A few facts on integrality

Darij Grinberg

Version 3 (18 August 2009, a mistake fixed later)

The purpose of this note is to collect some theorems and proofs related to integrality in commutative algebra. The note is subdivided into three parts.

Part 1 (Integrality over rings) consists of known facts (Theorems 1, 4, 5) and a generalized exercise from [1] (Corollary 3) with a few minor variations (Theorem 2 and Corollary 6).

Part 2 (Integrality over ideal semifiltrations) merges integrality over rings (as considered in Part 1) and integrality over ideals (a less-known but still very useful notion; the book [2] is devoted to it) into one general notion - that of integrality over ideal semifiltrations (Definition 9). This notion is very general, yet it can be reduced to the basic notion of integrality over rings by a suitable change of base ring (Theorem 7). This reduction allows to extend some standard properties of integrality over rings to the general case (Theorems 8 and 9).

Part 3 (Generalizing to two ideal semifiltrations) continues Part 2, adding one more layer of generality. Its main result is a "relative" version of Theorem 7 (Theorem 11) and a known fact generalized one more time (Theorem 13).

This note is supposed to be self-contained (only linear algebra and basic knowledge about rings, ideals and polynomials is assumed). The proofs are constructive. However, when writing down the proofs I focussed on maximal detail (to ensure correctness) rather than on clarity, so the proofs are probably a pain to read. I think of making a short version of this note with the obvious parts of proofs left out.

Preludium

Definitions and notations:

Definition 1. In the following, "ring" will always mean "commutative ring with unity". We denote the set $\{0, 1, 2, ...\}$ by $\mathbb{N}$, and the set $\{1, 2, 3, ...\}$ by $\mathbb{N}^+$.

Definition 2. Let A be a ring, and let $n \in \mathbb{N}$. Let M be an A-module. If m_1, m_2, ..., m_n are n elements of M, then we define an A-submodule $\langle m_1, m_2, ..., m_n \rangle_A$ of M by

$$\langle m_1, m_2, ..., m_n \rangle_A = \left\{ \sum_{i=1}^{n} a_i m_i \ \mid \ (a_1, a_2, ..., a_n) \in A^n \right\}.$$

Also, if S is a finite set, and m_s is an element of M for every $s \in S$, then we define an A-submodule $\langle m_s \ \mid \ s \in S \rangle_A$ of M by

$$\langle m_s \ \mid \ s \in S \rangle_A = \left\{ \sum_{s \in S} a_s m_s \ \mid \ (a_s)_{s \in S} \in A^S \right\}.$$

Of course, if $m_1, m_2, ..., m_n$ are n elements of M, then $\langle m_1, m_2, ..., m_n \rangle_A = \langle m_s \ \mid \ s \in \{1, 2, ..., n\} \rangle_A$.

Definition 3. Let A be a ring, and let $n \in \mathbb{N}$. Let M be an A-module. We say that the A-module M is *n-generated* if there exist n elements m_1, m_2, ..., m_n of M such that $M = \langle m_1, m_2, ..., m_n \rangle_A$. In other words, the A-module M is n-generated if and only if there exists a set S and an element m_s of M for every $s \in S$ such that $|S| = n$ and $M = \langle m_s \ \mid \ s \in S \rangle_A$.

Definition 4. Let A and B be two rings. We say that $A \subseteq B$ if and only if

$$(\text{the set } A \text{ is a subset of the set } B) \text{ and } (\text{the inclusion map } A \to B \text{ is a ring homomorphism}).$$

Now assume that $A \subseteq B$. Then, obviously, B is canonically an A-algebra (since $A \subseteq B$). If $u_1, u_2, ..., u_n$ are n elements of B, then we define an A-subalgebra $A[u_1, u_2, ..., u_n]$ of B by

$$A[u_1, u_2, ..., u_n] = \{P(u_1, u_2, ..., u_n) \mid P \in A[X_1, X_2, ..., X_n]\}.$$

In particular, if u is an element of B, then the A-subalgebra $A[u]$ of B is defined by

$$A[u] = \{P(u) \mid P \in A[X]\}.$$

Since $A[X] = \left\{\sum\limits_{i=0}^{m} a_i X^i \mid m \in \mathbb{N} \text{ and } (a_0, a_1, ..., a_m) \in A^{m+1}\right\}$, this becomes

$$\begin{aligned} A[u] &= \left\{\left(\sum_{i=0}^{m} a_i X^i\right)(u) \mid m \in \mathbb{N} \text{ and } (a_0, a_1, ..., a_m) \in A^{m+1}\right\} \\ &\qquad \left(\text{where } \left(\sum_{i=0}^{m} a_i X^i\right)(u) \text{ means the polynomial } \sum_{i=0}^{m} a_i X^i \text{ evaluated at } X = u\right) \\ &= \left\{\sum_{i=0}^{m} a_i u^i \mid m \in \mathbb{N} \text{ and } (a_0, a_1, ..., a_m) \in A^{m+1}\right\} \qquad \left(\text{because } \left(\sum_{i=0}^{m} a_i X^i\right)(u) = \sum_{i=0}^{m} a_i u^i\right). \end{aligned}$$

Obviously, $uA[u] \subseteq A[u]$ (since $A[u]$ is an A-algebra and $u \in A[u]$).

1. Integrality over rings

Theorem 1. Let A and B be two rings such that $A \subseteq B$. Obviously, B is canonically an A-module (since $A \subseteq B$). Let $n \in \mathbb{N}$. Let $u \in B$. Then, the following four assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are pairwise equivalent:

Assertion $\mathcal{A}$: There exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$.

Assertion $\mathcal{B}$: There exists an n-generated A-submodule U of B such that $uU \subseteq U$ and such that $v = 0$ for every $v \in B$ satisfying $vU = 0$.

Assertion $\mathcal{C}$: There exists an n-generated A-submodule U of B such that $1 \in U$ and $uU \subseteq U$.

Assertion $\mathcal{D}$: We have $A[u] = \langle u^0, u^1, ..., u^{n-1}\rangle_A$.

Definition 5. Let A and B be two rings such that $A \subseteq B$. Let $n \in \mathbb{N}$. Let $u \in B$. We say that the element u of B is *n-integral over A* if it satisfies the four equivalent assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of Theorem 1.

Hence, u is n-integral over A if and only if there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$.

Proof of Theorem 1. We will prove the implications $\mathcal{A} \Longrightarrow \mathcal{C}$, $\mathcal{C} \Longrightarrow \mathcal{B}$, $\mathcal{B} \Longrightarrow \mathcal{A}$, $\mathcal{A} \Longrightarrow \mathcal{D}$ and $\mathcal{D} \Longrightarrow \mathcal{C}$.

Proof of the implication $\mathcal{A} \Longrightarrow \mathcal{C}$. Assume that Assertion $\mathcal{A}$ holds. Then, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$. Since $P \in A[X]$ is a monic polynomial with $\deg P = n$, there exist elements a_0, a_1, ..., a_{n-1} of A such that $P(X) = X^n + \sum\limits_{k=0}^{n-1} a_k X^k$. Thus, $P(u) = u^n + \sum\limits_{k=0}^{n-1} a_k u^k$, so that $P(u) = 0$ becomes $u^n + \sum\limits_{k=0}^{n-1} a_k u^k = 0$. Hence, $u^n = -\sum\limits_{k=0}^{n-1} a_k u^k$.

Let U be the A-submodule $\langle u^0, u^1, ..., u^{n-1}\rangle_A$ of B. Then, U is an n-generated A-module (since u^0, u^1, ..., u^{n-1} are n elements of U). Besides, $1 = u^0 \in U$.

Now, $u \cdot u^k \in U$ for any $k \in \{0, 1, ..., n-1\}$ (since $k \in \{0, 1, ..., n-1\}$ yields either $0 \leq k < n-1$ or $k = n-1$, but $u \cdot u^k = u^{k+1} \in \langle u^0, u^1, ..., u^{n-1}\rangle_A = U$ if $0 \leq k < n-1$, and $u \cdot u^k = u \cdot u^{n-1} = u^n = -\sum\limits_{k=0}^{n-1} a_k u^k \in \langle u^0, u^1, ..., u^{n-1}\rangle_A = U$ if $k = n-1$, so that $u \cdot u^k \in U$ in both cases). Hence,

$$uU = u\left\langle u^0, u^1, ..., u^{n-1}\right\rangle_A = \left\langle u \cdot u^0, u \cdot u^1, ..., u \cdot u^{n-1}\right\rangle_A \subseteq U$$

(since $u \cdot u^k \in U$ for any $k \in \{0, 1, ..., n-1\}$).

Thus, Assertion $\mathcal{C}$ holds. Hence, we have proved that $\mathcal{A} \Longrightarrow \mathcal{C}$.

Proof of the implication $\mathcal{C} \Longrightarrow \mathcal{B}$. Assume that Assertion $\mathcal{C}$ holds. Then, there exists an n-generated A-submodule U of B such that $1 \in U$ and $uU \subseteq U$. We have $v = 0$ for every $v \in B$ satisfying $vU = 0$ (since $1 \in U$ and $vU = 0$ yield $v \cdot \underbrace{1}_{\in U} \in vU = 0$ and thus $v \cdot 1 = 0$, so that $v = 0$). Thus, Assertion $\mathcal{B}$ holds. Hence, we have proved that $\mathcal{C} \Longrightarrow \mathcal{B}$.

Proof of the implication $\mathcal{B} \Longrightarrow \mathcal{A}$. Assume that Assertion $\mathcal{B}$ holds. Then, there exists an n-generated A-submodule U of B such that $uU \subseteq U$ and such that $v = 0$ for every $v \in B$ satisfying $vU = 0$. Since the A-module U is n-generated, there exist n elements $m_1, m_2, ..., m_n$ of U such that $U = \langle m_1, m_2, ..., m_n\rangle_A$. For any $k \in \{1, 2, ..., n\}$, we have

$$\begin{aligned} um_k &\in uU \qquad \text{(since } m_k \in U\text{)} \\ &\subseteq U = \langle m_1, m_2, ..., m_n\rangle_A, \end{aligned}$$

so that there exist n elements $a_{k,1}$, $a_{k,2}$, ..., $a_{k,n}$ of A such that $um_k = \sum\limits_{i=1}^{n} a_{k,i} m_i$.

Define a vector $v \in B^n$ by $v_i = m_i$ for all $i \in \{1, 2, ..., n\}$. (Here, for any vector w and any integer x, we denote by w_x the entry of the vector w in the x-th row.)

Define a matrix $S \in A^{n \times n}$ by $S_{k,i} = a_{k,i}$ for all $k \in \{1, 2, ..., n\}$ and $i \in \{1, 2, ..., n\}$. (Here, for any matrix T and any integers x and y, we denote by $T_{x,y}$ the entry of the matrix T in the x-th row and the y-th column.) Then, for any $k \in \{1, 2, ..., n\}$, we have $u \underbrace{m_k}_{=v_k} = uv_k = (uv)_k$ and $\sum\limits_{i=1}^{n} \underbrace{a_{k,i}}_{=S_{k,i}} \underbrace{m_i}_{=v_i} = \sum\limits_{i=1}^{n} S_{k,i} v_i = (Sv)_k$, so that $um_k = \sum\limits_{i=1}^{n} a_{k,i} m_i$ becomes $(uv)_k = (Sv)_k$. Since this holds for every $k \in \{1, 2, ..., n\}$, we conclude that $uv = Sv$. Thus,

$$0 = uv - Sv = uI_n v - Sv = (uI_n - S)\, v.$$

Now, let $P \in A[X]$ be the characteristic polynomial of the matrix $S \in A^{n\times n}$. Then, P is monic, and $\deg P = n$. Besides, $P(X) = \det(XI_n - S)$, so that $P(u) = \det(uI_n - S)$. Thus,

$$\begin{aligned} P(u)\cdot v &= \det(uI_n - S)\cdot v = \underbrace{\det(uI_n - S)\, I_n}_{=\operatorname{adj}(uI_n-S)\cdot(uI_n-S)} \cdot v \\ &= \operatorname{adj}(uI_n - S)\cdot \underbrace{(uI_n - S)\, v}_{=0} = 0. \end{aligned}$$

Hence, for any $k \in \{1, 2, ..., n\}$, we have

$$P(u)\cdot \underbrace{m_k}_{=v_k} = P(u)\cdot v_k = \left(\underbrace{P(u)\cdot v}_{=0}\right)_k = 0,$$

so that

$$\begin{aligned} P(u)\cdot U &= P(u)\cdot \langle m_1, m_2, ..., m_n\rangle_A = \langle P(u)\cdot m_1, P(u)\cdot m_2, ..., P(u)\cdot m_n\rangle_A \\ &= \langle 0, 0, ..., 0\rangle_A \qquad \text{(since } P(u)\cdot m_k = 0 \text{ for any } k \in \{1, 2, ..., n\}) \\ &= 0. \end{aligned}$$

This implies $P(u) = 0$ (since $v = 0$ for every $v \in B$ satisfying $vU = 0$). Thus, Assertion $\mathcal{A}$ holds. Hence, we have proved that $\mathcal{B} \Longrightarrow \mathcal{A}$.

Proof of the implication $\mathcal{A} \Longrightarrow \mathcal{D}$. Assume that Assertion $\mathcal{A}$ holds. Then, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$. Since $P \in A[X]$ is a monic polynomial with $\deg P = n$, there exist elements a_0, a_1, ..., a_{n-1} of A such that $P(X) = X^n + \sum\limits_{k=0}^{n-1} a_k X^k$. Thus, $P(u) = u^n + \sum\limits_{k=0}^{n-1} a_k u^k$, so that $P(u) = 0$ becomes $u^n + \sum\limits_{k=0}^{n-1} a_k u^k = 0$. Hence, $u^n = -\sum\limits_{k=0}^{n-1} a_k u^k$.

Let U be the A-submodule $\langle u^0, u^1, ..., u^{n-1}\rangle_A$ of B. As in the Proof of the implication $\mathcal{A} \Longrightarrow \mathcal{C}$, we can show that U is an n-generated A-module, and that $1 \in U$ and $uU \subseteq U$.

Now, we are going to show that

$$u^i \in U \qquad \text{for any } i \in \mathbb{N}. \tag{1}$$

Proof of (1). We will prove (1) by induction over i:

Induction base: The assertion (1) holds for $i = 0$ (since $u^0 \in U$). This completes the induction base.

Induction step: Let $\tau \in \mathbb{N}$. If the assertion (1) holds for $i = \tau$, then the assertion (1) holds for $i = \tau + 1$ (because if the assertion (1) holds for $i = \tau$, then $u^\tau \in U$, so that $u^{\tau+1} = u \cdot \underbrace{u^\tau}_{\in U} \in uU \subseteq U$, so that $u^{\tau+1} \in U$, and thus the assertion (1) holds for $i = \tau + 1$). This completes the induction step.

Hence, the induction is complete, and (1) is proven.

Thus,

$$A[u] = \left\{ \sum_{i=0}^{m} a_i u^i \ \mid\ m \in \mathbb{N} \text{ and } (a_0, a_1, ..., a_m) \in A^{m+1} \right\} \subseteq U$$

(since $\sum\limits_{i=0}^{m} a_i u^i \in U$ for any $m \in \mathbb{N}$ and any $(a_0, a_1, ..., a_m) \in A^{m+1}$, because $a_i \in A$ and $u^i \in U$ for any $i \in \{0, 1, ..., m\}$ (by (1)) and U is an A-module). On the other hand, $U \subseteq A[u]$, since

$$U = \left\langle u^0, u^1, ..., u^{n-1} \right\rangle_A = \left\{ \sum_{i=0}^{n-1} a_i u^i \mid (a_0, a_1, ..., a_{n-1}) \in A^n \right\}$$
$$\subseteq \left\{ \sum_{i=0}^{m} a_i u^i \mid m \in \mathbb{N} \text{ and } (a_0, a_1, ..., a_m) \in A^{m+1} \right\} = A[u].$$

Thus, $U = A[u]$. In other words, $\left\langle u^0, u^1, ..., u^{n-1} \right\rangle_A = A[u]$. Thus, Assertion $\mathcal{D}$ holds. Hence, we have proved that $\mathcal{A} \Longrightarrow \mathcal{D}$.

Proof of the implication $\mathcal{D} \Longrightarrow \mathcal{C}$. Assume that Assertion $\mathcal{D}$ holds. Then, $A[u] = \left\langle u^0, u^1, ..., u^{n-1} \right\rangle_A$.

Let U be the A-submodule $\left\langle u^0, u^1, ..., u^{n-1} \right\rangle_A$ of B. Then, U is an n-generated A-module (since u^0, u^1, ..., u^{n-1} are n elements of U). Besides, $1 = u^0 \in U$.

Also,

$$uU = u \cdot \left\langle u^0, u^1, ..., u^{n-1} \right\rangle_A = u \cdot A[u] \subseteq A[u] = \left\langle u^0, u^1, ..., u^{n-1} \right\rangle_A = U.$$

Thus, Assertion $\mathcal{C}$ holds. Hence, we have proved that $\mathcal{D} \Longrightarrow \mathcal{C}$.

Now, we have proved the implications $\mathcal{A} \Longrightarrow \mathcal{D}$, $\mathcal{D} \Longrightarrow \mathcal{C}$, $\mathcal{C} \Longrightarrow \mathcal{B}$ and $\mathcal{B} \Longrightarrow \mathcal{A}$ above. Thus, all four assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are pairwise equivalent, and Theorem 1 is proven.

> **Theorem 2.** Let A and B be two rings such that $A \subseteq B$. Let $n \in \mathbb{N}$. Let $v \in B$. Let a_0, a_1, ..., a_n be $n+1$ elements of A such that $\sum\limits_{i=0}^{n} a_i v^i = 0$. Let $k \in \{0, 1, ..., n\}$. Then, $\sum\limits_{i=0}^{n-k} a_{i+k} v^i$ is n-integral over A.

Proof of Theorem 2. Let U be the A-submodule $\left\langle v^0, v^1, ..., v^{n-1} \right\rangle_A$ of B. Then, U is an n-generated A-module (since v^0, v^1, ..., v^{n-1} are n elements of U). Besides, $1 = v^0 \in U$.

Let $u = \sum\limits_{i=0}^{n-k} a_{i+k} v^i$. Then,

$$0 = \sum_{i=0}^{n} a_i v^i = \sum_{i=0}^{k-1} a_i v^i + \sum_{i=k}^{n} a_i v^i = \sum_{i=0}^{k-1} a_i v^i + \sum_{i=0}^{n-k} a_{i+k} \underbrace{v^{i+k}}_{=v^i v^k}$$
(here, we substituted $i+k$ for i in the second sum)
$$= \sum_{i=0}^{k-1} a_i v^i + v^k \underbrace{\sum_{i=0}^{n-k} a_{i+k} v^i}_{=u} = \sum_{i=0}^{k-1} a_i v^i + v^k u,$$

so that $v^k u = -\sum\limits_{i=0}^{k-1} a_i v^i$.

Now, we are going to show that

$$uv^t \in U \qquad \text{for any } t \in \{0, 1, ..., n-1\}. \tag{2}$$

Proof of (2). Since $t \in \{0, 1, ..., n-1\}$, one of the following two cases must hold:
Case 1: We have $t \in \{0, 1, ..., k-1\}$.
Case 2: We have $t \in \{k, k+1, ..., n-1\}$.
In Case 1, we have

$$\begin{aligned} uv^t &= \sum_{i=0}^{n-k} a_{i+k} \underbrace{v^i \cdot v^t}_{=v^{i+t}} = \sum_{i=0}^{n-k} a_{i+k} v^{i+t} \in \left\langle v^0, v^1, ..., v^{n-1} \right\rangle_A \\ &\qquad \left(\begin{array}{c} \text{since } t \in \{0, 1, ..., k-1\} \text{ yields } i+t \in \{0, 1, ..., n-1\} \text{ and thus} \\ v^{i+t} \in \{v^0, v^1, ..., v^{n-1}\} \text{ for any } i \in \{0, 1, ..., n-k\} \end{array} \right) \\ &= U. \end{aligned}$$

In Case 2, we have $t \in \{k, k+1, ..., n-1\}$, thus $t-k \in \{0, 1, ..., n-k-1\}$ and hence

$$\begin{aligned} uv^t &= u \underbrace{v^{k+(t-k)}}_{=v^k v^{t-k}} = v^k u \cdot v^{t-k} = -\sum_{i=0}^{k-1} a_i \underbrace{v^i \cdot v^{t-k}}_{=v^{i+(t-k)}} \qquad \left(\text{since } v^k u = -\sum_{i=0}^{k-1} a_i v^i \right) \\ &= -\sum_{i=0}^{k-1} a_i v^{i+(t-k)} \in \left\langle v^0, v^1, ..., v^{n-1} \right\rangle_A \\ &\qquad \left(\begin{array}{c} \text{since } t-k \in \{0, 1, ..., n-k-1\} \text{ yields } i+(t-k) \in \{0, 1, ..., n-1\} \text{ and thus} \\ v^{i+(t-k)} \in \{v^0, v^1, ..., v^{n-1}\} \text{ for any } i \in \{0, 1, ..., k-1\} \end{array} \right) \\ &= U. \end{aligned}$$

Hence, in both cases, we have $uv^t \in U$. Thus, $uv^t \in U$ always holds, and (2) is proven.

Now,

$$uU = u \left\langle v^0, v^1, ..., v^{n-1} \right\rangle_A = \left\langle uv^0, uv^1, ..., uv^{n-1} \right\rangle_A \subseteq U \qquad (\text{due to (2)}).$$

Altogether, U is an n-generated A-submodule of B such that $1 \in U$ and $uU \subseteq U$. Thus, $u \in B$ satisfies Assertion $\mathcal{C}$ of Theorem 1. Hence, $u \in B$ satisfies the four equivalent assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of Theorem 1. Consequently, u is n-integral over A. Since $u = \sum\limits_{i=0}^{n-k} a_{i+k} v^i$, this means that $\sum\limits_{i=0}^{n-k} a_{i+k} v^i$ is n-integral over A. This proves Theorem 2.

Corollary 3. Let A and B be two rings such that $A \subseteq B$. Let $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$. Let $u \in B$ and $v \in B$. Let s_0, s_1, ..., s_α be $\alpha + 1$ elements of A such that $\sum\limits_{i=0}^{\alpha} s_i v^i = u$. Let t_0, t_1, ..., t_β be $\beta + 1$ elements of A such that $\sum\limits_{i=0}^{\beta} t_i v^{\beta - i} = uv^\beta$. Then, u is $(\alpha + \beta)$-integral over A.

(This Corollary 3 generalizes Exercise 2-5 in [1].)

Proof of Corollary 3. Let $k = \beta$ and $n = \alpha + \beta$. Then, $k \in \{0, 1, ..., n\}$. Define $n+1$ elements a_0, a_1, ..., a_n of A by

$$a_i = \begin{cases} t_{\beta-i}, & \text{if } i < \beta; \\ t_0 - s_0, & \text{if } i = \beta; \\ -s_{i-\beta}, & \text{if } i > \beta; \end{cases} \qquad \text{for every } i \in \{0, 1, ..., n\}.$$

Then,

$$\sum_{i=0}^{n} a_i v^i = \sum_{i=0}^{\alpha+\beta} a_i v^i = \sum_{i=0}^{\beta-1} \underbrace{a_i}_{\substack{=t_{\beta-i},\\ \text{since}\\ i<\beta}} v^i + \sum_{i=\beta}^{\beta} \underbrace{a_i}_{\substack{=t_0-s_0,\\ \text{since}\\ i=\beta}} v^i + \sum_{i=\beta+1}^{\alpha+\beta} \underbrace{a_i}_{\substack{=-s_{i-\beta},\\ \text{since}\\ i>\beta}} v^i$$

$$= \sum_{i=0}^{\beta-1} t_{\beta-i} v^i + \underbrace{\sum_{i=\beta}^{\beta} (t_0 - s_0) v^i}_{\substack{=(t_0-s_0)v^\beta \\ =t_0 v^\beta - s_0 v^\beta}} + \underbrace{\sum_{i=\beta+1}^{\alpha+\beta} (-s_{i-\beta}) v^i}_{=-\sum_{i=\beta+1}^{\alpha+\beta} s_{i-\beta} v^i}$$

$$= \sum_{i=0}^{\beta-1} t_{\beta-i} v^i + t_0 v^\beta - s_0 v^\beta - \sum_{i=\beta+1}^{\alpha+\beta} s_{i-\beta} v^i = \sum_{i=0}^{\beta-1} t_{\beta-i} v^i + t_0 v^\beta - \left(s_0 v^\beta + \sum_{i=\beta+1}^{\alpha+\beta} s_{i-\beta} v^i \right)$$

$$= \sum_{i=0}^{\beta-1} t_{\beta-i} v^i + t_0 v^\beta - \left(s_0 v^\beta + \sum_{i=1}^{\alpha} \underbrace{s_{(i+\beta)-\beta}}_{=s_i} \underbrace{v^{i+\beta}}_{=v^i v^\beta} \right)$$

(here, we substituted $i + \beta$ for i in the second sum)

$$= \sum_{i=0}^{\beta-1} t_{\beta-i} v^i + t_0 v^\beta - \left(s_0 v^\beta + \sum_{i=1}^{\alpha} s_i v^i v^\beta \right)$$

$$= \sum_{i=1}^{\beta} \underbrace{t_{\beta-(\beta-i)}}_{=t_i} v^{\beta-i} + t_0 \underbrace{v^\beta}_{=v^{\beta-0}} - \left(s_0 \underbrace{v^\beta}_{=v^0 v^\beta} + \sum_{i=1}^{\alpha} s_i v^i v^\beta \right)$$

(here, we substituted $\beta - i$ for i in the first sum)

$$= \sum_{i=1}^{\beta} t_i v^{\beta-i} + t_0 v^{\beta-0} - \left(s_0 v^0 v^\beta + \sum_{i=1}^{\alpha} s_i v^i v^\beta \right)$$

$$= \underbrace{\sum_{i=1}^{\beta} t_i v^{\beta-i} + t_0 v^{\beta-0}}_{=\sum_{i=0}^{\beta} t_i v^{\beta-i} = u v^\beta} - \left(\underbrace{s_0 v^0 + \sum_{i=1}^{\alpha} s_i v^i}_{=\sum_{i=0}^{\alpha} s_i v^i = u} \right) v^\beta = u v^\beta - u v^\beta = 0.$$

Thus, Theorem 2 yields that $\sum\limits_{i=0}^{n-k} a_{i+k}v^i$ is n-integral over A. But

$$\sum_{i=0}^{n-k} a_{i+k}v^i = \sum_{i=0}^{n-\beta} a_{i+\beta}v^i = \sum_{i=0}^{0} \underbrace{a_{i+\beta}}_{\substack{=t_0-s_0,\\ \text{since}\\ i=0 \text{ yields}\\ i+\beta=\beta}} v^i + \sum_{i=1}^{n-\beta} \underbrace{a_{i+\beta}}_{\substack{=-s_{(i+\beta)-\beta},\\ \text{since}\\ i>0 \text{ yields}\\ i+\beta>\beta}} v^i$$

$$= \underbrace{\sum_{i=0}^{0} (t_0 - s_0) v^i}_{\substack{=(t_0-s_0)v^0\\ =t_0v^0-s_0v^0\\ =t_0-s_0v^0}} + \sum_{i=1}^{n-\beta} \left(- \underbrace{s_{(i+\beta)-\beta}}_{=s_i} \right) v^i$$

$$= t_0 - s_0v^0 + \sum_{i=1}^{n-\beta} (-s_i) v^i = t_0 - s_0v^0 - \sum_{i=1}^{n-\beta} s_iv^i$$

$$= t_0 - s_0v^0 - \sum_{i=1}^{\alpha} s_iv^i \qquad \text{(since } n = \alpha + \beta \text{ yields } n - \beta = \alpha\text{)}$$

$$= t_0 - \left(\underbrace{s_0v^0 + \sum_{i=1}^{\alpha} s_iv^i}_{=\sum\limits_{i=0}^{\alpha} s_iv^i = u} \right) = t_0 - u.$$

Thus, $t_0 - u$ is n-integral over A. On the other hand, $-t_0$ is 1-integral over A (by Theorem 5 **(a)** below, applied to $a = -t_0$). Thus, $(-t_0) + (t_0 - u)$ is $n \cdot 1$-integral over A (by Theorem 5 **(b)** below, applied to $x = -t_0$, $y = t_0 - u$ and $m = 1$). In other words, $-u$ is n-integral over A (since $(-t_0) + (t_0 - u) = -u$ and $n \cdot 1 = n$). On the other hand, -1 is 1-integral over A (by Theorem 5 **(a)** below, applied to $a = -1$). Thus, $(-1) \cdot (-u)$ is $n \cdot 1$-integral over A (by Theorem 5 **(c)** below, applied to $x = -1$, $y = -u$ and $m = 1$). In other words, u is $(\alpha + \beta)$-integral over A (since $(-1) \cdot (-u) = u$ and $n \cdot 1 = n = \alpha + \beta$). This proves Corollary 3.

Theorem 4. Let A and B be two rings such that $A \subseteq B$. Let $v \in B$ and $u \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that v is m-integral over A, and that u is n-integral over $A[v]$. Then, u is nm-integral over A.

Proof of Theorem 4. Since v is m-integral over A, we have $A[v] = \langle v^0, v^1, ..., v^{m-1} \rangle_A$ (this is the Assertion $\mathcal{D}$ of Theorem 1, stated for v and m in lieu of u and n).

Since u is n-integral over $A[v]$, we have $(A[v])[u] = \langle u^0, u^1, ..., u^{n-1} \rangle_{A[v]}$ (this is the Assertion $\mathcal{D}$ of Theorem 1, stated for $A[v]$ in lieu of A).

Let $S = \{0, 1, ..., n-1\} \times \{0, 1, ..., m-1\}$.

Let $x \in (A[v])[u]$. Then, there exist n elements $b_0, b_1, ..., b_{n-1}$ of $A[v]$ such that $x = \sum\limits_{i=0}^{n-1} b_iu^i$ (since $x \in (A[v])[u] = \langle u^0, u^1, ..., u^{n-1} \rangle_{A[v]}$). But for each $i \in \{0, 1, ..., n-1\}$, there exist m elements $a_{i,0}$, $a_{i,1}$, ..., $a_{i,m-1}$ of A such that $b_i = \sum\limits_{j=0}^{m-1} a_{i,j} v^j$ (because $b_i \in A[v] = \langle v^0, v^1, ..., v^{m-1} \rangle_A$). Thus,

$$x = \sum_{i=0}^{n-1} \underbrace{b_i}_{= \sum\limits_{j=0}^{m-1} a_{i,j} v^j} u^i = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} a_{i,j} v^j u^i = \sum_{(i,j) \in \{0,1,...,n-1\} \times \{0,1,...,m-1\}} a_{i,j} v^j u^i = \sum_{(i,j) \in S} a_{i,j} v^j u^i$$
$$\in \left\langle v^j u^i \mid (i,j) \in S \right\rangle_A \qquad \text{(since } a_{i,j} \in A \text{ for every } (i,j) \in S\text{)}$$

So we have proved that $x \in \langle v^j u^i \mid (i,j) \in S \rangle_A$ for every $x \in (A[v])[u]$. Thus, $(A[v])[u] \subseteq \langle v^j u^i \mid (i,j) \in S \rangle_A$. Conversely, $\langle v^j u^i \mid (i,j) \in S \rangle_A \subseteq (A[v])[u]$ (since $v^j \in A[v]$ for every $(i,j) \in S$, and thus $\underbrace{v^j}_{\in A[v]} u^i \in (A[v])[u]$ for every $(i,j) \in S$, and therefore

$$\left\langle v^j u^i \mid (i,j) \in S \right\rangle_A = \left\{ \underbrace{\sum_{(i,j) \in S} a_{i,j} v^j u^i}_{\substack{\in (A[v])[u], \text{ since} \\ v^j u^i \in (A[v])[u] \text{ for all } (i,j) \in S \\ \text{and } (A[v])[u] \text{ is an } A\text{-module}}} \mid (a_{i,j})_{(i,j) \in S} \in A^S \right\} \subseteq (A[v])[u]$$

). Hence, $(A[v])[u] = \langle v^j u^i \mid (i,j) \in S \rangle_A$. Thus, the A-module $(A[v])[u]$ is nm-generated (since

$$|S| = |\{0,1,...,n-1\} \times \{0,1,...,m-1\}| = \underbrace{|\{0,1,...,n-1\}|}_{=n} \cdot \underbrace{|\{0,1,...,m-1\}|}_{=m} = nm$$

).

Let $U = (A[v])[u]$. Then, the A-module U is nm-generated. Besides, U is an A-submodule of B, and we have $1 = u^0 \in (A[v])[u] = U$ and

$$uU = u(A[v])[u] \subseteq (A[v])[u] \qquad \text{(since } (A[v])[u] \text{ is an } A[v]\text{-algebra and } u \in (A[v])[u]\text{)}$$
$$= U.$$

Altogether, we now know that the A-submodule U of B is nm-generated and satisfies $1 \in U$ and $uU \subseteq U$.

Thus, the element u of B satisfies the Assertion $\mathcal{C}$ of Theorem 1 with n replaced by nm. Hence, $u \in B$ satisfies the four equivalent assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of Theorem 1, all with n replaced by nm. Thus, u is nm-integral over A. This proves Theorem 4.

Theorem 5. Let A and B be two rings such that $A \subseteq B$.

(a) Let $a \in A$. Then, a is 1-integral over A.

(b) Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m-integral over A, and that y is n-integral over A. Then, $x + y$ is nm-integral over A.

(c) Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m-integral over A, and that y is n-integral over A. Then, xy is nm-integral over A.

Proof of Theorem 5. **(a)** There exists a monic polynomial $P \in A[X]$ with $\deg P = 1$ and $P(a) = 0$ (namely, the polynomial $P \in A[X]$ defined by $P(X) = X - a$). Thus, a is 1-integral over A. This proves Theorem 5 **(a)**.

(b) Since y is n-integral over A, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(y) = 0$. Since $P \in A[X]$ is a monic polynomial with $\deg P = n$, there exists a polynomial $\widetilde{P} \in A[X]$ with $\deg \widetilde{P} < n$ and $P(X) = X^n + \widetilde{P}(X)$.

Now, define a polynomial $Q \in (A[x])[X]$ by $Q(X) = P(X - x)$. Then,

$$\begin{aligned}\deg Q &= \deg P \qquad \text{(since shifting the polynomial } P \text{ by the constant } x \text{ does not change its degree)} \\ &= n\end{aligned}$$

and $Q(x+y) = P((x+y) - x) = P(y) = 0$.

Define a polynomial $\widetilde{Q} \in (A[x])[X]$ by $\widetilde{Q}(X) = ((X-x)^n - X^n) + \widetilde{P}(X - x)$. Then, $\deg \widetilde{Q} < n$ (since

$$\begin{aligned}&\deg\left(\widetilde{P}(X-x)\right) = \deg\left(\widetilde{P}(X)\right) \\ &\qquad \left(\text{since shifting the polynomial } \widetilde{P} \text{ by the constant } x \text{ does not change its degree}\right) \\ &= \deg \widetilde{P} < n\end{aligned}$$

and

$$\begin{aligned}\deg\left((X-x)^n - X^n\right) &= \deg\left(((X-x) - X) \cdot \sum_{k=0}^{n-1} (X-x)^k X^{n-1-k}\right) \\ &\leq \underbrace{\deg((X-x) - X)}_{=\deg(-x)=0} + \underbrace{\deg\left(\sum_{k=0}^{n-1} (X-x)^k X^{n-1-k}\right)}_{\substack{\leq n-1, \text{ since} \\ \deg\left((X-x)^k X^{n-1-k}\right) \leq n-1 \\ \text{for any } k \in \{0,1,\ldots,n-1\}}} \\ &\leq 0 + (n-1) = n - 1 < n\end{aligned}$$

yield

$$\begin{aligned}\deg \widetilde{Q} &= \deg\left(\widetilde{Q}(X)\right) = \deg\left(((X-x)^n - X^n) + \widetilde{P}(X-x)\right) \\ &\leq \max\left\{\underbrace{\deg((X-x)^n - X^n)}_{<n}, \underbrace{\deg\left(\widetilde{P}(X-x)\right)}_{<n}\right\} < \max\{n, n\} = n\end{aligned}$$

). Thus, the polynomial Q is monic (since

$$\begin{aligned}Q(X) &= P(X-x) = (X-x)^n + \widetilde{P}(X-x) \qquad \left(\text{since } P(X) = X^n + \widetilde{P}(X)\right) \\ &= X^n + \underbrace{((X-x)^n - X^n) + \widetilde{P}(X-x)}_{=\widetilde{Q}(X)} = X^n + \widetilde{Q}(X)\end{aligned}$$

and $\deg \widetilde{Q} < n$).

Hence, there exists a monic polynomial $Q \in (A[x])[X]$ with $\deg Q = n$ and $Q(x+y) = 0$. Thus, $x+y$ is n-integral over $A[x]$. Thus, Theorem 4 (applied to $v = x$ and $u = x+y$) yields that $x+y$ is nm-integral over A. This proves Theorem 5 **(b)**.

(c) Since y is n-integral over A, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(y) = 0$. Since $P \in A[X]$ is a monic polynomial with $\deg P = n$, there exist elements a_0, a_1, ..., a_{n-1} of A such that $P(X) = X^n + \sum\limits_{k=0}^{n-1} a_k X^k$. Thus, $P(y) = y^n + \sum\limits_{k=0}^{n-1} a_k y^k$.

Now, define a polynomial $Q \in (A[x])[X]$ by $Q(X) = X^n + \sum\limits_{k=0}^{n-1} x^{n-k} a_k X^k$. Then,

$$Q(xy) = \underbrace{(xy)^n}_{=x^n y^n} + \sum_{k=0}^{n-1} x^{n-k} \underbrace{a_k (xy)^k}_{\substack{=a_k x^k y^k \\ =x^k a_k y^k}} = x^n y^n + \sum_{k=0}^{n-1} \underbrace{x^{n-k} x^k}_{=x^n} a_k y^k$$

$$= x^n y^n + \sum_{k=0}^{n-1} x^n a_k y^k = x^n \left(\underbrace{y^n + \sum_{k=0}^{n-1} a_k y^k}_{=P(y)=0} \right) = 0.$$

Also, the polynomial $Q \in (A[x])[X]$ is monic and $\deg Q = n$ (since $Q(X) = X^n + \sum\limits_{k=0}^{n-1} x^{n-k} a_k X^k$). Thus, there exists a monic polynomial $Q \in (A[x])[X]$ with $\deg Q = n$ and $Q(xy) = 0$. Thus, xy is n-integral over $A[x]$. Hence, Theorem 4 (applied to $v = x$ and $u = xy$) yields that xy is nm-integral over A. This proves Theorem 5 **(c)**.

> **Corollary 6.** Let A and B be two rings such that $A \subseteq B$. Let $n \in \mathbb{N}^+$ and $m \in \mathbb{N}$. Let $v \in B$. Let b_0, b_1, ..., b_{n-1} be n elements of A, and let $u = \sum\limits_{i=0}^{n-1} b_i v^i$. Assume that vu is m-integral over A. Then, u is nm-integral over A.

Proof of Corollary 6. Define $n+1$ elements a_0, a_1, ..., a_n of $A[vu]$ by

$$a_i = \begin{cases} -vu, & \text{if } i = 0; \\ b_{i-1}, & \text{if } i > 0 \end{cases} \qquad \text{for every } i \in \{0, 1, ..., n\}.$$

Then, $a_0 = -vu$. Let $k = 1$. Then,

$$\sum_{i=0}^{n} a_i v^i = \underbrace{a_0}_{=-vu} \underbrace{v^0}_{=1} + \sum_{i=1}^{n} \underbrace{a_i}_{\substack{=b_{i-1}, \\ \text{since} \\ i>0}} \underbrace{v^i}_{=v^{i-1}v} = -vu + \sum_{i=1}^{n} b_{i-1} v^{i-1} v = -vu + \underbrace{\sum_{i=0}^{n-1} b_i v^i}_{=u} v$$

(here, we substituted i for $i-1$ in the sum)

$$= -vu + uv = 0.$$

Now, $A[vu]$ and B are two rings such that $A[vu] \subseteq B$. The $n+1$ elements a_0, a_1, ..., a_n of $A[vu]$ satisfy $\sum\limits_{i=0}^{n} a_i v^i = 0$. We have $k = 1 \in \{0, 1, ..., n\}$.

Hence, Theorem 2 (applied to the ring $A[vu]$ in lieu of A) yields that $\sum\limits_{i=0}^{n-k} a_{i+k} v^i$ is n-integral over $A[vu]$. But

$$\sum_{i=0}^{n-k} a_{i+k} v^i = \sum_{i=0}^{n-1} \underbrace{a_{i+1}}_{\substack{=b_{(i+1)-1}, \\ \text{since } i+1>0}} v^i = \sum_{i=0}^{n-1} b_{(i+1)-1} v^i = \sum_{i=0}^{n-1} b_i v^i = u.$$

Hence, u is n-integral over $A[vu]$. But vu is m-integral over A. Thus, Theorem 4 (applied to vu in lieu of v) yields that u is nm-integral over A. This proves Corollary 6.

2. Integrality over ideal semifiltrations

Definitions:

Definition 6. Let A be a ring, and let $(I_\rho)_{\rho \in \mathbb{N}}$ be a sequence of ideals of A. Then, $(I_\rho)_{\rho \in \mathbb{N}}$ is called an *ideal semifiltration* of A if and only if it satisfies the two conditions

$$I_0 = A;$$
$$I_a I_b \subseteq I_{a+b} \qquad \text{for every } a \in \mathbb{N} \text{ and } b \in \mathbb{N}.$$

Definition 7. Let A and B be two rings such that $A \subseteq B$. Then, we identify the polynomial ring $A[Y]$ with a subring of the polynomial ring $B[Y]$ (in fact, every element of $A[Y]$ has the form $\sum\limits_{i=0}^{m} a_i Y^i$ for some $m \in \mathbb{N}$ and $(a_0, a_1, ..., a_m) \in A^{m+1}$, and thus can be seen as an element of $B[Y]$ by regarding a_i as an element of B for every $i \in \{0, 1, ..., m\}$).

Definition 8. Let A be a ring, and let $(I_\rho)_{\rho \in \mathbb{N}}$ be an ideal semifiltration of A. Consider the polynomial ring $A[Y]$. Let $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$ denote the A-submodule $\sum\limits_{i \in \mathbb{N}} I_i Y^i$ of the A-algebra $A[Y]$. Then,

$$A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right] = \sum_{i \in \mathbb{N}} I_i Y^i$$
$$= \left\{ \sum_{i \in \mathbb{N}} a_i Y^i \mid (a_i \in I_i \text{ for all } i \in \mathbb{N}), \text{ and } (\text{only finitely many } i \in \mathbb{N} \text{ satisfy } a_i \neq 0) \right\}$$
$$= \{P \in A[Y] \mid \text{the } i\text{-th coefficient of the polynomial } P \text{ lies in } I_i \text{ for every } i \in \mathbb{N}\}.$$

Now, $1 \in A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$ (because $1 = \underbrace{1}_{\in A = I_0} \cdot Y^0 \in I_0 Y^0 \subseteq \sum\limits_{i \in \mathbb{N}} I_i Y^i = A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$).

Also, the A-submodule $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ of $A[Y]$ is closed under multiplication (since

$$\begin{aligned} A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \cdot A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] &= \sum_{i\in\mathbb{N}} I_i Y^i \cdot \sum_{i\in\mathbb{N}} I_i Y^i = \sum_{i\in\mathbb{N}} I_i Y^i \cdot \sum_{j\in\mathbb{N}} I_j Y^j \\ &\qquad \text{(here we renamed } i \text{ as } j \text{ in the second sum)} \\ &= \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} I_i Y^i I_j Y^j = \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} \underbrace{I_i I_j}_{\substack{\subseteq I_{i+j},\\ \text{since } (I_\rho)_{\rho\in\mathbb{N}} \\ \text{is an ideal} \\ \text{semifiltration}}} \underbrace{Y^i Y^j}_{=Y^{i+j}} \\ &\subseteq \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} I_{i+j} Y^{i+j} \subseteq \sum_{k\in\mathbb{N}} I_k Y^k = \sum_{i\in\mathbb{N}} I_i Y^i \\ &\qquad \text{(here we renamed } k \text{ as } i \text{ in the sum)} \\ &= A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \end{aligned}$$

). Hence, $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ is an A-subalgebra of the A-algebra $A[Y]$. This A-subalgebra $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ is called the *Rees algebra* of the ideal semifiltration $(I_\rho)_{\rho\in\mathbb{N}}$.

Clearly, $A \subseteq A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$, since $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] = \sum_{i\in\mathbb{N}} I_i Y^i \supseteq \underbrace{I_0}_{=A} \underbrace{Y^0}_{=1} = A \cdot 1 = A$.

Definition 9. Let A and B be two rings such that $A \subseteq B$. Let $(I_\rho)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $n \in \mathbb{N}$. Let $u \in B$.

We say that the element u of B is n*-integral over* $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$ if there exists some $(a_0, a_1, ..., a_n) \in A^{n+1}$ such that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i} \text{ for every } i \in \{0, 1, ..., n\}.$$

We start with a theorem which reduces the question of n-integrality over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$ to that of n-integrality over a ring[1]:

> **Theorem 7.** Let A and B be two rings such that $A \subseteq B$. Let $(I_\rho)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $n \in \mathbb{N}$. Let $u \in B$.
>
> Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ defined in Definition 8.
>
> Then, the element u of B is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$ if and only if the element uY of the polynomial ring $B[Y]$ is n-integral over the ring $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$. (Here, $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq B[Y]$ because $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq A[Y]$ and we consider $A[Y]$ as a subring of $B[Y]$ as explained in Definition 7).

[1] Theorem 7 is inspired by Proposition 5.2.1 in [2].

Proof of Theorem 7. In order to verify Theorem 7, we have to prove the following two lemmata:

Lemma $\mathcal{E}$: If u is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$, then uY is n-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$.

Lemma $\mathcal{F}$: If uY is n-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$, then u is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$.

Proof of Lemma $\mathcal{E}$: Assume that u is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$. Then, by Definition 9, there exists some $(a_0, a_1, ..., a_n) \in A^{n+1}$ such that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i} \text{ for every } i \in \{0, 1, ..., n\}.$$

Note that $a_k Y^{n-k} \in A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ for every $k \in \{0, 1, ..., n\}$ (because $\underbrace{a_k}_{\in I_{n-k}} Y^{n-k} \in I_{n-k} Y^{n-k} \subseteq \sum\limits_{i\in\mathbb{N}} I_i Y^i = A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$). Thus, we can define a polynomial $P \in \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[X]$ by $P(X) = \sum\limits_{k=0}^{n} a_k Y^{n-k} X^k$. This polynomial P satisfies $\deg P \leq n$, and its coefficient before X^n is $\underbrace{a_n}_{=1} \underbrace{Y^{n-n}}_{=Y^0=1} = 1$. Hence, this polynomial P is monic and satisfies $\deg P = n$. Also, $P(X) = \sum\limits_{k=0}^{n} a_k Y^{n-k} X^k$ yields

$$P(uY) = \sum_{k=0}^{n} a_k Y^{n-k} (uY)^k = \sum_{k=0}^{n} a_k Y^{n-k} u^k Y^k = \sum_{k=0}^{n} a_k u^k \underbrace{Y^{n-k} Y^k}_{=Y^n} = Y^n \cdot \underbrace{\sum_{k=0}^{n} a_k u^k}_{=0} = 0.$$

Thus, there exists a monic polynomial $P \in \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[X]$ with $\deg P = n$ and $P(uY) = 0$. Hence, uY is n-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$. This proves Lemma $\mathcal{E}$.

Proof of Lemma $\mathcal{F}$: Assume that uY is n-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$. Then, there exists a monic polynomial $P \in \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[X]$ with $\deg P = n$ and $P(uY) = 0$. Since $P \in \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[X]$ satisfies $\deg P = n$, there exists $(p_0, p_1, ..., p_n) \in \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)^{n+1}$ such that $P(X) = \sum\limits_{k=0}^{n} p_k X^k$. Besides, $p_n = 1$, since P is monic and $\deg P = n$.

For every $k \in \{0, 1, ..., n\}$, we have $p_k \in A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] = \sum\limits_{i\in\mathbb{N}} I_i Y^i$, and thus, there exists a sequence $(p_{k,i})_{i\in\mathbb{N}} \in A^{\mathbb{N}}$ such that $p_k = \sum\limits_{i\in\mathbb{N}} p_{k,i} Y^i$, such that $p_{k,i} \in I_i$ for every $i \in \mathbb{N}$, and such that only finitely many $i \in \mathbb{N}$ satisfy $p_{k,i} \neq 0$. Thus, $P(X) = \sum\limits_{k=0}^{n} p_k X^k$

becomes $P(X)=\sum_{k=0}^{n}\sum_{i\in\mathbb{N}}p_{k,i}Y^iX^k$ (since $p_k=\sum_{i\in\mathbb{N}}p_{k,i}Y^i$). Hence,

$$\begin{aligned}P(uY)&=\sum_{k=0}^{n}\sum_{i\in\mathbb{N}}p_{k,i}Y^i\underbrace{(uY)^k}_{\substack{=u^kY^k\\=Y^ku^k}}=\sum_{k=0}^{n}\sum_{i\in\mathbb{N}}p_{k,i}\underbrace{Y^iY^k}_{=Y^{i+k}}u^k\\&=\sum_{k=0}^{n}\sum_{i\in\mathbb{N}}p_{k,i}Y^{i+k}u^k=\sum_{k\in\{0,1,...,n\}}\sum_{i\in\mathbb{N}}p_{k,i}Y^{i+k}u^k\\&=\sum_{(k,i)\in\{0,1,...,n\}\times\mathbb{N}}p_{k,i}Y^{i+k}u^k=\sum_{\ell\in\mathbb{N}}\sum_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N};\\i+k=\ell}}p_{k,i}\underbrace{Y^{i+k}}_{=Y^\ell}u^k\\&=\sum_{\ell\in\mathbb{N}}\sum_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N};\\i+k=\ell}}p_{k,i}Y^\ell u^k=\sum_{\ell\in\mathbb{N}}\sum_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N};\\i+k=\ell}}p_{k,i}u^kY^\ell.\end{aligned}$$

Hence, $P(uY)=0$ becomes $\sum_{\ell\in\mathbb{N}}\sum_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N};\\i+k=\ell}}p_{k,i}u^kY^\ell=0$. In other words, the polynomial $\sum_{\ell\in\mathbb{N}}\underbrace{\sum_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N};\\i+k=\ell}}p_{k,i}u^k}_{\in B}Y^\ell\in B[Y]$ equals 0. Hence, its coefficient before Y^n equals 0 as well. But its coefficient before Y^n is $\sum_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N};\\i+k=n}}p_{k,i}u^k$. Hence, $\sum_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N};\\i+k=n}}p_{k,i}u^k$ equals 0.

Thus,

$$0=\sum_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N};\\i+k=n}}p_{k,i}u^k=\sum_{k\in\{0,1,...,n\}}\sum_{\substack{i\in\mathbb{N};\\i+k=n}}p_{k,i}u^k=\sum_{k\in\{0,1,...,n\}}p_{k,n-k}u^k$$

$$\left(\begin{array}{c}\text{since }\{i\in\mathbb{N}\mid i+k=n\}=\{i\in\mathbb{N}\mid i=n-k\}=\{n-k\}\text{ (because }n-k\in\mathbb{N},\\ \text{since }k\in\{0,1,...,n\}\text{) yields }\sum_{\substack{i\in\mathbb{N};\\i+k=n}}p_{k,i}u^k=\sum_{i\in\{n-k\}}p_{k,i}u^k=p_{k,n-k}u^k\end{array}\right).$$

Note that

$$\begin{aligned}\sum_{i\in\mathbb{N}}p_{n,i}Y^i&=p_n\qquad\left(\text{since }\sum_{i\in\mathbb{N}}p_{k,i}Y^i=p_k\text{ for every }k\in\{0,1,...,n\}\right)\\&=1=1\cdot Y^0\end{aligned}$$

in $A[Y]$, and thus the coefficient of the polynomial $\sum_{i\in\mathbb{N}}p_{n,i}Y^i\in A[Y]$ before Y^0 is 1; but the coefficient of the polynomial $\sum_{i\in\mathbb{N}}p_{n,i}Y^i\in A[Y]$ before Y^0 is $p_{n,0}$; hence, $p_{n,0}=1$.

Define an $(n+1)$-tuple $(a_0,a_1,...,a_n)\in A^{n+1}$ by $a_k=p_{k,n-k}$ for every $k\in\{0,1,...,n\}$. Then, $a_n=p_{n,n-n}=p_{n,0}=1$. Besides,

$$\sum_{k=0}^{n}a_ku^k=\sum_{k=0}^{n}p_{k,n-k}u^k=\sum_{k\in\{0,1,...,n\}}p_{k,n-k}u^k=0.$$

Finally, $a_k = p_{k,n-k} \in I_{n-k}$ (since $p_{k,i} \in I_i$ for every $i \in \mathbb{N}$) for every $k \in \{0, 1, ..., n\}$. In other words, $a_i \in I_{n-i}$ for every $i \in \{0, 1, ..., n\}$.

Altogether, we now know that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i} \text{ for every } i \in \{0, 1, ..., n\}.$$

Thus, by Definition 9, the element u is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$. This proves Lemma $\mathcal{F}$.

Combining Lemmata $\mathcal{E}$ and $\mathcal{F}$, we obtain that u is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$ if and only if uY is n-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$. This proves Theorem 7.

The next theorem is an analogue of Theorem 5 for integrality over ideal semifiltrations:

Theorem 8. Let A and B be two rings such that $A \subseteq B$. Let $(I_\rho)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A.

(a) Let $u \in A$. Then, u is 1-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$ if and only if $u \in I_1$.

(b) Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$, and that y is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$. Then, $x + y$ is nm-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$.

(c) Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$, and that y is n-integral over A. Then, xy is nm-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$.

Proof of Theorem 8. **(a)** In order to verify Theorem 8 **(a)**, we have to prove the following two lemmata:

Lemma $\mathcal{G}$: If u is 1-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$, then $u \in I_1$.

Lemma $\mathcal{H}$: If $u \in I_1$, then u is 1-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$.

Proof of Lemma $\mathcal{G}$: Assume that u is 1-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$. Then, by Definition 9 (applied to $n = 1$), there exists some $(a_0, a_1) \in A^2$ such that

$$\sum_{k=0}^{1} a_k u^k = 0, \qquad a_1 = 1, \qquad \text{and} \qquad a_i \in I_{1-i} \text{ for every } i \in \{0, 1\}.$$

Thus, $a_0 \in I_{1-0}$ (since $a_i \in I_{1-i}$ for every $i \in \{0, 1\}$). Also,

$$0 = \sum_{k=0}^{1} a_k u^k = a_0 \underbrace{u^0}_{=1} + \underbrace{a_1}_{=1} \underbrace{u^1}_{=u} = a_0 + u,$$

so that $u = -\underbrace{a_0}_{\in I_{1-0} = I_1} \in I_1$ (since I_1 is an ideal). This proves Lemma $\mathcal{G}$.

Proof of Lemma $\mathcal{H}$: Assume that $u \in I_1$. Then, $-u \in I_1$ (since I_1 is an ideal). Set $a_0 = -u$ and $a_1 = 1$. Then, $\sum\limits_{k=0}^{1} a_k u^k = \underbrace{a_0}_{=-u} \underbrace{u^0}_{=1} + \underbrace{a_1}_{=1} \underbrace{u^1}_{=u} = -u + u = 0$. Also, $a_i \in I_{1-i}$ for every $i \in \{0, 1\}$ (since $a_0 = -u \in I_1 = I_{1-0}$ and $a_1 = 1 \in A = I_0 = I_{1-1}$). Altogether, we now know that $(a_0, a_1) \in A^2$ and

$$\sum_{k=0}^{1} a_k u^k = 0, \qquad a_1 = 1, \qquad \text{and} \qquad a_i \in I_{1-i} \text{ for every } i \in \{0, 1\}.$$

Thus, by Definition 9 (applied to $n = 1$), the element u is 1-integral over $\left(A, (I_\rho)_{\rho \in \mathbb{N}}\right)$. This proves Lemma $\mathcal{H}$.

Combining Lemmata $\mathcal{G}$ and $\mathcal{H}$, we obtain that u is 1-integral over $\left(A, (I_\rho)_{\rho \in \mathbb{N}}\right)$ if and only if $u \in I_1$. This proves Theorem 8 **(a)**.

(b) Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$. Theorem 7 (applied to x and m instead of u and n) yields that xY is m-integral over $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$ (since x is m-integral over $\left(A, (I_\rho)_{\rho \in \mathbb{N}}\right)$). Also, Theorem 7 (applied to y instead of u) yields that yY is n-integral over $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$ (since y is n-integral over $\left(A, (I_\rho)_{\rho \in \mathbb{N}}\right)$). Hence, Theorem 5 **(b)** (applied to $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$, $B[Y]$, xY and yY instead of A, B, x and y, respectively) yields that $xY + yY$ is nm-integral over $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$. Since $xY + yY = (x + y) Y$, this means that $(x + y) Y$ is nm-integral over $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$. Hence, Theorem 7 (applied to $x + y$ and nm instead of u and n) yields that $x + y$ is nm-integral over $\left(A, (I_\rho)_{\rho \in \mathbb{N}}\right)$. This proves Theorem 8 **(b)**.

(c) First, a trivial observation:

Lemma $\mathcal{I}$: Let A, A' and B' be three rings such that $A \subseteq A' \subseteq B'$. Let $v \in B'$. Let $n \in \mathbb{N}$. If v is n-integral over A, then v is n-integral over A'.

Proof of Lemma $\mathcal{I}$: Assume that v is n-integral over A. Then, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(v) = 0$. Since $A \subseteq A'$, we can identify the polynomial ring $A[X]$ with a subring of the polynomial ring $A'[X]$ (as explained in Definition 7). Thus, $P \in A[X]$ yields $P \in A'[X]$. Hence, there exists a monic polynomial $P \in A'[X]$ with $\deg P = n$ and $P(v) = 0$. Thus, v is n-integral over A'. This proves Lemma $\mathcal{I}$.

Now let us prove Theorem 8 **(c)**.

Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$. Theorem 7 (applied to x and m instead of u and n) yields that xY is m-integral over $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$ (since x is m-integral over $\left(A, (I_\rho)_{\rho \in \mathbb{N}}\right)$). On the other hand, Lemma $\mathcal{I}$ (applied to $A' = A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$, $B' = B[Y]$ and $v = y$) yields that y is n-integral over $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$ (since y is n-integral over A, and $A \subseteq A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right] \subseteq B[Y]$). Hence, Theorem 5 **(c)** (applied to $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$, $B[Y]$ and xY instead of A, B and x, respectively) yields that $xY \cdot y$ is nm-integral over $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$. Since $xY \cdot y = xyY$, this means that xyY is nm-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$. Hence, Theorem 7 (applied to xy and nm instead of u and n) yields that xy is nm-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$. This proves Theorem 8 **(c)**.

The next theorem imitates Theorem 4 for integrality over ideal semifiltrations:

Theorem 9. Let A and B be two rings such that $A \subseteq B$. Let $(I_\rho)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A.

Let $v \in B$ and $u \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$.

(a) Then, $(I_\rho A[v])_{\rho\in\mathbb{N}}$ is an ideal semifiltration of $A[v]$.

(b) Assume that v is m-integral over A, and that u is n-integral over $\left(A[v], (I_\rho A[v])_{\rho\in\mathbb{N}}\right)$. Then, u is nm-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$.

Proof of Theorem 9. **(a)** More generally:

Lemma $\mathcal{J}$: Let A and A' be two rings such that $A \subseteq A'$. Let $(I_\rho)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Then, $(I_\rho A')_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A'.

Proof of Lemma $\mathcal{J}$: Since $(I_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A, the set I_ρ is an ideal of A for every $\rho \in \mathbb{N}$, and we have

$$I_0 = A;$$
$$I_a I_b \subseteq I_{a+b} \qquad \text{for every } a \in \mathbb{N} \text{ and } b \in \mathbb{N}.$$

Now, the set $I_\rho A'$ is an ideal of A' for every $\rho \in \mathbb{N}$ (since I_ρ is an ideal of A), and we have

$$I_0 A' = AA' = A';$$
$$I_a A' \cdot I_b A' = I_a I_b A' \subseteq I_{a+b} A' \ \left(\text{since } I_a I_b \subseteq I_{a+b}\right) \qquad \text{for every } a \in \mathbb{N} \text{ and } b \in \mathbb{N}.$$

Thus, $(I_\rho A')_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A'. This proves Lemma $\mathcal{J}$.

Now let us prove Theorem 9 **(a)**. In fact, Lemma $\mathcal{J}$ (applied to $A' = A[v]$) yields that $(I_\rho A[v])_{\rho\in\mathbb{N}}$ is an ideal semifiltration of $A[v]$. This proves Theorem 9 **(a)**.

(b) First, we will show a simple fact:

Lemma $\mathcal{K}$: Let A, A' and B' be three rings such that $A \subseteq A' \subseteq B'$. Let $v \in B'$. Then, $A' \cdot A[v] = A'[v]$.

Proof of Lemma $\mathcal{K}$: We have $\underbrace{A'}_{\subseteq A'[v]} \cdot \underbrace{A[v]}_{\substack{\subseteq A'[v], \\ \text{since } A \subseteq A'}} \subseteq A'[v] \cdot A'[v] = A'[v]$ (since $A'[v]$ is a ring). On the other hand, let x be an element of $A'[v]$. Then, there exists some $n \in \mathbb{N}$ and some $(a_0, a_1, ..., a_n) \in (A')^{n+1}$ such that $x = \sum\limits_{k=0}^{n} a_k v^k$. Thus,

$$x = \sum_{k=0}^{n} \underbrace{a_k}_{\in A'} \underbrace{v^k}_{\in A[v]} \in \sum_{k=0}^{n} A' \cdot A[v] \subseteq A' \cdot A[v] \qquad \left(\text{since } A' \cdot A[v] \text{ is an additive group}\right).$$

Thus, we have proved that $x \in A' \cdot A[v]$ for every $x \in A'[v]$. Therefore, $A'[v] \subseteq A' \cdot A[v]$. Combined with $A' \cdot A[v] \subseteq A'[v]$, this yields $A' \cdot A[v] = A'[v]$. Hence, we have established Lemma $\mathcal{K}$.

Now let us prove Theorem 9 **(b)**. In fact, consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$. We have $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq A[Y]$, and (as explained in Definition 7) we can identify the polynomial ring $A[Y]$ with a subring of $(A[v])[Y]$ (since $A \subseteq A[v]$). Hence, $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq (A[v])[Y]$. On the other hand, $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right] \subseteq (A[v])[Y]$.

Now, we will show that $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right] = \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[v]$.

In fact, Definition 8 yields

$$\begin{aligned}(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right] &= \sum_{i\in\mathbb{N}} I_i A[v] \cdot Y^i = \sum_{i\in\mathbb{N}} I_i Y^i \cdot A[v] = A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \cdot A[v] \\ &\qquad \left(\text{since } \sum_{i\in\mathbb{N}} I_i Y^i = A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right) \\ &= \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[v]\end{aligned}$$

(by Lemma $\mathcal{K}$ (applied to $A' = A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ and $B' = (A[v])[Y]$)).

Note that (as explained in Definition 7) we can identify the polynomial ring $(A[v])[Y]$ with a subring of $B[Y]$ (since $A[v] \subseteq B$). Thus, $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq (A[v])[Y]$ yields $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq B[Y]$.

Besides, Lemma $\mathcal{I}$ (applied to $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$, $B[Y]$ and m instead of A', B' and n) yields that v is m-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ (since v is m-integral over A, and $A \subseteq A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq B[Y]$).

Now, Theorem 7 (applied to $A[v]$ and $(I_\rho A[v])_{\rho\in\mathbb{N}}$ instead of A and $(I_\rho)_{\rho\in\mathbb{N}}$) yields that uY is n-integral over $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right]$ (since u is n-integral over $\left(A[v], (I_\rho A[v])_{\rho\in\mathbb{N}}\right)$). Since $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right] = \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[v]$, this means that uY is n-integral over $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[v]$. Now, Theorem 4 (applied to $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$, $B[Y]$ and uY instead of A, B and u) yields that uY is nm-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ (since v is m-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$, and uY is n-integral over $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[v]$). Thus, Theorem 7 (applied to nm instead of n) yields that u is nm-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$. This proves Theorem 9 **(b)**.

3. Generalizing to two ideal semifiltrations

Theorem 10. Let A be a ring.

(a) Then, $(A)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A.

(b) Let $(I_\rho)_{\rho\in\mathbb{N}}$ and $(J_\rho)_{\rho\in\mathbb{N}}$ be two ideal semifiltrations of A. Then, $(I_\rho J_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A.

Proof of Theorem 10. **(a)** Clearly, $(A)_{\rho\in\mathbb{N}}$ is a sequence of ideals of A. Hence, in order to prove that $(A)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A, it is enough to verify that it satisfies the two conditions

$$\begin{aligned} A &= A; \\ AA &\subseteq A \qquad \text{for every } a\in\mathbb{N} \text{ and } b\in\mathbb{N}. \end{aligned}$$

But these two conditions are obviously satisfied. Hence, $(A)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A. This proves Theorem 10 **(a)**.

(b) Since $(I_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A, it is a sequence of ideals of A, and it satisfies the two conditions

$$\begin{aligned} I_0 &= A; \\ I_aI_b &\subseteq I_{a+b} \qquad \text{for every } a\in\mathbb{N} \text{ and } b\in\mathbb{N}. \end{aligned}$$

Since $(J_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A, it is a sequence of ideals of A, and it satisfies the two conditions

$$\begin{aligned} J_0 &= A; \\ J_aJ_b &\subseteq J_{a+b} \qquad \text{for every } a\in\mathbb{N} \text{ and } b\in\mathbb{N}. \end{aligned}$$

Now, $I_\rho J_\rho$ is an ideal of A for every $\rho\in\mathbb{N}$ (since I_ρ and J_ρ are ideals of A for every $\rho\in\mathbb{N}$, and the product of any two ideals of A is an ideal of A). Hence, $(I_\rho J_\rho)_{\rho\in\mathbb{N}}$ is a sequence of ideals of A. Thus, in order to prove that $(I_\rho J_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A, it is enough to verify that it satisfies the two conditions

$$\begin{aligned} I_0J_0 &= A; \\ I_aJ_a\cdot I_bJ_b &\subseteq I_{a+b}J_{a+b} \qquad \text{for every } a\in\mathbb{N} \text{ and } b\in\mathbb{N}. \end{aligned}$$

But these two conditions are satisfied, since

$$\begin{aligned} &\underbrace{I_0}_{=A}\underbrace{J_0}_{=A} = AA = A; \\ &I_aJ_a\cdot I_bJ_b = \underbrace{I_aI_b}_{\subseteq I_{a+b}}\underbrace{J_aJ_b}_{\subseteq J_{a+b}} \subseteq I_{a+b}J_{a+b} \qquad \text{for every } a\in\mathbb{N} \text{ and } b\in\mathbb{N}. \end{aligned}$$

Hence, $(I_\rho J_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A. This proves Theorem 10 **(b)**.

Now let us generalize Theorem 7:

> **Theorem 11.** Let A and B be two rings such that $A\subseteq B$. Let $(I_\rho)_{\rho\in\mathbb{N}}$ and $(J_\rho)_{\rho\in\mathbb{N}}$ be two ideal semifiltrations of A. Let $n\in\mathbb{N}$. Let $u\in B$.
>
> We know that $(I_\rho J_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A (according to Theorem 10 **(b)**).
>
> Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$.
>
> We will abbreviate the ring $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ by $A_{[I]}$.

By Lemma $\mathcal{J}$ (applied to $A_{[I]}$ and $(J_\tau)_{\tau\in\mathbb{N}}$ instead of A' and $(I_\rho)_{\rho\in\mathbb{N}}$), the sequence $\left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}$ is an ideal semifiltration of $A_{[I]}$ (since $A \subseteq A_{[I]}$ and since $(J_\tau)_{\tau\in\mathbb{N}} = (J_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A).

Then, the element u of B is n-integral over $\left(A, (I_\rho J_\rho)_{\rho\in\mathbb{N}}\right)$ if and only if the element uY of the polynomial ring $B[Y]$ is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$.

(Here, $A_{[I]} \subseteq B[Y]$ because $A_{[I]} = A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq A[Y]$ and we consider $A[Y]$ as a subring of $B[Y]$ as explained in Definition 7.)

Proof of Theorem 11. First, note that

$$\sum_{\ell\in\mathbb{N}} I_\ell Y^\ell = \sum_{i\in\mathbb{N}} I_i Y^i \qquad \text{(here we renamed } \ell \text{ as } i \text{ in the sum)}$$
$$= A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] = A_{[I]}.$$

In order to verify Theorem 11, we have to prove the following two lemmata:

Lemma $\mathcal{E}'$: If u is n-integral over $\left(A, (I_\rho J_\rho)_{\rho\in\mathbb{N}}\right)$, then uY is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$.

Lemma $\mathcal{F}'$: If uY is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$, then u is n-integral over $\left(A, (I_\rho J_\rho)_{\rho\in\mathbb{N}}\right)$.

Proof of Lemma $\mathcal{E}'$: Assume that u is n-integral over $\left(A, (I_\rho J_\rho)_{\rho\in\mathbb{N}}\right)$. Then, by Definition 9 (applied to $(I_\rho J_\rho)_{\rho\in\mathbb{N}}$ instead of $(I_\rho)_{\rho\in\mathbb{N}}$), there exists some $(a_0, a_1, ..., a_n) \in A^{n+1}$ such that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i} J_{n-i} \text{ for every } i \in \{0, 1, ..., n\}.$$

Note that $a_k Y^{n-k} \in A_{[I]}$ for every $k \in \{0, 1, ..., n\}$ (because $a_k \in I_{n-k} J_{n-k} \subseteq I_{n-k}$ (since I_{n-k} is an ideal of A) and thus $a_k Y^{n-k} \in I_{n-k} Y^{n-k} \subseteq \sum\limits_{i\in\mathbb{N}} I_i Y^i = A_{[I]}$). Thus, we can define an $(n+1)$-tuple $(b_0, b_1, ..., b_n) \in \left(A_{[I]}\right)^{n+1}$ by $b_k = a_k Y^{n-k}$ for every $k \in \{0, 1, ..., n\}$. Then,

$$\sum_{k=0}^{n} b_k \cdot (uY)^k = \sum_{k=0}^{n} a_k Y^{n-k} \cdot (uY)^k = \sum_{k=0}^{n} a_k Y^{n-k} u^k Y^k = \sum_{k=0}^{n} a_k u^k \underbrace{Y^{n-k} Y^k}_{=Y^n} = Y^n \cdot \underbrace{\sum_{k=0}^{n} a_k u^k}_{=0} = 0;$$

$$b_n = \underbrace{a_n}_{=1} \underbrace{Y^{n-n}}_{=Y^0=1} = 1,$$

and

$$b_i = \underbrace{a_i}_{\substack{\in I_{n-i} J_{n-i} \\ = J_{n-i} I_{n-i}}} Y^{n-i} \in J_{n-i} \underbrace{I_{n-i} Y^{n-i}}_{\substack{\subseteq \sum_{\ell\in\mathbb{N}} I_\ell Y^\ell \\ = A_{[I]}}} \subseteq J_{n-i} A_{[I]}$$

for every $i \in \{0, 1, ..., n\}$.

Altogether, we now know that $(b_0, b_1, ..., b_n) \in \left(A_{[I]}\right)^{n+1}$ and

$$\sum_{k=0}^{n} b_k \cdot (uY)^k = 0, \qquad b_n = 1, \qquad \text{and} \qquad b_i \in J_{n-i} A_{[I]} \text{ for every } i \in \{0, 1, ..., n\}.$$

Hence, by Definition 9 (applied to $A_{[I]}$, $B[Y]$, $\left(J_\tau A_{[I]}\right)_{\tau \in \mathbb{N}}$, uY and $(b_0, b_1, ..., b_n)$ instead of A, B, $(I_\rho)_{\rho \in \mathbb{N}}$, u and $(a_0, a_1, ..., a_n)$), the element uY is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau \in \mathbb{N}}\right)$. This proves Lemma $\mathcal{E}'$.

Proof of Lemma $\mathcal{F}'$: Assume that uY is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau \in \mathbb{N}}\right)$. Then, by Definition 9 (applied to $A_{[I]}$, $B[Y]$, $\left(J_\tau A_{[I]}\right)_{\tau \in \mathbb{N}}$, uY and $(p_0, p_1, ..., p_n)$ instead of A, B, $(I_\rho)_{\rho \in \mathbb{N}}$, u and $(a_0, a_1, ..., a_n)$), there exists some $(p_0, p_1, ..., p_n) \in \left(A_{[I]}\right)^{n+1}$ such that

$$\sum_{k=0}^{n} p_k \cdot (uY)^k = 0, \qquad p_n = 1, \qquad \text{and} \qquad p_i \in J_{n-i} A_{[I]} \text{ for every } i \in \{0, 1, ..., n\}.$$

For every $k \in \{0, 1, ..., n\}$, we have

$$\begin{aligned} p_k \in J_{n-k} A_{[I]} &= J_{n-k} \sum_{i \in \mathbb{N}} I_i Y^i \qquad \left(\text{since } A_{[I]} = \sum_{i \in \mathbb{N}} I_i Y^i\right) \\ &= \sum_{i \in \mathbb{N}} J_{n-k} I_i Y^i = \sum_{i \in \mathbb{N}} I_i J_{n-k} Y^i, \end{aligned}$$

and thus, there exists a sequence $(p_{k,i})_{i \in \mathbb{N}} \in A^{\mathbb{N}}$ such that $p_k = \sum_{i \in \mathbb{N}} p_{k,i} Y^i$, such that $p_{k,i} \in I_i J_{n-k}$ for every $i \in \mathbb{N}$, and such that only finitely many $i \in \mathbb{N}$ satisfy $p_{k,i} \neq 0$. Thus,

$$\begin{aligned} \sum_{k=0}^{n} p_k \cdot (uY)^k &= \sum_{k=0}^{n} \sum_{i \in \mathbb{N}} p_{k,i} Y^i \cdot \underbrace{(uY)^k}_{\substack{=u^k Y^k \\ =Y^k u^k}} \qquad \left(\text{since } p_k = \sum_{i \in \mathbb{N}} p_{k,i} Y^i\right) \\ &= \sum_{k=0}^{n} \sum_{i \in \mathbb{N}} p_{k,i} \underbrace{Y^i \cdot Y^k}_{=Y^{i+k}} u^k \\ &= \sum_{k=0}^{n} \sum_{i \in \mathbb{N}} p_{k,i} Y^{i+k} u^k = \sum_{k \in \{0,1,...,n\}} \sum_{i \in \mathbb{N}} p_{k,i} Y^{i+k} u^k \\ &= \sum_{(k,i) \in \{0,1,...,n\} \times \mathbb{N}} p_{k,i} Y^{i+k} u^k = \sum_{\ell \in \mathbb{N}} \sum_{\substack{(k,i) \in \{0,1,...,n\} \times \mathbb{N}; \\ i+k=\ell}} p_{k,i} \underbrace{Y^{i+k}}_{=Y^\ell} u^k \\ &= \sum_{\ell \in \mathbb{N}} \sum_{\substack{(k,i) \in \{0,1,...,n\} \times \mathbb{N}; \\ i+k=\ell}} p_{k,i} Y^\ell u^k = \sum_{\ell \in \mathbb{N}} \sum_{\substack{(k,i) \in \{0,1,...,n\} \times \mathbb{N}; \\ i+k=\ell}} p_{k,i} u^k Y^\ell. \end{aligned}$$

Hence, $\sum_{k=0}^{n} p_k \cdot (uY)^k = 0$ becomes $\sum_{\ell \in \mathbb{N}} \sum_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N};\\ i+k=\ell}} p_{k,i} u^k Y^\ell = 0$. In other words, the polynomial $\sum_{\ell\in\mathbb{N}} \underbrace{\sum_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N};\\ i+k=\ell}} p_{k,i} u^k}_{\in B} Y^\ell \in B[Y]$ equals 0. Hence, its coefficient before Y^n equals 0 as well. But its coefficient before Y^n is $\sum_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N};\\ i+k=n}} p_{k,i} u^k$. Hence, $\sum_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N};\\ i+k=n}} p_{k,i} u^k$ equals 0.

Thus,

$$0 = \sum_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N};\\ i+k=n}} p_{k,i} u^k = \sum_{k\in\{0,1,...,n\}} \sum_{\substack{i\in\mathbb{N};\\ i+k=n}} p_{k,i} u^k = \sum_{k\in\{0,1,...,n\}} p_{k,n-k} u^k$$

$$\left(\begin{array}{c}\text{since } \{i\in\mathbb{N} \mid i+k=n\} = \{i\in\mathbb{N} \mid i=n-k\} = \{n-k\} \text{ (because } n-k\in\mathbb{N}, \\ \text{since } k\in\{0,1,...,n\}\text{) yields } \sum_{\substack{i\in\mathbb{N};\\ i+k=n}} p_{k,i}u^k = \sum_{i\in\{n-k\}} p_{k,i}u^k = p_{k,n-k}u^k\end{array}\right).$$

Note that

$$\sum_{i\in\mathbb{N}} p_{n,i} Y^i = p_n \qquad \left(\text{since } \sum_{i\in\mathbb{N}} p_{k,i}Y^i = p_k \text{ for every } k\in\{0,1,...,n\}\right)$$
$$= 1 = 1\cdot Y^0$$

in $A[Y]$, and thus the coefficient of the polynomial $\sum_{i\in\mathbb{N}} p_{n,i}Y^i \in A[Y]$ before Y^0 is 1; but the coefficient of the polynomial $\sum_{i\in\mathbb{N}} p_{n,i}Y^i \in A[Y]$ before Y^0 is $p_{n,0}$; hence, $p_{n,0} = 1$.

Define an $(n+1)$-tuple $(a_0, a_1, ..., a_n) \in A^{n+1}$ by $a_k = p_{k,n-k}$ for every $k \in \{0,1,...,n\}$. Then, $a_n = p_{n,n-n} = p_{n,0} = 1$. Besides,

$$\sum_{k=0}^{n} a_k u^k = \sum_{k=0}^{n} p_{k,n-k} u^k = \sum_{k\in\{0,1,...,n\}} p_{k,n-k} u^k = 0.$$

Finally, $a_k = p_{k,n-k} \in I_{n-k}J_{n-k}$ (since $p_{k,i} \in I_i J_{n-k}$ for every $i\in\mathbb{N}$) for every $k \in \{0,1,...,n\}$. In other words, $a_i \in I_{n-i}J_{n-i}$ for every $i\in\{0,1,...,n\}$.

Altogether, we now know that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i}J_{n-i} \text{ for every } i\in\{0,1,...,n\}.$$

Thus, by Definition 9 (applied to $(I_\rho J_\rho)_{\rho\in\mathbb{N}}$ instead of $(I_\rho)_{\rho\in\mathbb{N}}$), the element u is n-integral over $\left(A, (I_\rho J_\rho)_{\rho\in\mathbb{N}}\right)$. This proves Lemma $\mathcal{F}'$.

Combining Lemmata $\mathcal{E}'$ and $\mathcal{F}'$, we obtain that u is n-integral over $\left(A, (I_\rho J_\rho)_{\rho\in\mathbb{N}}\right)$ if and only if uY is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$. This proves Theorem 11.

For the sake of completeness, we mention the following trivial fact (which shows why Theorem 11 generalizes Theorem 7):

Theorem 12. Let A and B be two rings such that $A \subseteq B$. Let $n \in \mathbb{N}$. Let $u \in B$.

We know that $(A)_{\rho \in \mathbb{N}}$ is an ideal semifiltration of A (according to Theorem 10 **(a)**).

Then, the element u of B is n-integral over $\left(A, (A)_{\rho \in \mathbb{N}}\right)$ if and only if u is n-integral over A.

Proof of Theorem 12. In order to verify Theorem 12, we have to prove the following two lemmata:

Lemma $\mathcal{L}$: If u is n-integral over $\left(A, (A)_{\rho \in \mathbb{N}}\right)$, then u is n-integral over A.

Lemma $\mathcal{M}$: If u is n-integral over A, then u is n-integral over $\left(A, (A)_{\rho \in \mathbb{N}}\right)$.

Proof of Lemma $\mathcal{L}$: Assume that u is n-integral over $\left(A, (A)_{\rho \in \mathbb{N}}\right)$. Then, by Definition 9 (applied to $(A)_{\rho \in \mathbb{N}}$ instead of $(I_\rho)_{\rho \in \mathbb{N}}$), there exists some $(a_0, a_1, ..., a_n) \in A^{n+1}$ such that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in A \text{ for every } i \in \{0, 1, ..., n\}.$$

Define a polynomial $P \in A[X]$ by $P(X) = \sum_{k=0}^{n} a_k X^k$. Then, $P(X) = \sum_{k=0}^{n} a_k X^k = \underbrace{a_n}_{=1} X^n + \sum_{k=0}^{n-1} a_k X^k = X^n + \sum_{k=0}^{n-1} a_k X^k$. Hence, the polynomial P is monic, and $\deg P = n$. Besides, $P(u) = 0$ (since $P(X) = \sum_{k=0}^{n} a_k X^k$ yields $P(u) = \sum_{k=0}^{n} a_k u^k = 0$). Thus, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$. Hence, u is n-integral over A. This proves Lemma $\mathcal{L}$.

Proof of Lemma $\mathcal{M}$: Assume that u is n-integral over A. Then, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$. Since $\deg P = n$, there exists some $(n+1)$-tuple $(a_0, a_1, ..., a_n) \in A^{n+1}$ such that $P(X) = \sum_{k=0}^{n} a_k X^k$. Thus, $a_n = 1$ (since P is monic, and $\deg P = n$). Also, $\sum_{k=0}^{n} a_k X^k = P(X)$ yields $\sum_{k=0}^{n} a_k u^k = P(u) = 0$. Altogether, we now know that $(a_0, a_1, ..., a_n) \in A^{n+1}$ and

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in A \text{ for every } i \in \{0, 1, ..., n\}.$$

Hence, by Definition 9 (applied to $(A)_{\rho \in \mathbb{N}}$ instead of $(I_\rho)_{\rho \in \mathbb{N}}$), the element u is n-integral over $\left(A, (A)_{\rho \in \mathbb{N}}\right)$. This proves Lemma $\mathcal{M}$.

Combining Lemmata $\mathcal{L}$ and $\mathcal{M}$, we obtain that u is n-integral over $\left(A, (A)_{\rho \in \mathbb{N}}\right)$ if and only if u is n-integral over A. This proves Theorem 12.

Finally, let us generalize Theorem 8 **(c)**:

Theorem 13. Let A and B be two rings such that $A \subseteq B$. Let $(I_\rho)_{\rho\in\mathbb{N}}$ and $(J_\rho)_{\rho\in\mathbb{N}}$ be two ideal semifiltrations of A.

Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$, and that y is n-integral over $\left(A, (J_\rho)_{\rho\in\mathbb{N}}\right)$. Then, xy is nm-integral over $\left(A, (I_\rho J_\rho)_{\rho\in\mathbb{N}}\right)$.

Proof of Theorem 13. First, a trivial observation:

Lemma $\mathcal{I}'$: Let A, A' and B' be three rings such that $A \subseteq A' \subseteq B'$. Let $(I_\rho)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $v \in B'$. Let $n \in \mathbb{N}$. If v is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$, then v is n-integral over $\left(A', (I_\rho A')_{\rho\in\mathbb{N}}\right)$. (Note that $(I_\rho A')_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A', according to Lemma $\mathcal{J}$.)

Proof of Lemma $\mathcal{I}'$: Assume that v is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$. Then, by Definition 9 (applied to B' and v instead of B and u), there exists some $(a_0, a_1, ..., a_n) \in A^{n+1}$ such that

$$\sum_{k=0}^{n} a_k v^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i} \text{ for every } i \in \{0, 1, ..., n\}.$$

But $(a_0, a_1, ..., a_n) \in A^{n+1}$ yields $(a_0, a_1, ..., a_n) \in (A')^{n+1}$ (since $A \subseteq A'$), and $a_i \in I_{n-i}$ yields $a_i \in I_{n-i}A'$ (since $I_{n-i} \subseteq I_{n-i}A'$) for every $i \in \{0, 1, ..., n\}$. Thus, $(a_0, a_1, ..., a_n) \in (A')^{n+1}$ and

$$\sum_{k=0}^{n} a_k v^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i}A' \text{ for every } i \in \{0, 1, ..., n\}.$$

Hence, by Definition 9 (applied to B', A', $(I_\rho A')_{\rho\in\mathbb{N}}$ and v instead of B, A, $(I_\rho)_{\rho\in\mathbb{N}}$ and u), the element v is n-integral over $\left(A', (I_\rho A')_{\rho\in\mathbb{N}}\right)$. This proves Lemma $\mathcal{I}'$.

Now let us prove Theorem 13.

We have $(J_\rho)_{\rho\in\mathbb{N}} = (J_\tau)_{\tau\in\mathbb{N}}$. Hence, y is n-integral over $\left(A, (J_\tau)_{\tau\in\mathbb{N}}\right)$ (since y is n-integral over $\left(A, (J_\rho)_{\rho\in\mathbb{N}}\right)$).

Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$. We will abbreviate the ring $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ by $A_{[I]}$. We have $A_{[I]} \subseteq B[Y]$, because $A_{[I]} = A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq A[Y]$ and we consider $A[Y]$ as a subring of $B[Y]$ as explained in Definition 7.

Theorem 7 (applied to x and m instead of u and n) yields that xY is m-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ (since x is m-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$). In other words, xY is m-integral over $A_{[I]}$ (since $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] = A_{[I]}$).

On the other hand, Lemma $\mathcal{I}'$ (applied to $A_{[I]}$, $B[Y]$, $(J_\tau)_{\tau\in\mathbb{N}}$ and y instead of A', B', $(I_\rho)_{\rho\in\mathbb{N}}$ and v) yields that y is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$ (since y is n-integral over $\left(A, (J_\tau)_{\tau\in\mathbb{N}}\right)$, and $A \subseteq A_{[I]} \subseteq B[Y]$).

Hence, Theorem 8 **(c)** (applied to $A_{[I]}$, $B[Y]$, $\left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}$, y, xY, m and n instead of A, B, $\left(I_\rho\right)_{\rho\in\mathbb{N}}$, x, y, n and m respectively) yields that $y \cdot xY$ is mn-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$ (since y is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$, and xY is m-integral over $A_{[I]}$). Since $y \cdot xY = xyY$ and $mn = nm$, this means that xyY is nm-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$. Hence, Theorem 11 (applied to xy and nm instead of u and n) yields that xy is nm-integral over $\left(A, \left(I_\rho J_\rho\right)_{\rho\in\mathbb{N}}\right)$. This proves Theorem 13.

References

[1] J. S. Milne, *Algebraic Number Theory*, version 3.02.
`http://jmilne.org/math/CourseNotes/math676.html`

[2] Craig Huneke and Irena Swanson, *Integral Closure of Ideals, Rings, and Modules*, London Mathematical Society Lecture Note Series, 336. Cambridge University Press, Cambridge, 2006.