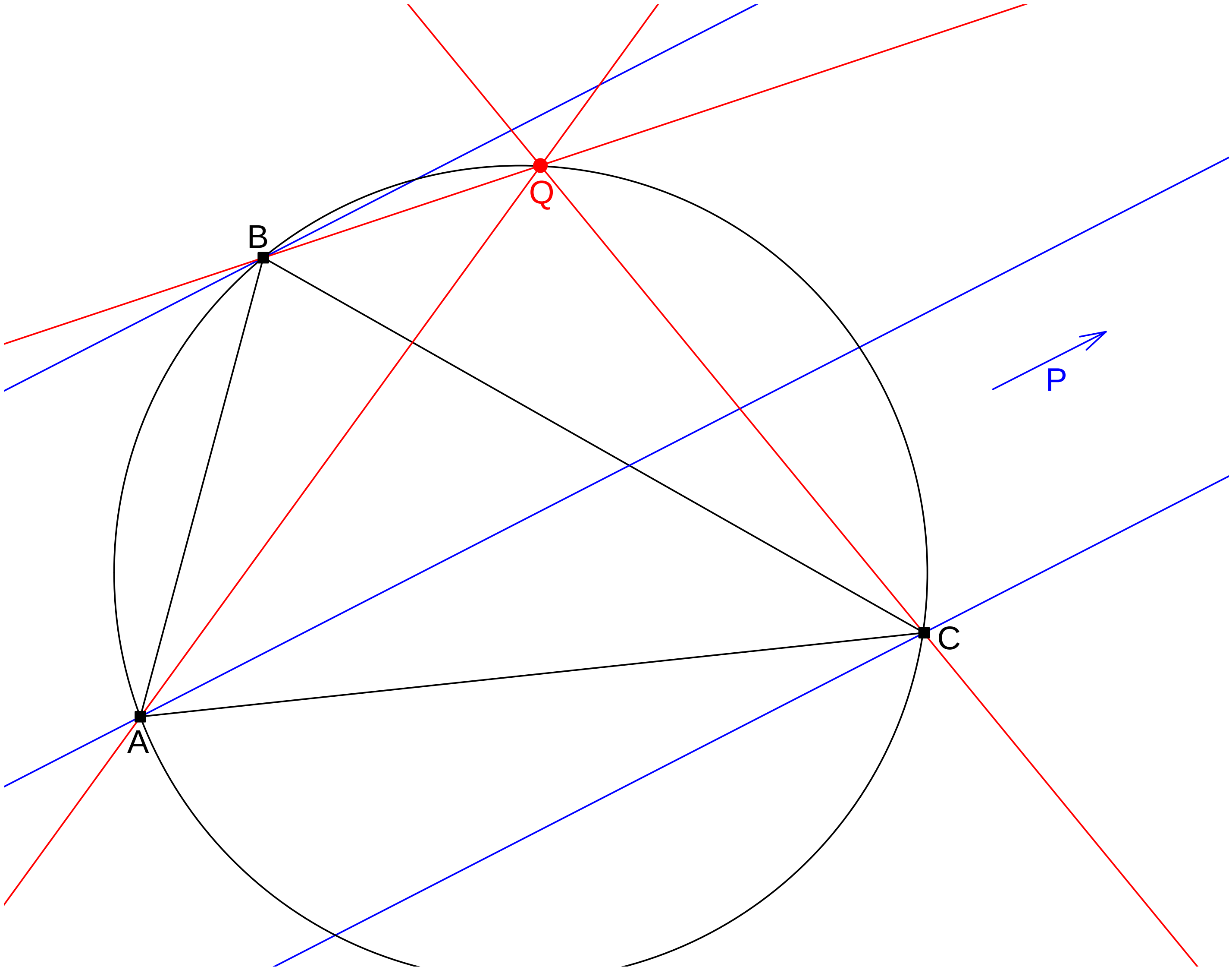


Fig. 10

*Proof of Theorem 4.* In order to show Theorem 4, we have to prove two assertions:

*Assertion 1:* If the point $P$ is an infinite point, then the point $Q$ lies on the circumcircle of triangle $ABC$.

*Assertion 2:* If the point $Q$ lies on the circumcircle of triangle $ABC$, then the point $P$ is an infinite point.

*Proof of Assertion 1.* The point $Q$ is the isogonal conjugate of $P$ wrt triangle $ABC$, that is, the point of intersection of the isogonals of the lines $AP$, $BP$, $CP$ wrt the angles $CAB$, $ABC$, $BCA$. But since the point $P$ is an infinite point, this point of intersection must lie on the circumcircle of triangle $ABC$, as we saw in the proof of Theorem 3 in Case 2. Thus, $Q$ lies on the circumcircle of triangle $ABC$, and Assertion 1 is proven.

*First proof of Assertion 2.* Let $P_1$ be the point of intersection of the line $AP$ with the line at infinity. Let $Q_1$ be the isogonal conjugate of this point $P_1$ wrt triangle $ABC$. Then, the line $AQ_1$ is the isogonal of the line $AP_1$ wrt the angle $CAB$. But since the point $Q$ is the isogonal conjugate of the point $P$ wrt triangle $ABC$, the line $AQ$ is the isogonal of the line $AP$ wrt the angle $CAB$. Since the lines $AP_1$ and $AP$ coincide, their isogonals wrt the angle $CAB$ must also coincide; i. e., the lines $AQ_1$ and $AQ$ coincide. This means that the point $Q_1$ lies on the line $AQ$. Since $P_1$ is an infinite point, the already established Assertion 1 shows that its isogonal conjugate $Q_1$ lies on the circumcircle of triangle $ABC$. Thus, the point $Q_1$ is the point of intersection of the line $AQ$ with the circumcircle of triangle $ABC$ (different from $A$). But since the point $Q$ lies on the circumcircle of triangle $ABC$, the point $Q$ itself is the point of intersection of the line $AQ$ with the circumcircle of triangle $ABC$ (different from $A$). Hence, the points $Q_1$ and $Q$ coincide. Thus, the isogonal conjugates of these points $Q_1$ and $Q$ wrt triangle $ABC$ must also coincide; but the isogonal conjugate of $Q_1$ is the point $P_1$ (since we have defined $Q_1$ as the isogonal conjugate of $P_1$ wrt triangle $ABC$), and the isogonal conjugate of $Q$ is the point $P$. Hence, the points $P_1$ and $P$ coincide. Since $P_1$ is an infinite point, it follows that the point $P$ is an infinite point, and Assertion 2 is proven.

*Second proof of Assertion 2.* We will show the following assertion, which is nothing but Assertion 2 with $P$ and $Q$ interchanged:

*Assertion 2':* If the point $P$ lies on the circumcircle of triangle $ABC$, then the point $Q$ is an infinite point.

This assertion 2' can be derived from our proof of Theorem 3 as follows: Since the point $P$ lies on the circumcircle of triangle $ABC$, it is an Euclidean point; thus, according to the proof of Theorem 3 in Case 1, the isogonals of the lines $AP$, $BP$, $CP$ wrt the angles $CAB$, $ABC$, $BCA$ are the perpendicular bisectors of the segments $Y'Z'$, $Z'X'$, $X'Y'$. Since the point $Q$ is the isogonal conjugate of the point $P$ wrt triangle $ABC$, it is the point of intersection of these isogonals, thus the point of intersection of the perpendicular bisectors of the segments $Y'Z'$, $Z'X'$, $X'Y'$. Now, as the point $P$ lies on the circumcircle of triangle $ABC$, according to a well-known fact (Steiner line theorem), the reflections $X'$, $Y'$, $Z'$ of the point $P$ in the lines $BC$, $CA$, $AB$ lie on one line; the perpendicular bisectors of the segments $Y'Z'$, $Z'X'$, $X'Y'$ are all perpendicular to this line $X'Y'Z'$ and thus all parallel to each other. The point $Q$ is the point of intersection of these parallel perpendicular bisectors, therefore an infinite point (actually, it is the infinite point of intersection of all lines perpendicular to the line $X'Y'Z'$ and can be regarded as the "circumcenter" of the degenerate triangle $X'Y'Z'$). Thus, Assertion 2' is proven, and therefore Assertion 2 as well. This completes our proof of Theorem 4.

## 4. Reflections and pedal circles

Another almost trivial consequence of the proof of Theorem 3 is the following fact (Fig. 11):

**Theorem 5.** If $P$ is an Euclidean point in the plane of triangle $ABC$, and $X'$, $Y'$, $Z'$ are the reflections of the point $P$ in the lines $BC$, $CA$, $AB$, then the isogonal conjugate $Q$ of the point $P$ wrt the triangle $ABC$ is the circumcenter of triangle $X'Y'Z'$, and the lines $AQ$, $BQ$, $CQ$ are the perpendicular bisectors of the sides $Y'Z'$, $Z'X'$, $X'Y'$ of this triangle.

*Proof of Theorem 5.* Since $P$ is an Euclidean point, we can retrieve from our proof of Theorem 3 in Case 1 the observation that the isogonals of the lines $AP$, $BP$, $CP$ wrt the angles $CAB$, $ABC$, $BCA$ are the perpendicular bisectors of the segments $Y'Z'$, $Z'X'$, $X'Y'$. But as the point $Q$ is the isogonal conjugate of $P$ wrt triangle $ABC$, the lines $AQ$, $BQ$, $CQ$ are the isogonals of the lines $AP$, $BP$, $CP$ wrt the angles $CAB$, $ABC$, $BCA$. Hence, the lines $AQ$, $BQ$, $CQ$ are the perpendicular bisectors of the segments $Y'Z'$, $Z'X'$, $X'Y'$. The point $Q$, being the point of intersection of the lines $AQ$, $BQ$, $CQ$, is therefore the point of intersection of the perpendicular bisectors of the segments $Y'Z'$, $Z'X'$, $X'Y'$, thus the circumcenter of triangle $X'Y'Z'$. This proves Theorem 5.

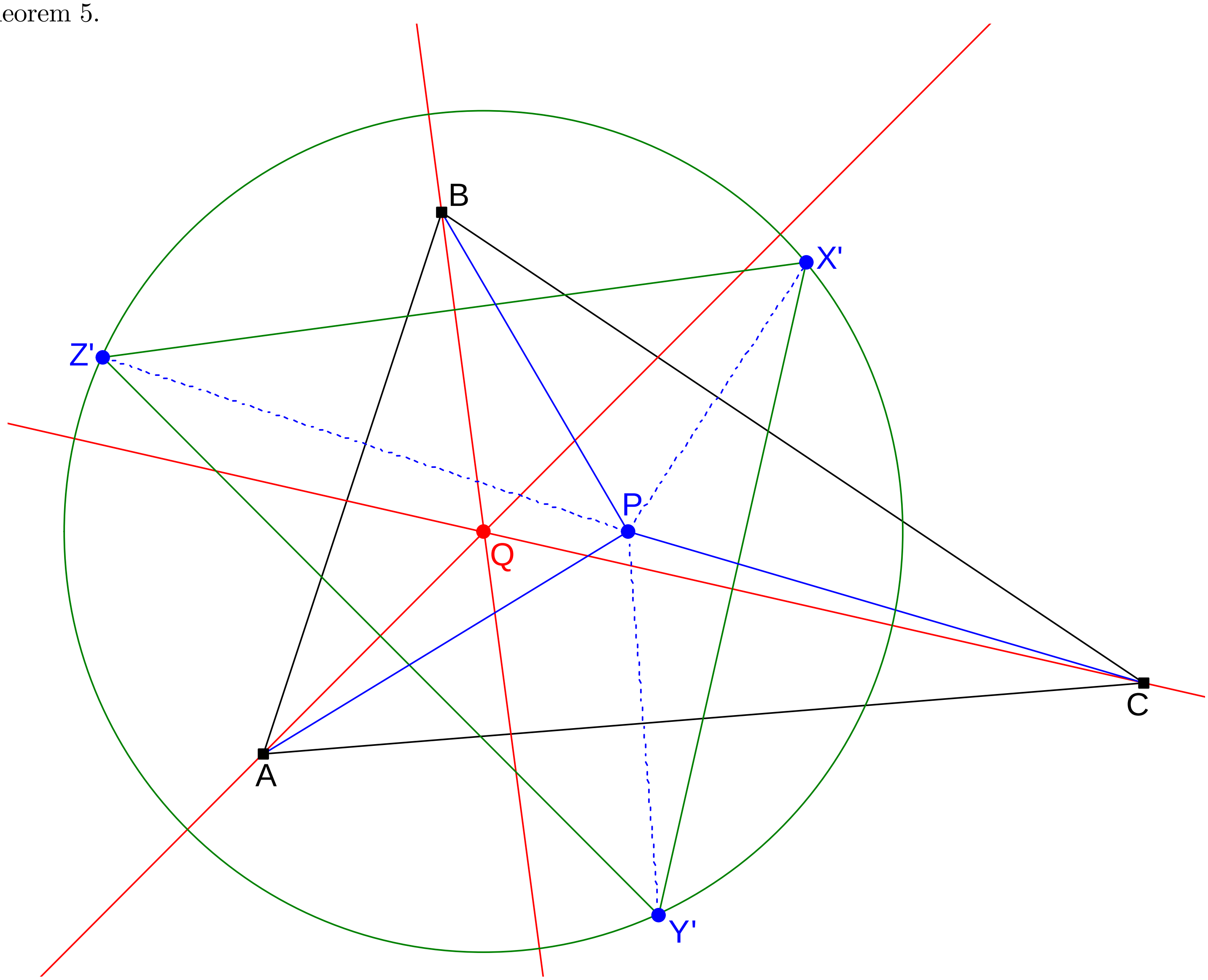


Fig. 11

Another easily accessible property of isogonals is the following one (Fig. 12):

**Theorem 6.** If $P$ is an Euclidean point in the plane of a triangle $ABC$, if $X$, $Y$, $Z$ are the orthogonal projections of the point $P$ on the lines $BC$, $CA$, $AB$, and if $Q$ is the isogonal conjugate of the point $P$ wrt the triangle $ABC$, then $AQ \perp YZ$, $BQ \perp ZX$, $CQ \perp XY$.

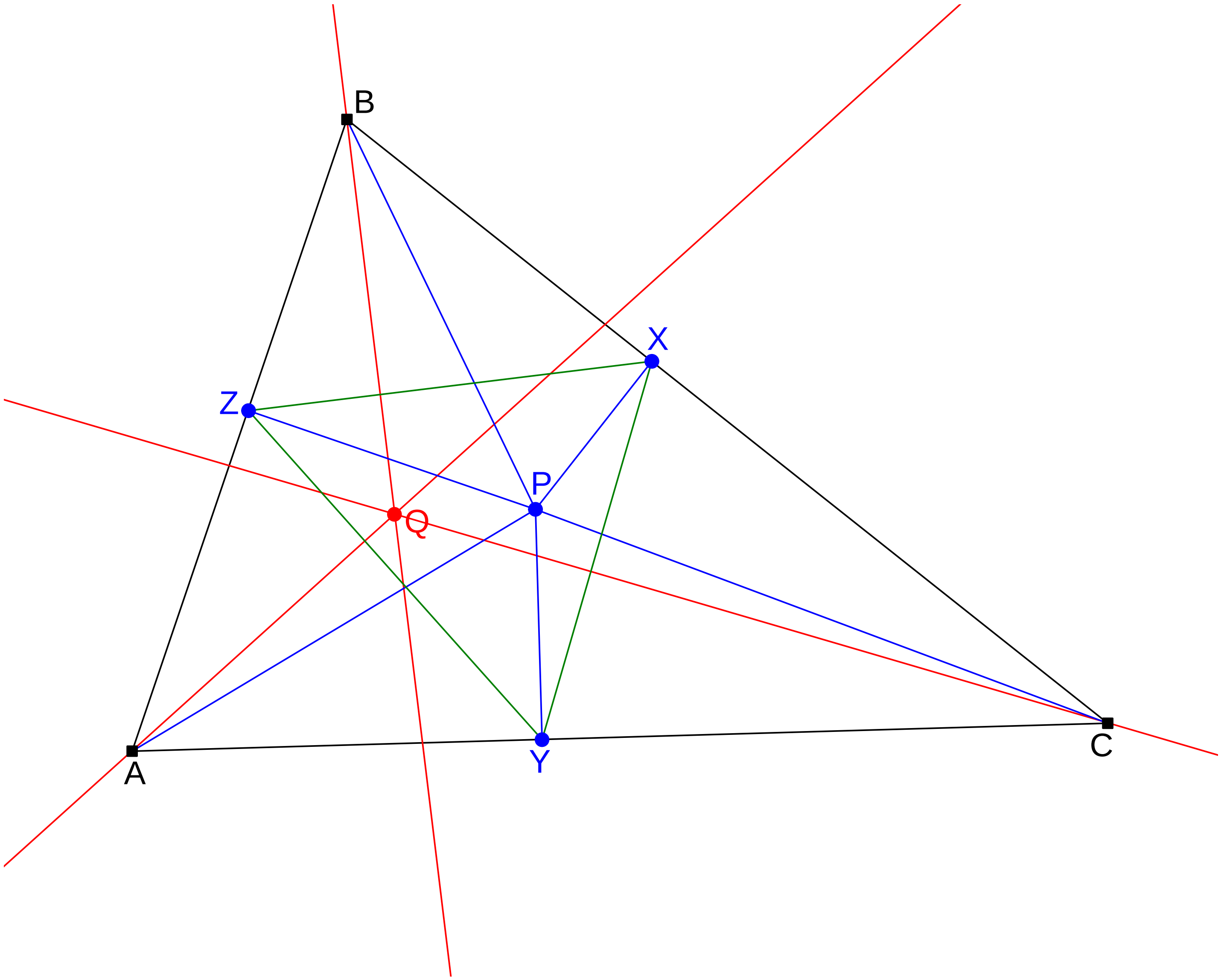


Fig. 12

*Proof of Theorem 6.* Since the points $Y$ and $Z$ are the orthogonal projections of the point $P$ on the lines $CA$ and $AB$, according to Theorem 2 **a)** we see that the line $YZ$ is perpendicular to the isogonal of the line $AP$ wrt the lines $CA$ and $AB$, that is, to the isogonal of the line $AP$ wrt the angle $CAB$. But as the point $Q$ is the isogonal conjugate of $P$ wrt triangle $ABC$, the line $AQ$ is the isogonal of the line $AP$ wrt the angle $CAB$. Hence, the line $YZ$ is perpendicular to the line $AQ$. Similarly, the lines $ZX$ and $XY$ are perpendicular to the lines $BQ$ and $CQ$, and Theorem 6 is verified.

We slowly move to deeper waters and show two well-known but less obvious properties of isogonal conjugates (Fig. 13):

**Theorem 7.** Let $P$ and $Q$ be two Euclidean isogonally conjugate points wrt a triangle $ABC$. Let $X$, $Y$, $Z$ be the orthogonal projections of the point $P$ on the lines $BC$, $CA$, $AB$, and let $U$, $V$, $W$ be the orthogonal projections of the point $Q$ on the lines $BC$, $CA$, $AB$.

**a)** We have $PX \cdot QU = PY \cdot QV = PZ \cdot QW$.

**b)** The points $X$, $Y$, $Z$, $U$, $V$, $W$ lie on one circle centered at the midpoint $M$ of the segment $PQ$.

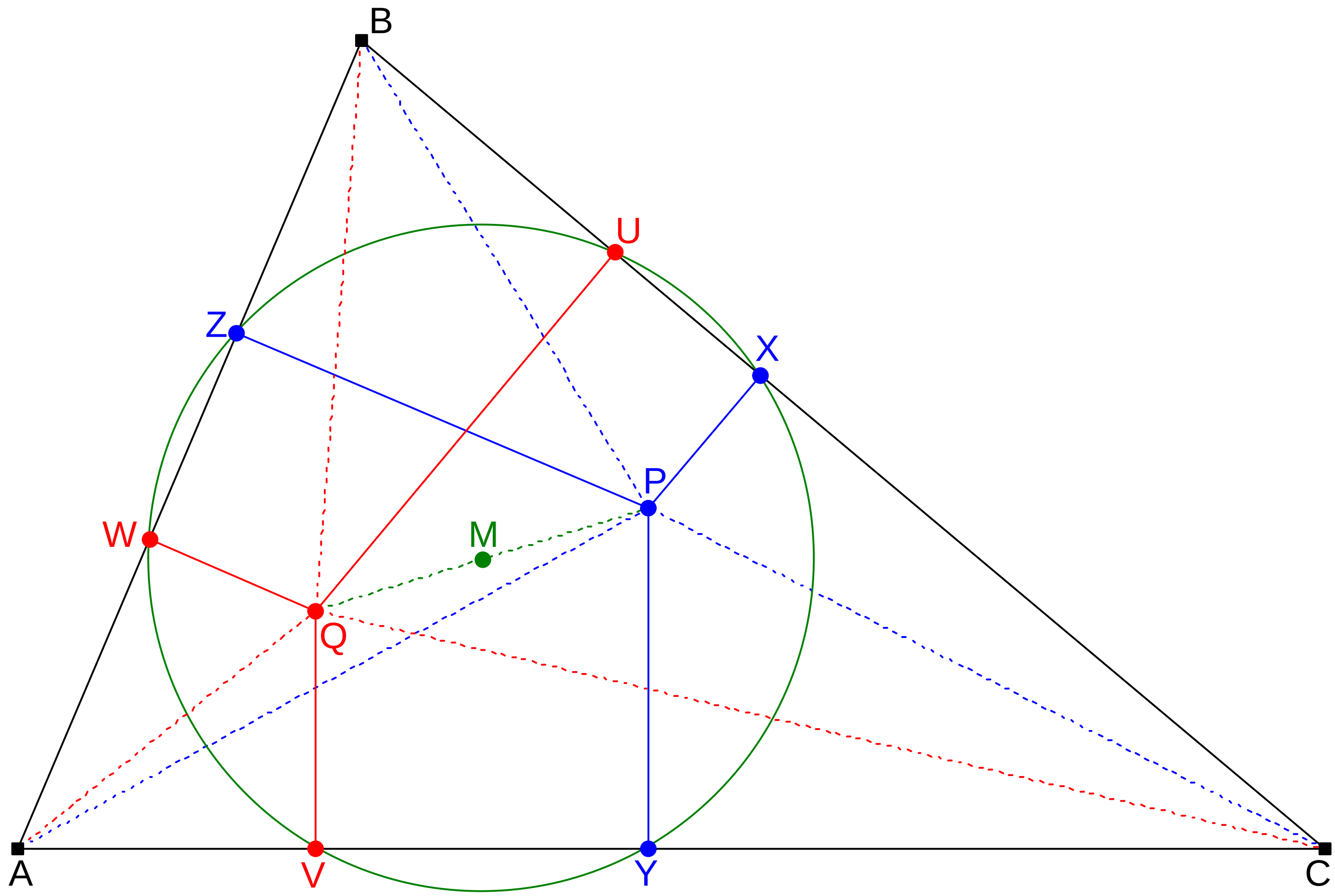


Fig. 13

*Proof of Theorem 7.* **a)** Since $P$ is the isogonal conjugate of $Q$ wrt the triangle $ABC$, the line $AP$ is the isogonal of the line $AQ$ wrt the angle $CAB$. Thus, $\measuredangle (CA;\ AP) = -\measuredangle (AB;\ AQ)$, that is, $\measuredangle YAP = -\measuredangle WAQ$. Furthermore, $\measuredangle AYP = -\measuredangle AWQ$ (since $\measuredangle AYP = 90^\circ$ and $\measuredangle AWQ = 90^\circ$, and since we are working with directed angles modulo $180^\circ$, we have $90^\circ = -90^\circ$). Thus, triangle $AYP$ is oppositely similar to triangle $AWQ$. Similarly, triangle $AZP$ is oppositely similar to triangle $AVQ$. Thus, the quadrilateral $AYPZ$, being formed by the triangles $AYP$ and $AZP$, is oppositely similar to the quadrilateral $AWQV$, being formed by the triangles $AWQ$ and $AVQ$. This similitude yields $PY : PZ = QW : QV$, hence $PY \cdot QV = PZ \cdot QW$. Analogous considerations lead to $PX \cdot QU = PY \cdot QV$. Thus, $PX \cdot QU = PY \cdot QV = PZ \cdot QW$, and Theorem 7 **a)** is proven.

**b)** (See Fig. 14.) The point $Q$ is the isogonal conjugate of $P$ wrt the triangle $ABC$. Thus, according to Theorem 5, the point $Q$ is the circumcenter of triangle $X'Y'Z'$, where $X'$, $Y'$, $Z'$ are the reflections of the point $P$ in the lines $BC$, $CA$, $AB$. Hence, $QX' = QY' = QZ'$.

Since $X$ is the orthogonal projection of the point $P$ on the line $BC$, while $X'$ is the reflection of $P$ in $BC$, we see that $X$ is the midpoint of the segment $PX'$. On the other hand, $M$ is the midpoint of the segment $PQ$. Thus, the segment $MX$ is a midparallel in triangle $PQX'$, so that $MX = \dfrac{QX'}{2}$. Similarly, $MY = \dfrac{QY'}{2}$ and $MZ = \dfrac{QZ'}{2}$. Therefore, $QX' = QY' = QZ'$ entails $MX = MY = MZ$.

As the configuration is symmetric in the points $P$ and $Q$, we similarly obtain $MU = MV = MW$.

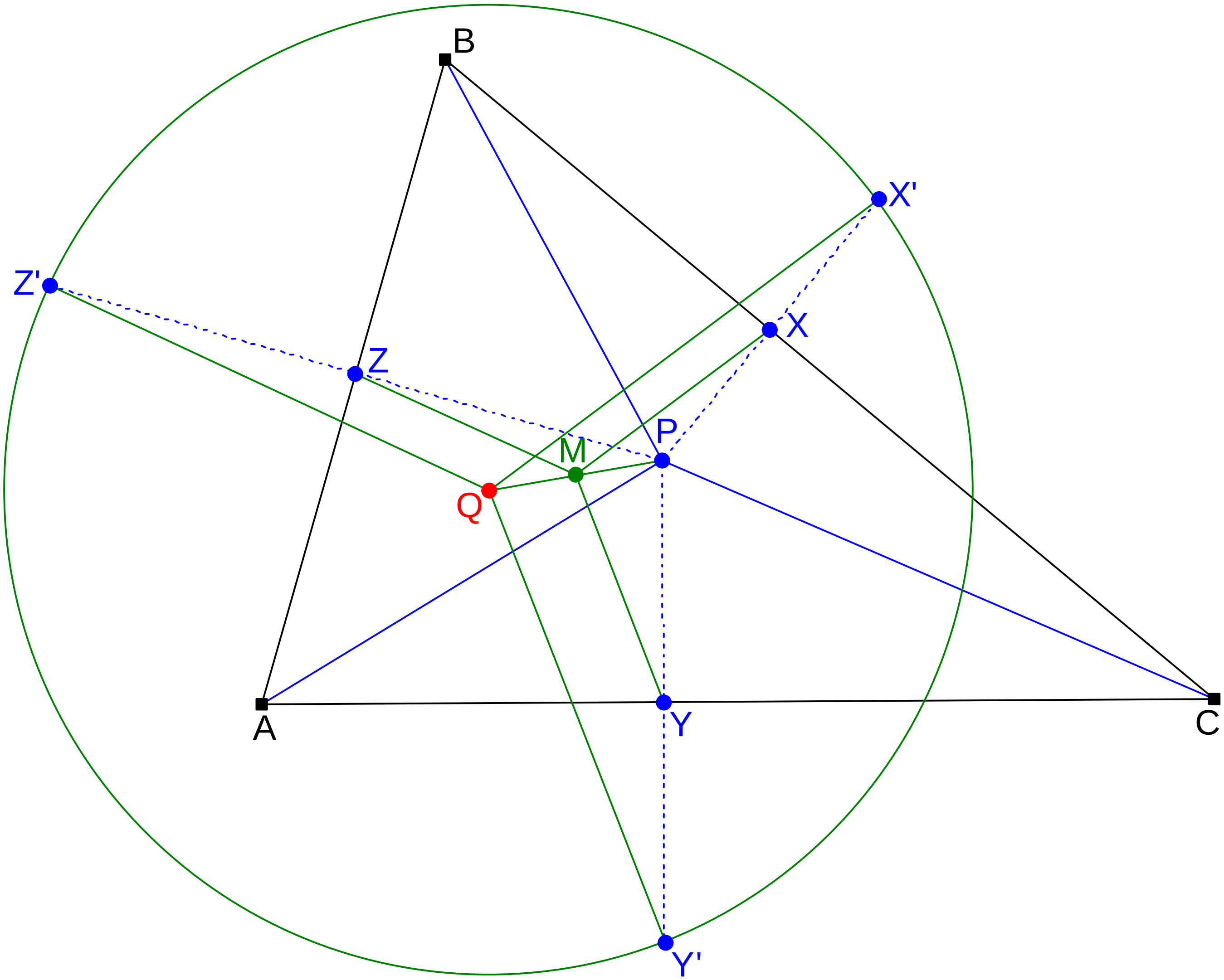


Fig. 14

(See Fig. 15.) Since $PX \perp UX$ and $QU \perp UX$, we have $PX \parallel QU$; thus, the quadrilateral $PXUQ$ is a trapezoid with the bases $PX$ and $QU$. Hence, the line through the midpoints of its legs $PQ$ and $UX$ is the midparallel of this trapezoid, therefore is parallel to its base $PX$, and thus, since $PX \perp UX$, perpendicular to the line $UX$. Hence, this midparallel passes through the midpoint of the segment $UX$ and is perpendicular to the line $UX$; thus, it is the perpendicular bisector of the segment $UX$. Since the midpoint of the segment $PQ$ lies on this midparallel, we thus obtain that the midpoint of the segment $PQ$ lies on the perpendicular bisector of the segment $UX$. The midpoint of the segment $PQ$ is $M$; thus, the point $M$ lies on the perpendicular bisector of the segment $UX$. This yields $MU = MX$. Together with $MX = MY = MZ$ and $MU = MV = MW$, this yields $MX = MY = MZ = MU = MV = MW$; in other words, the points $X$, $Y$, $Z$, $U$, $V$, $W$ lie on one circle centered at $M$. This proves Theorem 7 **b)**.

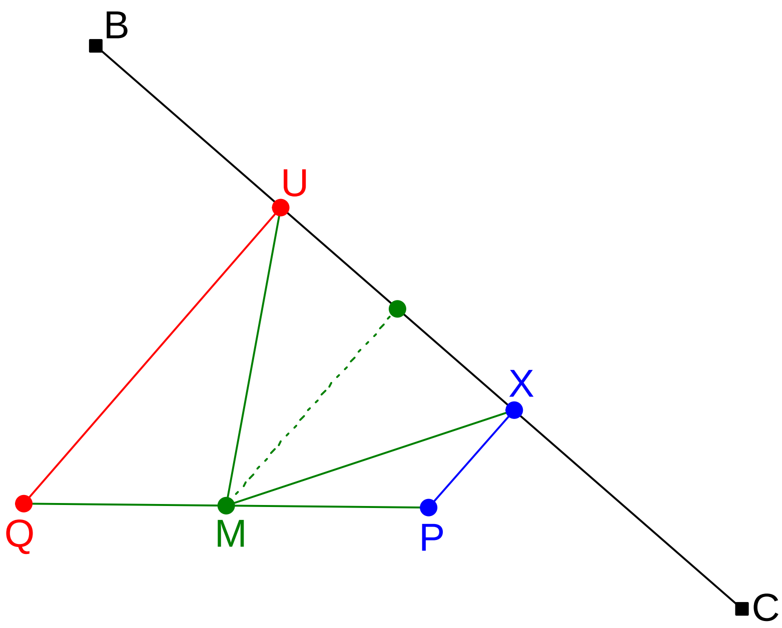


Fig. 15

Theorem 7 **b)** is known as the **pedal circle theorem**, and the proof we have given here is apparently new.

### 5. A result by Hatzipolakis, Yiu and Ehrmann

The following observations are mostly due to Jean-Pierre Ehrmann ([6]) and, to smaller amounts, to Peter Scholze and me. We start with an interesting result of Ehrmann (Fig. 16):

**Theorem 8.** Let $P$ and $Q$ be two Euclidean isogonally conjugate points wrt a triangle $ABC$. Then, the isogonal of the line $AP$ wrt the angle $BPC$ and the isogonal of the line $AQ$ wrt the angle $BQC$ are symmetric to each other wrt the line $BC$.

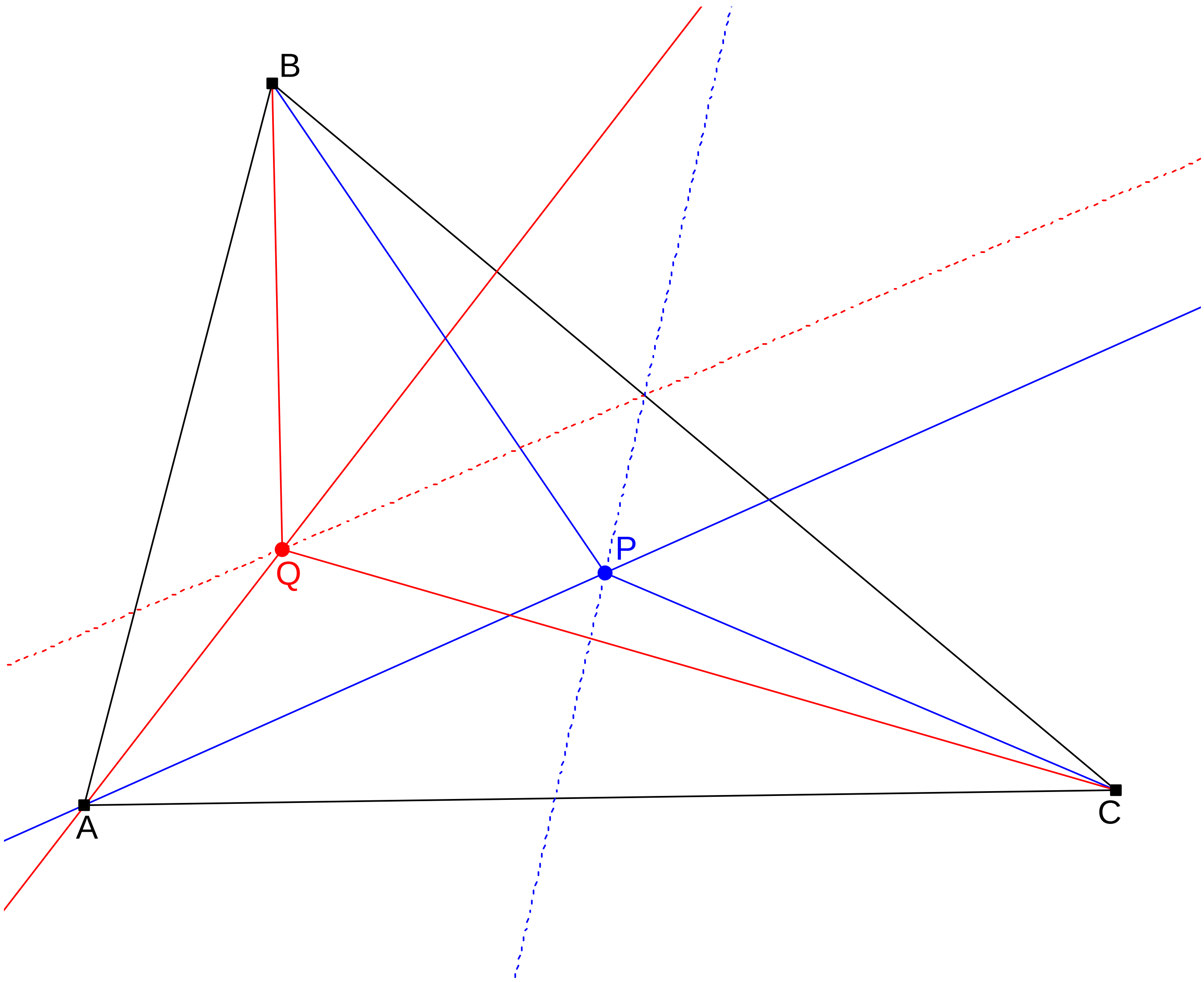


Fig. 16

*Proof of Theorem 8.* (See Fig. 17.) Let $X'$, $Y'$, $Z'$ be the reflections of the point $P$ in the lines $BC$, $CA$, $AB$, and let $U'$ be the reflection of the point $Q$ in the line $BC$. Then, the point $Q$ is, in turn, the reflection of $U'$ in the line $BC$. Since the points $Q$ and $X'$ are the reflections of the points $U'$ and $P$ in the line $BC$, the line $QX'$ is the reflection of the line $U'P$ (or, in other words, of the line $PU'$) in the line $BC$. This means that the lines $PU'$ and $QX'$ are symmetric to each other wrt the line $BC$.

According to Theorem 5, the point $Q$ is the circumcenter of triangle $X'Y'Z'$, thus the center of the circle through the points $X'$, $Y'$, $Z'$. Hence, by the central angle theorem for directed angles modulo $180°$, we have $\measuredangle QX'Z' = 90° - \measuredangle Z'Y'X'$. In other words, $\measuredangle (QX';\ Z'X') = 90° - \measuredangle (Y'Z';\ X'Y')$. Furthermore, Theorem 5 tells us that the lines $AQ$, $BQ$, $CQ$ are the perpendicular bisectors of the segments $Y'Z'$, $Z'X'$, $X'Y'$; this yields $AQ \perp Y'Z'$, $BQ \perp Z'X'$, $CQ \perp X'Y'$, and thus $\measuredangle (Y'Z';\ AQ) = 90°$,

$\measuredangle(BQ;\ Z'X') = 90^\circ$ and $\measuredangle(X'Y';\ CQ) = 90^\circ$. Hence,

$$\begin{aligned}\measuredangle(BQ;\ QX') &= \measuredangle(BQ;\ Z'X') - \measuredangle(QX';\ Z'X') = 90^\circ - (90^\circ - \measuredangle(Y'Z';\ X'Y')) \\ &= \measuredangle(Y'Z';\ X'Y') = \measuredangle(Y'Z';\ AQ) - \measuredangle(X'Y';\ AQ) \\ &= 90^\circ - \measuredangle(X'Y';\ AQ) = \measuredangle(X'Y';\ CQ) - \measuredangle(X'Y';\ AQ) = \measuredangle(AQ;\ CQ) \\ &= -\measuredangle(CQ;\ AQ)\,.\end{aligned}$$

Thus, the line $QX'$ is the isogonal of the line $AQ$ wrt the angle $BQC$. Similarly, the line $PU'$ is the isogonal of the line $AP$ wrt the angle $BPC$. Since we know that the lines $PU'$ and $QX'$ are symmetric to each other wrt the line $BC$, we have thus proven that the isogonal of the line $AP$ wrt the angle $BPC$ and the isogonal of the line $AQ$ wrt the angle $BQC$ are symmetric to each other wrt the line $BC$. Theorem 8 is thus established.

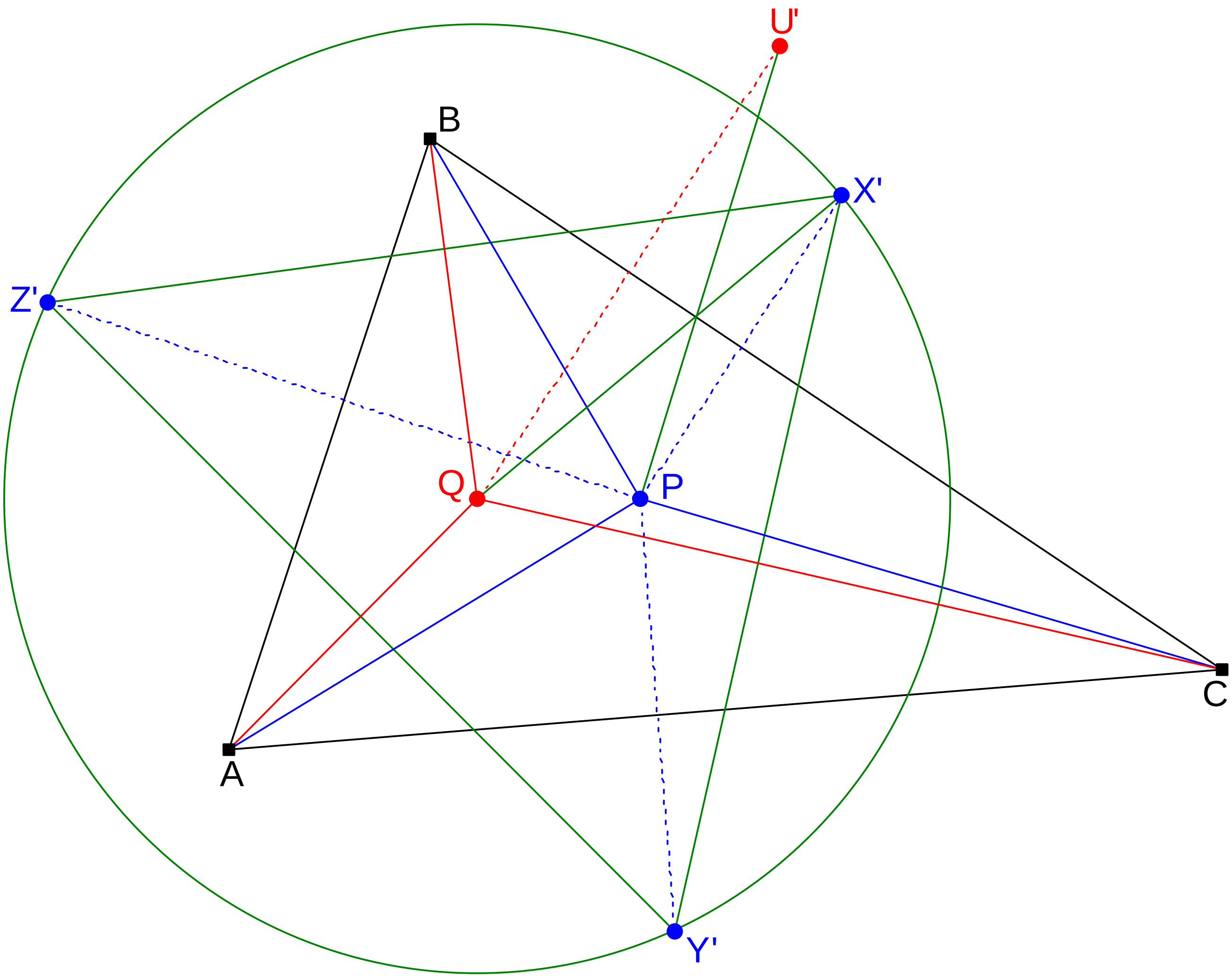


Fig. 17

The preceeding theorem is a crucial lemma in the proof of a result that was conjectured by Antreas P. Hatzipolakis, verified using computer algebra by Paul Yiu ([7]) and proven synthetically by Jean-Pierre Ehrmann ([6]):

**Theorem 9.** Let $P$ be a point in the plane of a triangle $ABC$. The lines $AP$, $BP$, $CP$ intersect the lines $BC$, $CA$, $AB$ at the points $A'$, $B'$, $C'$. Let $Q$ be the isogonal conjugate of the point $P$ wrt the triangle $ABC$. Then, the reflections of the lines $AQ$,

$BQ$, $CQ$ in the lines $B'C'$, $C'A'$, $A'B'$ concur at one point. (See Fig. 18.)

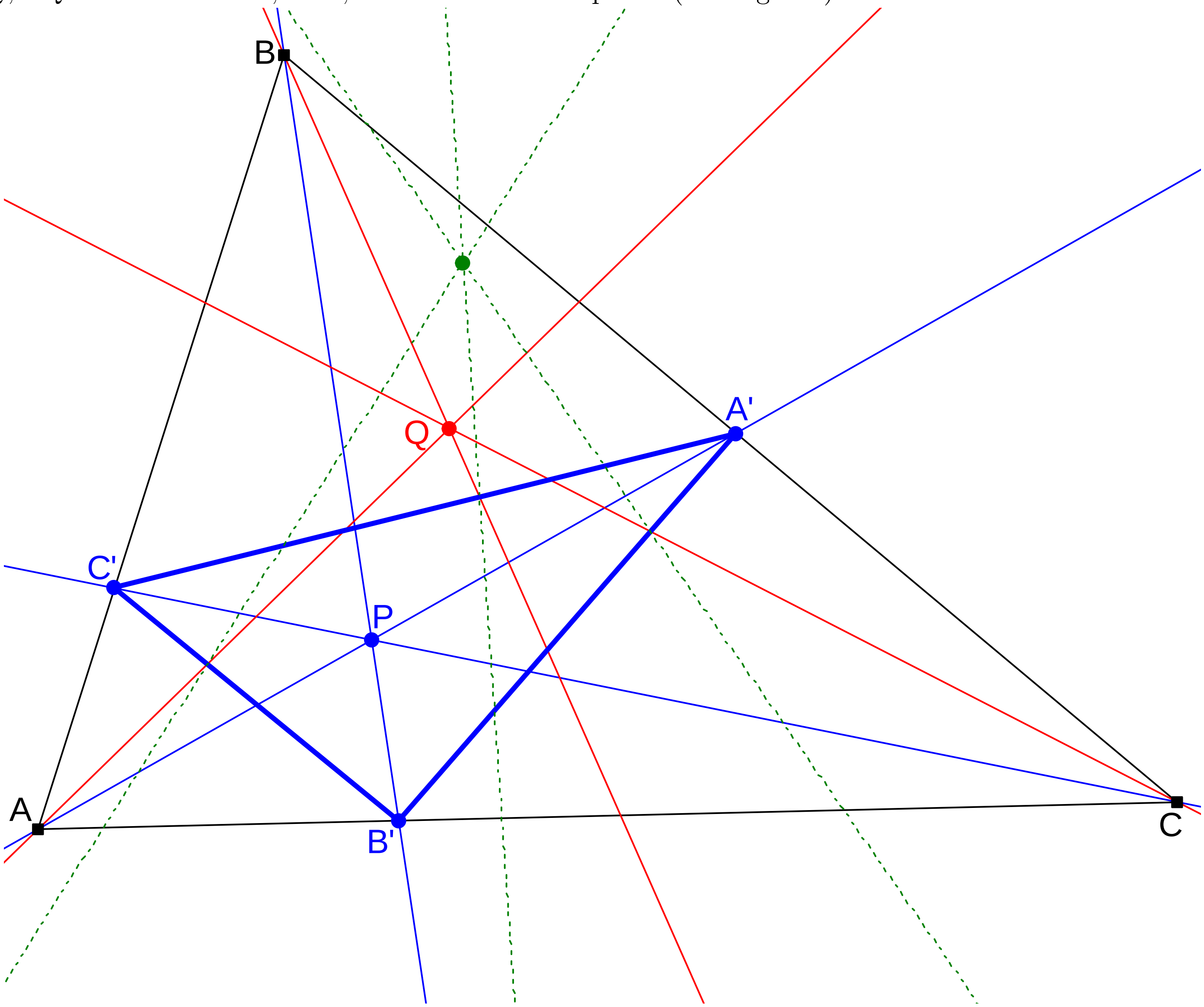


Fig. 18

*Proof of Theorem 9.* Again, we follow the proof given by Ehrmann in [6], rewriting it in a more elementary fashion.

(See Fig. 19.) Let $A_1$, $B_1$, $C_1$, $P_1$ be the isogonal conjugates of the points $A$, $B$, $C$, $P$ wrt the triangle $A'B'C'$.

Since $P_1$ is the isogonal conjugate of $P$ wrt triangle $A'B'C'$, the line $A'P_1$ is the isogonal of the line $A'P$ wrt the angle $C'A'B'$. Since $A_1$ is the isogonal conjugate of $A$ wrt triangle $A'B'C'$, the line $A'A_1$ is the isogonal of the line $A'A$ wrt the angle $C'A'B'$. Since the lines $A'P$ and $A'A$ coincide, their isogonals wrt the angle $C'A'B'$ must also coincide; i. e., the lines $A'P_1$ and $A'A_1$ coincide. Hence, the points $A'$, $A_1$, $P_1$ are collinear. Similarly, the points $B'$, $B_1$ and $P_1$ are collinear, and the points $C'$, $C_1$ and $P_1$ are collinear.

Since $B_1$ is the isogonal conjugate of $B$ wrt triangle $A'B'C'$, the line $A'B_1$ is the isogonal of the line $A'B$ wrt the angle $C'A'B'$. Since $C_1$ is the isogonal conjugate of $C$ wrt triangle $A'B'C'$, the line $A'C_1$ is the isogonal of the line $A'C$ wrt the angle $C'A'B'$. Since the lines $A'B$ and $A'C$ coincide, their isogonals wrt the angle $C'A'B'$ must also coincide; this means that the lines $A'B_1$ and $A'C_1$ coincide. In other words, the points $A'$, $B_1$, $C_1$ are collinear. Similarly, the points $B'$, $C_1$, $A_1$ are collinear, and the points $C'$, $A_1$, $B_1$ are collinear.

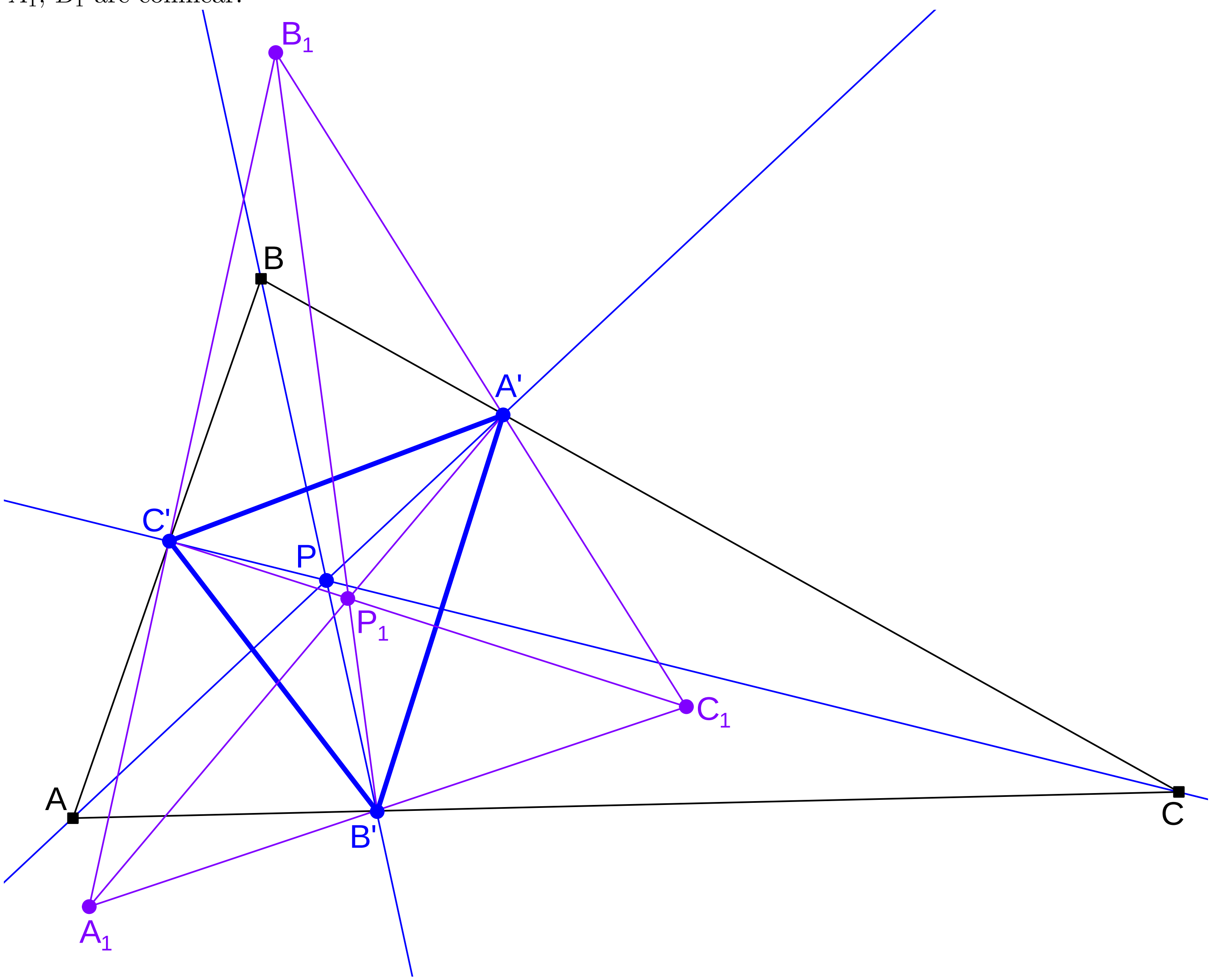


Fig. 19

Since $Q$ is the isogonal conjugate of the point $P$ wrt triangle $ABC$, the line $AQ$ is the isogonal of the line $AP$ wrt the angle $CAB$.

(See Fig. 20.) Since $A$ and $A_1$ are isogonally conjugate points wrt the triangle $A'B'C'$, Theorem 8 yields that the isogonal of the line $A'A$ wrt the angle $B'AC'$ and the isogonal of the line $A'A_1$ wrt the angle $B'A_1C'$ are symmetric to each other wrt the line $B'C'$. Now, the isogonal of the line $A'A$ wrt the angle $B'AC'$ is the isogonal of the line $AP$ wrt the angle $CAB$ (since the line $A'A$ is the line $A'P$, and the angle $B'AC'$ is the angle $CAB$), and this is the line $AQ$. Further, the isogonal of the line $A'A_1$ wrt the angle $B'A_1C'$ is the isogonal of the line $A_1P_1$ wrt the angle $C_1A_1B_1$ (since the line $A'A_1$ is the line $A_1P_1$, and the angle $B'A_1C'$ is the angle $C_1A_1B_1$). Thus we obtained that the line $AQ$ and the isogonal of the line $A_1P_1$ wrt the angle $C_1A_1B_1$ are symmetric to each other wrt the line $B'C'$. In other words, the reflection of the line $AQ$ in the line $B'C'$ is the isogonal of the line $A_1P_1$ wrt the angle $C_1A_1B_1$. Similarly, the reflections of the lines $BQ$ and $CQ$ in the lines $C'A'$ and $A'B'$ are the isogonals of the lines $B_1P_1$ and $C_1P_1$ wrt the angles $A_1B_1C_1$ and $B_1C_1A_1$. Altogether, the reflections of the lines $AQ$, $BQ$, $CQ$ in the lines $B'C'$, $C'A'$, $A'B'$ are the isogonals of the lines $A_1P_1$, $B_1P_1$, $C_1P_1$ wrt the angles $C_1A_1B_1$, $A_1B_1C_1$, $B_1C_1A_1$, and thus they concur at one point - at the isogonal conjugate of the point $P_1$ wrt the triangle $A_1B_1C_1$. Theorem 9 is proven.

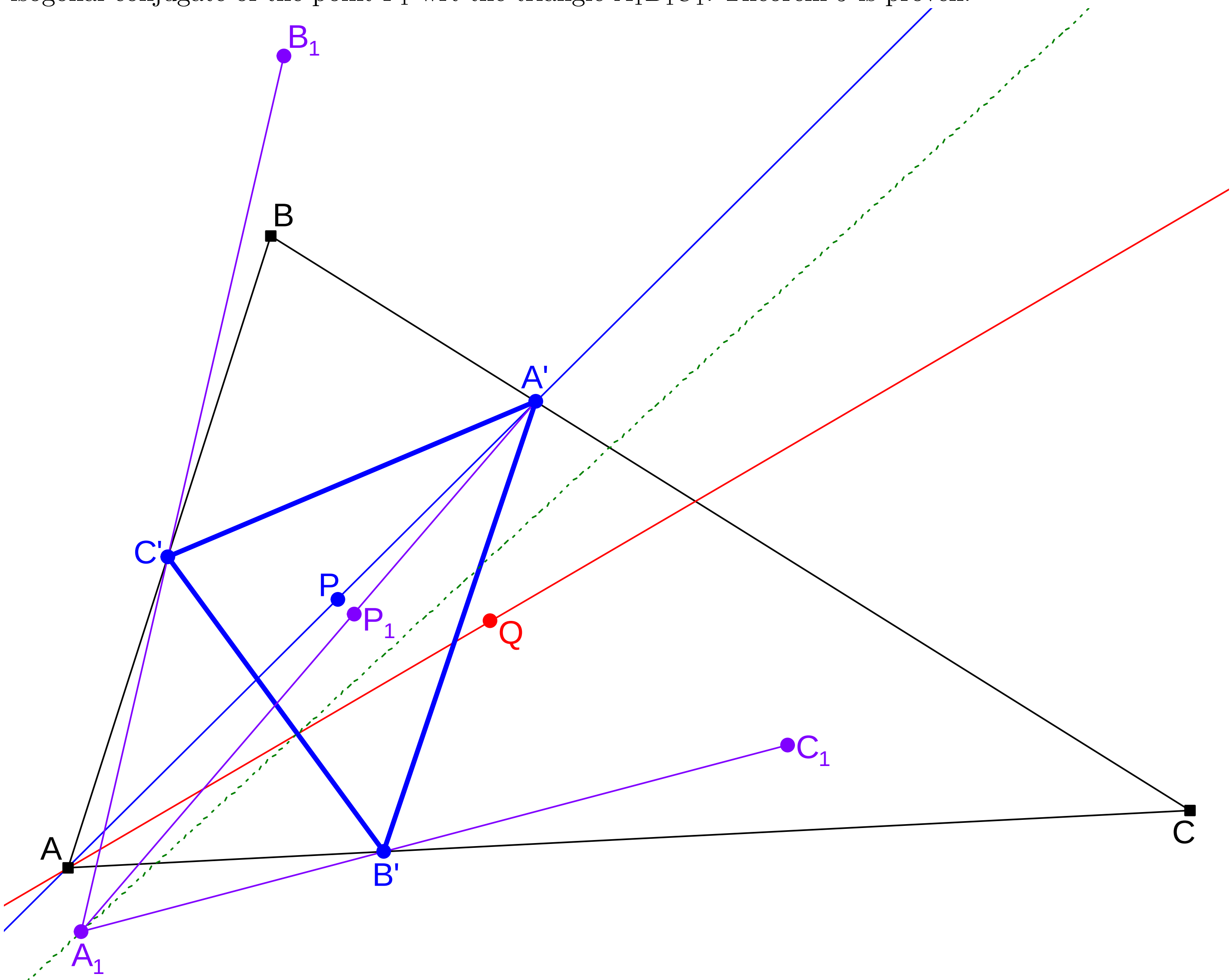

Fig. 20

**6. Reflections in perpendicular bisectors**

As an application of isogonal conjugates, we are now going to prove some properties of the reflections of a point in the perpendicular bisectors of a triangle noted by José Carlos Chávez Sandoval in [5]. We start with some trivial facts:

**Theorem 10.** Let $P$ be an Euclidean point in the plane of a triangle $ABC$, and let $D$, $E$, $F$ be the reflections of this point $P$ in the perpendicular bisectors of the segments $BC$, $CA$, $AB$.

**a)** The points $P$, $D$, $E$, $F$ lie on one circle centered at the circumcenter $O$ of triangle $ABC$.

**b)** Triangle $DEF$ is oppositely similar to triangle $ABC$. (See Fig. 21.)

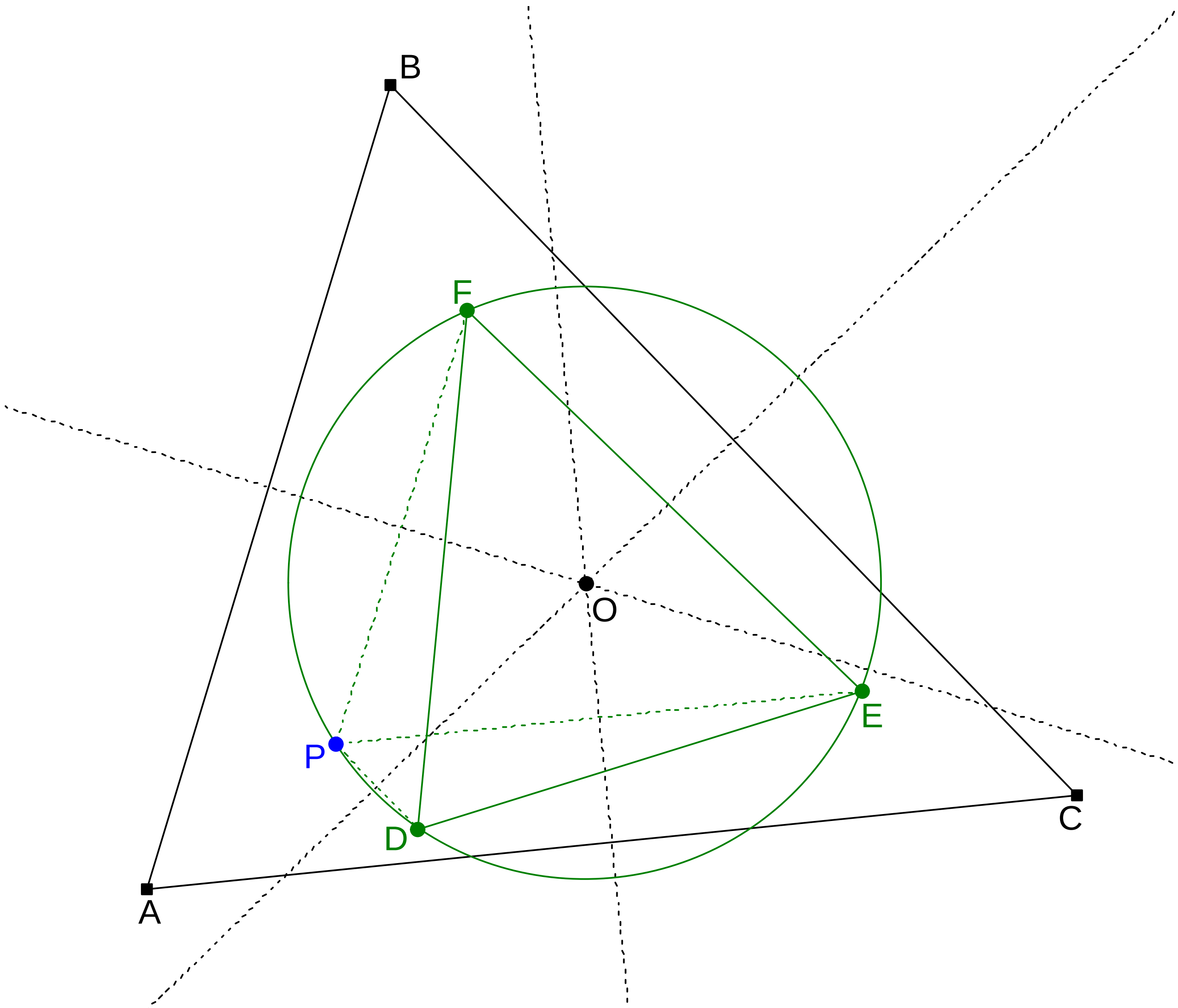


Fig. 21

*Proof of Theorem 10.* **a)** The circumcenter $O$ of triangle $ABC$ lies on the perpendicular bisector of its side $BC$. The point $D$ is the reflection of the point $P$ in this perpendicular bisector. Thus, $OD = OP$. Similarly, $OE = OP$ and $OF = OP$. Consequently, $OP = OD = OE = OF$, so that the points $P$, $D$, $E$, $F$ lie on one circle centered at $O$, and Theorem 10 **a)** is proven.

**b)** As the points $P$, $D$, $E$, $F$ lie on one circle, we have $\measuredangle FDE = \measuredangle FPE$. But as the point $E$ is the reflection of $P$ in the perpendicular bisector of $CA$, the line $PE$ is perpendicular to the perpendicular bisector of $CA$. In turn, the perpendicular bisector of $CA$ is perpendicular to the line $CA$. Hence, the line $PE$ is parallel to the line $CA$. Similarly, the line $PF$ is parallel to the line $AB$. Thus, $\measuredangle(PF;\ PE) = \measuredangle(AB;\ CA)$, what becomes $\measuredangle FPE = \measuredangle BAC$. Hence, $\measuredangle FDE = \measuredangle FPE = \measuredangle BAC = -\measuredangle CAB$. Similarly, $\measuredangle DEF = -\measuredangle ABC$. Consequently, the triangles $DEF$ and $ABC$ are oppositely similar, and Theorem 10 **b)** is proven.

Now we come to a nontrivial property of triangle $DEF$. Before we formulate it we define a traditional notion in triangle geometry:

If $S$ is the centroid of triangle $ABC$, and $T$ is an arbitrary point in the plane, then the image of the point $T$ under the homothety with center $S$ and factor $-\dfrac{1}{2}$ is called the **complement** of the point $T$ wrt the triangle $ABC$.

Now we show a theorem by José Carlos Chávez Sandoval ([5]):

**Theorem 11.** Let $P$ be an Euclidean point in the plane of a triangle $ABC$, and let $D$, $E$, $F$ be the reflections of this point $P$ in the perpendicular bisectors of the segments $BC$, $CA$, $AB$. Denote by $A_M$, $B_M$, $C_M$ the midpoints of the segments $BC$, $CA$, $AB$, and by $D_M$, $E_M$, $F_M$ the midpoints of the segments $EF$, $FD$, $DE$. Let $Q$ be the isogonal conjugate of the point $P$ wrt the triangle $ABC$, and let $Q'$ be the complement of the point $Q$ wrt the triangle $ABC$. Then, the lines $A_MD_M$, $B_ME_M$, $C_MF_M$ pass through the point $Q'$. (See Fig. 22.)

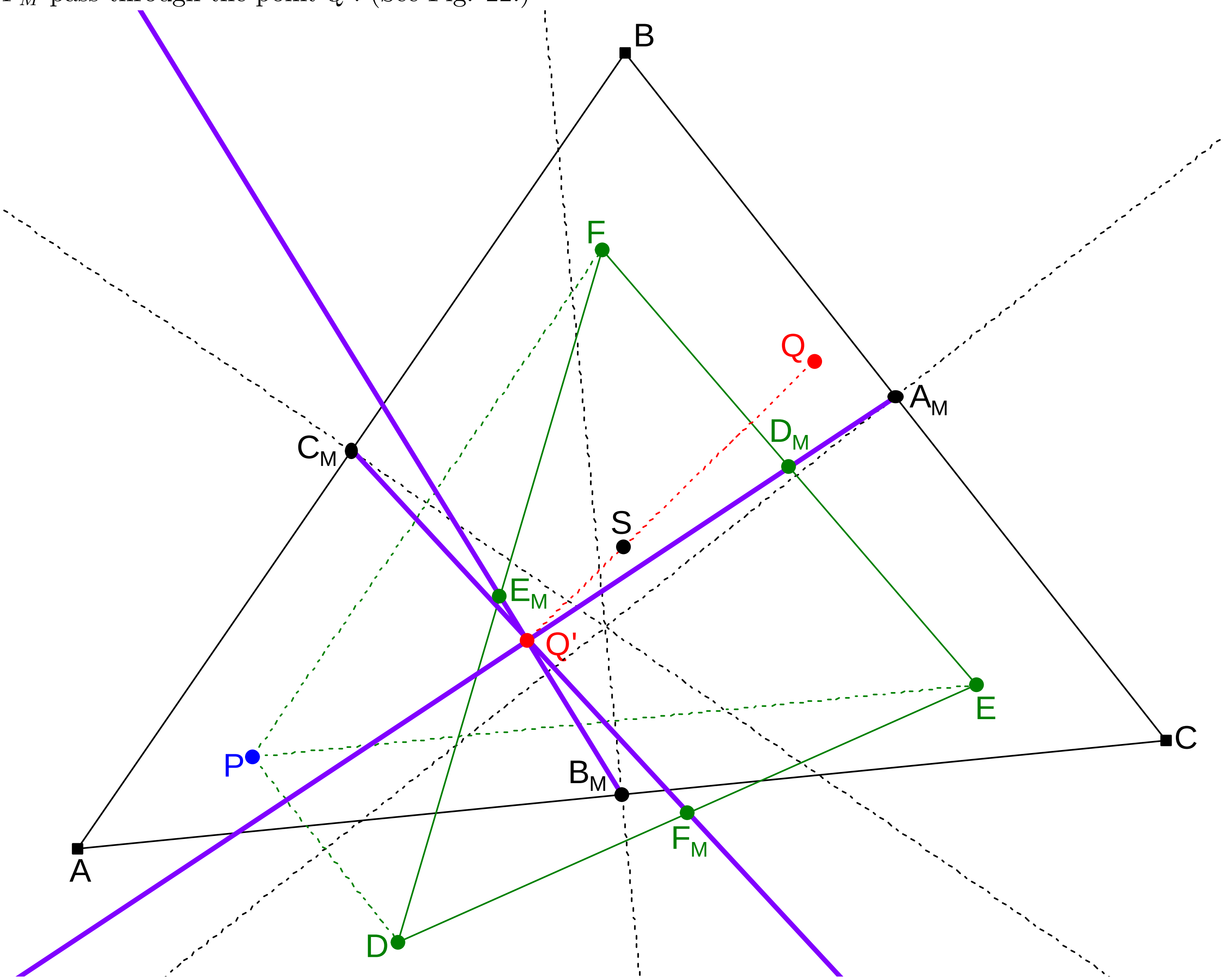


Fig. 22

*Proof of Theorem 11.* (See Fig. 23.) Since $A_M$ is the midpoint of the side $BC$ of triangle $ABC$, the segment $AA_M$ is the $A$-median of this triangle; thus, it passes through the centroid $S$ of triangle $ABC$ and is divided by this centroid in the ratio $2:1$. This means, we have $\dfrac{AS}{SA_M} = 2$ (with directed segments). On the other hand, the point $Q'$ is the complement of the point $Q$ wrt triangle $ABC$, thus the image of the point $Q$ under the homothety with center $S$ and factor $-\frac{1}{2}$; this means that the point $Q'$ lies on the line $SQ$ and satisfies $SQ' = -\frac{1}{2} \cdot SQ$, so that $2 \cdot SQ' = -SQ = QS$, and $\frac{QS}{SQ'} = 2$. Comparing this with $\frac{AS}{SA_M} = 2$, we get $\frac{AS}{SA_M} = \frac{QS}{SQ'}$, so that, by Thales, $A_MQ' \parallel AQ$.

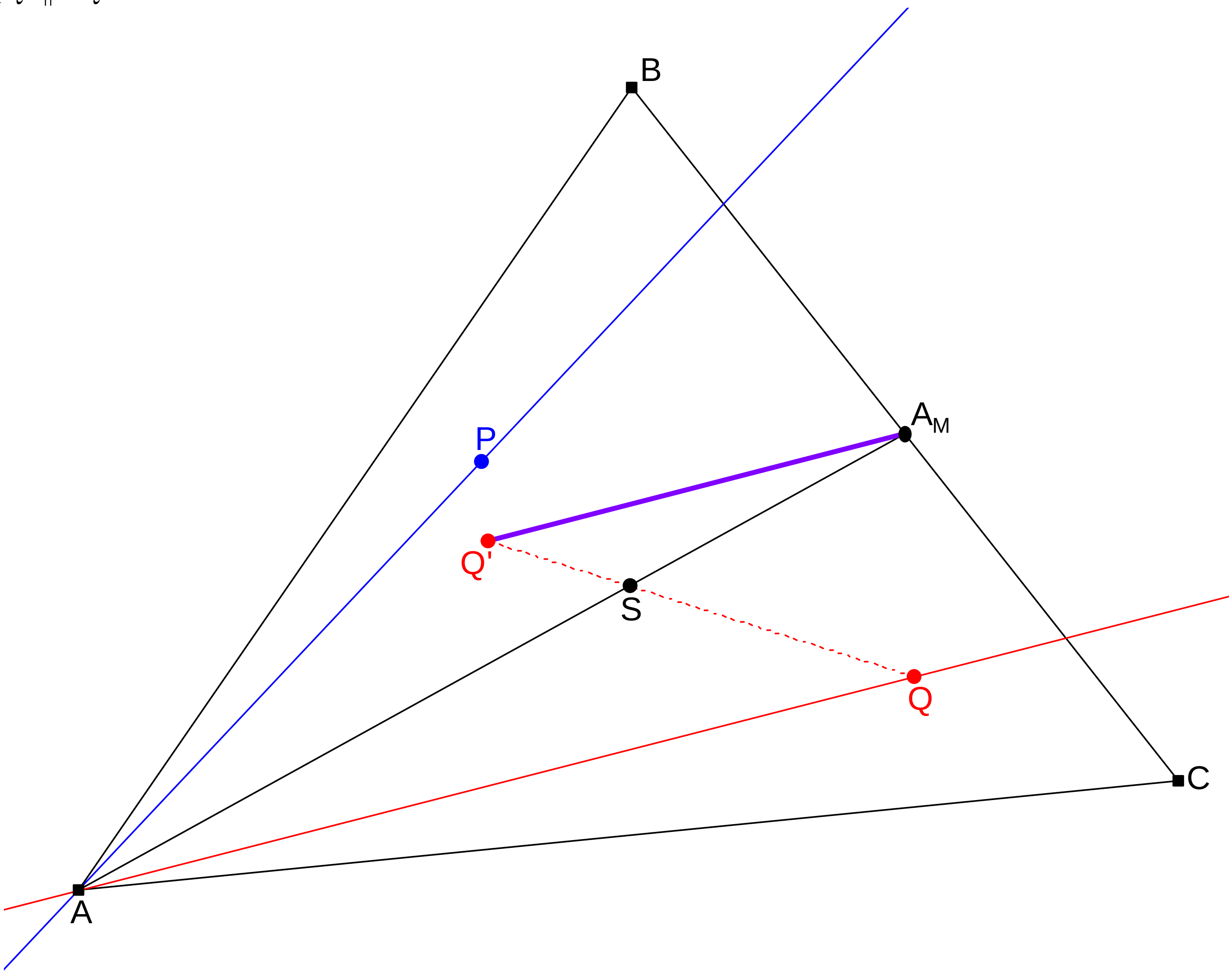


Fig. 23

(See Fig. 24.) Now, let $Z'$ be the reflection of the point $P$ in the line $AB$. Then, $\measuredangle Z'AB = -\measuredangle PAB$. On the other hand, the reflection wrt the perpendicular bisector of the segment $AB$ maps the points $A$ and $B$ to the points $B$ and $A$, respectively, and the point $P$ to the point $F$ (because the point $F$ was defined as the reflection of the point $P$ in the perpendicular bisector of the segment $AB$). Since reflection wrt a line changes the sign of angles (but leaves them invariant in other respects), we thus have $\measuredangle FBA = -\measuredangle PAB$. Together with $\measuredangle Z'AB = -\measuredangle PAB$, this results in $\measuredangle FBA = \measuredangle Z'AB$, that means, $\measuredangle (BF;\ AB) = \measuredangle (AZ';\ AB)$. Therefore, $BF \parallel AZ'$. Similarly, $AF \parallel BZ'$. Thus, the quadrilateral $AFBZ'$ is a parallelogram. Using vectors, this rewrites as $\overrightarrow{AZ'} = \overrightarrow{FB}$.

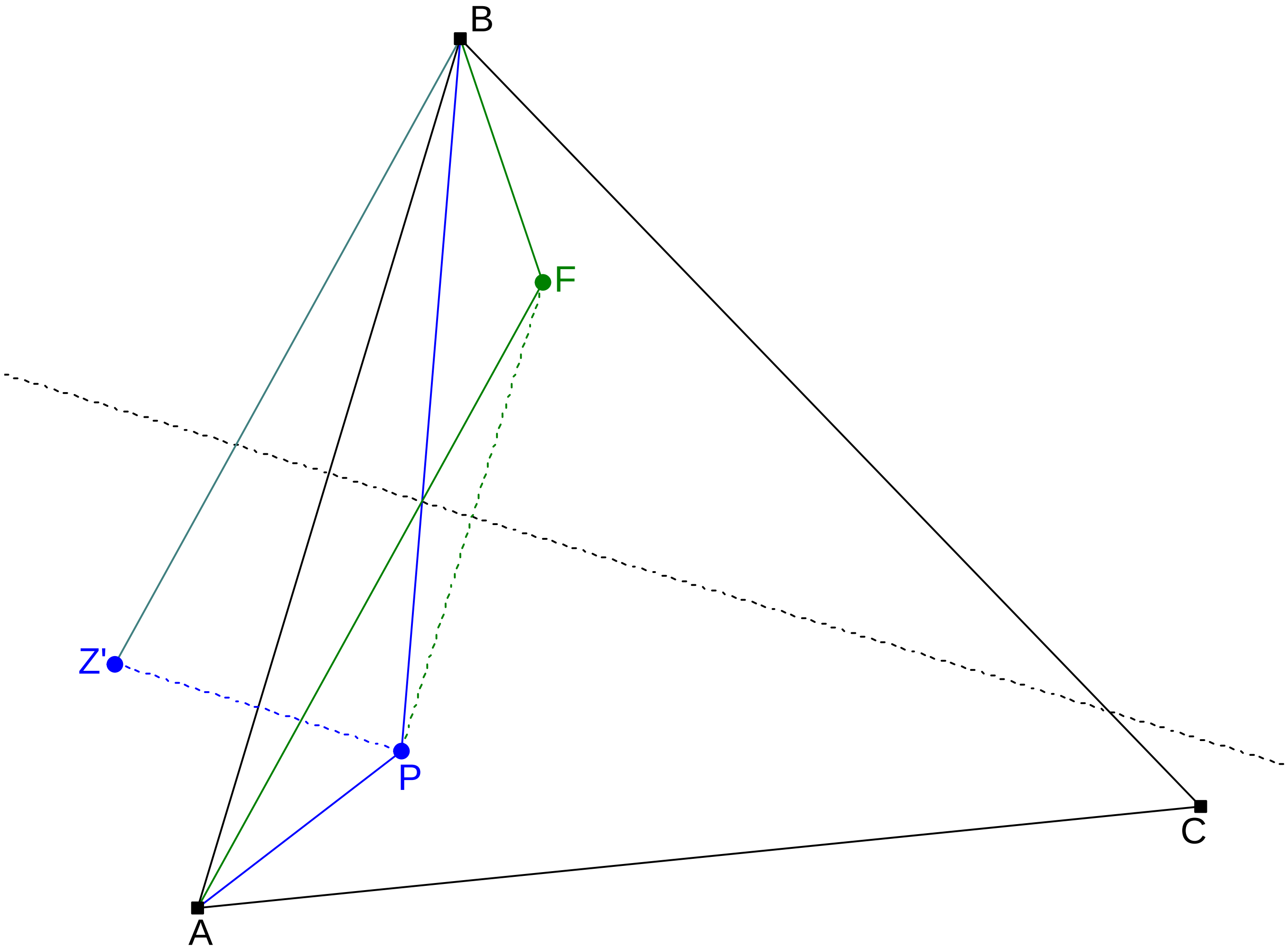


Fig. 24

(See Fig. 25.) After we have introduced the reflection $Z'$ of the point $P$ in the line $AB$, we also denote by $Y'$ the reflection of the point $P$ in the line $CA$. Similarly to $\overrightarrow{AZ'} = \overrightarrow{FB}$, we can then show $\overrightarrow{AY'} = \overrightarrow{EC}$. Furthermore, according to Theorem 5, the line $AQ$ is the perpendicular bisector of the segment $Y'Z'$; consequently, the midpoint $X'_M$ of the segment $Y'Z'$ lies on $AQ$.

On the other hand, since $X'_M$ is the midpoint of the segment $Y'Z'$, we have $\overrightarrow{AX'_M} = \dfrac{\overrightarrow{AY'} + \overrightarrow{AZ'}}{2}$. Since $\overrightarrow{AY'} = \overrightarrow{EC}$ and $\overrightarrow{AZ'} = \overrightarrow{FB}$, this becomes $\overrightarrow{AX'_M} = \dfrac{\overrightarrow{EC} + \overrightarrow{FB}}{2}$.

Since $A_M$ is the midpoint of the segment $BC$, we have

$$\overrightarrow{D_M A_M} = \frac{\overrightarrow{D_M B} + \overrightarrow{D_M C}}{2} = \frac{\left(\overrightarrow{D_M F} + \overrightarrow{FB}\right) + \left(\overrightarrow{EC} - \overrightarrow{ED_M}\right)}{2}.$$

But since $D_M$ is the midpoint of the segment $EF$, we have $\overrightarrow{D_M F} = \overrightarrow{ED_M}$, so this simplifies to

$$\overrightarrow{D_M A_M} = \frac{\left(\overrightarrow{ED_M} + \overrightarrow{FB}\right) + \left(\overrightarrow{EC} - \overrightarrow{ED_M}\right)}{2} = \frac{\overrightarrow{EC} + \overrightarrow{FB}}{2} = \overrightarrow{AX'_M}.$$

Thus, $D_M A_M \parallel AX'_M$. Since the line $AX'_M$ coincides with the line $AQ$, we thus get $D_M A_M \parallel AQ$. On the other hand, we know that $A_M Q' \parallel AQ$. Therefore, $D_M A_M \parallel$

$A_M Q'$. Now, the lines $D_M A_M$ and $A_M Q'$ have a common point (the point $A_M$), and thus can only be parallel if they coincide. As a consequence, the lines $D_M A_M$ and $A_M Q'$ coincide, i. e., the point $Q'$ lies on the line $A_M D_M$. Similarly, we may show that the point $Q'$ lies on the lines $B_M E_M$ and $C_M F_M$, and the proof of Theorem 11 is finished.

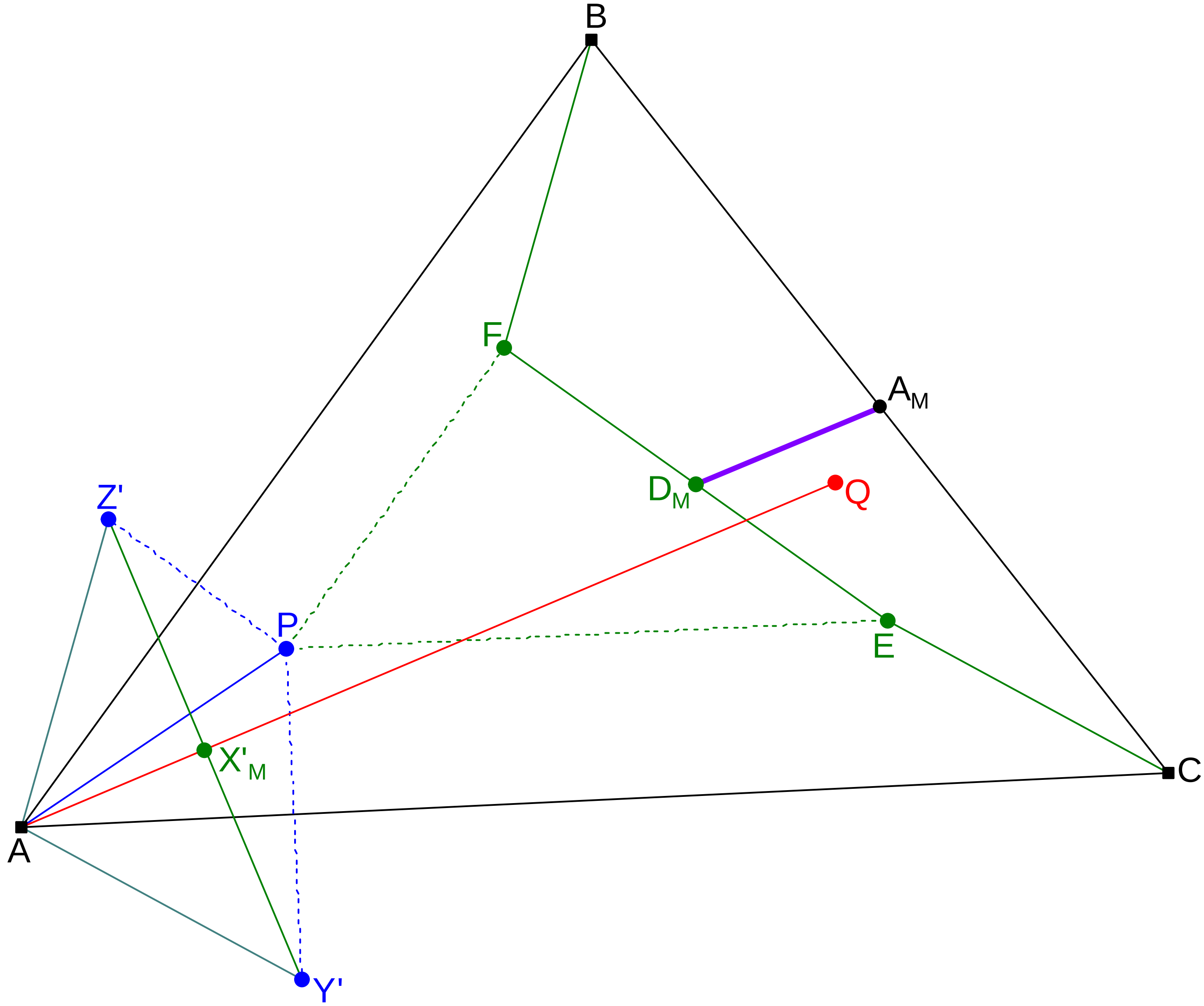


Fig. 25

## 7. More on isogonal conjugates and reflections

The following theorem sheds light on a familiar configuration - reflections of a point in the sides of a triangle and its isogonal conjugate, see Theorem 5 - from a new, symmetric viewpoint:[2]

**Theorem 12.** Let $A$, $B$, $C$, $D$ be four distinct points in the plane, and let $A'$, $B'$, $C'$, $D'$ be four distinct points in the plane. Then, the following four assertions $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$ are pairwisely equivalent:

[2]Of course, the point $D$ in the following has nothing to do with the point $D$ from Theorems 10 and 11, and the points $A'$, $B'$, $C'$ in the following have nothing to do with the points $A'$, $B'$, $C'$ from Theorem 9.

**Assertion $\mathcal{B}_1$:** The points $A$, $B$, $C$, $D$ are the circumcenters of triangles $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$.

**Assertion $\mathcal{B}_2$:** The lines $AB$, $BC$, $CD$, $DA$, $AC$, $BD$ are the perpendicular bisectors of the segments $C'D'$, $D'A'$, $A'B'$, $B'C'$, $B'D'$, $A'C'$.

**Assertion $\mathcal{B}_3$:** The points $A'$, $A'$, $A'$, $B'$, $B'$, $B'$, $C'$, $C'$, $C'$, $D'$, $D'$, $D'$ are the reflections of the points $B'$, $C'$, $D'$, $C'$, $D'$, $A'$, $D'$, $A'$, $B'$, $A'$, $B'$, $C'$ in the lines $CD$, $DB$, $BC$, $DA$, $AC$, $CD$, $AB$, $BD$, $DA$, $BC$, $CA$, $AB$.

**Assertion $\mathcal{B}_4$:** The points $A'$, $B'$, $C'$, $D'$ are the isogonal conjugates of the points $A$, $B$, $C$, $D$ wrt the triangles $BCD$, $CDA$, $DAB$, $ABC$.

(See Fig. 26.)

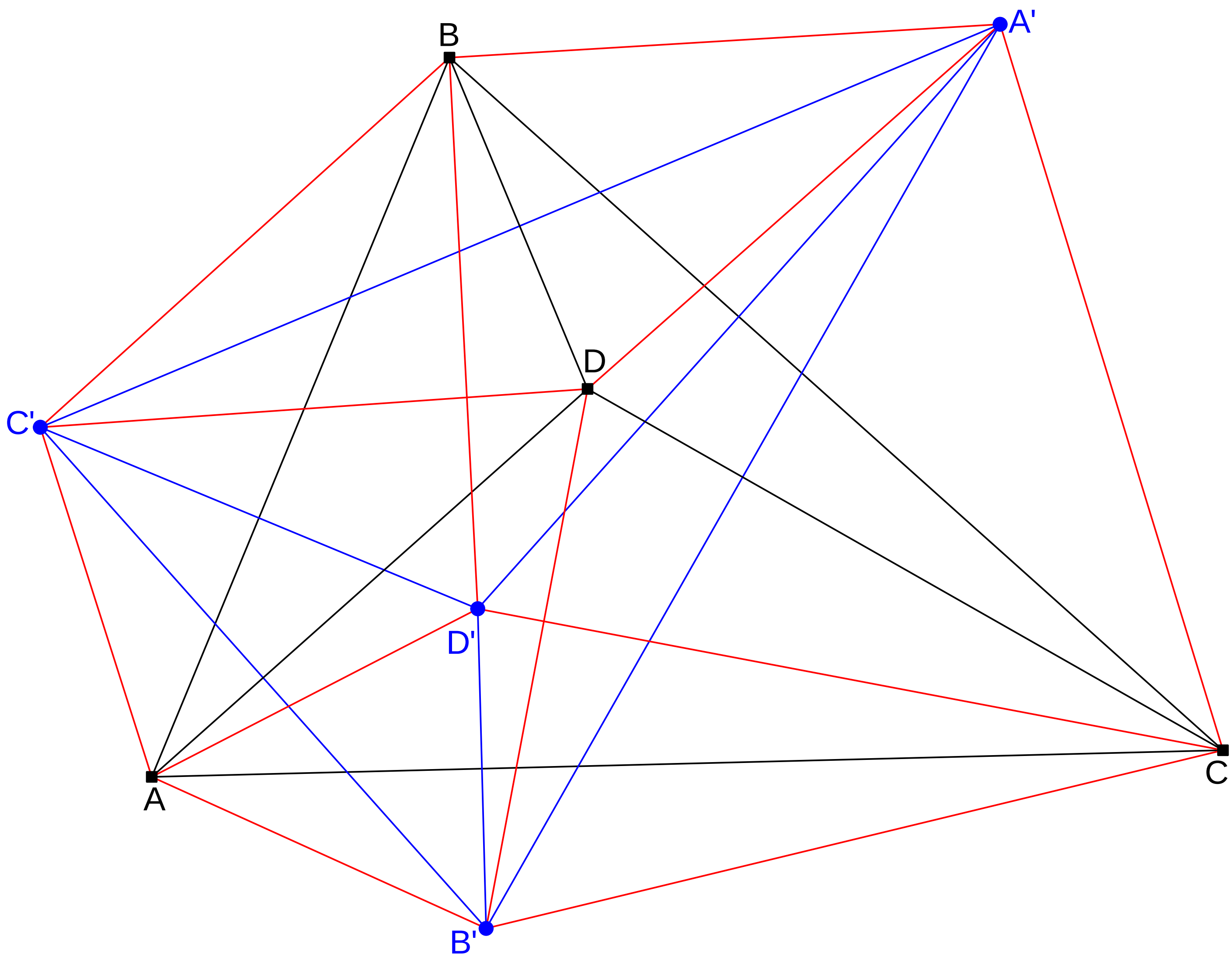


Fig. 26

*Proof of Theorem 12.* First, we show the equivalence of the assertions $\mathcal{B}_1$ and $\mathcal{B}_2$:

If Assertion $\mathcal{B}_1$ holds, then the points $A$, $B$, $C$, $D$ are the circumcenters of triangles $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$. Then, since the circumcenter of a triangle lies on the perpendicular bisectors of its sides, the point $A$, being the circumcenter of triangle $B'C'D'$, must lie on the perpendicular bisector of the segment $C'D'$. Similarly, the point $B$ must lie on the perpendicular bisector of the segment $C'D'$. Hence, the line $AB$ is the perpendicular bisector of the segment $C'D'$. Similarly, the lines $BC$, $CD$,

$DA$, $AC$, $BD$ are the perpendicular bisectors of the segments $D'A'$, $A'B'$, $B'C'$, $B'D'$, $A'C'$, and thus Assertion $\mathcal{B}_2$ is fulfilled.

Conversely, if Assertion $\mathcal{B}_2$ holds, then the lines $AB$, $BC$, $CD$, $DA$, $AC$, $BD$ are the perpendicular bisectors of the segments $C'D'$, $D'A'$, $A'B'$, $B'C'$, $B'D'$, $A'C'$. Hence, the point $A$, being the point of intersection of the lines $AB$, $AC$, $DA$, must be the point of intersection of the perpendicular bisectors of the segments $C'D'$, $B'D'$, $B'C'$, and thus the circumcenter of the triangle $B'C'D'$. Similarly, the points $B$, $C$, $D$ are the circumcenters of the triangles $C'D'A'$, $D'A'B'$, $A'B'C'$. Thus, Assertion $\mathcal{B}_1$ must hold.

Hence, we have shown that the assertions $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent.

The equivalence of the assertions $\mathcal{B}_3$ and $\mathcal{B}_2$ evidently follows from the following obvious fact: If $P_1$ and $P_2$ are two distinct points and $g_1$ is a line, then the point $P_2$ is the reflection of the point $P_1$ in the line $g_1$ if and only if the line $g_1$ is the perpendicular bisector of the segment $P_1P_2$.

Altogether we have now shown that the assertions $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ are pairwisely equivalent. In order to verify the equivalence of all four assertions $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$, it remains to prove that the assertions $\mathcal{B}_3$ and $\mathcal{B}_4$ are equivalent. In order to prove this, we have to establish two auxiliary results:

*Auxiliary result 1.* If Assertion $\mathcal{B}_4$ holds, then so does Assertion $\mathcal{B}_3$.

*Auxiliary result 2.* If Assertion $\mathcal{B}_3$ holds, then so does Assertion $\mathcal{B}_4$.

*Proof of Auxiliary result 1.* Assume that Assertion $\mathcal{B}_4$ holds. Then, particularly, the point $B'$ is the isogonal conjugate of the point $B$ wrt triangle $CDA$. Thus, the line $AB'$ is the isogonal of the line $AB$ wrt the angle $DAC$. Hence, $\measuredangle(AC;\ AB') = -\measuredangle(DA;\ AB)$. Equivalently, $\measuredangle CAB' = -\measuredangle DAB$. Similarly, $\measuredangle CAD' = -\measuredangle BAD$. Thus, $\measuredangle CAB' = -\measuredangle DAB = -(-\measuredangle BAD) = -\measuredangle CAD'$. Now, if $B_1'$ is the reflection of the point $D'$ in the line $AC$, then $\measuredangle CAB_1' = -\measuredangle CAD'$, so that $\measuredangle CAB_1' = \measuredangle CAB'$. Hence, the point $B_1'$ lies on the line $AB'$. Similarly, the point $B_1'$ lies on the line $CB'$. But the lines $AB'$ and $CB'$ have only one point in common, namely the point $B'$. Thus, since the point $B_1'$ lies on both of these lines, we must have $B_1' = B'$. Since we have introduced the point $B_1'$ as the reflection of the point $D'$ in the line $AC$, we thus conclude that the point $B'$ is the reflection of the point $D'$ in the line $AC$. Similarly, the points $A'$, $A'$, $A'$, $B'$, $B'$, $C'$, $C'$, $C'$, $D'$, $D'$, $D'$ are the reflections of the points $B'$, $C'$, $D'$, $C'$, $A'$, $D'$, $A'$, $B'$, $A'$, $B'$, $C'$ in the lines $CD$, $DB$, $BC$, $DA$, $CD$, $AB$, $BD$, $DA$, $BC$, $CA$, $AB$. In other words, Assertion $\mathcal{B}_3$ holds. This proves Auxiliary result 1.

*First proof of Auxiliary result 2.* Assume that Assertion $\mathcal{B}_3$ is valid. Since the assertions $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ are equivalent, it thus follows that Assertions $\mathcal{B}_1$ and $\mathcal{B}_2$ hold as well; i. e., the points $A$, $B$, $C$, $D$ are the circumcenters of triangles $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$, and the lines $AB$, $BC$, $CD$, $DA$, $AC$, $BD$ are the perpendicular bisectors of the segments $C'D'$, $D'A'$, $A'B'$, $B'C'$, $B'D'$, $A'C'$.

Since the lines $DA$, $AB$, $AC$ are the perpendicular bisectors of the segments $B'C'$, $C'D'$, $B'D'$, we have $DA \perp B'C'$, $AB \perp C'D'$, $AC \perp B'D'$, so that $\measuredangle(DA;\ B'C') = 90^\circ$, $\measuredangle(AB;\ C'D') = 90^\circ$, $\measuredangle(B'D';\ AC) = 90^\circ$. Thus,

$$\begin{aligned}\measuredangle(DA;\ AB) &= \measuredangle(DA;\ B'C') + \measuredangle(B'C';\ AB) = 90^\circ + \measuredangle(B'C';\ AB) \\ &= \measuredangle(AB;\ C'D') + \measuredangle(B'C';\ AB) = \measuredangle(B'C';\ C'D') = \measuredangle B'C'D'.\end{aligned}$$

On the other hand, since $A$ is the circumcenter of triangle $B'C'D'$, thus the center of a circle through the points $B'$, $C'$, $D'$, the central angle theorem yields $\measuredangle D'B'A =$

$90^\circ - \measuredangle B'C'D'$. Hence,

$$\begin{aligned}\measuredangle (AC;\ AB') &= \measuredangle (B'D';\ AB') - \measuredangle (B'D';\ AC) = \measuredangle D'B'A - 90^\circ \\ &= (90^\circ - \measuredangle B'C'D') - 90^\circ = -\measuredangle B'C'D'.\end{aligned}$$

Comparing this with $\measuredangle (DA;\ AB) = \measuredangle B'C'D'$, we infer $\measuredangle (AC;\ AB') = -\measuredangle (DA;\ AB)$. Thus, the line $AB'$ is the isogonal of the line $AB$ wrt the angle $DAC$. Similarly, the lines $CB'$ and $DB'$ are the isogonals of the lines $CB$ and $DB$ wrt the angles $ACD$ and $CDA$. Thus, the point $B'$ is the point of intersection of the isogonals of the lines $CB$, $DB$, $AB$ wrt the angles $ACD$, $CDA$, $DAC$. In other words, the point $B'$ is the isogonal conjugate of the point $B$ wrt the triangle $CDA$. Similarly, the points $C'$, $D'$, $A'$ are the isogonal conjugates of the points $C$, $D$, $A$ wrt the triangles $DAB$, $ABC$, $BCD$. Hence, Assertion $\mathcal{B}_4$ holds; this proves Auxiliary result 2.

*Second proof of Auxiliary result 2.* It is particularly easy to prove Auxiliary result 2 basing on Theorem 5:

Assume that Assertion $\mathcal{B}_3$ holds. Then, in particular, the point $A'$ is the reflection of the point $B'$ in the line $CD$. Hence, $DA' = DB'$. Similarly, $DB' = DC'$. Thus, $DA' = DB' = DC'$, and this signifies that the point $D$ is the circumcenter of triangle $A'B'C'$.

Since we assumed Assertion $\mathcal{B}_3$ to hold, the points $A'$, $B'$, $C'$ are the reflections of the point $D'$ in the lines $BC$, $CA$, $AB$. Thus, after Theorem 5, the isogonal conjugate of the point $D'$ wrt triangle $ABC$ is the circumcenter of triangle $A'B'C'$. But as we know that the point $D$ is the circumcenter of triangle $A'B'C'$, it follows that the point $D$ is the isogonal conjugate of the point $D'$ wrt triangle $ABC$. Hence, in turn, the point $D'$ is the isogonal conjugate of the point $D$ wrt triangle $ABC$. Similarly, the points $A'$, $B'$, $C'$ are the isogonal conjugates of the points $A$, $B$, $C$ wrt triangles $BCD$, $CDA$, $DAB$. Therefore, Assertion $\mathcal{B}_4$ holds. This again proves Auxiliary result 2.

This completes the proof of Theorem 12.

An easy consequence of Theorem 12 is (Fig. 27):

**Theorem 13.** Let $P$ be an Euclidean point in the plane of triangle $ABC$, and let $X'$, $Y'$, $Z'$ be the reflections of this point $P$ in the lines $BC$, $CA$, $AB$. Let $Q$ be the isogonal conjugate of the point $P$ wrt triangle $ABC$. Then, the points $X'$, $Y'$, $Z'$ are the isogonal conjugates of the points $A$, $B$, $C$ wrt the triangles $BQC$, $CQA$, $AQB$.

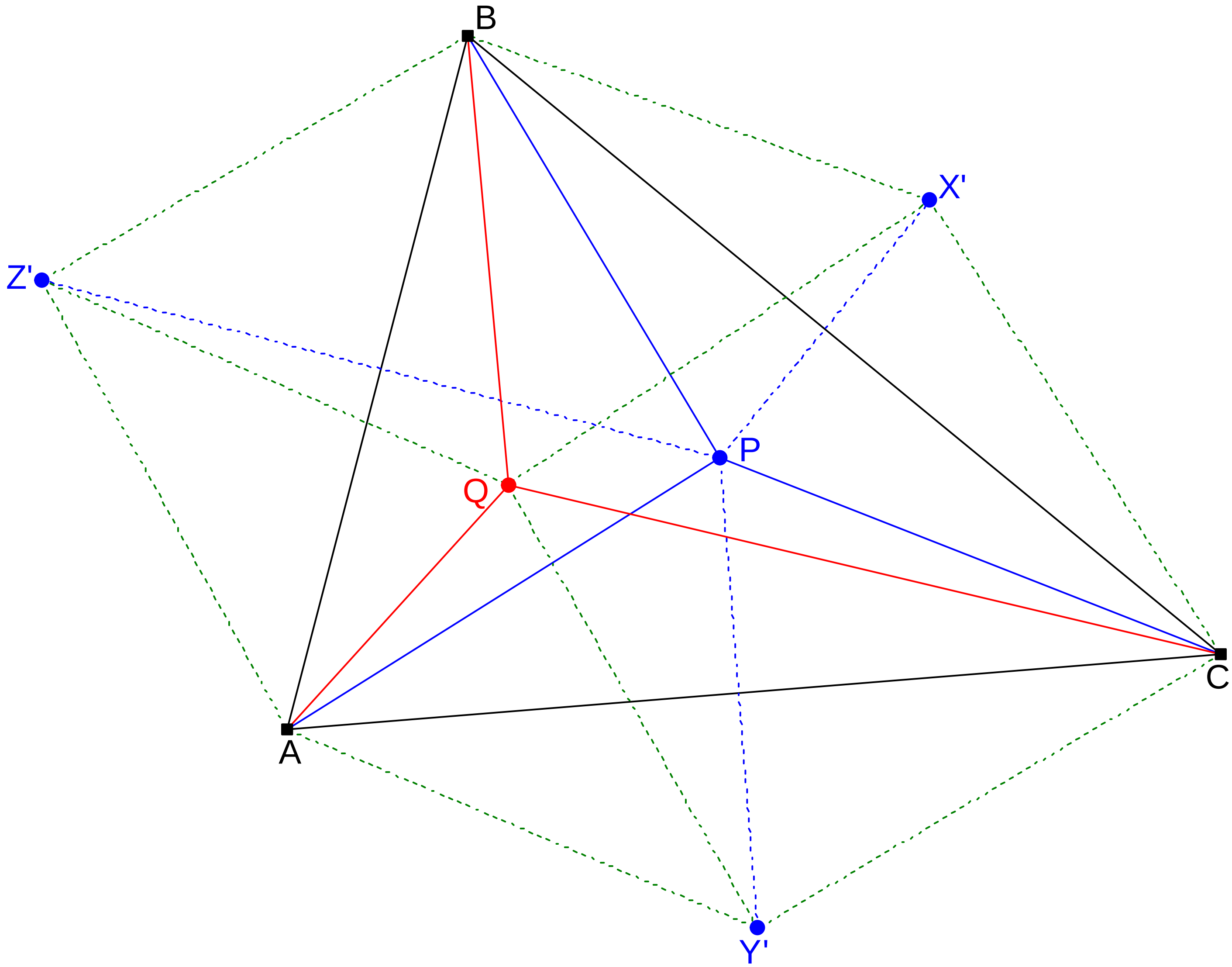


Fig. 27

*Proof of Theorem 13.* (See Fig. 28.) Since $Y'$ is the reflection of the point $P$ in the line $CA$, we have $AY' = AP$. Similarly, $AZ' = AP$. Therefore, $AY' = AP = AZ'$; thus, the point $A$ is the circumcenter of triangle $Y'Z'P$. Similarly, $BZ' = BP = BX'$ and $CX' = CP = CY'$, what yields that the points $B$ and $C$ are the circumcenters of triangles $Z'PX'$ and $PX'Y'$. Further, according to Theorem 5, the point $Q$ is the circumcenter of triangle $X'Y'Z'$. Altogether, we see that the points $A$, $B$, $C$, $Q$ are the circumcenters of triangles $Y'Z'P$, $Z'PX'$, $PX'Y'$, $X'Y'Z'$. Thus, the four distinct points $A$, $B$, $C$, $Q$ and the four distinct points $X'$, $Y'$, $Z'$, $P$ fulfill the Assertion $\mathcal{B}_1$ of Theorem 12. But since, according to Theorem 12, the Assertion $\mathcal{B}_1$ is equivalent to Assertion $\mathcal{B}_4$, these points must therefore also satisfy Assertion $\mathcal{B}_4$. In other words, the points $X'$, $Y'$, $Z'$, $P$ are the isogonal conjugates of the points $A$, $B$, $C$, $Q$ wrt the triangles $BCQ$, $CQA$, $QAB$, $ABC$. Equivalently, the points $X'$, $Y'$, $Z'$, $P$ are the isogonal conjugates of the points $A$, $B$, $C$, $Q$ wrt the triangles $BQC$, $CQA$, $AQB$, $ABC$. This implies Theorem 13.

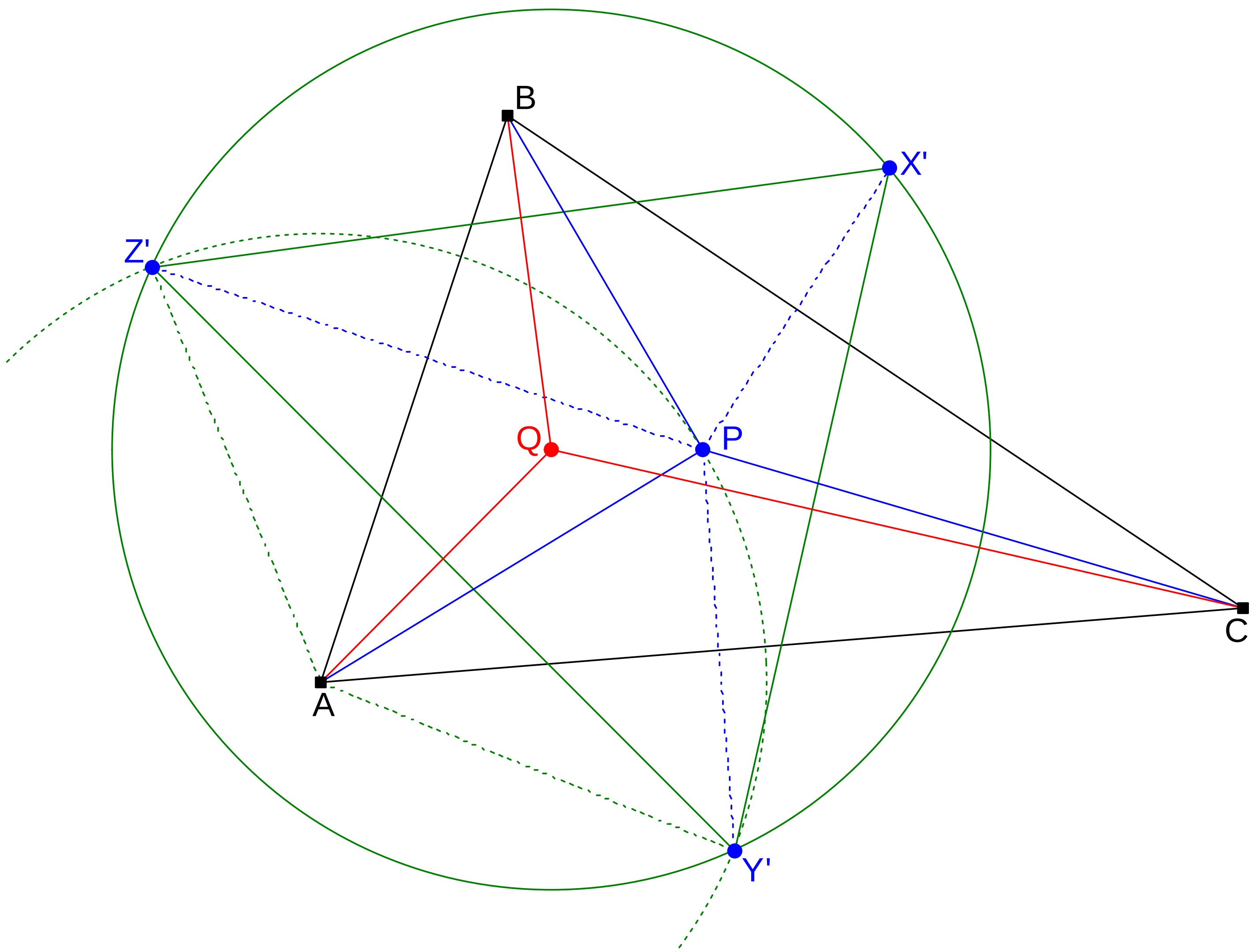


Fig. 28

Theorem 13 simplifies our proof of Theorem 8 given above. In fact, according to Theorem 13, the point $X'$ is the isogonal conjugate of the point $A$ wrt triangle $BQC$, what immediately yields that the line $QX'$ is the isogonal of the line $QA$ (that is, the line $AQ$) wrt the angle $BQC$; this was a crucial result in the proof of Theorem 8. [Thanks to Marcello Tarquini for reminding me of this shortcut.]

## 8. Isogonal conjugates of basic centers

It is useful to identify the isogonal conjugates of known triangle centers. We start with a trivial fact:

**Theorem 14.** Let $P$ be a point in the plane of a triangle $ABC$. Then, the isogonal conjugate of the point $P$ wrt triangle $ABC$ coincides with the point $P$ if and only if the point $P$ is the incenter or one of the three excenters of triangle $ABC$. (See Fig. 29.)

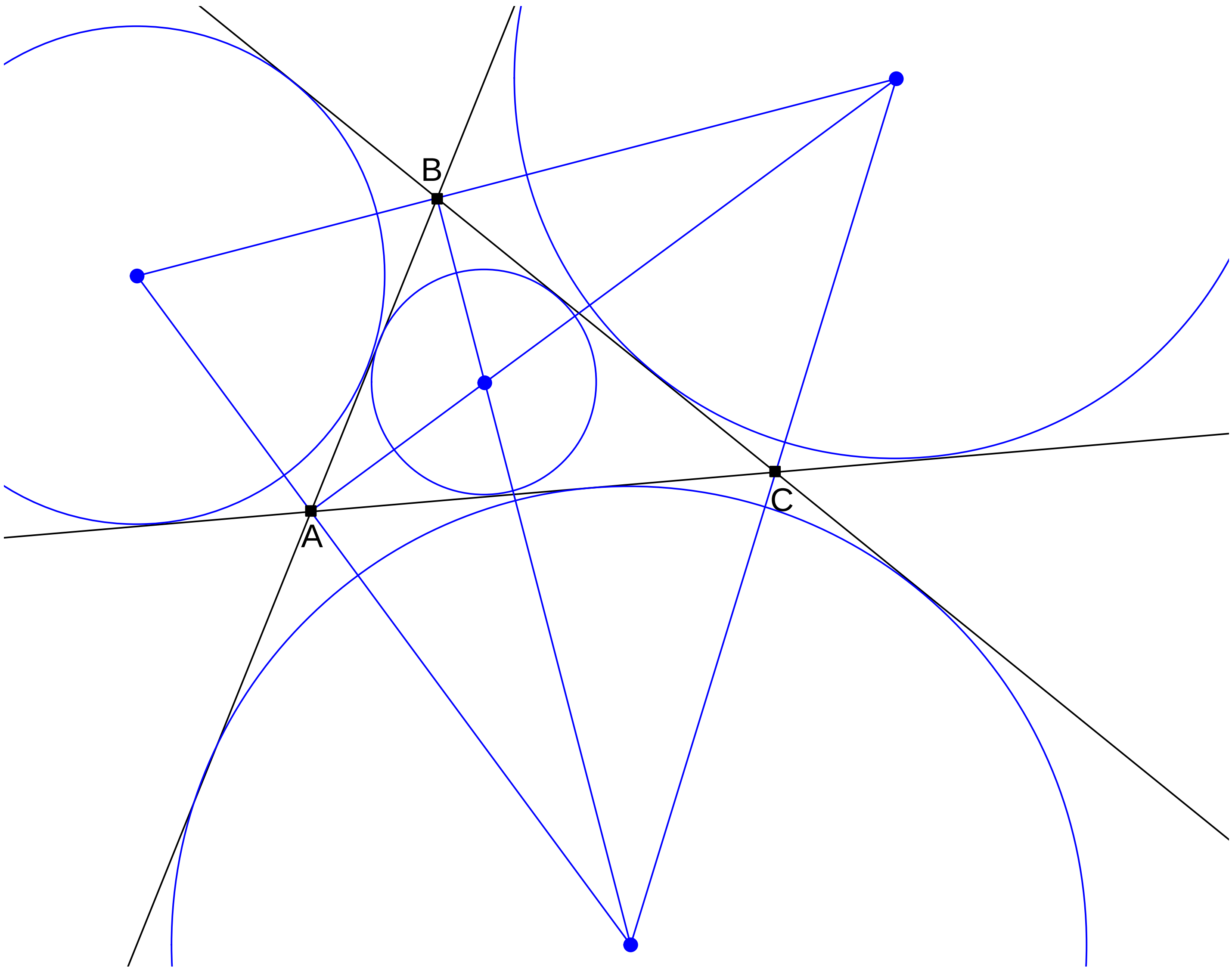


Fig. 29

*Proof of Theorem 14.* According to the definition of isogonals, the isogonal of the line $AP$ wrt the angle $CAB$ coincides with the line $AP$ if and only if $\measuredangle(AB;\ AP) = -\measuredangle(CA;\ AP)$. But this is equivalent to the line $AP$ being either the internal or the external angle bisector of the angle $CAB$. Hence, the isogonal of the line $AP$ wrt the angle $CAB$ coincides with the line $AP$ if and only if the line $AP$ is either the internal or the external angle bisector of the angle $CAB$. Similarly, the isogonal of the line $BP$ wrt the angle $ABC$ coincides with the line $BP$ if and only if the line $BP$ is either the internal or the external angle bisector of the angle $ABC$, and the isogonal of the line $CP$ wrt the angle $BCA$ coincides with the line $CP$ if and only if the line $CP$ is either the internal or the external angle bisector of the angle $BCA$.

The isogonal conjugate of the point $P$ wrt triangle $ABC$ is the point of intersection of the isogonals of the lines $AP$, $BP$, $CP$ wrt the angles $CAB$, $ABC$, $BCA$. Consequently, the isogonal conjugate of the point $P$ wrt triangle $ABC$ coincides with the point $P$ if and only if the isogonals of the lines $AP$, $BP$, $CP$ wrt the angles $CAB$, $ABC$, $BCA$ intersect at the point $P$, i. e. if and only if the isogonals of the lines $AP$, $BP$, $CP$ wrt the angles $CAB$, $ABC$, $BCA$ coincide with the lines $AP$, $BP$, $CP$. As we know, this is equivalent to the lines $AP$, $BP$, $CP$ being the internal or external angle bisectors of the angles $CAB$, $ABC$, $BCA$. But there are exactly four points $P$ in the plane of triangle $ABC$ such that the lines $AP$, $BP$, $CP$ are the internal or external angle bisectors of the angles $CAB$, $ABC$, $BCA$, and these points are the incenter and the three excenters of triangle $ABC$. Thus it is proven that the isogonal conjugate of the point $P$ wrt triangle $ABC$ coincides with the point $P$ if and only if the point $P$ is the incenter or one of the three excenters of triangle $ABC$. This proves Theorem 14.

A slightly less trivial fact is the following (Fig. 30):

**Theorem 15.** Let $O$ be the circumcenter and $H$ the orthocenter of a triangle $ABC$. Then, the points $O$ and $H$ are isogonally conjugate points wrt triangle $ABC$.

In brief: The circumcenter and the orthocenter of any triangle are isogonally conjugate points wrt this triangle.

*First proof of Theorem 15.* Being the circumcenter of triangle $ABC$, the point $O$ is the center of a circle through the points $A$, $B$, $C$. Thus, according to the central angle theorem, $\measuredangle CAO = 90^\circ - \measuredangle ABC$. On the other hand, since $H$ is the orthocenter of triangle $ABC$, we know that $AH \perp BC$, so that $\measuredangle\left(AH;\ BC\right) = 90^\circ$, and therefore

$$\begin{aligned}\measuredangle\left(AB;\ AH\right) &= \measuredangle\left(AB;\ BC\right) - \measuredangle\left(AH;\ BC\right) = \measuredangle ABC - 90^\circ = -\left(90^\circ - \measuredangle ABC\right) \\ &= -\measuredangle CAO = -\measuredangle\left(CA;\ AO\right).\end{aligned}$$

Hence, the line $AH$ is the isogonal of the line $AO$ wrt the angle $CAB$. Similarly, the lines $BH$ and $CH$ are the isogonals of the lines $BO$ and $CO$ wrt the angles $ABC$ and $BCA$. Thus, the point $H$, being the point of intersection of these lines $AH$, $BH$, $CH$, must be the isogonal conjugate of the point $O$ wrt triangle $ABC$. This proves Theorem 15.

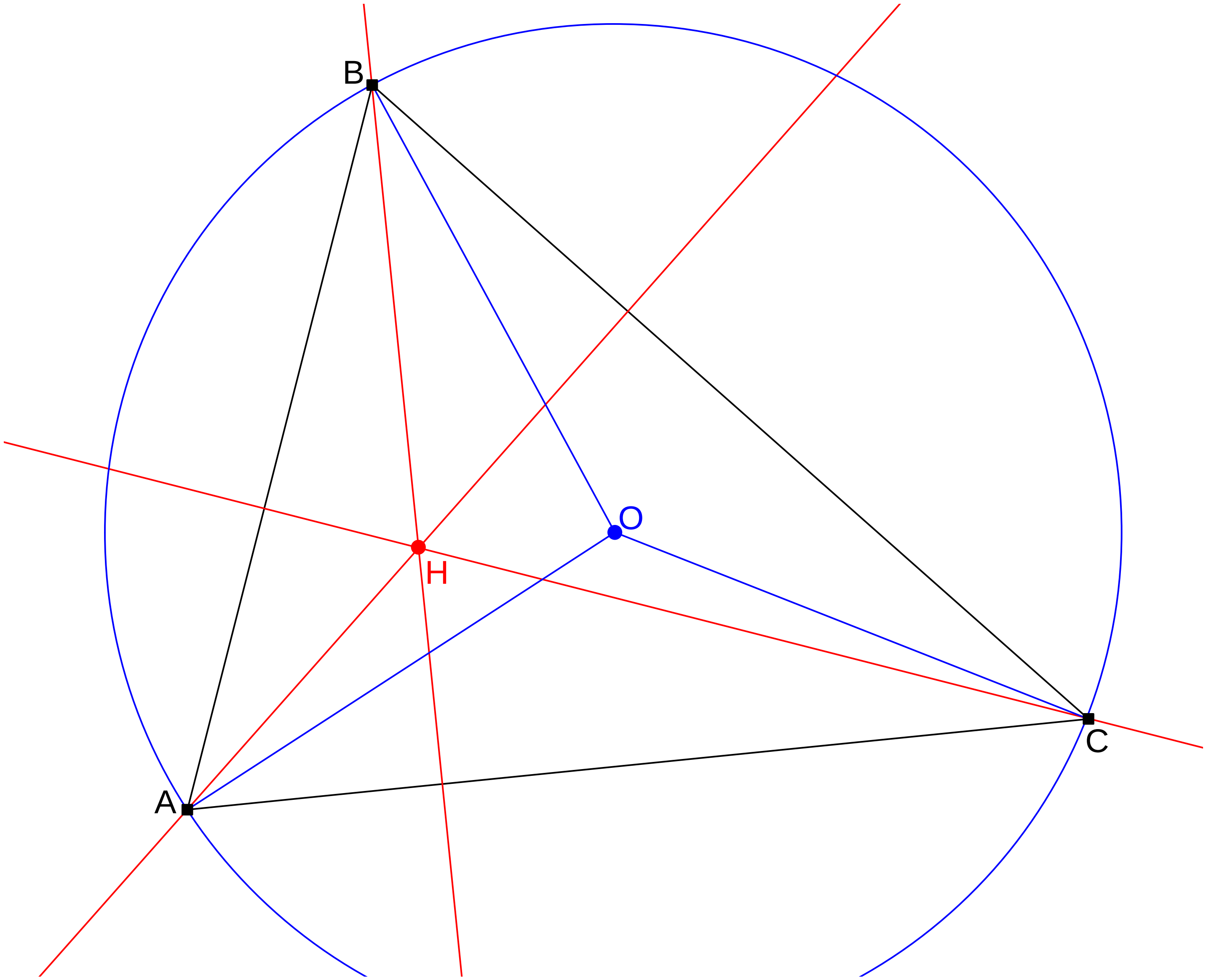


Fig. 30

*Second proof of Theorem 15.* (See Fig. 31.) Let $B_M$ and $C_M$ be the midpoints of the sides $CA$ and $AB$ of triangle $ABC$. Then, $B_MC_M \parallel BC$. On the other hand, since the circumcenter $O$ of triangle $ABC$ lies on the perpendicular bisectors of its sides $CA$ and $AB$, the midpoints $B_M$ and $C_M$ of these sides $CA$ and $AB$ must be the orthogonal projections of the point $O$ on the lines $CA$ and $AB$. Hence, by Theorem 2 **a)** (applied to the two lines $CA$ and $AB$ and the point $O$ in the plane), the line $B_MC_M$ is perpendicular to the isogonal of the line $AO$ wrt the lines $CA$ and $AB$, i. e. wrt the angle $CAB$. In other words: The isogonal of the line $AO$ wrt the angle $CAB$ is perpendicular to the line $B_MC_M$. Since $B_MC_M \parallel BC$, this isogonal must therefore be perpendicular to the line $BC$, and since this isogonal passes through the point $A$, we can conclude that it is the $A$-altitude of triangle $ABC$ and thus passes through its orthocenter $H$. So the point $H$ lies on the isogonal of the line $AO$ wrt the angle $CAB$. Similarly, the point $H$ lies on the isogonals of the lines $BO$ and $CO$ wrt the angles $ABC$ and $BCA$. Thus, the point $H$ is the point of intersection of these isogonals, i. e. the isogonal conjugate of the point $O$ wrt triangle $ABC$. Once again Theorem 15 is proven.

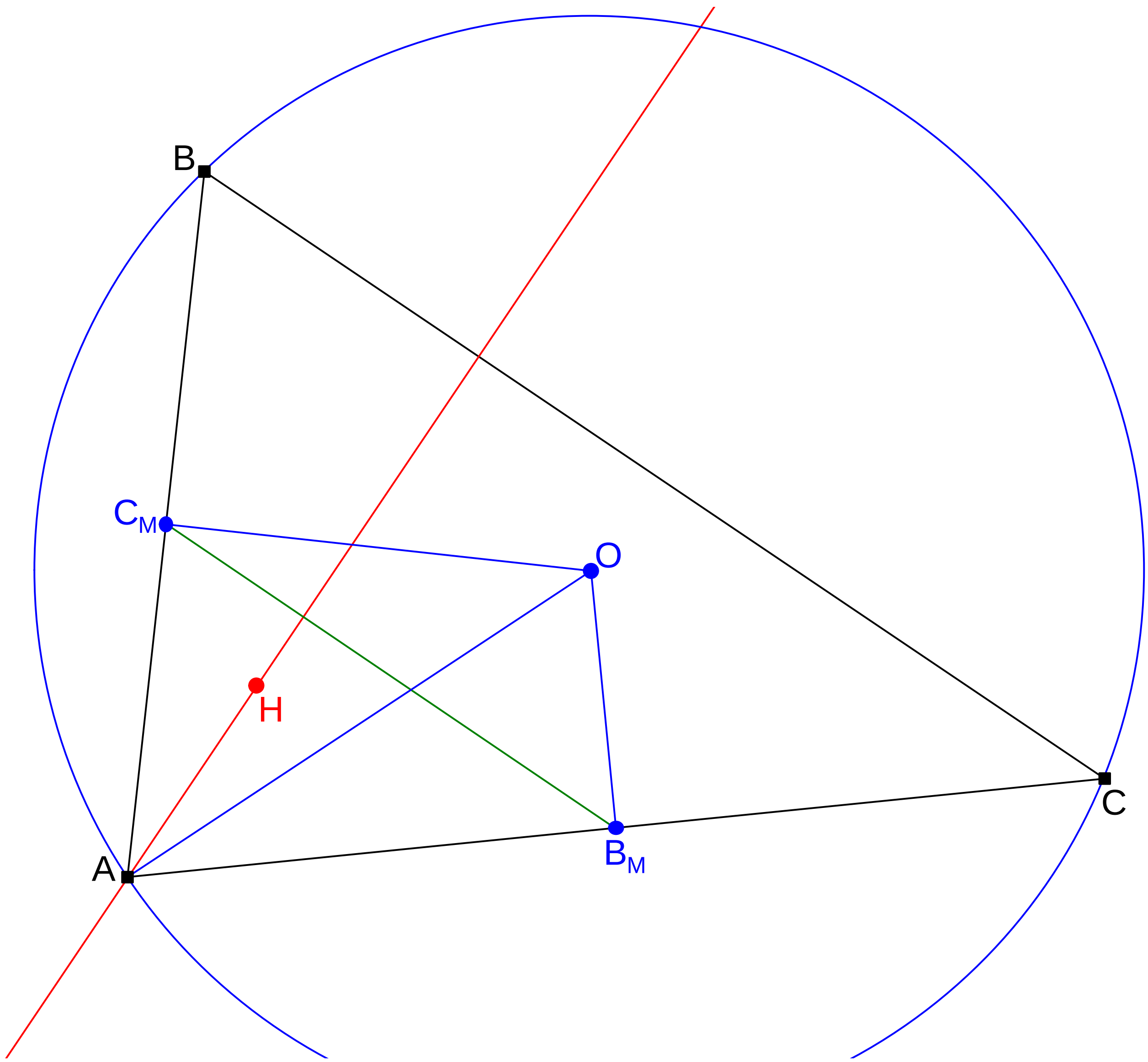


Fig. 31

At this point, we mention an easy consequence of Theorem 15 which will be of use to us later (Fig. 32):

**Theorem 16.** Let $A_H$, $B_H$, $C_H$ be the feet of the altitudes of triangle $ABC$ from the vertices $A$, $B$, $C$. Let $O$ be the circumcenter of triangle $ABC$. Then, $AO \perp B_H C_H$, $BO \perp C_H A_H$, $CO \perp A_H B_H$.

*Proof of Theorem 16.* Obviously, the orthocenter $H$ of triangle $ABC$ lies on the altitude of triangle $ABC$ from the vertex $A$. The point $A_H$ is the foot of this altitude. Thus, the point $A_H$ is the orthogonal projection of the point $H$ on the line $BC$. Similarly, the points $B_H$ and $C_H$ are the orthogonal projections of the point $H$ on the lines $CA$ and $AB$. By Theorem 15, the point $O$ is the isogonal conjugate of the point $H$ wrt triangle $ABC$. Thus, applying Theorem 6 to the point $H$ in the plane of triangle $ABC$, we get $AO \perp B_H C_H$, $BO \perp C_H A_H$, $CO \perp A_H B_H$. This proves Theorem 16.

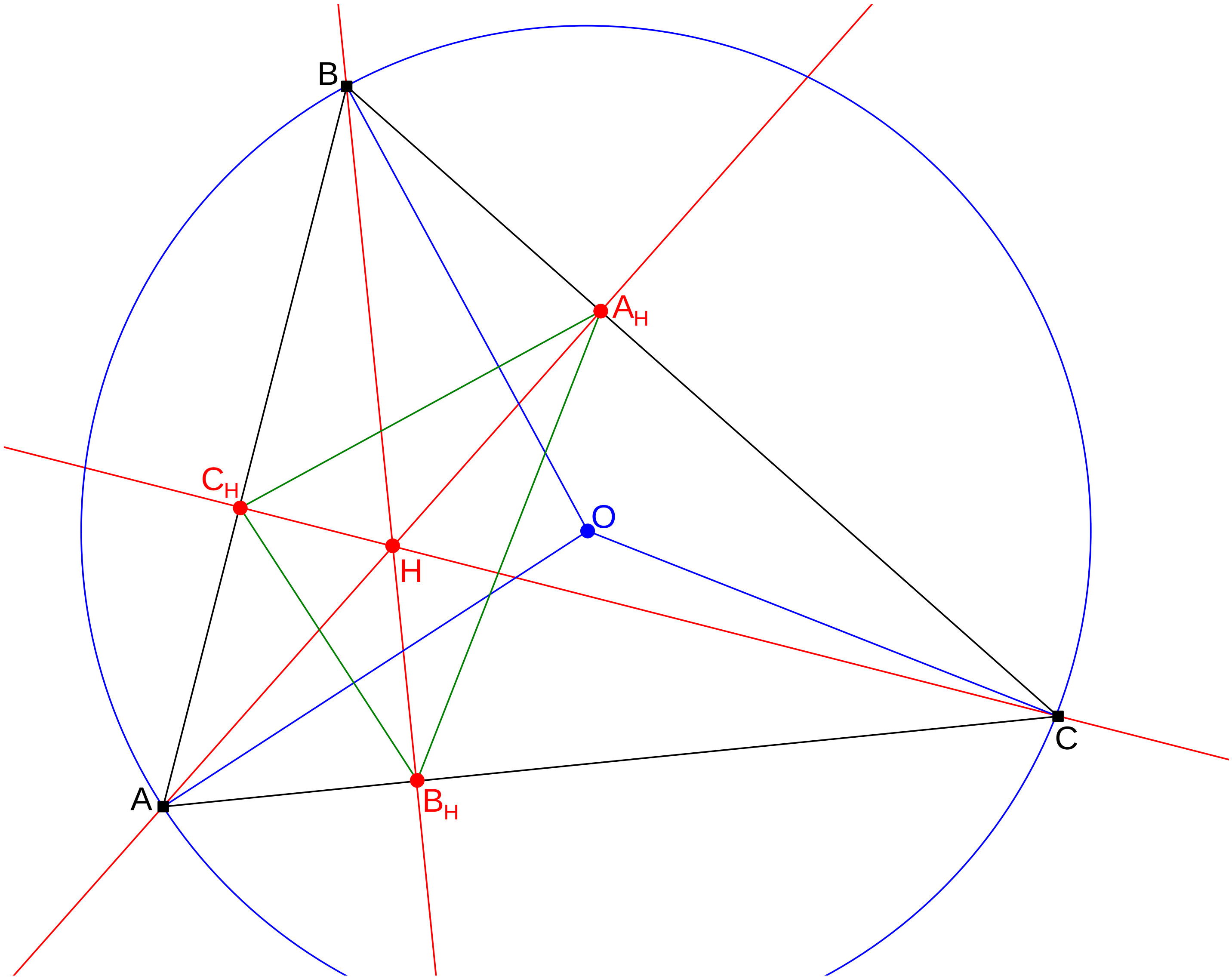


Fig. 32

Theorem 16 can also be directly shown through an angle chase.

## 9. Symmedians and antiparallels

(See Fig. 33.) Let $S$ be the centroid of triangle $ABC$. Then, the lines $AS$, $BS$, and $CS$ are the $A$-median, the $B$-median, and the $C$-median of triangle $ABC$, respectively. The isogonals of these medians $AS$, $BS$, $CS$ wrt the angles $CAB$, $ABC$, $BCA$ are called the **$A$-symmedian**, **$B$-symmedian**, **$C$-symmedian** of triangle $ABC$, respectively, and altogether referred to as the three **symmedians** of triangle $ABC$. As a consequence of their definition, these symmedians of triangle $ABC$ intersect at one point, namely at the isogonal conjugate of the point $S$ wrt triangle $ABC$. This isogonal conjugate of the point $S$ wrt triangle $ABC$ is called the **symmedian point** of triangle $ABC$ and will be denoted by $K$ in the following. Then, as the $A$-symmedian, the $B$-symmedian, and the $C$-symmedian of triangle $ABC$ intersect at the point $K$, they are the lines $AK$, $BK$, and $CK$.

In brief: The symmedian point of a triangle is the point of intersection of its symmedians, and it is the isogonal conjugate of the centroid of this triangle wrt this triangle.

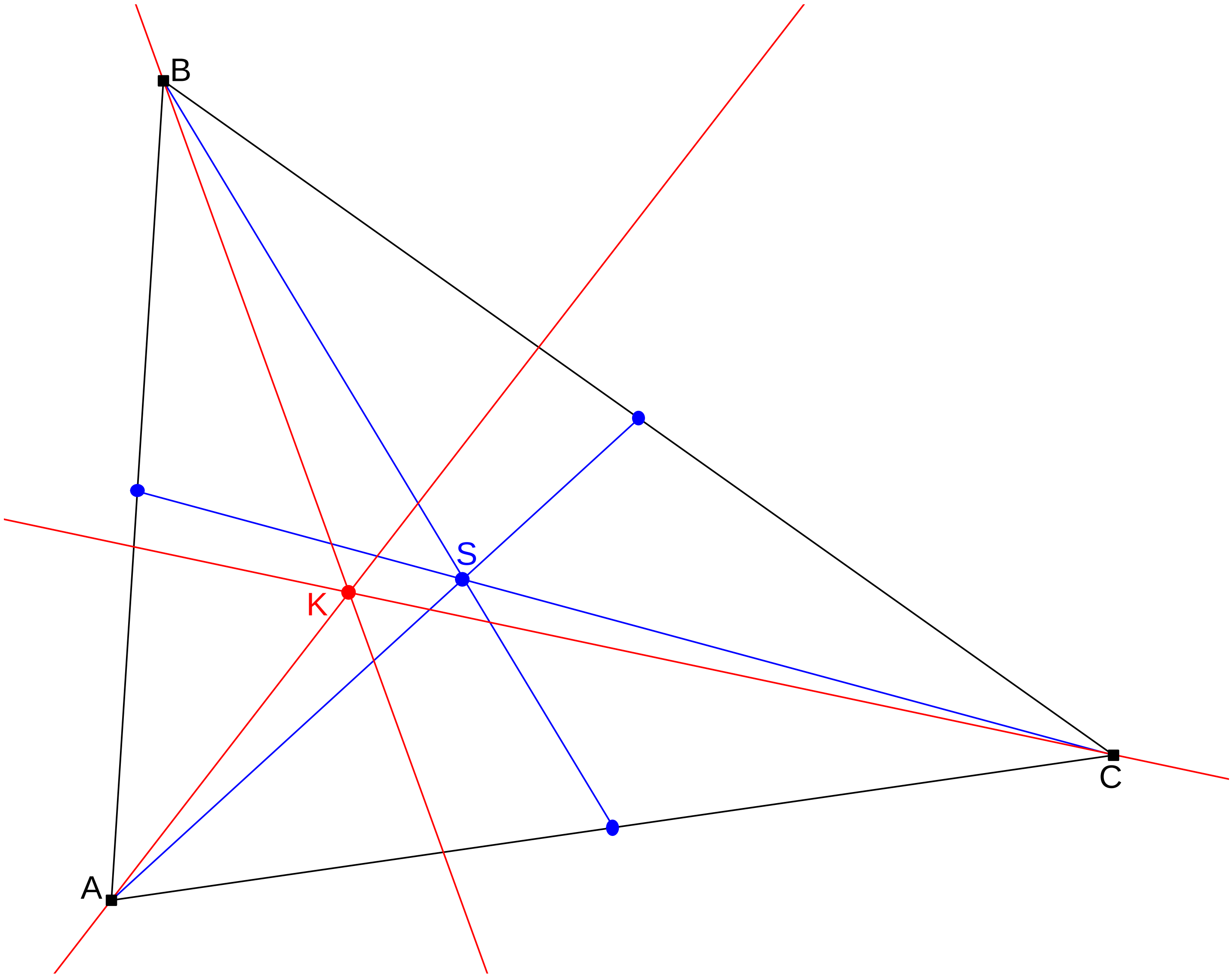


Fig. 33

We won't discuss further properties of the symmedian point here, but we use the occasion to give an introduction into the often utilized concept of antiparallels wrt a side of a triangle:

Let $ABC$ be a triangle, and $g$ a line in the plane. The line $g$ is said to be **antiparallel** to $BC$ wrt the triangle $ABC$ if and only if it is parallel to the tangent to the circumcircle of triangle $ABC$ at the point $A$. In this case, one also uses to say that the line $g$ is an **antiparallel** to the side $BC$ of triangle $ABC$.

Similarly, we define when a line is called antiparallel to $CA$ or to $AB$ wrt triangle $ABC$, or, in other words, when it is an antiparallel to the side $CA$ or to the side $AB$ of triangle $ABC$.

An important remark about this notion of antiparallelism is that the meaning of the word "antiparallel", in contrast to the meaning of the word "parallel", depends on the triangle $ABC$: Just to say that two lines are antiparallel to each other doesn't make sense; one can only say that a line is antiparallel to a side of a triangle wrt this triangle. Accordingly, in the formulation "the line $g$ is an antiparallel to the side $BC$ of triangle $ABC$ ", one cannot omit the clause "of triangle $ABC$ ", since it specifies the triangle of reference for the notion "antiparallel".

Clearly, through any point there exists exactly one antiparallel to the side $BC$ of triangle $ABC$ (in fact, by its definition, an antiparallel to the side $BC$ of triangle $ABC$ means a parallel to the tangent to the circumcircle of triangle $ABC$ at the point $A$, and through any point there exists exactly one parallel to this tangent). Similarly, through any point there exists exactly one antiparallel to the side $CA$ of triangle $ABC$ and exactly one antiparallel to the side $AB$ of triangle $ABC$.

The basic advantage of the notion of antiparallelism are the many equivalent criteria for a line to be antiparallel to a side of a triangle. Some of these criteria are given by the following theorem[3]:

**Theorem 17.** Let $ABC$ be a triangle, and let $g$ be a line in the plane. Consider the following ten assertions:

**Assertion $\mathcal{E}_0$:** The line $g$ is antiparallel to $BC$ wrt triangle $ABC$.

**Assertion $\mathcal{E}_1$:** The line $g$ is parallel to the tangent to the circumcircle of triangle $ABC$ at the point $A$.

**Assertion $\mathcal{E}_2$:** The line $g$ is perpendicular to the line $AO$, where $O$ is the circumcenter of triangle $ABC$.

**Assertion $\mathcal{E}_3$:** We have $\measuredangle(CA;\ g) = \measuredangle(BC;\ AB)$.

**Assertion $\mathcal{E}_4$:** We have $\measuredangle(AB;\ g) = \measuredangle(BC;\ CA)$.

**Assertion $\mathcal{E}_5$:** The line $g$ is parallel to the line $B_HC_H$, where $B_H$ and $C_H$ are the feet of the $B$-altitude and the $C$-altitude of triangle $ABC$. [4]

**Assertion $\mathcal{E}_6$:** If $P$ and $Q$ are the points of intersection of the line $g$ with the lines $CA$ and $AB$, then the triangles $APQ$ and $ABC$ are oppositely similar.

**Assertion $\mathcal{E}_7$:** If $P$ and $Q$ are the points of intersection of the line $g$ with the lines $CA$ and $AB$, then there exists a circle which meets the line $CA$ at the points $C$ and $P$ and meets the line $AB$ at the points $B$ and $Q$. [5]

**Assertion $\mathcal{E}_8$:** If $P$ and $Q$ are the points of intersection of the line $g$ with the lines $CA$ and $AB$, then $AB \cdot AQ = AC \cdot AP$, where the segments are directed.

**Assertion $\mathcal{E}_9$:** If $P$ and $Q$ are the points of intersection of the line $g$ with the lines $CA$ and $AB$, then the midpoint of the segment $PQ$ lies on the $A$-symmedian of triangle

[3] Of course, the points $P$ and $Q$ from Theorem 17 have nothing to do with the points $P$ and $Q$ from the above results on isogonal conjugates.

[4] If triangle $ABC$ is right-angled at $A$, then these feet $B_H$ and $C_H$ both coincide with the vertex $A$; in this case, the line $B_HC_H$ is to be understood as the tangent to the circumcircle of triangle $ABC$ at the point $A$.

[5] Hereby, the following convention applies:

If a circle $k$ touches a line $g$ at a point $T$, then we say that the circle $k$ meets the line $g$ at the points $T$ and $T$.

This convention clarifies how the Assertion $\mathcal{E}_7$ is to be understood if e. g. the point $P$ coincides with the point $C$. In this case, Assertion $\mathcal{E}_7$ states that there exists a circle which meets the line $CA$ at the points $C$ and $C$ (that is, touches the line $CA$ at the point $C$) and meets the line $AB$ at the points $B$ and $Q$.

This is the reason why the formulation "there exists a circle which meets the line $CA$ at the points $C$ and $P$ and meets the line $AB$ at the points $B$ and $Q$ " is superior to the shorter formulation "the points $B$, $C$, $P$, $Q$ lie on one circle". In fact, if $P \neq C$ and $Q \neq B$, these formulations are equivalent, but e. g. in the case when the point $P$ coincides with the point $C$, the points $B$, $C$, $P$, $Q$ always lie on one circle; hence, if we had used the formulation "the points $B$, $C$, $P$, $Q$ lie on one circle" for Assertion $\mathcal{E}_7$, then this Assertion $\mathcal{E}_7$ would not be equivalent to $\mathcal{E}_0$ in the case $P = C$ (and similarly in the case $Q = B$ as well).

$ABC$.

Then, we have:

**a)** The six assertions $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$ and $\mathcal{E}_5$ are pairwisely equivalent.

**b)** If the line $g$ doesn't pass through the point $A$, then the ten assertions $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$, $\mathcal{E}_5$, $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$ and $\mathcal{E}_9$ are pairwisely equivalent.

(See Fig. 34 for Assertions $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$, Fig. 36 for Assertion $\mathcal{E}_5$, Fig. 37 for Assertions $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$, and Fig. 38 for Assertion $\mathcal{E}_9$.)

Theorem 17 yields, altogether, nine criteria for a line to be antiparallel to $BC$ wrt triangle $ABC$. Similar criteria hold for antiparallelism to $CA$ or to $AB$.

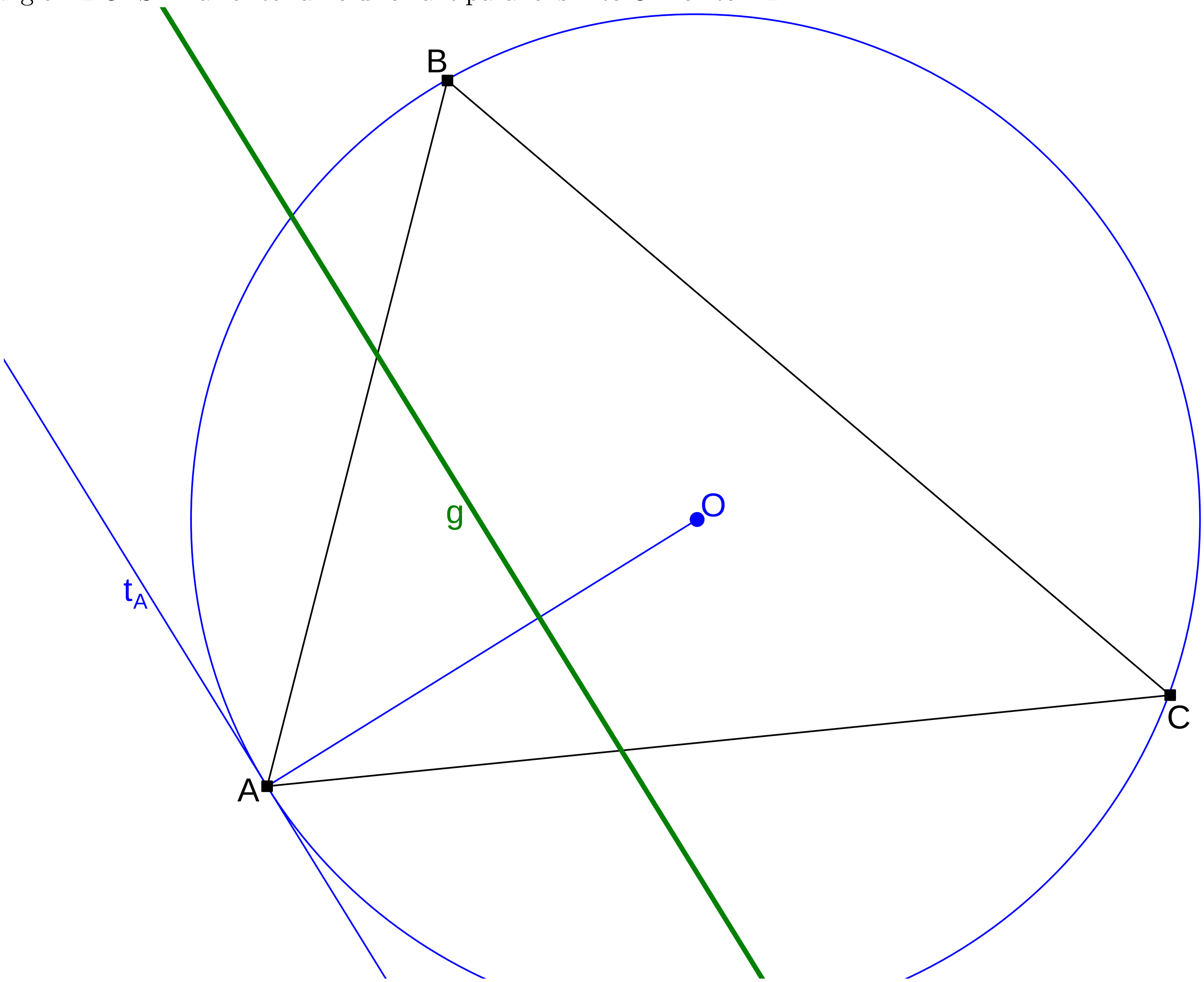


Fig. 34

Our proof of Theorem 17 will require a fact which slightly extends the intersecting chords theorem, the intersecting secants theorem, and the intersecting secant and tangent theorem:

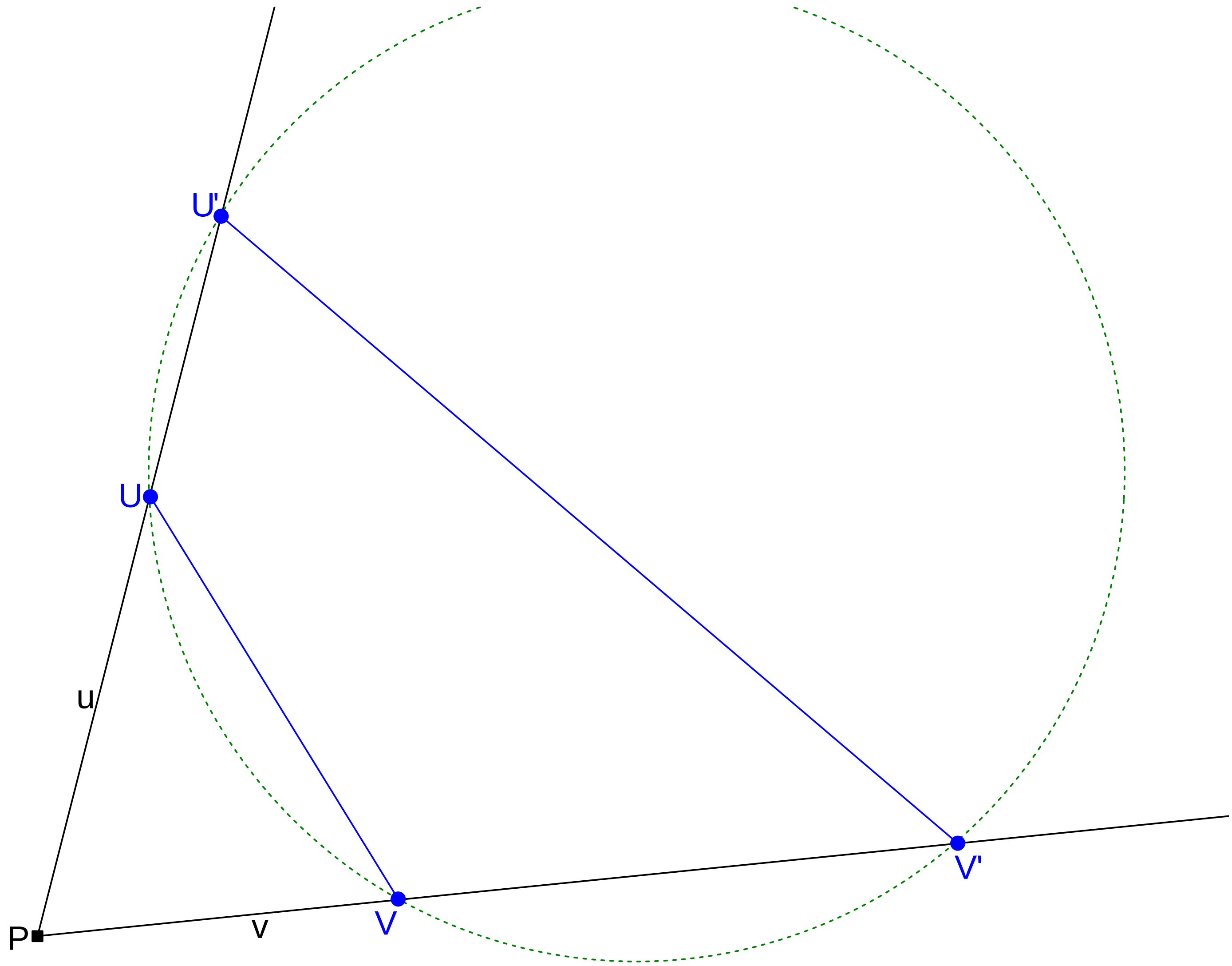


Fig. 35

**Theorem 18.** Let $u$ and $v$ be two lines which intersect at an Euclidean point $P$. Let $U$ and $U'$ be two points on the line $u$ distinct from $P$, and let $V$ and $V'$ be two points on the line $v$ distinct from $P$. Then, the following three assertions $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ are pairwisely equivalent:

**Assertion $\mathcal{D}_1$:** The triangles $PUV$ and $PV'U'$ are oppositely similar.

**Assertion $\mathcal{D}_2$:** There exists a circle which meets the line $u$ at the points $U$ and $U'$ and meets the line $v$ at the points $V$ and $V'$. [6]

**Assertion $\mathcal{D}_3$:** We have $PU \cdot PU' = PV \cdot PV'$, where the segments are directed.

(See Fig. 35.)

*Proof of Theorem 18.*[7] We consider only the case when $U \neq U'$ and $V \neq V'$. The

[6] Hereby, the same convention applies as in Assertion $\mathcal{E}_7$ of Theorem 17.

[7] This proof is given here but for the sake of consequence in our application of directed angles modulo 180°. In fact, the equivalence of Assertions $\mathcal{D}_1$ and $\mathcal{D}_3$ is a trivial corollary of a well-known similitude criterion which states that two triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ are oppositely similar if and only if $\measuredangle P_1P_2P_3 = -\measuredangle Q_1Q_2Q_3$ und $P_1P_2 : P_2P_3 = Q_1Q_2 : Q_2Q_3$. But here, the sign $\measuredangle$ stands for directed angles modulo 360°; such a similitude criterion cannot hold for directed angles modulo 180°. As we aim at using directed angles modulo 180° throughout this paper, we will prove this equivalence in a different way.

cases $U = U'$ and $V = V'$ can be handled in the same way, with the only difference that here and there instead of chordal angles, we have angles between a chord and a tangent.

First we show the equivalence of the Assertions $\mathcal{D}_1$ and $\mathcal{D}_2$:

Assertion $\mathcal{D}_1$ states that triangles $PUV$ and $PV'U'$ are oppositely similar. This is equivalent to $\measuredangle PUV = -\measuredangle PV'U'$ (in fact, if triangles $PUV$ and $PV'U'$ are oppositely similar, then $\measuredangle PUV = -\measuredangle PV'U'$, and conversely, if $\measuredangle PUV = -\measuredangle PV'U'$, then, together with $\measuredangle UPV = -\measuredangle V'PU'$, this yields the opposite similarity of triangles $PUV$ and $PV'U'$). But $\measuredangle PUV = -\measuredangle PV'U'$ rewrites as $\measuredangle U'UV = \measuredangle U'V'V$, and this equation holds if and only if the points $U$, $V$, $U'$ and $V'$ lie on one circle. Since $U \neq U'$ and $V \neq V'$, the points $U$, $V$, $U'$ and $V'$ lie on one circle if and only if there exists a circle which meets the line $u$ at the points $U$ and $U'$ and meets the line $v$ at the points $V$ and $V'$. The latter is Assertion $\mathcal{D}_2$. Combining these steps, we realize that we have proven the equivalence of Assertions $\mathcal{D}_1$ and $\mathcal{D}_2$.

Now we are going to prove the equivalence of Assertions $\mathcal{D}_2$ and $\mathcal{D}_3$:

If Assertion $\mathcal{D}_2$ holds, then there exists a circle which meets the line $u$ at the points $U$ and $U'$ and meets the line $v$ at the points $V$ and $V'$. The power of the point $P$ wrt this circle equals $PU \cdot PU'$ on the one hand, and equals $PV \cdot PV'$ on the other hand. Thus, $PU \cdot PU' = PV \cdot PV'$, so that Assertion $\mathcal{D}_3$ is valid.

Conversely: If we assume Assertion $\mathcal{D}_3$ to hold, then $PU \cdot PU' = PV \cdot PV'$. Now, let $V_1'$ be the point of intersection of the circle through the points $U$, $U'$ and $V$ with the line $v$ different from $V$. [8] Then, this circle meets the line $u$ at the points $U$ and $U'$ and meets the line $v$ at the points $V$ and $V_1'$. Consequently, the power of the point $P$ wrt this circle equals $PU \cdot PU'$ on the one hand, and equals $PV \cdot PV_1'$ on the other hand. Thus, $PU \cdot PU' = PV \cdot PV_1'$. Comparison with $PU \cdot PU' = PV \cdot PV'$ yields $PV \cdot PV_1' = PV \cdot PV'$, thus $PV_1' = PV'$ (since $PV \neq 0$). Since the points $V_1'$ and $V'$ both lie on the line $v$ and since we use directed segments, this entails that the points $V_1'$ and $V'$ coincide. The fact that the circle through the points $U$, $U'$ and $V$ meets the line $u$ at the points $U$ and $U'$ and meets the line $v$ at the points $V$ and $V_1'$ can now be rewritten as follows: The circle through the points $U$, $U'$ and $V$ meets the line $u$ at the points $U$ and $U'$ and meets the line $v$ at the points $V$ and $V'$. Thus, Assertion $\mathcal{D}_2$ is fulfilled.

Hence, we have shown the equivalence of Assertions $\mathcal{D}_2$ and $\mathcal{D}_3$. Altogether, we subsume that all three Assertions $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ are equivalent, and Theorem 18 is proven.

*Proof of Theorem 17.* **a)** The equivalence of Assertions $\mathcal{E}_0$ and $\mathcal{E}_1$ is a paraphrase of the definition of antiparallelism.

In order to show the equivalence of Assertions $\mathcal{E}_1$ and $\mathcal{E}_2$, it is enough to show that the tangent to the circumcircle of triangle $ABC$ at the point $A$ is perpendicular to the line $AO$. But this is clear, since $O$ is the center of this circumcircle.

[8] If this circle happens to touch the line $v$, then we set $V_1' = V$.

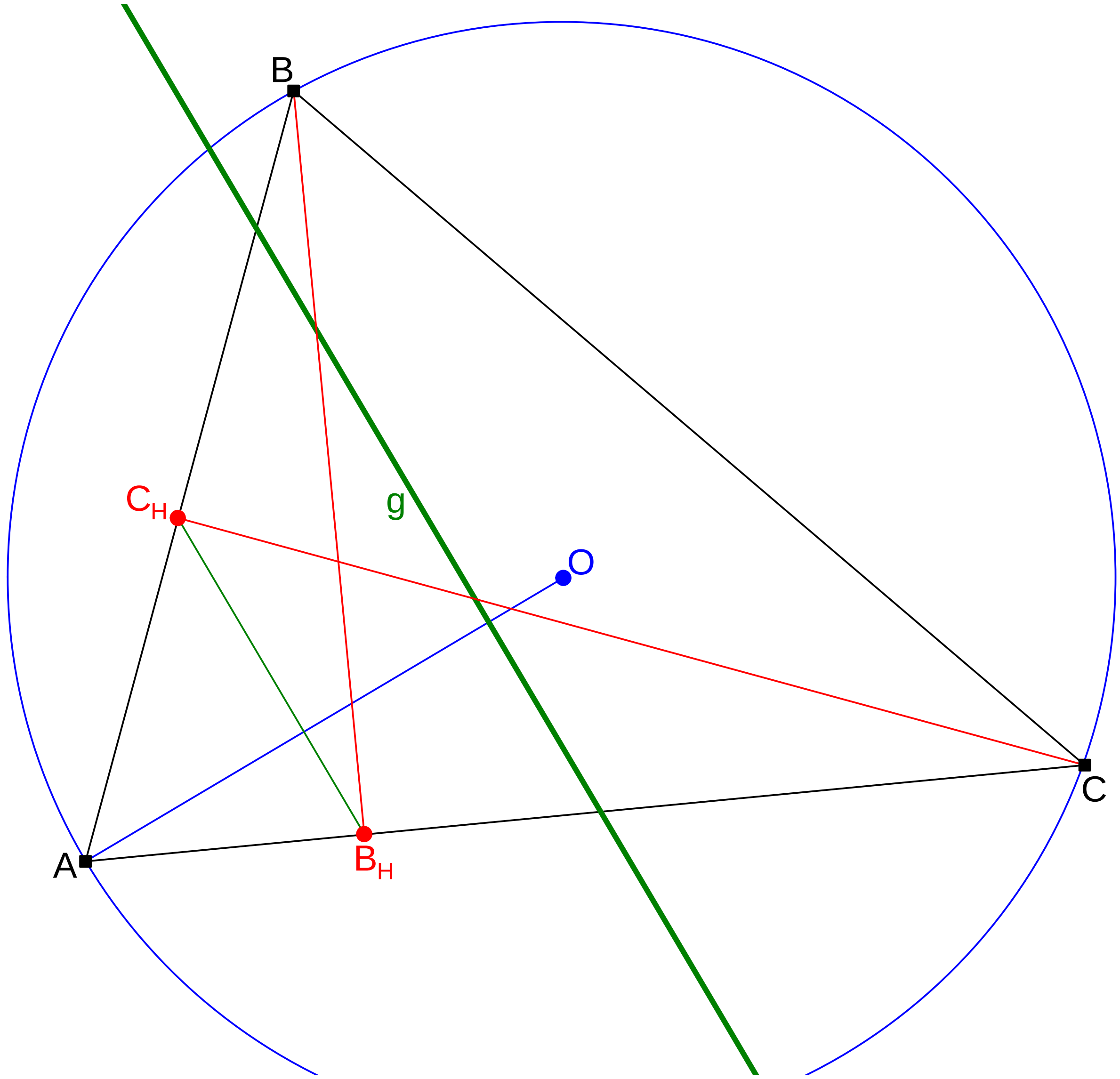


Fig. 36

The equivalence of Assertions $\mathcal{E}_1$ and $\mathcal{E}_3$ can be proven as follows: (See Fig. 35.) Let $t_A$ be the tangent to the circumcircle of triangle $ABC$ at the point $A$. Then, Assertion $\mathcal{E}_1$ states that $g \parallel t_A$. This is equivalent to $\measuredangle (CA;\ g) = \measuredangle (CA;\ t_A)$. But since $t_A$ is the tangent to the circumcircle of triangle $ABC$ at the point $A$, while $\measuredangle CBA$ is the chordal angle of the chord $CA$ in this circumcircle, we have $\measuredangle (CA;\ t_A) = \measuredangle CBA$, what rewrites as $\measuredangle (CA;\ t_A) = \measuredangle (BC;\ AB)$. Hence, the equation $\measuredangle (CA;\ g) = \measuredangle (CA;\ t_A)$ is equivalent to the equation $\measuredangle (CA;\ g) = \measuredangle (BC;\ AB)$. But this equation is Assertion $\mathcal{E}_3$. Hence it is shown that Assertion $\mathcal{E}_1$ is equivalent to Assertion $\mathcal{E}_3$.

An analogous argument shows the equivalence of the Assertions $\mathcal{E}_1$ and $\mathcal{E}_4$.

In order to prove the equivalence of the Assertions $\mathcal{E}_2$ and $\mathcal{E}_5$, it is obviously sufficient to verify that $AO \perp B_H C_H$. But this follows from Theorem 16.[9]

Altogether, we have proven all six Assertions $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$ and $\mathcal{E}_5$ to be equivalent. Thus, the proof of Theorem 17 **a)** is complete.

[9] The relation $AO \perp B_H C_H$ also holds in the case when triangle $ABC$ is right-angled at $A$. In fact, in this case we have specified that the line $B_H C_H$ is the tangent to the circumcircle of triangle $ABC$ at the point $A$, and this tangent is perpendicular to the line $AO$ (as we already saw).

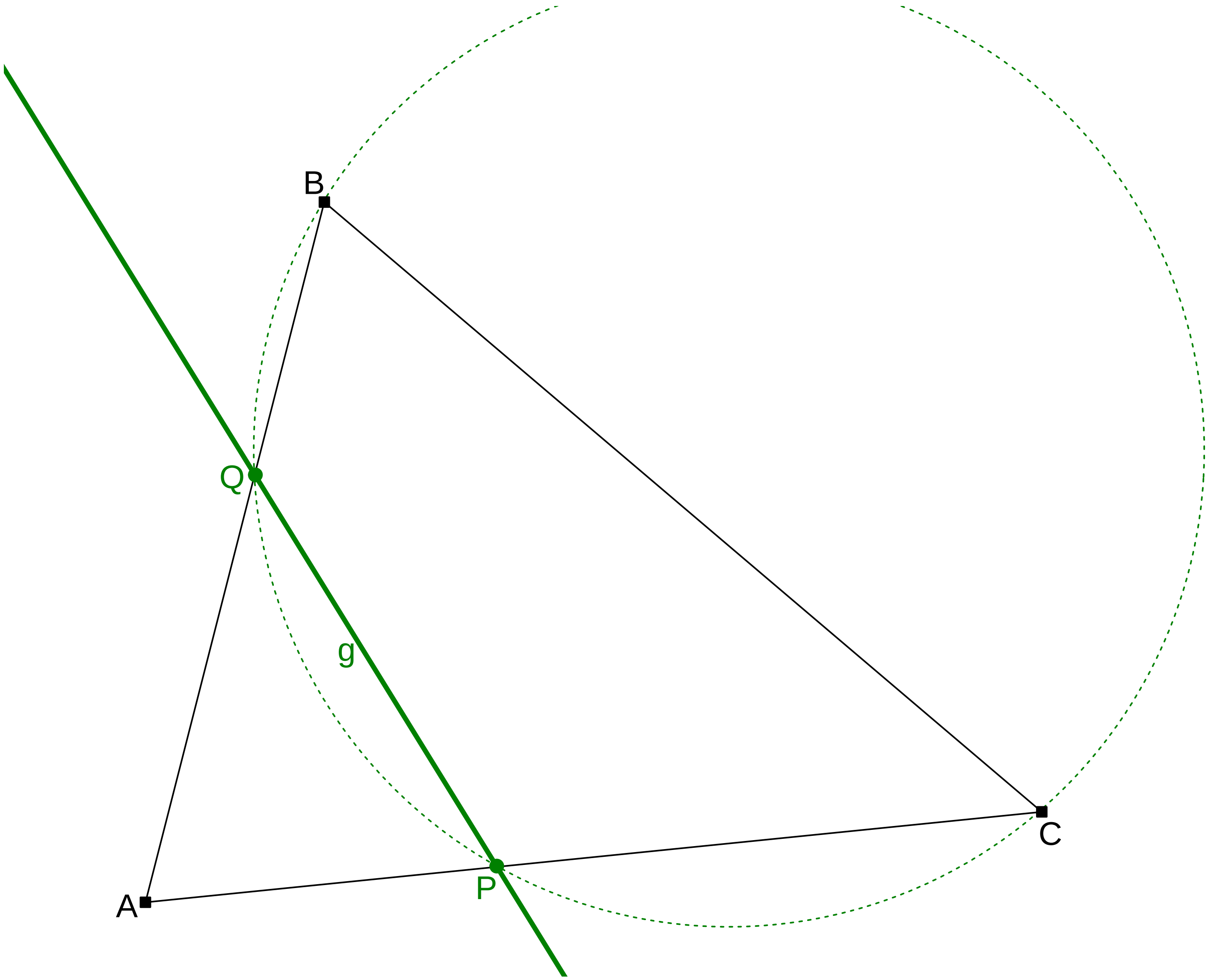


Fig. 37

**b)** As Theorem 17 **a)** is already demonstrated, we know that the Assertions $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$ and $\mathcal{E}_5$ are all equivalent. It just remains to prove the equivalence of the Assertions $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$ and $\mathcal{E}_9$ to these Assertions in the case when the line $g$ doesn't pass through the point $A$.

First we establish the equivalence of Assertions $\mathcal{E}_3$ and $\mathcal{E}_6$:

If Assertion $\mathcal{E}_3$ holds, then $\measuredangle (CA;\ g) = \measuredangle (BC;\ AB)$; this rewrites as $\measuredangle APQ = -\measuredangle ABC$. On the other hand, $\measuredangle PAQ = -\measuredangle BAC$. Hence, the triangles $APQ$ and $ABC$ are oppositely similar, so that Assertion $\mathcal{E}_6$ is valid.

Conversely: If Assertion $\mathcal{E}_6$ holds, then triangles $APQ$ and $ABC$ are oppositely similar, what yields $\measuredangle APQ = -\measuredangle ABC$. In other words, $\measuredangle (CA;\ g) = \measuredangle (BC;\ AB)$. Hence, Assertion $\mathcal{E}_3$ is valid.

Thus we have shown the equivalence of the Assertions $\mathcal{E}_3$ and $\mathcal{E}_6$.

The equivalence of the Assertions $\mathcal{E}_6$, $\mathcal{E}_7$ and $\mathcal{E}_8$ follows from Theorem 18, applied to the two lines $CA$ and $AB$ which intersect at the Euclidean point $A$, the two points $P$ and $C$ on the line $CA$ distinct from $A$, and the two points $Q$ and $B$ on the line $AB$ distinct from $A$.

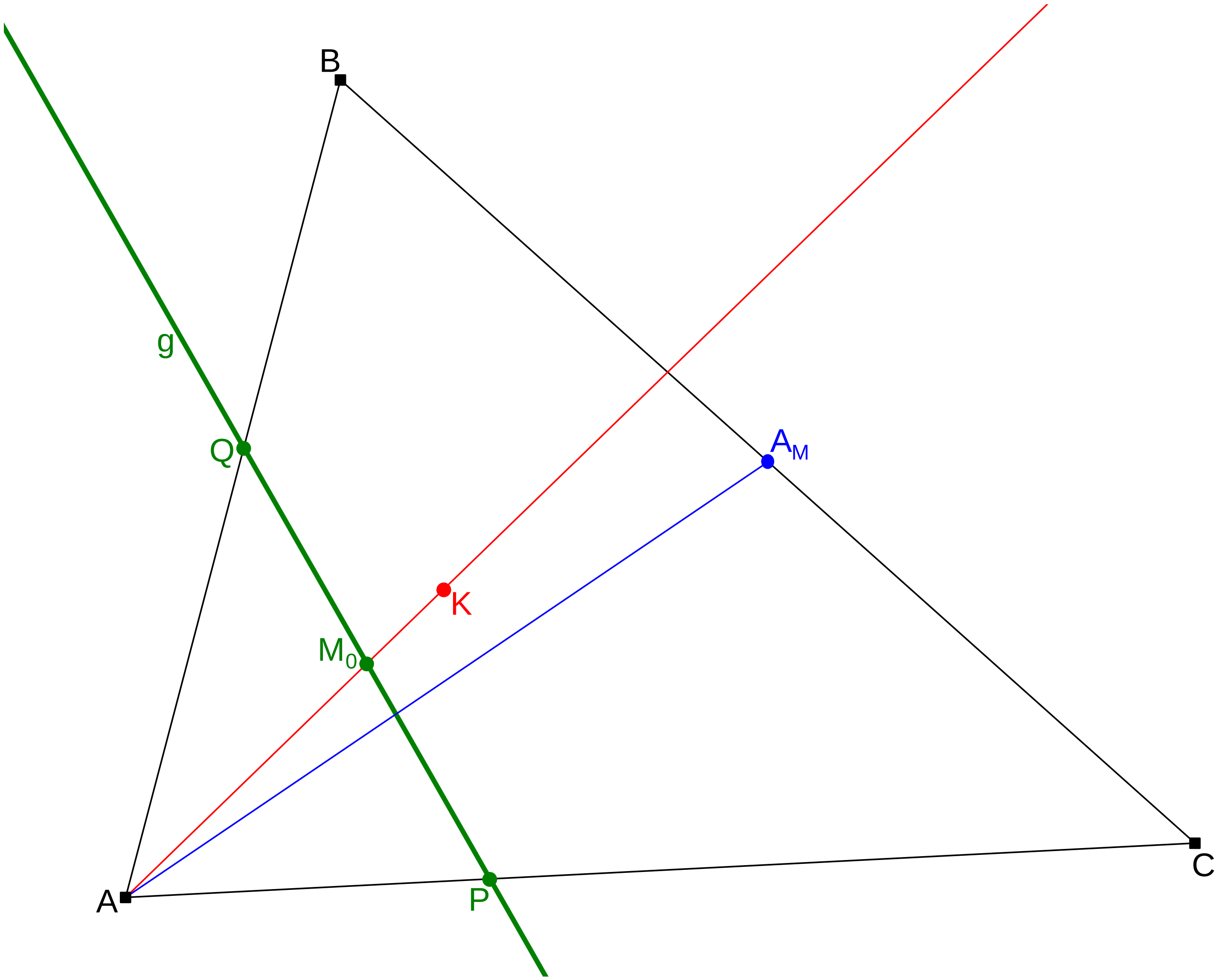


Fig. 38

Altogether, we now know that the nine Assertions $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$, $\mathcal{E}_5$, $\mathcal{E}_6$, $\mathcal{E}_7$ and $\mathcal{E}_8$ are all pairwisely equivalent. In order to prove the equivalence of all ten Assertions $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$, $\mathcal{E}_5$, $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$ and $\mathcal{E}_9$, it therefore suffices to prove the equivalence of the Assertions $\mathcal{E}_0$ and $\mathcal{E}_9$. This proof is obtained by showing the following two auxiliary results:

*Auxiliary result 1.* If Assertion $\mathcal{E}_0$ holds, then Assertion $\mathcal{E}_9$ holds.

*Auxiliary result 2.* If Assertion $\mathcal{E}_9$ holds, then Assertion $\mathcal{E}_0$ holds.

*Proof of Auxiliary result 1.* (See Fig. 38.) Assume that Assertion $\mathcal{E}_0$ holds. As we already know that Assertion $\mathcal{E}_0$ is equivalent to Assertion $\mathcal{E}_6$, it thus follows that Assertion $\mathcal{E}_6$ holds as well. That is, the triangles $APQ$ and $ABC$ are oppositely similar. In oppositely similar triangles, corresponding points form oppositely equal angles. Let $M_0$ be the midpoint of the segment $PQ$, and $A_M$ the midpoint of the segment $BC$. Then, the points $M_0$ and $A_M$ are corresponding points in the oppositely similar triangles $APQ$ and $ABC$ (being the midpoints of their respective sides $PQ$ and $BC$). Hence, they form oppositely equal angles; particularly, $\measuredangle PAM_0 = -\measuredangle BAA_M$. In other words, $\measuredangle (CA;\ AM_0) = -\measuredangle (AB;\ AA_M)$. Hence, the line $AM_0$ is the isogonal of the line $AA_M$

wrt the angle $CAB$. But since $A_M$ is the midpoint of the segment $BC$, the line $AA_M$ is the $A$-median of triangle $ABC$. Therefore, the line $AM_0$ is the isogonal of the $A$-median of triangle $ABC$ wrt the angle $CAB$, hence the $A$-symmedian of triangle $ABC$. Consequently, the point $M_0$ lies on the $A$-symmedian of triangle $ABC$. Since $M_0$ is the midpoint of the segment $PQ$, this is exactly Assertion $\mathcal{E}_9$. Thus, Auxiliary result 1 is proven.

*Proof of Auxiliary result 2.* (See Fig. 39. This figure is intentionally drawn wrong in order not to tempt to unfounded conclusions.) Assume that Assertion $\mathcal{E}_9$ is valid. In other words, the midpoint $M_0$ of the segment $PQ$ lies on the $A$-symmedian of triangle $ABC$.

Let the antiparallel to the side $BC$ of triangle $ABC$ through the point $M_0$ intersect the lines $CA$ and $AB$ at the points $P'$ and $Q'$. Since the line $P'Q'$ is antiparallel to $BC$ wrt triangle $ABC$, it fulfills Assertion $\mathcal{E}_0$ of Theorem 17. As we have already shown that Assertion $\mathcal{E}_0$ implies Assertion $\mathcal{E}_9$ (this was our Auxiliary result 1), we thus conclude that this line $P'Q'$ fulfills Assertion $\mathcal{E}_9$. That is: The midpoint of the segment $P'Q'$ lies on the $A$-symmedian of triangle $ABC$.

Hence, the midpoint of the segment $P'Q'$ is the point of intersection of the line $P'Q'$ with the $A$-symmedian of triangle $ABC$. But this point of intersection is the point $M_0$. Therefore, the midpoint of the segment $P'Q'$ must be the point $M_0$.

Since $M_0$ is the midpoint of the segment $PQ$, the reflection with respect to the point $M_0$ maps the point $P$ to the point $Q$. Since $M_0$ is the midpoint of the segment $P'Q'$, the reflection with respect to the point $M_0$ maps the point $P'$ to the point $Q'$. Now, if the points $P$ and $P'$ were distinct, then the points $Q$ and $Q'$ would therefore be distinct as well (since a reflection maps distinct points to distinct points), and we would have $QQ' \parallel PP'$ (since a reflection maps a line to a parallel line); but this cannot be true, since the lines $QQ'$ and $PP'$ are the lines $AB$ and $CA$, and we have $AB \nparallel CA$. Hence, the points $P$ and $P'$ cannot be distinct, i. e. we must have $P = P'$. Similarly, $Q = Q'$. As we know that the line $P'Q'$ is antiparallel to $BC$ wrt triangle $ABC$, we can thus conclude that the line $PQ$ is antiparallel to $BC$ wrt triangle $ABC$. This means that Assertion $\mathcal{E}_0$ holds. Thus, Auxiliary result 2 is proven.

This completes the proof of Theorem 17.

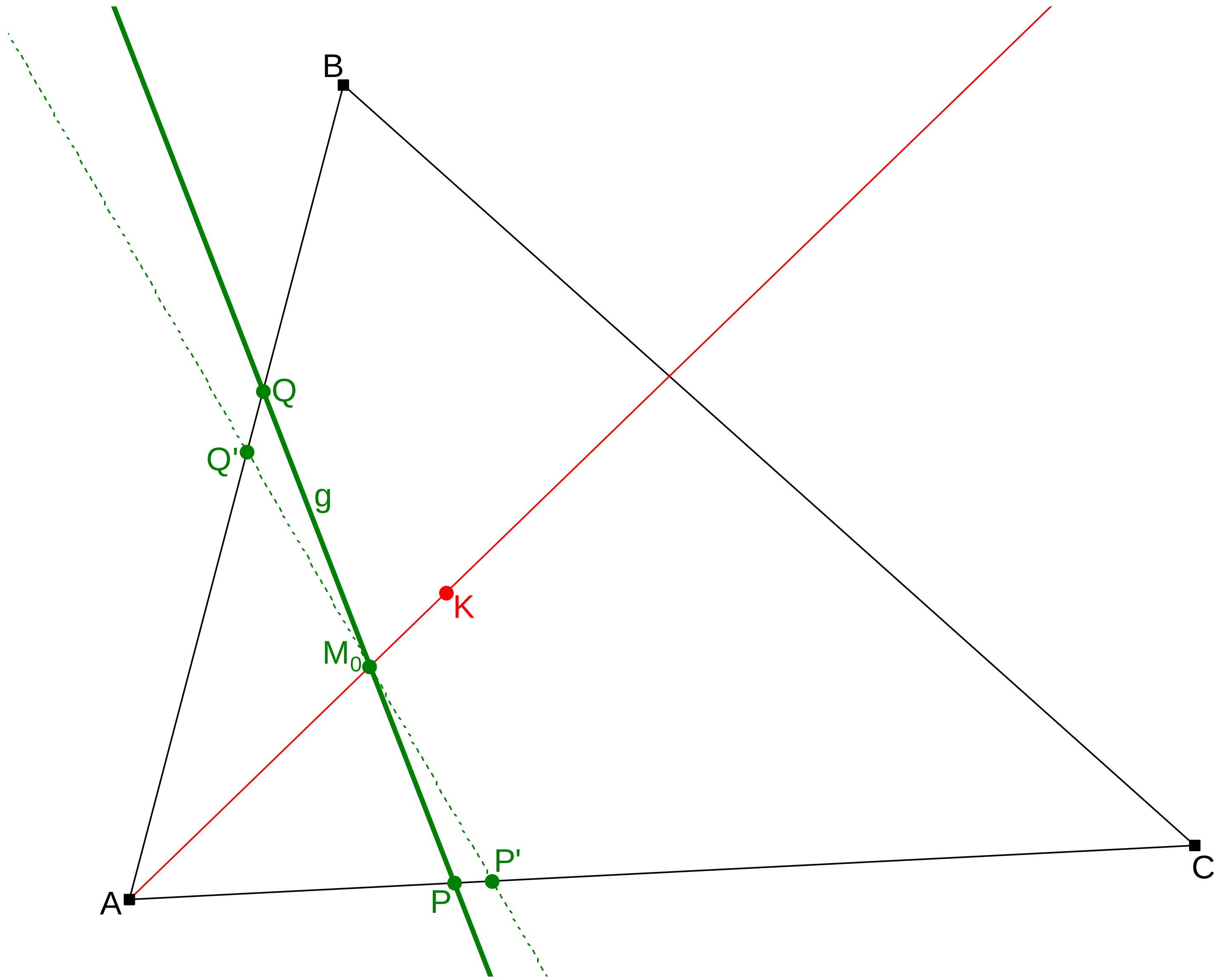


Fig. 39

The notion of antiparallelism is mostly applied in the theory of Tucker circles, but can also often be used to simplify some standard ways of conclusion in elementary geometry.

## 10. Isogonal conjugates of Kiepert points

A means for identification of isogonal conjugates is the following fact (Fig. 40):

**Theorem 19.** Let $ABC$ be a triangle, and let $P$ and $Q$ be two points on the perpendicular bisector of the segment $BC$.

**a)** The following five assertions $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$ and $\mathcal{F}_5$ are pairwisely equivalent:

**Assertion $\mathcal{F}_1$:** We have $\measuredangle ABP = \measuredangle ACQ$.

**Assertion $\mathcal{F}_2$:** We have $\measuredangle ACP = \measuredangle ABQ$.

**Assertion $\mathcal{F}_3$:** We have $\measuredangle BCP + \measuredangle BCQ = \measuredangle BAC$.

**Assertion $\mathcal{F}_4$:** We have $\measuredangle PBC + \measuredangle QBC = \measuredangle BAC$.

**Assertion $\mathcal{F}_5$:** The points $P$ and $Q$ are inverse to each other wrt the circumcircle of triangle $ABC$.

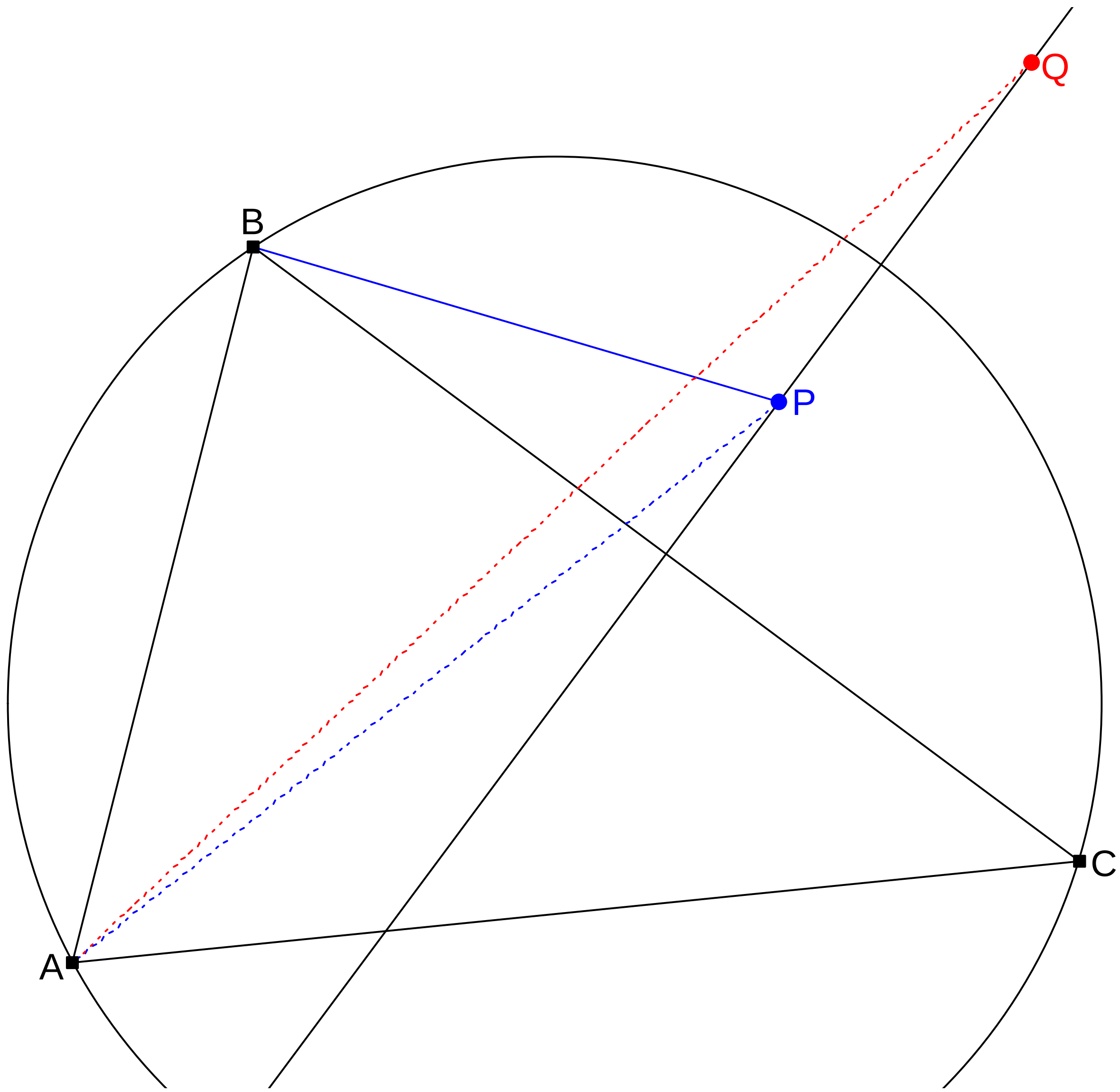


Fig. 40

**b)** If one of the five Assertions $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$ and $\mathcal{F}_5$ holds, then the lines $AP$ and $AQ$ are isogonal to each other wrt the angle $CAB$.

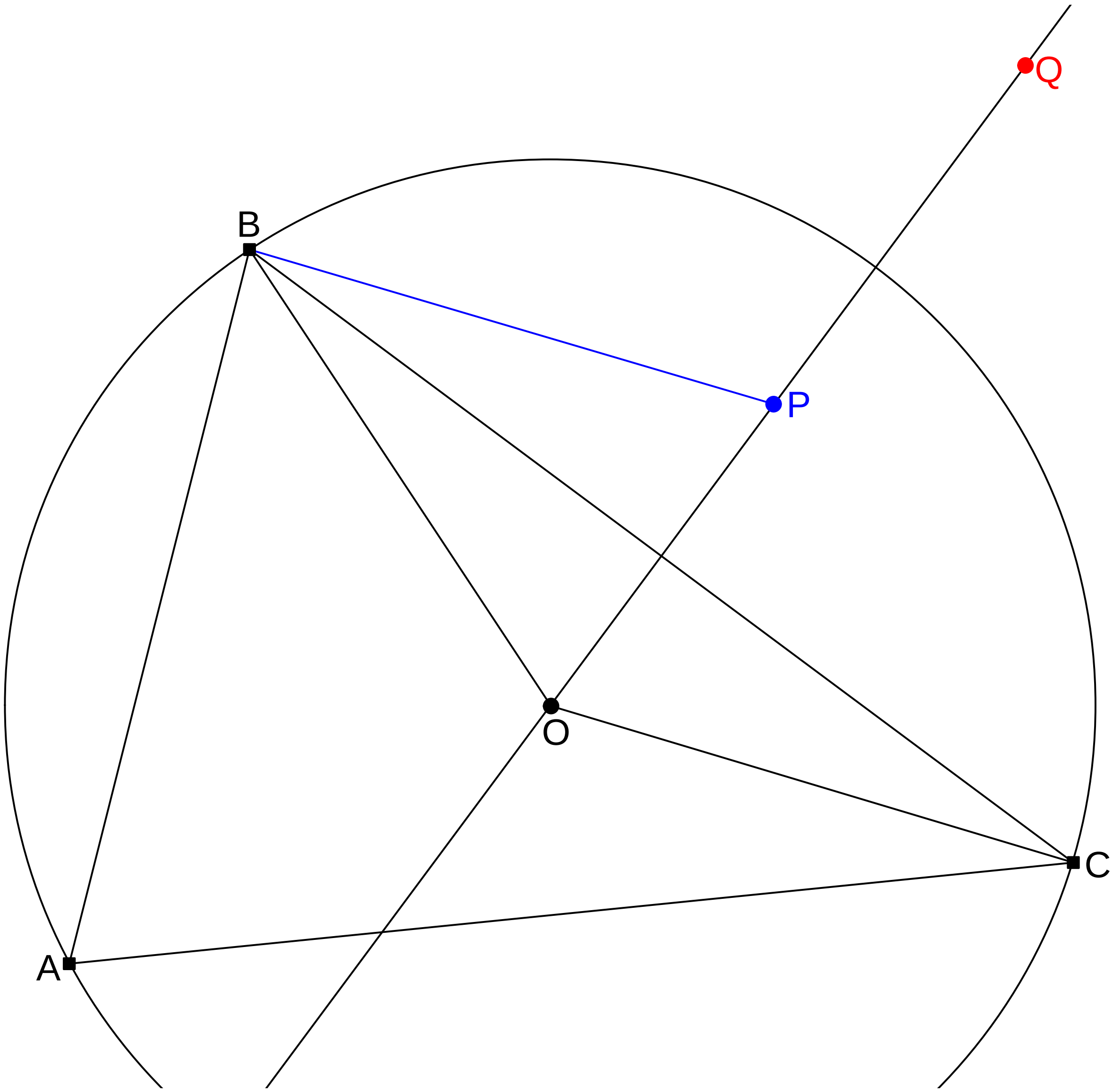


Fig. 41

*Proof of Theorem 19.* **a)** (See Fig. 41.) Let $O$ be the center of the circumcircle of triangle $ABC$. Then, $O$ is the center of a circle through the points $A$, $B$, $C$; hence, by the central angle theorem, $\measuredangle ABO = 90^\circ - \measuredangle BCA$.

The circumcenter $O$ of triangle $ABC$ obviously lies on the perpendicular bisector of its side $BC$. On the other hand, we know that the points $P$ and $Q$ lie on this perpendicular bisector. Hence, the points $O$, $P$, $Q$ lie on one line, namely on the perpendicular bisector of the segment $BC$. Thus, $OQ \perp BC$, so that $\measuredangle (OQ;\ BC) = 90^\circ$. Hence,

$$\measuredangle BQO = \measuredangle (BQ;\ OQ) = \measuredangle (BQ;\ BC) - \measuredangle (OQ;\ BC) = \measuredangle QBC - 90^\circ,$$

thus $\measuredangle QBC = \measuredangle BQO + 90^\circ$. Since the point $Q$ lies on the perpendicular bisector of the segment $BC$, we have $BQ = CQ$; this means that triangle $BQC$ is isosceles, and satisfies $\measuredangle QBC = \measuredangle BCQ$. Hence, $\measuredangle QBC = \measuredangle BQO + 90^\circ$ becomes $\measuredangle BCQ =$

$\measuredangle BQO + 90°$. Consequently,

$$\begin{aligned}\measuredangle ACQ &= \measuredangle BCQ - \measuredangle BCA = (\measuredangle BQO + 90°) - \measuredangle BCA \\ &= (90° - \measuredangle BCA) + \measuredangle BQO = \measuredangle ABO + \measuredangle BQO = \measuredangle ABO - \measuredangle OQB.\end{aligned}$$

Consider the two lines $OP$ and $OB$ which intersect at the Euclidean point $O$; further, consider the two points $P$ and $Q$ on the line $OP$ distinct from $O$, and the two points $B$ and $B$ on the line $OB$ distinct from $O$. We can apply Theorem 18 to these lines and points. Assertion $\mathcal{D}_1$ of Theorem 18 then states that the triangles $OPB$ and $OBQ$ are oppositely similar, and Assertion $\mathcal{D}_3$ states that $OP \cdot OQ = OB \cdot OB$ (where the segments are directed). As Theorem 18 ensures that the Assertions $\mathcal{D}_1$ and $\mathcal{D}_3$ are equivalent, we thus get:

*Auxiliary result 1.* The triangles $OPB$ and $OBQ$ are oppositely similar if and only if $OP \cdot OQ = OB \cdot OB$.

Now we can show the equivalence of the Assertions $\mathcal{F}_1$ and $\mathcal{F}_5$:

If Assertion $\mathcal{F}_1$ holds, then $\measuredangle ABP = \measuredangle ACQ$. Since $\measuredangle ABP = \measuredangle ABO + \measuredangle OBP$ and $\measuredangle ACQ = \measuredangle ABO - \measuredangle OQB$, this becomes $\measuredangle ABO + \measuredangle OBP = \measuredangle ABO - \measuredangle OQB$, so that $\measuredangle OBP = -\measuredangle OQB$. Furthermore, it is evident that $\measuredangle BOP = -\measuredangle QOB$. Hence, the triangles $OPB$ and $OBQ$ are oppositely similar. According to Auxiliary result 1, this entails $OP \cdot OQ = OB \cdot OB$, that is, $OP \cdot OQ = OB^2$. Now, we know that the points $O$, $P$, $Q$ lie on one line, and that $O$ is the center and $OB$ is the radius of the circumcircle of triangle $ABC$. Hence, the equation $OP \cdot OQ = OB^2$ signifies that the points $P$ and $Q$ are inverse to each other wrt the circumcircle of triangle $ABC$. Thus, Assertion $\mathcal{F}_5$ must hold.

Conversely: Assume that Assertion $\mathcal{F}_5$ holds. This means that the points $P$ and $Q$ are inverse to each other wrt the circumcircle of triangle $ABC$. Since $O$ is the center and $OB$ is the radius of this circumcircle, this yields $OP \cdot OQ = OB^2$. In other words, $OP \cdot OQ = OB \cdot OB$. According to Auxiliary result 1, we thus conclude that the triangles $OPB$ and $OBQ$ are oppositely similar, so that $\measuredangle OBP = -\measuredangle OQB$. Hence, $\measuredangle ABP = \measuredangle ABO + \measuredangle OBP = \measuredangle ABO - \measuredangle OQB = \measuredangle ACQ$, and thus Assertion $\mathcal{F}_1$ is valid.

Thus we have shown that Assertions $\mathcal{F}_1$ and $\mathcal{F}_5$ are equivalent. In a similar way we can prove that Assertions $\mathcal{F}_2$ and $\mathcal{F}_5$ are equivalent.

As seen above, $\measuredangle QBC = \measuredangle BCQ$. Thus,

$$\measuredangle (BQ;\ CP) = \measuredangle (BC;\ CP) + \measuredangle (BQ;\ BC) = \measuredangle BCP + \measuredangle QBC = \measuredangle BCP + \measuredangle BCQ,$$

so that

$$\begin{aligned}\measuredangle ACP - \measuredangle ABQ &= \measuredangle (AC;\ CP) - \measuredangle (AB;\ BQ) = (\measuredangle (AC;\ BQ) + \measuredangle (BQ;\ CP)) - \measuredangle (AB;\ BQ) \\ &= \measuredangle (BQ;\ CP) - (\measuredangle (AB;\ BQ) - \measuredangle (AC;\ BQ)) = \measuredangle (BQ;\ CP) - \measuredangle (AB;\ AC) \\ &= (\measuredangle BCP + \measuredangle BCQ) - \measuredangle BAC.\end{aligned}$$

Consequently, we have $\measuredangle ACP = \measuredangle ABQ$ if and only if $\measuredangle BCP + \measuredangle BCQ = \measuredangle BAC$. In other words, Assertion $\mathcal{F}_2$ is equivalent to Assertion $\mathcal{F}_3$.

We have $\measuredangle QBC = \measuredangle BCQ$ and similarly $\measuredangle PBC = \measuredangle BCP$. Thus, the equation $\measuredangle BCP + \measuredangle BCQ = \measuredangle BAC$ is equivalent to the equation $\measuredangle PBC + \measuredangle QBC = \measuredangle BAC$. In other words, Assertion $\mathcal{F}_3$ is equivalent to Assertion $\mathcal{F}_4$.

Altogether, we have proven the equivalence of all five Assertions $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$ and $\mathcal{F}_5$; thus, Theorem 19 **a)** is verified.

**b)** Assume that one of the five Assertions $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$ and $\mathcal{F}_5$ holds. Since, according to Theorem 19 **a)**, all these five Assertions are equivalent, we can therefore conclude that Assertion $\mathcal{F}_1$ holds, i. e. we have $\measuredangle ABP = \measuredangle ACQ$.

(See Fig. 42.) Since the point $P$ lies on the perpendicular bisector of the segment $BC$, we have $PB = PC$. The circle with center $P$ and radius $PB = PC$ passes through the points $B$ and $C$; let $B_P$ and $C_P$ be the points of intersection of this circle with the lines $AB$ and $CA$ different from $B$ and $C$. Then, $PB_P = PC_P = PB = PC$.

Since $PB_P = PB$, the triangle $BPB_P$ is isosceles, so that $\measuredangle BB_PP = \measuredangle PBB_P$. Hence, $\measuredangle AB_PP = \measuredangle BB_PP = \measuredangle PBB_P = -\measuredangle ABP = -\measuredangle ACQ$.

On the other hand, the points $B$, $C$, $B_P$ and $C_P$ lie on one circle (namely, on the circle with center $P$ and radius $PB = PC$); thus, $\measuredangle BB_PC_P = \measuredangle BCC_P$. This yields $\measuredangle AB_PC_P = \measuredangle BB_PC_P = \measuredangle BCC_P = -\measuredangle ACB$. Similarly, $\measuredangle AC_PB_P = -\measuredangle ABC$. Consequently, the triangles $AC_PB_P$ and $ABC$ are oppositely similar.

From $\measuredangle AB_PP = -\measuredangle ACQ$ and $\measuredangle AB_PC_P = -\measuredangle ACB$, it follows that

$$\measuredangle C_PB_PP = \measuredangle AB_PP - \measuredangle AB_PC_P = (-\measuredangle ACQ) - (-\measuredangle ACB) = -(\measuredangle ACQ - \measuredangle ACB) = -\measuredangle BCQ.$$

Similarly, $\measuredangle B_PC_PP = -\measuredangle CBQ$. Thus, the triangles $PC_PB_P$ and $QBC$ are oppositely similar.

Since triangle $AC_PB_P$ is oppositely similar to triangle $ABC$, and triangle $PC_PB_P$ is oppositely similar to triangle $QBC$, the quadrilateral $AC_PPB_P$ formed by the triangles $AC_PB_P$ and $PC_PB_P$ is oppositely similar to the quadrilateral $ABQC$ formed by the triangles $ABC$ and $QBC$. Consequently, $\measuredangle C_PAP = -\measuredangle BAQ$. In other words, $\measuredangle (CA;\ AP) = -\measuredangle (AB;\ AQ)$. Thus, the line $AP$ is the isogonal of the line $AQ$ wrt the angle $CAB$. This proves Theorem 19 **b)**, and thus concludes the proof of Theorem 19.

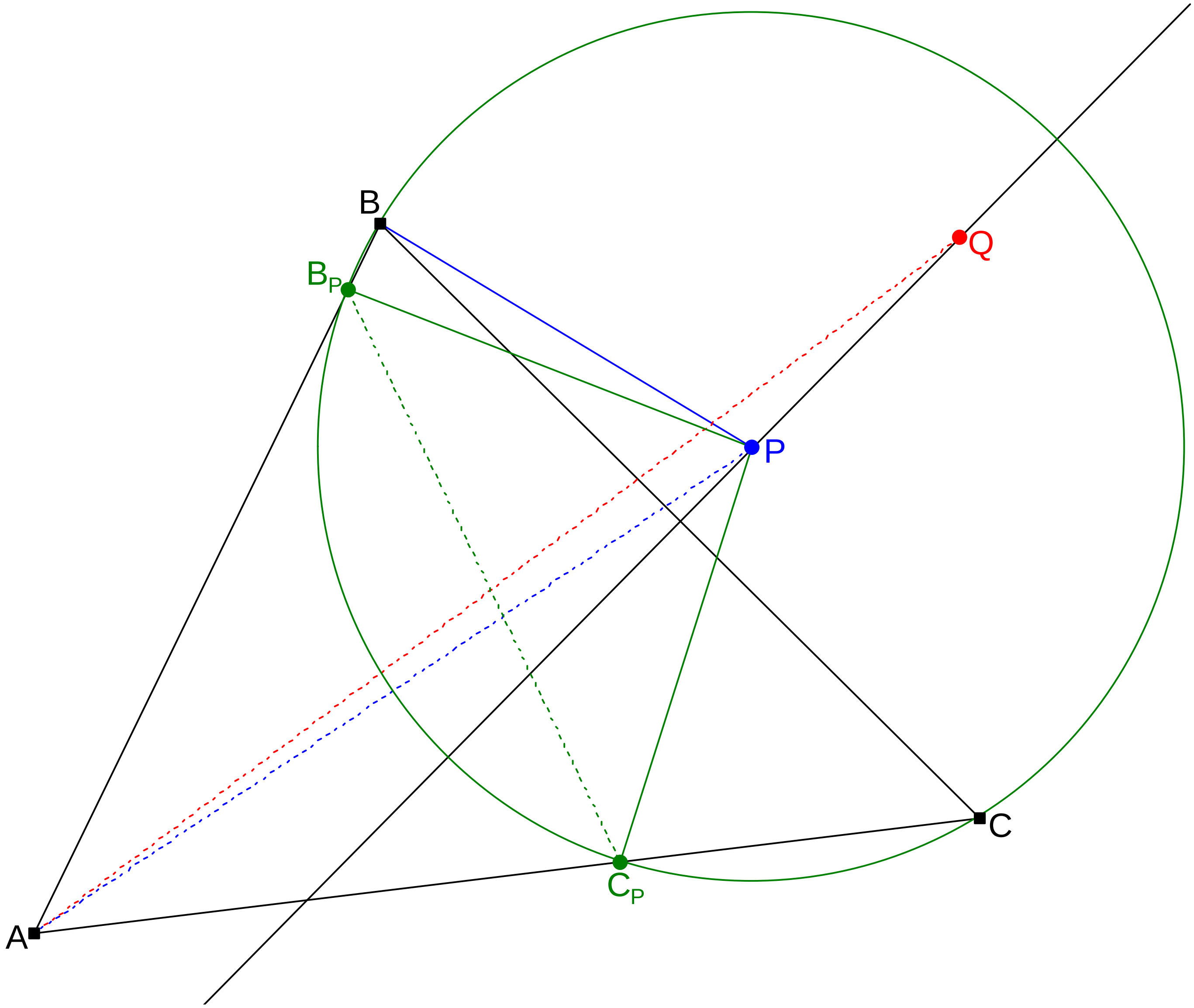


Fig. 42

Theorem 19 can be used to obtain a result about the isogonal conjugates of the so-called *Kiepert points* of the triangle. These points are defined as follows:

(See Fig. 43.) Let $\varphi$ be an arbitrary angle. On the sides $BC$, $CA$, $AB$ of triangle $ABC$, we erect isosceles triangles $BA_\varphi C$, $CB_\varphi A$, $AC_\varphi B$ with the bases $BC$, $CA$, $AB$ and the equal base angle

$$\measuredangle BCA_\varphi = \measuredangle A_\varphi BC = \measuredangle CAB_\varphi = \measuredangle B_\varphi CA = \measuredangle ABC_\varphi = \measuredangle C_\varphi AB = \varphi.$$

The **Kiepert theorem** states that the lines $AA_\varphi$, $BB_\varphi$, $CC_\varphi$ concur at one point. We denote this point by $K_\varphi$, and call it the **$\varphi$-Kiepert point** of triangle $ABC$. Occasionally, triangle $A_\varphi B_\varphi C_\varphi$ is referred to as the **$\varphi$-Kiepert triangle** of triangle $ABC$.

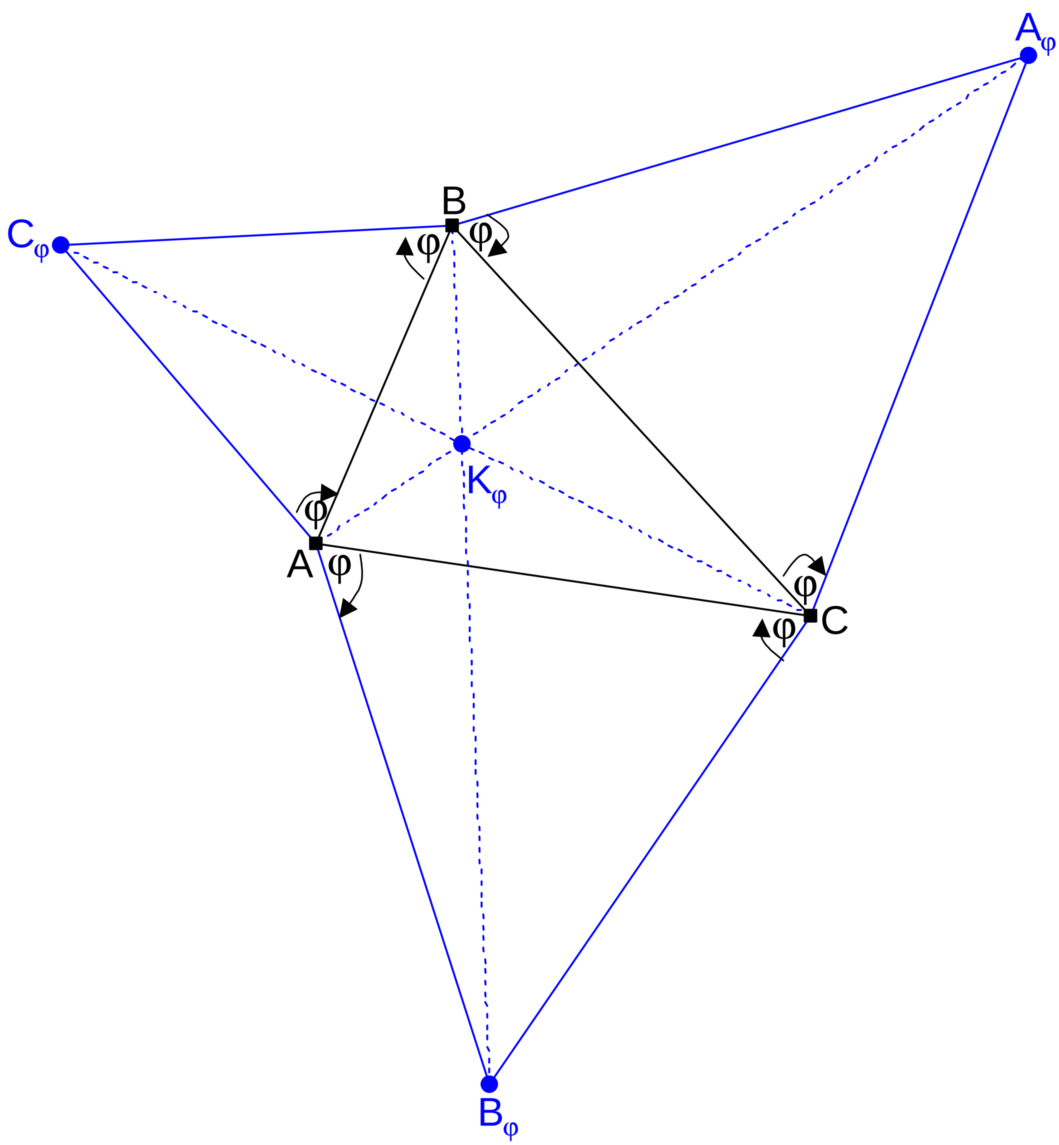


Fig. 43

(See Fig. 44.) Let $A'_\varphi$, $B'_\varphi$, $C'_\varphi$ be the inverses of the points $A_\varphi$, $B_\varphi$, $C_\varphi$ in the circumcircle of triangle $ABC$. Let $O$ be the center of this circumcircle.

Since triangle $BA_\varphi C$ is isosceles with base $BC$, we have $BA_\varphi = CA_\varphi$; thus, the point $A_\varphi$ lies on the perpendicular bisector of the segment $BC$. The circumcenter $O$ of triangle $ABC$ also lies on the perpendicular bisector of its side $BC$. Thus, the line $OA_\varphi$ is the perpendicular bisector of the segment $BC$. Now, the point $A'_\varphi$ is the inverse of the point $A_\varphi$ in the circumcircle of triangle $ABC$ and therefore lies on the line $OA_\varphi$ (since $O$ is the center of this circumcircle). Thus, the point $A'_\varphi$ lies on the perpendicular bisector of the segment $BC$.

Now as we know that the points $A_\varphi$ and $A'_\varphi$ both lie on the perpendicular bisector of the segment $BC$, we can apply Theorem 19 to them. Since the points $A_\varphi$ and $A'_\varphi$ are inverse to each other wrt the circumcircle of triangle $ABC$, they fulfill Assertion $\mathcal{F}_5$ of Theorem 19. Since, according to Theorem 19 **a)**, the Assertions $\mathcal{F}_3$ and $\mathcal{F}_5$ are equivalent, they therefore also fulfill Assertion $\mathcal{F}_3$; that is, we have $\measuredangle BCA_\varphi + \measuredangle BCA'_\varphi = \measuredangle BAC$. Furthermore, according to Theorem 19 **b)**, the validity of Assertion $\mathcal{F}_5$ implies that the lines $AA_\varphi$ and $AA'_\varphi$ are isogonal to each other wrt the angle $CAB$.

Since $\measuredangle BCA_\varphi = \varphi$, the equation $\measuredangle BCA_\varphi + \measuredangle BCA'_\varphi = \measuredangle BAC$ becomes $\varphi +$

$\measuredangle BCA'_\varphi = \measuredangle BAC$, thus $\measuredangle BCA'_\varphi = \measuredangle BAC - \varphi$. Since the point $A'_\varphi$ lies on the perpendicular bisector of the segment $BC$, we have $BA'_\varphi = CA'_\varphi$; thus, triangle $BA'_\varphi C$ is isosceles with base $BC$, and this yields $\measuredangle BCA'_\varphi = \measuredangle A'_\varphi BC$. Combining this with $\measuredangle BCA'_\varphi = \measuredangle BAC - \varphi$, we obtain $\measuredangle BCA'_\varphi = \measuredangle A'_\varphi BC = \measuredangle BAC - \varphi$. Similarly, triangle $CB'_\varphi A$ is isosceles with base $CA$ and fulfills $\measuredangle CAB'_\varphi = \measuredangle B'_\varphi CA = \measuredangle CBA - \varphi$, and triangle $AC'_\varphi B$ is isosceles with base $AB$ and fulfills $\measuredangle ABC'_\varphi = \measuredangle C'_\varphi AB = \measuredangle ACB - \varphi$.

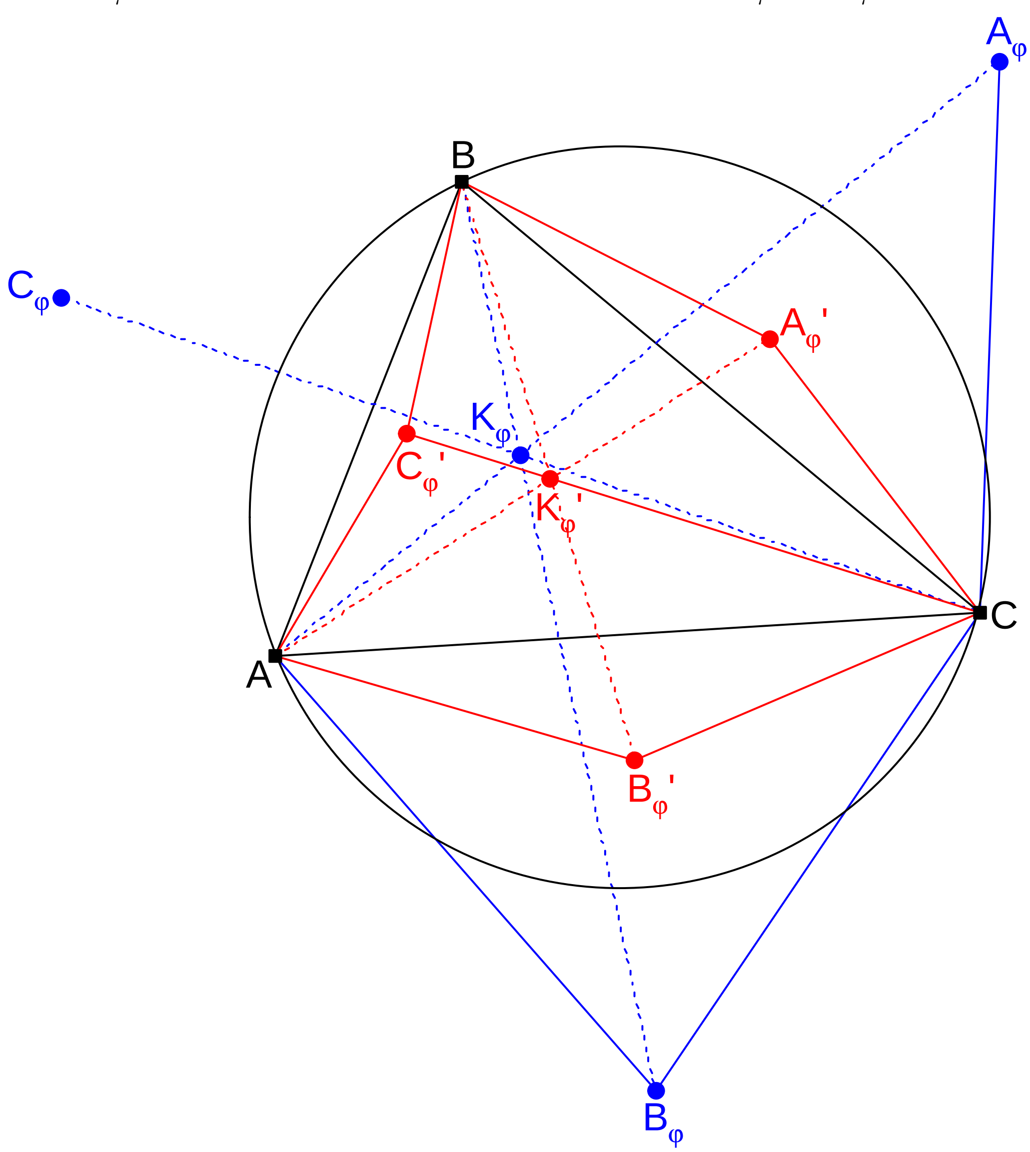


Fig. 44

Since the lines $AA_\varphi$ and $AA'_\varphi$ are isogonal to each other wrt the angle $CAB$, the line $AA'_\varphi$ is the isogonal of the line $AA_\varphi$ wrt the angle $CAB$. Now, the line $AA_\varphi$ is the line $AK_\varphi$. Hence, the line $AA'_\varphi$ is the isogonal of the line $AK_\varphi$ wrt the angle $CAB$. Similarly, the lines $BB'_\varphi$ and $CC'_\varphi$ are the isogonals of the lines $BK_\varphi$ and $CK_\varphi$ wrt the angles $ABC$ and $BCA$. Thus, altogether, the lines $AA'_\varphi$, $BB'_\varphi$, $CC'_\varphi$ are the isogonals of the lines $AK_\varphi$, $BK_\varphi$, $CK_\varphi$ wrt the angles $CAB$, $ABC$, $BCA$, and hence they concur at one point, namely at the isogonal conjugate of the point $K_\varphi$ wrt triangle $ABC$.

Summing up, we see:

**Theorem 20.** Let $ABC$ be a triangle, and $\varphi$ an arbitary angle. On the sides $BC$, $CA$, $AB$ of triangle $ABC$, we erect isosceles triangles $BA_\varphi C$, $CB_\varphi A$, $AC_\varphi B$ with the bases $BC$, $CA$, $AB$ and the equal base angle

$$\measuredangle BCA_\varphi = \measuredangle A_\varphi BC = \measuredangle CAB_\varphi = \measuredangle B_\varphi CA = \measuredangle ABC_\varphi = \measuredangle C_\varphi AB = \varphi.$$

Then, the lines $AA_\varphi$, $BB_\varphi$, $CC_\varphi$ concur at one point: the $\varphi$-Kiepert point $K_\varphi$ of triangle $ABC$.

Let $A'_\varphi$, $B'_\varphi$, $C'_\varphi$ be the inverses of the points $A_\varphi$, $B_\varphi$, $C_\varphi$ in the circumcircle of triangle $ABC$. Then, the triangles $BA'_\varphi C$, $CB'_\varphi A$, $AC'_\varphi B$ are isosceles with the bases $BC$, $CA$, $AB$ and the base angles $\measuredangle BCA'_\varphi = \measuredangle A'_\varphi BC = \measuredangle BAC - \varphi$, $\measuredangle CAB'_\varphi = \measuredangle B'_\varphi CA = \measuredangle CBA - \varphi$, $\measuredangle ABC'_\varphi = \measuredangle C'_\varphi AB = \measuredangle ACB - \varphi$. The lines $AA'_\varphi$, $BB'_\varphi$, $CC'_\varphi$ concur at one point, namely at the isogonal conjugate of the point $K_\varphi$ wrt the triangle $ABC$.

Using the theory of Tucker circles, we can show that this isogonal conjugate lies on the Brocard axis of triangle $ABC$, which is the line joining the circumcenter and the symmedian point of triangle $ABC$.

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