A few facts on integrality *DETAILED VERSION*

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The purpose of this note is to collect some theorems and proofs related to integrality in commutative algebra. The note is subdivided into four parts.

Part 1 (Integrality over rings) consists of known facts (Theorems 1, 4, 5) and a generalized exercise from [1] (Corollary 3) with a few minor variations (Theorem 2 and Corollary 6).

Part 2 (Integrality over ideal semifiltrations) merges integrality over rings (as considered in Part 1) and integrality over ideals (a less-known but still very useful notion; the book [2] is devoted to it) into one general notion - that of integrality over ideal semifiltrations (Definition 9). This notion is very general, yet it can be reduced to the basic notion of integrality over rings by a suitable change of base ring (Theorem 7). This reduction allows to extend some standard properties of integrality over rings to the general case (Theorems 8 and 9).

Part 3 (Generalizing to two ideal semifiltrations) continues Part 2, adding one more layer of generality. Its main result is a "relative" version of Theorem 7 (Theorem 11) and a known fact generalized one more time (Theorem 13).

Part 4 (Accelerating ideal semifiltrations) generalizes Theorem 11 (and thus also Theorem 7) a bit further by considering a generalization of powers of an ideal.

Part 5 (Generalizing a lemma by Lombardi) is about an auxiliary result Lombardi used in [3] to prove Kronecker's Theorem[1]. We extend this auxiliary result here.

This note is supposed to be self-contained (only linear algebra and basic knowledge about rings, ideals and polynomials is assumed). The proofs are constructive. However, when writing down the proofs I focussed on maximal detail (to ensure correctness) rather than on clarity, so the proofs are probably a pain to read. I think of making a short version of this note with the obvious parts of proofs left out.

This is the long version of this paper, with all proofs maximally detailed. For all practical purposes, the brief version [4] should be totally enough, and probably better as it is much easier to read.

Preludium

Definitions and notations:

Definition 1. In the following, "ring" will always mean "commutative ring with unity". We denote the set $\{0, 1, 2, ...\}$ by $\mathbb{N}$, and the set $\{1, 2, 3, ...\}$ by $\mathbb{N}^+$.

Definition 2. Let A be a ring. Let M be an A-module. If $n \in \mathbb{N}$, and if m_1, m_2, ..., m_n are n elements of M, then we define an A-submodule $\langle m_1, m_2, ..., m_n \rangle_A$ of M by

$$\langle m_1, m_2, ..., m_n \rangle_A = \left\{ \sum_{i=1}^{n} a_i m_i \ \mid \ (a_1, a_2, ..., a_n) \in A^n \right\}.$$

[1] **Kronecker's Theorem.** Let B be a ring ("ring" always means "commutative ring with unity" in this paper). Let g and h be two elements of the polynomial ring $B[X]$. Let g_α be any coefficient of the polynomial g. Let h_β be any coefficient of the polynomial h. Let A be a subring of B which contains all coefficients of the polynomial gh. Then, the element $g_\alpha h_\beta$ of B is integral over the subring A.

Also, if S is a finite set, and m_s is an element of M for every $s \in S$, then we define an A-submodule $\langle m_s \mid s \in S \rangle_A$ of M by

$$\langle m_s \mid s \in S \rangle_A = \left\{ \sum_{s \in S} a_s m_s \mid (a_s)_{s \in S} \in A^S \right\}.$$

Of course, if $m_1, m_2, ..., m_n$ are n elements of M, then $\langle m_1, m_2, ..., m_n \rangle_A = \langle m_s \mid s \in \{1, 2, ..., n\} \rangle_A$.

We notice something almost trivial:

> **Module inclusion lemma.** Let A be a ring. Let M be an A-module. Let N be an A-submodule of M. If S is a finite set, and m_s is an element of N for every $s \in S$, then $\langle m_s \mid s \in S \rangle_A \subseteq N$. [2]

Definition 3. Let A be a ring, and let $n \in \mathbb{N}$. Let M be an A-module. We say that the A-module M is *n-generated* if there exist n elements $m_1, m_2, ..., m_n$ of M such that $M = \langle m_1, m_2, ..., m_n \rangle_A$. In other words, the A-module M is n-generated if and only if there exists a set S and an element m_s of M for every $s \in S$ such that $|S| = n$ and $M = \langle m_s \mid s \in S \rangle_A$.

Definition 4. Let A and B be two rings. We say that $A \subseteq B$ if and only if

$$(\text{the set } A \text{ is a subset of the set } B) \text{ and } (\text{the inclusion map } A \to B \text{ is a ring homomorphism}).$$

Now assume that $A \subseteq B$. Then, obviously, B is canonically an A-algebra (since $A \subseteq B$). If $u_1, u_2, ..., u_n$ are n elements of B, then we define an A-subalgebra $A[u_1, u_2, ..., u_n]$ of B by

$$A[u_1, u_2, ..., u_n] = \{P(u_1, u_2, ..., u_n) \mid P \in A[X_1, X_2, ..., X_n]\}.$$

In particular, if u is an element of B, then the A-subalgebra $A[u]$ of B is defined by

$$A[u] = \{P(u) \mid P \in A[X]\}.$$

Since $A[X] = \left\{ \sum_{i=0}^{m} a_i X^i \mid m \in \mathbb{N} \text{ and } (a_0, a_1, ..., a_m) \in A^{m+1} \right\}$, this becomes

$$\begin{aligned} A[u] &= \left\{ \left(\sum_{i=0}^{m} a_i X^i \right)(u) \mid m \in \mathbb{N} \text{ and } (a_0, a_1, ..., a_m) \in A^{m+1} \right\} \\ &\qquad \left(\text{where } \left(\sum_{i=0}^{m} a_i X^i \right)(u) \text{ means the polynomial } \sum_{i=0}^{m} a_i X^i \text{ evaluated at } X = u \right) \\ &= \left\{ \sum_{i=0}^{m} a_i u^i \mid m \in \mathbb{N} \text{ and } (a_0, a_1, ..., a_m) \in A^{m+1} \right\} \qquad \left(\text{because } \left(\sum_{i=0}^{m} a_i X^i \right)(u) = \sum_{i=0}^{m} a_i u^i \right). \end{aligned}$$

Obviously, $uA[u] \subseteq A[u]$ (since $A[u]$ is an A-algebra and $u \in A[u]$).

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[2] *Proof.* We have

$$\langle m_s \mid s \in S \rangle_A = \left\{ \sum_{s \in S} a_s m_s \mid (a_s)_{s \in S} \in A^S \right\} \subseteq N,$$

since $\sum_{s \in S} a_s m_s \in N$ for every $(a_s)_{s \in S} \in A^S$ (because $m_s \in N$ for every $s \in S$, and because N is an A-module).

1. Integrality over rings

Theorem 1. Let A and B be two rings such that $A \subseteq B$. Obviously, B is canonically an A-module (since $A \subseteq B$). Let $n \in \mathbb{N}$. Let $u \in B$. Then, the following four assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are pairwise equivalent:

Assertion $\mathcal{A}$*:* There exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$.

Assertion $\mathcal{B}$*:* There exist a B-module C and an n-generated A-submodule U of C such that $uU \subseteq U$ and such that every $v \in B$ satisfying $vU = 0$ satisfies $v = 0$. (Here, C is an A-module, since C is a B-module and $A \subseteq B$.)

Assertion $\mathcal{C}$*:* There exists an n-generated A-submodule U of B such that $1 \in U$ and $uU \subseteq U$.

Assertion $\mathcal{D}$*:* We have $A[u] = \langle u^0, u^1, ..., u^{n-1} \rangle_A$.

Definition 5. Let A and B be two rings such that $A \subseteq B$. Let $n \in \mathbb{N}$. Let $u \in B$. We say that the element u of B is *n-integral over A* if it satisfies the four equivalent assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of Theorem 1.

Hence, in particular, the element u of B is n-integral over A if and only if it satisfies the assertion $\mathcal{A}$ of Theorem 1. In other words, u is n-integral over A if and only if there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$.

Proof of Theorem 1. We will prove the implications $\mathcal{A} \Longrightarrow \mathcal{C}$, $\mathcal{C} \Longrightarrow \mathcal{B}$, $\mathcal{B} \Longrightarrow \mathcal{A}$, $\mathcal{A} \Longrightarrow \mathcal{D}$ and $\mathcal{D} \Longrightarrow \mathcal{C}$.

Proof of the implication $\mathcal{A} \Longrightarrow \mathcal{C}$*.* Assume that Assertion $\mathcal{A}$ holds. Then, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$. Since $P \in A[X]$ is a monic polynomial with $\deg P = n$, there exist elements a_0, a_1, ..., a_{n-1} of A such that $P(X) = X^n + \sum\limits_{k=0}^{n-1} a_k X^k$. Thus, $P(u) = u^n + \sum\limits_{k=0}^{n-1} a_k u^k$, so that $P(u) = 0$ becomes $u^n + \sum\limits_{k=0}^{n-1} a_k u^k = 0$. Hence, $u^n = -\sum\limits_{k=0}^{n-1} a_k u^k$.

Let U be the A-submodule $\langle u^0, u^1, ..., u^{n-1} \rangle_A$ of B. Then, U is an n-generated A-module (since u^0, u^1, ..., u^{n-1} are n elements of U). Besides, $1 = u^0 \in U$.

Now, $u \cdot u^k \in U$ for any $k \in \{0, 1, ..., n-1\}$ (since $k \in \{0, 1, ..., n-1\}$ yields either $0 \leq k < n-1$ or $k = n-1$, but $u \cdot u^k = u^{k+1} \in \langle u^0, u^1, ..., u^{n-1} \rangle_A = U$ if $0 \leq k < n-1$, and $u \cdot u^k = u \cdot u^{n-1} = u^n = -\sum\limits_{k=0}^{n-1} a_k u^k \in \langle u^0, u^1, ..., u^{n-1} \rangle_A = U$ if $k = n-1$, so that $u \cdot u^k \in U$ in both cases). Hence,

$$uU = u \left\langle u^0, u^1, ..., u^{n-1} \right\rangle_A = \left\langle u \cdot u^0, u \cdot u^1, ..., u \cdot u^{n-1} \right\rangle_A \subseteq U$$

(since $u \cdot u^k \in U$ for any $k \in \{0, 1, ..., n-1\}$).

Thus, Assertion $\mathcal{C}$ holds. Hence, we have proved that $\mathcal{A} \Longrightarrow \mathcal{C}$.

Proof of the implication $\mathcal{C} \Longrightarrow \mathcal{B}$*.* Assume that Assertion $\mathcal{C}$ holds. Then, there exists an n-generated A-submodule U of B such that $1 \in U$ and $uU \subseteq U$. Every $v \in B$ satisfying $vU = 0$ satisfies $v = 0$ (since $1 \in U$ and $vU = 0$ yield $v \cdot \underbrace{1}_{\in U} \in vU = 0$

and thus $v \cdot 1 = 0$, so that $v = 0$). Set $C = B$. Then, C is a B-module, and U is an n-generated A-submodule of C (since U is an n-generated A-submodule of B, and $C = B$). Thus, Assertion $\mathcal{B}$ holds. Hence, we have proved that $\mathcal{C} \Longrightarrow \mathcal{B}$.

Proof of the implication $\mathcal{B} \Longrightarrow \mathcal{A}$. Assume that Assertion $\mathcal{B}$ holds. Then, there exist a B-module C and an n-generated A-submodule U of C such that $uU \subseteq U$ (where C is an A-module, since C is a B-module and $A \subseteq B$), and such that every $v \in B$ satisfying $vU = 0$ satisfies $v = 0$.

Since the A-module U is n-generated, there exist n elements m_1, m_2, ..., m_n of U such that $U = \langle m_1, m_2, ..., m_n \rangle_A$. For any $k \in \{1, 2, ..., n\}$, we have

$$\begin{aligned} um_k &\in uU \qquad \text{(since } m_k \in U\text{)} \\ &\subseteq U = \langle m_1, m_2, ..., m_n \rangle_A , \end{aligned}$$

so that there exist n elements $a_{k,1}$, $a_{k,2}$, ..., $a_{k,n}$ of A such that $um_k = \sum\limits_{i=1}^{n} a_{k,i} m_i$.

We introduce two notations:

- For any matrix T and any integers x and y, we denote by $T_{x,y}$ the entry of the matrix T in the x-th row and the y-th column.

- For any assertion $\mathcal{U}$, we denote by $[\mathcal{U}]$ the Boolean value of the assertion $\mathcal{U}$ (that is, $[\mathcal{U}] = \begin{cases} 1, & \text{if } \mathcal{U} \text{ is true;} \\ 0, & \text{if } \mathcal{U} \text{ is false} \end{cases}$).

Clearly, the $n \times n$ identity matrix I_n satisfies $(I_n)_{\tau,i} = [\tau = i]$ for every $\tau \in \{1, 2, ..., n\}$ and $i \in \{1, 2, ..., n\}$.

Note that for every $\tau \in \{1, 2, ..., n\}$, we have

$$\sum_{i=1}^{n} (I_n)_{\tau,i} m_i = m_\tau, \tag{1}$$

since

$$\begin{aligned} \sum_{i=1}^{n} \underbrace{(I_n)_{\tau,i}}_{=[\tau=i]=[i=\tau]} m_i &= \sum_{i=1}^{n} [i = \tau] m_i = \sum_{i \in \{1,2,...,n\}} [i = \tau] m_i \\ &= \sum_{\substack{i \in \{1,2,...,n\} \\ \text{such that } i=\tau}} \underbrace{[i = \tau]}_{\substack{=1, \text{ since} \\ i=\tau \text{ is true}}} m_i + \sum_{\substack{i \in \{1,2,...,n\} \\ \text{such that } i\neq\tau}} \underbrace{[i = \tau]}_{\substack{=0, \text{ since} \\ i=\tau \text{ is false,} \\ \text{since } i\neq\tau}} m_i \\ &= \sum_{\substack{i \in \{1,2,...,n\} \\ \text{such that } i=\tau}} \underbrace{1 m_i}_{=m_i} + \underbrace{\sum_{\substack{i \in \{1,2,...,n\} \\ \text{such that } i\neq\tau}} 0 m_i}_{=0} = \sum_{\substack{i \in \{1,2,...,n\} \\ \text{such that } i=\tau}} m_i + 0 \\ &= \sum_{\substack{i \in \{1,2,...,n\} \\ \text{such that } i=\tau}} m_i = \sum_{i \in \{\tau\}} m_i \qquad \left(\begin{array}{c} \text{since } \{i \in \{1, 2, ..., n\} \mid i = \tau\} = \{\tau\}, \\ \text{because } \tau \in \{1, 2, ..., n\} \end{array} \right) \\ &= m_\tau . \end{aligned}$$

Hence, for every $k \in \{1, 2, ..., n\}$, we have

$$\sum_{i=1}^{n} \left(u \left(I_n \right)_{k,i} - a_{k,i} \right) m_i = \sum_{i=1}^{n} \left(u \left(I_n \right)_{k,i} m_i - a_{k,i} m_i \right) = u \underbrace{\sum_{i=1}^{n} \left(I_n \right)_{k,i} m_i}_{\substack{=m_k, \text{ by (1)} \\ \text{(applied to } \tau = k)}} - \sum_{i=1}^{n} a_{k,i} m_i$$

$$= u m_k - \sum_{i=1}^{n} a_{k,i} m_i = 0$$

(since $u m_k = \sum\limits_{i=1}^{n} a_{k,i} m_i$).

Define a matrix $S \in A^{n \times n}$ by ($S_{k,i} = a_{k,i}$ for all $k \in \{1, 2, ..., n\}$ and $i \in \{1, 2, ..., n\}$).

Define a matrix $T \in B^{n \times n}$ by $T = \operatorname{adj} \left(u I_n - S \right)$ (where S is considered as an element of $B^{n \times n}$, because $S \in A^{n \times n}$ and $A \subseteq B$).

Let $P \in A \left[X \right]$ be the characteristic polynomial of the matrix $S \in A^{n \times n}$. Then, P is monic, and $\deg P = n$. Besides, $P \left(X \right) = \det \left(X I_n - S \right)$, so that $P \left(u \right) = \det \left(u I_n - S \right)$. Then,

$$P \left(u \right) \cdot I_n = \det \left(u I_n - S \right) \cdot I_n = \underbrace{\operatorname{adj} \left(u I_n - S \right)}_{=T} \cdot \left(u I_n - S \right) = T \cdot \left(u I_n - S \right).$$

Now, for every $\tau \in \{1, 2, ..., n\}$, we have

$$\begin{aligned}
P \left(u \right) m_\tau &= P \left(u \right) \sum_{i=1}^{n} \left(I_n \right)_{\tau,i} m_i \qquad \left(\text{since (1) yields } m_\tau = \sum_{i=1}^{n} \left(I_n \right)_{\tau,i} m_i \right) \\
&= \sum_{i=1}^{n} \underbrace{P \left(u \right) \cdot \left(I_n \right)_{\tau,i}}_{= \left(P(u) \cdot I_n \right)_{\tau,i}} m_i = \sum_{i=1}^{n} \left(\underbrace{P \left(u \right) \cdot I_n}_{= T \cdot (u I_n - S)} \right)_{\tau,i} m_i = \sum_{i=1}^{n} \underbrace{\left(T \cdot \left(u I_n - S \right) \right)_{\tau,i}}_{= \sum\limits_{k=1}^{n} T_{\tau,k} \left(u I_n - S \right)_{k,i}} m_i \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} T_{\tau,k} \left(u I_n - S \right)_{k,i} m_i = \sum_{k=1}^{n} T_{\tau,k} \sum_{i=1}^{n} \underbrace{\left(u I_n - S \right)_{k,i}}_{= u \left(I_n \right)_{k,i} - S_{k,i}} m_i \\
&= \sum_{k=1}^{n} T_{\tau,k} \sum_{i=1}^{n} \left(u \left(I_n \right)_{k,i} - \underbrace{S_{k,i}}_{= a_{k,i}} \right) m_i = \sum_{k=1}^{n} T_{\tau,k} \underbrace{\sum_{i=1}^{n} \left(u \left(I_n \right)_{k,i} - a_{k,i} \right) m_i}_{=0} = 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
P \left(u \right) \cdot U &= P \left(u \right) \cdot \left\langle m_1, m_2, ..., m_n \right\rangle_A = \left\langle P \left(u \right) \cdot m_1, P \left(u \right) \cdot m_2, ..., P \left(u \right) \cdot m_n \right\rangle_A \\
&= \left\langle 0, 0, ..., 0 \right\rangle_A \qquad \left(\text{since } P \left(u \right) \cdot m_\tau = 0 \text{ for any } \tau \in \{1, 2, ..., n\} \right) \\
&= 0.
\end{aligned}$$

This implies $P \left(u \right) = 0$ (since every $v \in B$ satisfying $vU = 0$ satisfies $v = 0$). Thus, Assertion $\mathcal{A}$ holds. Hence, we have proved that $\mathcal{B} \Longrightarrow \mathcal{A}$.

Proof of the implication $\mathcal{A} \Longrightarrow \mathcal{D}$. Assume that Assertion $\mathcal{A}$ holds. Then, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$. Since $P \in A[X]$ is a monic polynomial with $\deg P = n$, there exist elements a_0, a_1, ..., a_{n-1} of A such that $P(X) = X^n + \sum\limits_{k=0}^{n-1} a_k X^k$. Thus, $P(u) = u^n + \sum\limits_{k=0}^{n-1} a_k u^k$, so that $P(u) = 0$ becomes $u^n + \sum\limits_{k=0}^{n-1} a_k u^k = 0$. Hence, $u^n = -\sum\limits_{k=0}^{n-1} a_k u^k$.

Let U be the A-submodule $\langle u^0, u^1, ..., u^{n-1}\rangle_A$ of B. As in the Proof of the implication $\mathcal{A} \Longrightarrow \mathcal{C}$, we can show that U is an n-generated A-module, and that $1 \in U$ and $uU \subseteq U$.

Now, we are going to show that

$$u^i \in U \qquad \text{for any } i \in \mathbb{N}. \tag{2}$$

Proof of (2). We will prove (2) by induction over i:

Induction base: The assertion (2) holds for $i = 0$ (since $u^0 \in U$). This completes the induction base.

Induction step: Let $\tau \in \mathbb{N}$. If the assertion (2) holds for $i = \tau$, then the assertion (2) holds for $i = \tau + 1$ (because if the assertion (2) holds for $i = \tau$, then $u^\tau \in U$, so that $u^{\tau+1} = u \cdot \underbrace{u^\tau}_{\in U} \in uU \subseteq U$, so that $u^{\tau+1} \in U$, and thus the assertion (2) holds for $i = \tau + 1$). This completes the induction step.

Hence, the induction is complete, and (2) is proven.

Thus,

$$A[u] = \left\{ \sum_{i=0}^{m} a_i u^i \ \mid \ m \in \mathbb{N} \text{ and } (a_0, a_1, ..., a_m) \in A^{m+1} \right\} \subseteq U$$

(since $\sum\limits_{i=0}^{m} a_i u^i \in U$ for any $m \in \mathbb{N}$ and any $(a_0, a_1, ..., a_m) \in A^{m+1}$, because $a_i \in A$ and $u^i \in U$ for any $i \in \{0, 1, ..., m\}$ (by (2)) and U is an A-module). On the other hand, $U \subseteq A[u]$, since

$$\begin{aligned} U = \left\langle u^0, u^1, ..., u^{n-1} \right\rangle_A &= \left\{ \sum_{i=0}^{n-1} a_i u^i \ \mid \ (a_0, a_1, ..., a_{n-1}) \in A^n \right\} \\ &\subseteq \left\{ \sum_{i=0}^{m} a_i u^i \ \mid \ m \in \mathbb{N} \text{ and } (a_0, a_1, ..., a_m) \in A^{m+1} \right\} = A[u]. \end{aligned}$$

Thus, $U = A[u]$. In other words, $\langle u^0, u^1, ..., u^{n-1}\rangle_A = A[u]$. Thus, Assertion $\mathcal{D}$ holds. Hence, we have proved that $\mathcal{A} \Longrightarrow \mathcal{D}$.

Proof of the implication $\mathcal{D} \Longrightarrow \mathcal{C}$. Assume that Assertion $\mathcal{D}$ holds. Then, $A[u] = \langle u^0, u^1, ..., u^{n-1}\rangle_A$.

Let U be the A-submodule $\langle u^0, u^1, ..., u^{n-1}\rangle_A$ of B. Then, U is an n-generated A-module (since u^0, u^1, ..., u^{n-1} are n elements of U). Besides, $1 = u^0 \in U$.

Also,

$$uU = u \cdot \left\langle u^0, u^1, ..., u^{n-1} \right\rangle_A = u \cdot A[u] \subseteq A[u] = \left\langle u^0, u^1, ..., u^{n-1} \right\rangle_A = U.$$

Thus, Assertion $\mathcal{C}$ holds. Hence, we have proved that $\mathcal{D} \Longrightarrow \mathcal{C}$.

Now, we have proved the implications $\mathcal{A} \Longrightarrow \mathcal{D}$, $\mathcal{D} \Longrightarrow \mathcal{C}$, $\mathcal{C} \Longrightarrow \mathcal{B}$ and $\mathcal{B} \Longrightarrow \mathcal{A}$ above. Thus, all four assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are pairwise equivalent, and Theorem 1 is proven.

Theorem 2. Let A and B be two rings such that $A \subseteq B$. Let $n \in \mathbb{N}$. Let $v \in B$. Let a_0, a_1, ..., a_n be $n+1$ elements of A such that $\sum\limits_{i=0}^{n} a_i v^i = 0$. Let $k \in \{0, 1, ..., n\}$. Then, $\sum\limits_{i=0}^{n-k} a_{i+k} v^i$ is n-integral over A.

Proof of Theorem 2. Let U be the A-submodule $\langle v^0, v^1, ..., v^{n-1} \rangle_A$ of B. Then, U is an n-generated A-module (since v^0, v^1, ..., v^{n-1} are n elements of U). Besides, $1 = v^0 \in U$.

Let $u = \sum\limits_{i=0}^{n-k} a_{i+k} v^i$. Then,

$$0 = \sum_{i=0}^{n} a_i v^i = \sum_{i=0}^{k-1} a_i v^i + \sum_{i=k}^{n} a_i v^i = \sum_{i=0}^{k-1} a_i v^i + \sum_{i=0}^{n-k} a_{i+k} \underbrace{v^{i+k}}_{=v^i v^k}$$

$$\text{(here, we substituted } i+k \text{ for } i \text{ in the second sum)}$$

$$= \sum_{i=0}^{k-1} a_i v^i + v^k \underbrace{\sum_{i=0}^{n-k} a_{i+k} v^i}_{=u} = \sum_{i=0}^{k-1} a_i v^i + v^k u,$$

so that $v^k u = -\sum\limits_{i=0}^{k-1} a_i v^i$.

Now, we are going to show that

$$uv^t \in U \qquad \text{for any } t \in \{0, 1, ..., n-1\}. \tag{3}$$

Proof of (3). Since $t \in \{0, 1, ..., n-1\}$, one of the following two cases must hold:

Case 1: We have $t \in \{0, 1, ..., k-1\}$.

Case 2: We have $t \in \{k, k+1, ..., n-1\}$.

In Case 1, we have

$$\begin{aligned} uv^t &= \sum_{i=0}^{n-k} a_{i+k} \underbrace{v^i \cdot v^t}_{=v^{i+t}} \qquad \left(\text{since } u = \sum_{i=0}^{n-k} a_{i+k} v^i\right) \\ &= \sum_{i=0}^{n-k} a_{i+k} v^{i+t} \in \left\langle v^0, v^1, ..., v^{n-1} \right\rangle_A \\ &\qquad \left(\begin{array}{c} \text{since } t \in \{0, 1, ..., k-1\} \text{ yields } i+t \in \{0, 1, ..., n-1\} \text{ and thus} \\ v^{i+t} \in \{v^0, v^1, ..., v^{n-1}\} \text{ for any } i \in \{0, 1, ..., n-k\} \end{array}\right) \\ &= U. \end{aligned}$$

In Case 2, we have $t \in \{k, k+1, ..., n-1\}$, thus $t - k \in \{0, 1, ..., n-k-1\}$ and hence

$$uv^t = u\underbrace{v^{k+(t-k)}}_{=v^k v^{t-k}} = v^k u \cdot v^{t-k} = -\sum_{i=0}^{k-1} a_i \underbrace{v^i \cdot v^{t-k}}_{=v^{i+(t-k)}} \qquad \left(\text{since } v^k u = -\sum_{i=0}^{k-1} a_i v^i\right)$$
$$= -\sum_{i=0}^{k-1} a_i v^{i+(t-k)} \in \left\langle v^0, v^1, ..., v^{n-1}\right\rangle_A$$
$$\left(\begin{array}{c}\text{since } t-k \in \{0,1,...,n-k-1\} \text{ yields } i+(t-k) \in \{0,1,...,n-1\} \text{ and thus} \\ v^{i+(t-k)} \in \{v^0, v^1, ..., v^{n-1}\} \text{ for any } i \in \{0,1,...,k-1\}\end{array}\right)$$
$$= U.$$

Hence, in both cases, we have $uv^t \in U$. Thus, $uv^t \in U$ always holds, and (3) is proven.

Now,

$$uU = u\left\langle v^0, v^1, ..., v^{n-1}\right\rangle_A = \left\langle uv^0, uv^1, ..., uv^{n-1}\right\rangle_A \subseteq U \qquad \text{(due to (3))}.$$

Altogether, U is an n-generated A-submodule of B such that $1 \in U$ and $uU \subseteq U$. Thus, $u \in B$ satisfies Assertion $\mathcal{C}$ of Theorem 1. Hence, $u \in B$ satisfies the four equivalent assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of Theorem 1. Consequently, u is n-integral over A. Since $u = \sum\limits_{i=0}^{n-k} a_{i+k} v^i$, this means that $\sum\limits_{i=0}^{n-k} a_{i+k} v^i$ is n-integral over A. This proves Theorem 2.

Corollary 3. Let A and B be two rings such that $A \subseteq B$. Let $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$. Let $u \in B$ and $v \in B$. Let s_0, s_1, ..., s_α be $\alpha + 1$ elements of A such that $\sum\limits_{i=0}^{\alpha} s_i v^i = u$. Let t_0, t_1, ..., t_β be $\beta + 1$ elements of A such that $\sum\limits_{i=0}^{\beta} t_i v^{\beta - i} = uv^\beta$. Then, u is $(\alpha + \beta)$-integral over A.

(This Corollary 3 generalizes Exercise 2-5 in [1].)

First proof of Corollary 3. Let $k = \beta$ and $n = \alpha + \beta$. Then, $k \in \{0, 1, ..., n\}$. Define $n + 1$ elements a_0, a_1, ..., a_n of A by

$$a_i = \begin{cases} t_{\beta - i}, & \text{if } i < \beta; \\ t_0 - s_0, & \text{if } i = \beta; \\ -s_{i-\beta}, & \text{if } i > \beta; \end{cases} \qquad \text{for every } i \in \{0, 1, ..., n\}.$$

Then,

$$\sum_{i=0}^{n} a_i v^i = \sum_{i=0}^{\alpha+\beta} a_i v^i = \sum_{i=0}^{\beta-1} \underbrace{a_i}_{\substack{=t_{\beta-i} \text{ (by the} \\ \text{definition of } a_i, \\ \text{since } i<\beta)}} v^i + \sum_{i=\beta}^{\beta} \underbrace{a_i}_{\substack{=t_0-s_0 \text{ (by the} \\ \text{definition of } a_i, \\ \text{since } i=\beta)}} v^i + \sum_{i=\beta+1}^{\alpha+\beta} \underbrace{a_i}_{\substack{=-s_{i-\beta} \text{ (by the} \\ \text{definition of } a_i, \\ \text{since } i>\beta)}} v^i$$

$$= \sum_{i=0}^{\beta-1} t_{\beta-i} v^i + \underbrace{\sum_{i=\beta}^{\beta} (t_0 - s_0) v^i}_{\substack{=(t_0-s_0)v^\beta \\ =t_0 v^\beta - s_0 v^\beta}} + \underbrace{\sum_{i=\beta+1}^{\alpha+\beta} (-s_{i-\beta}) v^i}_{=-\sum_{i=\beta+1}^{\alpha+\beta} s_{i-\beta} v^i}$$

$$= \sum_{i=0}^{\beta-1} t_{\beta-i} v^i + t_0 v^\beta - s_0 v^\beta - \sum_{i=\beta+1}^{\alpha+\beta} s_{i-\beta} v^i = \sum_{i=0}^{\beta-1} t_{\beta-i} v^i + t_0 v^\beta - \left(s_0 v^\beta + \sum_{i=\beta+1}^{\alpha+\beta} s_{i-\beta} v^i \right)$$

$$= \sum_{i=0}^{\beta-1} t_{\beta-i} v^i + t_0 v^\beta - \left(s_0 v^\beta + \sum_{i=1}^{\alpha} \underbrace{s_{(i+\beta)-\beta}}_{=s_i} \underbrace{v^{i+\beta}}_{=v^i v^\beta} \right)$$

(here, we substituted $i+\beta$ for i in the second sum)

$$= \sum_{i=0}^{\beta-1} t_{\beta-i} v^i + t_0 v^\beta - \left(s_0 v^\beta + \sum_{i=1}^{\alpha} s_i v^i v^\beta \right)$$

$$= \sum_{i=1}^{\beta} \underbrace{t_{\beta-(\beta-i)}}_{=t_i} v^{\beta-i} + t_0 \underbrace{v^\beta}_{=v^{\beta-0}} - \left(s_0 \underbrace{v^\beta}_{=v^0 v^\beta} + \sum_{i=1}^{\alpha} s_i v^i v^\beta \right)$$

(here, we substituted $\beta - i$ for i in the first sum)

$$= \sum_{i=1}^{\beta} t_i v^{\beta-i} + t_0 v^{\beta-0} - \left(s_0 v^0 v^\beta + \sum_{i=1}^{\alpha} s_i v^i v^\beta \right)$$

$$= \underbrace{\sum_{i=1}^{\beta} t_i v^{\beta-i} + t_0 v^{\beta-0}}_{=\sum_{i=0}^{\beta} t_i v^{\beta-i} = u v^\beta} - \left(\underbrace{s_0 v^0 + \sum_{i=1}^{\alpha} s_i v^i}_{=\sum_{i=0}^{\alpha} s_i v^i = u} \right) v^\beta = u v^\beta - u v^\beta = 0.$$

Thus, Theorem 2 yields that $\sum\limits_{i=0}^{n-k} a_{i+k}v^i$ is n-integral over A. But

$$\sum_{i=0}^{n-k} a_{i+k}v^i = \sum_{i=0}^{n-\beta} a_{i+\beta}v^i = \sum_{i=0}^{0} \underbrace{a_{i+\beta}}_{\substack{=t_0-s_0 \text{ (by the} \\ \text{definition of } a_{i+\beta}, \\ \text{since } i=0 \text{ yields} \\ i+\beta=\beta)}} v^i + \sum_{i=1}^{n-\beta} \underbrace{a_{i+\beta}}_{\substack{=-s_{(i+\beta)-\beta} \text{ (by the} \\ \text{definition of } a_{i+\beta}, \\ \text{since } i>0 \text{ yields} \\ i+\beta>\beta)}} v^i$$

$$= \underbrace{\sum_{i=0}^{0} (t_0 - s_0) v^i}_{\substack{=(t_0-s_0)v^0 \\ =t_0v^0-s_0v^0 \\ =t_0-s_0v^0}} + \sum_{i=1}^{n-\beta} \left(- \underbrace{s_{(i+\beta)-\beta}}_{=s_i} \right) v^i = t_0 - s_0v^0 + \sum_{i=1}^{n-\beta} (-s_i) v^i$$

$$= t_0 - s_0v^0 - \sum_{i=1}^{n-\beta} s_iv^i = t_0 - s_0v^0 - \sum_{i=1}^{\alpha} s_iv^i \qquad \text{(since } n = \alpha + \beta \text{ yields } n - \beta = \alpha)$$

$$= t_0 - \left(\underbrace{s_0v^0 + \sum_{i=1}^{\alpha} s_iv^i}_{=\sum\limits_{i=0}^{\alpha} s_iv^i = u} \right) = t_0 - u.$$

Thus, $t_0 - u$ is n-integral over A. On the other hand, $-t_0$ is 1-integral over A (by Theorem 5 **(a)** below, applied to $a = -t_0$). Thus, $(-t_0) + (t_0 - u)$ is $n \cdot 1$-integral over A (by Theorem 5 **(b)** below, applied to $x = -t_0$, $y = t_0 - u$ and $m = 1$). In other words, $-u$ is n-integral over A (since $(-t_0) + (t_0 - u) = -u$ and $n \cdot 1 = n$). On the other hand, -1 is 1-integral over A (by Theorem 5 **(a)** below, applied to $a = -1$). Thus, $(-1) \cdot (-u)$ is $n \cdot 1$-integral over A (by Theorem 5 **(c)** below, applied to $x = -1$, $y = -u$ and $m = 1$). In other words, u is $(\alpha + \beta)$-integral over A (since $(-1) \cdot (-u) = u$ and $n \cdot 1 = n = \alpha + \beta$). This proves Corollary 3.

We will provide a second proof of Corollary 3 in Part 5.

> **Theorem 4.** Let A and B be two rings such that $A \subseteq B$. Let $v \in B$ and $u \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that v is m-integral over A, and that u is n-integral over $A[v]$. Then, u is nm-integral over A.

Proof of Theorem 4. Since v is m-integral over A, we have $A[v] = \langle v^0, v^1, ..., v^{m-1} \rangle_A$ (this is the Assertion $\mathcal{D}$ of Theorem 1, stated for v and m in lieu of u and n).

Since u is n-integral over $A[v]$, we have $(A[v])[u] = \langle u^0, u^1, ..., u^{n-1} \rangle_{A[v]}$ (this is the Assertion $\mathcal{D}$ of Theorem 1, stated for $A[v]$ in lieu of A).

Let $S = \{0, 1, ..., n-1\} \times \{0, 1, ..., m-1\}$.

Let $x \in (A[v])[u]$. Then, there exist n elements $b_0, b_1, ..., b_{n-1}$ of $A[v]$ such that $x = \sum\limits_{i=0}^{n-1} b_iu^i$ (since $x \in (A[v])[u] = \langle u^0, u^1, ..., u^{n-1} \rangle_{A[v]}$). But for each $i \in \{0, 1, ..., n-1\}$, there exist m elements $a_{i,0}, a_{i,1}, ..., a_{i,m-1}$ of A such that $b_i = \sum\limits_{j=0}^{m-1} a_{i,j}v^j$ (because

$b_i \in A[v] = \langle v^0, v^1, ..., v^{m-1} \rangle_A$). Thus,

$$x = \sum_{i=0}^{n-1} \underbrace{b_i}_{= \sum_{j=0}^{m-1} a_{i,j} v^j} u^i = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} a_{i,j} v^j u^i = \sum_{(i,j) \in \{0,1,...,n-1\} \times \{0,1,...,m-1\}} a_{i,j} v^j u^i = \sum_{(i,j) \in S} a_{i,j} v^j u^i$$
$$\in \left\langle v^j u^i \mid (i,j) \in S \right\rangle_A \qquad \left(\text{since } a_{i,j} \in A \text{ for every } (i,j) \in S\right)$$

So we have proved that $x \in \langle v^j u^i \mid (i,j) \in S \rangle_A$ for every $x \in (A[v])[u]$. Thus, $(A[v])[u] \subseteq \langle v^j u^i \mid (i,j) \in S \rangle_A$. Conversely, $\langle v^j u^i \mid (i,j) \in S \rangle_A \subseteq (A[v])[u]$ (since $v^j \in A[v]$ for every $(i,j) \in S$, and thus $\underbrace{v^j}_{\in A[v]} u^i \in (A[v])[u]$ for every $(i,j) \in S$, and therefore

$$\left\langle v^j u^i \mid (i,j) \in S \right\rangle_A = \left\{ \underbrace{\sum_{(i,j) \in S} a_{i,j} v^j u^i}_{\substack{\in (A[v])[u], \text{ since} \\ v^j u^i \in (A[v])[u] \text{ for all } (i,j) \in S \\ \text{and } (A[v])[u] \text{ is an } A\text{-module}}} \mid (a_{i,j})_{(i,j) \in S} \in A^S \right\} \subseteq (A[v])[u]$$

). Hence, $(A[v])[u] = \langle v^j u^i \mid (i,j) \in S \rangle_A$. Thus, the A-module $(A[v])[u]$ is nm-generated (since

$$|S| = |\{0,1,...,n-1\} \times \{0,1,...,m-1\}| = \underbrace{|\{0,1,...,n-1\}|}_{=n} \cdot \underbrace{|\{0,1,...,m-1\}|}_{=m} = nm$$

).

Let $U = (A[v])[u]$. Then, the A-module $U = (A[v])[u]$ is nm-generated. Besides, U is an A-submodule of B, and we have $1 = u^0 \in (A[v])[u] = U$ and

$$uU = u(A[v])[u] \subseteq (A[v])[u] \qquad \left(\text{since } (A[v])[u] \text{ is an } A[v]\text{-algebra and } u \in (A[v])[u]\right)$$
$$= U.$$

Altogether, we now know that the A-submodule U of B is nm-generated and satisfies $1 \in U$ and $uU \subseteq U$.

Thus, the element u of B satisfies the Assertion $\mathcal{C}$ of Theorem 1 with n replaced by nm. Hence, $u \in B$ satisfies the four equivalent assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of Theorem 1, all with n replaced by nm. Thus, u is nm-integral over A. This proves Theorem 4.

Theorem 5. Let A and B be two rings such that $A \subseteq B$.

(a) Let $a \in A$. Then, a is 1-integral over A.

(b) Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m-integral over A, and that y is n-integral over A. Then, $x + y$ is nm-integral over A.

(c) Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m-integral over A, and that y is n-integral over A. Then, xy is nm-integral over A.

Proof of Theorem 5. **(a)** There exists a monic polynomial $P \in A[X]$ with $\deg P = 1$ and $P(a) = 0$ (namely, the polynomial $P \in A[X]$ defined by $P(X) = X - a$). Thus, a is 1-integral over A. This proves Theorem 5 **(a)**.

(b) Since y is n-integral over A, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(y) = 0$. Since $P \in A[X]$ is a monic polynomial with $\deg P = n$, there exists a polynomial $\widetilde{P} \in A[X]$ with $\deg \widetilde{P} < n$ and $P(X) = X^n + \widetilde{P}(X)$.

Now, define a polynomial $Q \in (A[x])[X]$ by $Q(X) = P(X - x)$. Then,

$$\begin{aligned}\deg Q &= \deg P \qquad \text{(since shifting the polynomial } P \text{ by the constant } x \text{ does not change its degree)}\\ &= n,\end{aligned}$$

and $Q(X) = P(X - x)$ yields $Q(x+y) = P((x+y) - x) = P(y) = 0$.

Define a polynomial $\widetilde{Q} \in (A[x])[X]$ by $\widetilde{Q}(X) = ((X-x)^n - X^n) + \widetilde{P}(X - x)$. Then, $\deg \widetilde{Q} < n$ (since

$$\begin{aligned}&\deg\left(\widetilde{P}(X-x)\right) = \deg\left(\widetilde{P}(X)\right)\\ &\qquad \left(\text{since shifting the polynomial } \widetilde{P} \text{ by the constant } x \text{ does not change its degree}\right)\\ &= \deg \widetilde{P} < n\end{aligned}$$

and

$$\begin{aligned}\deg((X-x)^n - X^n) &= \deg\left(((X-x)-X)\cdot\sum_{k=0}^{n-1}(X-x)^k X^{n-1-k}\right)\\ &\leq \underbrace{\deg((X-x)-X)}_{=\deg(-x)=0} + \underbrace{\deg\left(\sum_{k=0}^{n-1}(X-x)^k X^{n-1-k}\right)}_{\substack{\leq n-1,\text{ since}\\ \deg\left((X-x)^k X^{n-1-k}\right)\leq n-1\\ \text{for any } k\in\{0,1,\ldots,n-1\}}}\\ &\leq 0 + (n-1) = n-1 < n\end{aligned}$$

yield

$$\begin{aligned}\deg \widetilde{Q} &= \deg\left(\widetilde{Q}(X)\right) = \deg\left(((X-x)^n - X^n) + \widetilde{P}(X-x)\right)\\ &\leq \max\left\{\underbrace{\deg((X-x)^n - X^n)}_{<n}, \underbrace{\deg\left(\widetilde{P}(X-x)\right)}_{<n}\right\} < \max\{n,n\} = n\end{aligned}$$

). Thus, the polynomial Q is monic (since

$$\begin{aligned}Q(X) &= P(X-x) = (X-x)^n + \widetilde{P}(X-x) \qquad \left(\text{since } P(X) = X^n + \widetilde{P}(X)\right)\\ &= X^n + \underbrace{((X-x)^n - X^n) + \widetilde{P}(X-x)}_{=\widetilde{Q}(X)} = X^n + \widetilde{Q}(X)\end{aligned}$$

and $\deg \widetilde{Q} < n$).

Hence, there exists a monic polynomial $Q \in (A[x])[X]$ with $\deg Q = n$ and $Q(x+y) = 0$. Thus, $x+y$ is n-integral over $A[x]$. Thus, Theorem 4 (applied to $v = x$ and $u = x+y$) yields that $x+y$ is nm-integral over A. This proves Theorem 5 **(b)**.

(c) Since y is n-integral over A, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(y) = 0$. Since $P \in A[X]$ is a monic polynomial with $\deg P = n$, there exist elements a_0, a_1, ..., a_{n-1} of A such that $P(X) = X^n + \sum\limits_{k=0}^{n-1} a_k X^k$. Thus, $P(y) = y^n + \sum\limits_{k=0}^{n-1} a_k y^k$.

Now, define a polynomial $Q \in (A[x])[X]$ by $Q(X) = X^n + \sum\limits_{k=0}^{n-1} x^{n-k} a_k X^k$. Then,

$$Q(xy) = \underbrace{(xy)^n}_{=x^n y^n} + \sum_{k=0}^{n-1} x^{n-k} \underbrace{a_k (xy)^k}_{\substack{=a_k x^k y^k \\ =x^k a_k y^k}} = x^n y^n + \sum_{k=0}^{n-1} \underbrace{x^{n-k} x^k}_{=x^n} a_k y^k$$

$$= x^n y^n + \sum_{k=0}^{n-1} x^n a_k y^k = x^n \left(\underbrace{y^n + \sum_{k=0}^{n-1} a_k y^k}_{=P(y)=0} \right) = 0.$$

Also, the polynomial $Q \in (A[x])[X]$ is monic and $\deg Q = n$ (since $Q(X) = X^n + \sum\limits_{k=0}^{n-1} x^{n-k} a_k X^k$). Thus, there exists a monic polynomial $Q \in (A[x])[X]$ with $\deg Q = n$ and $Q(xy) = 0$. Thus, xy is n-integral over $A[x]$. Hence, Theorem 4 (applied to $v = x$ and $u = xy$) yields that xy is nm-integral over A. This proves Theorem 5 **(c)**.

Corollary 6. Let A and B be two rings such that $A \subseteq B$. Let $n \in \mathbb{N}^+$ and $m \in \mathbb{N}$. Let $v \in B$. Let b_0, b_1, ..., b_{n-1} be n elements of A, and let $u = \sum\limits_{i=0}^{n-1} b_i v^i$. Assume that vu is m-integral over A. Then, u is nm-integral over A.

Proof of Corollary 6. Define $n+1$ elements a_0, a_1, ..., a_n of $A[vu]$ by

$$a_i = \begin{cases} -vu, & \text{if } i = 0; \\ b_{i-1}, & \text{if } i > 0 \end{cases} \qquad \text{for every } i \in \{0, 1, ..., n\}.$$

Then, $a_0 = -vu$. Let $k = 1$. Then,

$$\sum_{i=0}^{n} a_i v^i = \underbrace{a_0}_{=-vu} \underbrace{v^0}_{=1} + \sum_{i=1}^{n} \underbrace{a_i}_{\substack{=b_{i-1}, \\ \text{since} \\ i>0}} \underbrace{v^i}_{=v^{i-1}v} = -vu + \sum_{i=1}^{n} b_{i-1} v^{i-1} v = -vu + \underbrace{\sum_{i=0}^{n-1} b_i v^i}_{=u} v$$

$$\text{(here, we substituted } i \text{ for } i-1 \text{ in the sum)}$$

$$= -vu + uv = 0.$$

Now, $A[vu]$ and B are two rings such that $A[vu] \subseteq B$. The $n+1$ elements a_0, a_1, ..., a_n of $A[vu]$ satisfy $\sum_{i=0}^{n} a_i v^i = 0$. We have $k = 1 \in \{0, 1, ..., n\}$.

Hence, Theorem 2 (applied to the ring $A[vu]$ in lieu of A) yields that $\sum_{i=0}^{n-k} a_{i+k} v^i$ is n-integral over $A[vu]$. But

$$\sum_{i=0}^{n-k} a_{i+k} v^i = \sum_{i=0}^{n-1} \underbrace{a_{i+1}}_{\substack{=b_{(i+1)-1}, \\ \text{since } i+1>0}} v^i = \sum_{i=0}^{n-1} b_{(i+1)-1} v^i = \sum_{i=0}^{n-1} b_i v^i = u.$$

Hence, u is n-integral over $A[vu]$. But vu is m-integral over A. Thus, Theorem 4 (applied to vu in lieu of v) yields that u is nm-integral over A. This proves Corollary 6.

2. Integrality over ideal semifiltrations

Definitions:

Definition 6. Let A be a ring, and let $(I_\rho)_{\rho\in\mathbb{N}}$ be a sequence of ideals of A. Then, $(I_\rho)_{\rho\in\mathbb{N}}$ is called an *ideal semifiltration* of A if and only if it satisfies the two conditions

$$I_0 = A;$$
$$I_a I_b \subseteq I_{a+b} \qquad \text{for every } a \in \mathbb{N} \text{ and } b \in \mathbb{N}.$$

Definition 7. Let A and B be two rings such that $A \subseteq B$. Then, we identify the polynomial ring $A[Y]$ with a subring of the polynomial ring $B[Y]$ (in fact, every element of $A[Y]$ has the form $\sum_{i=0}^{m} a_i Y^i$ for some $m \in \mathbb{N}$ and $(a_0, a_1, ..., a_m) \in A^{m+1}$, and thus can be seen as an element of $B[Y]$ by regarding a_i as an element of B for every $i \in \{0, 1, ..., m\}$).

Definition 8. Let A be a ring, and let $(I_\rho)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Consider the polynomial ring $A[Y]$. Let $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ denote the A-submodule $\sum_{i\in\mathbb{N}} I_i Y^i$ of the A-algebra $A[Y]$. Then,

$$A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] = \sum_{i\in\mathbb{N}} I_i Y^i$$
$$= \left\{ \sum_{i\in\mathbb{N}} a_i Y^i \mid (a_i \in I_i \text{ for all } i \in \mathbb{N}), \text{ and } (\text{only finitely many } i \in \mathbb{N} \text{ satisfy } a_i \neq 0) \right\}$$
$$= \{P \in A[Y] \mid \text{the } i\text{-th coefficient of the polynomial } P \text{ lies in } I_i \text{ for every } i \in \mathbb{N}\}.$$

Now, $1 \in A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ (because $1 = \underbrace{1}_{\in A = I_0} \cdot Y^0 \in I_0 Y^0 \subseteq \sum_{i\in\mathbb{N}} I_i Y^i = A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$).

Also, the A-submodule $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ of $A\left[Y\right]$ is closed under multiplication (since

$$\begin{aligned}
A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right] \cdot A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right] &= \sum_{i\in\mathbb{N}} I_i Y^i \cdot \sum_{i\in\mathbb{N}} I_i Y^i = \sum_{i\in\mathbb{N}} I_i Y^i \cdot \sum_{j\in\mathbb{N}} I_j Y^j \\
&\qquad \text{(here we renamed } i \text{ as } j \text{ in the second sum)} \\
&= \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} I_i Y^i I_j Y^j = \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} \underbrace{I_i I_j}_{\substack{\subseteq I_{i+j}, \\ \text{since } \left(I_\rho\right)_{\rho\in\mathbb{N}} \\ \text{is an ideal} \\ \text{semifiltration}}} \underbrace{Y^i Y^j}_{=Y^{i+j}} \\
&\subseteq \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} I_{i+j} Y^{i+j} \subseteq \sum_{k\in\mathbb{N}} I_k Y^k = \sum_{i\in\mathbb{N}} I_i Y^i \\
&\qquad \text{(here we renamed } k \text{ as } i \text{ in the sum)} \\
&= A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]
\end{aligned}$$

). Hence, $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ is an A-subalgebra of the A-algebra $A\left[Y\right]$. This A-subalgebra $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ is called the *Rees algebra* of the ideal semifiltration $\left(I_\rho\right)_{\rho\in\mathbb{N}}$.

Clearly, $A \subseteq A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$, since $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right] = \sum_{i\in\mathbb{N}} I_i Y^i \supseteq \underbrace{I_0}_{=A} \underbrace{Y^0}_{=1} = A \cdot 1 = A$.

Definition 9. Let A and B be two rings such that $A \subseteq B$. Let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $n \in \mathbb{N}$. Let $u \in B$.

We say that the element u of B is n*-integral over* $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$ if there exists some $\left(a_0, a_1, ..., a_n\right) \in A^{n+1}$ such that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i} \text{ for every } i \in \left\{0, 1, ..., n\right\}.$$

We start with a theorem which reduces the question of n-integrality over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$ to that of n-integrality over a ring[3]:

> **Theorem 7.** Let A and B be two rings such that $A \subseteq B$. Let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $n \in \mathbb{N}$. Let $u \in B$.
>
> Consider the polynomial ring $A\left[Y\right]$ and its A-subalgebra $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ defined in Definition 8.
>
> Then, the element u of B is n-integral over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$ if and only if the element uY of the polynomial ring $B\left[Y\right]$ is n-integral over the ring $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. (Here, $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right] \subseteq B\left[Y\right]$ because $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right] \subseteq A\left[Y\right]$ and we consider $A\left[Y\right]$ as a subring of $B\left[Y\right]$ as explained in Definition 7).

[3]Theorem 7 is inspired by Proposition 5.2.1 in [2].

Proof of Theorem 7. In order to verify Theorem 7, we have to prove the following two lemmata:

Lemma $\mathcal{E}$: If u is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$, then uY is n-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$.

Lemma $\mathcal{F}$: If uY is n-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$, then u is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$.

Proof of Lemma $\mathcal{E}$: Assume that u is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$. Then, by Definition 9, there exists some $(a_0, a_1, ..., a_n) \in A^{n+1}$ such that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i} \text{ for every } i \in \{0, 1, ..., n\}.$$

Note that $a_k Y^{n-k} \in A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ for every $k \in \{0, 1, ..., n\}$ (because $\underbrace{a_k}_{\in I_{n-k}} Y^{n-k} \in I_{n-k}Y^{n-k} \subseteq \sum\limits_{i\in\mathbb{N}} I_i Y^i = A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$). Thus, we can define a polynomial $P \in \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[X]$ by $P(X) = \sum\limits_{k=0}^{n} a_k Y^{n-k} X^k$. This polynomial P satisfies $\deg P \leq n$, and its coefficient before X^n is $\underbrace{a_n}_{=1} \underbrace{Y^{n-n}}_{=Y^0=1} = 1$. Hence, this polynomial P is monic and satisfies $\deg P = n$. Also, $P(X) = \sum\limits_{k=0}^{n} a_k Y^{n-k} X^k$ yields

$$P(uY) = \sum_{k=0}^{n} a_k Y^{n-k} (uY)^k = \sum_{k=0}^{n} a_k Y^{n-k} u^k Y^k = \sum_{k=0}^{n} a_k u^k \underbrace{Y^{n-k} Y^k}_{=Y^n} = Y^n \cdot \underbrace{\sum_{k=0}^{n} a_k u^k}_{=0} = 0.$$

Thus, there exists a monic polynomial $P \in \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[X]$ with $\deg P = n$ and $P(uY) = 0$. Hence, uY is n-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$. This proves Lemma $\mathcal{E}$.

Proof of Lemma $\mathcal{F}$: Assume that uY is n-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$. Then, there exists a monic polynomial $P \in \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[X]$ with $\deg P = n$ and $P(uY) = 0$. Since $P \in \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[X]$ satisfies $\deg P = n$, there exists $(p_0, p_1, ..., p_n) \in \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)^{n+1}$ such that $P(X) = \sum\limits_{k=0}^{n} p_k X^k$. Besides, $p_n = 1$, since P is monic and $\deg P = n$.

For every $k \in \{0, 1, ..., n\}$, we have $p_k \in A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] = \sum\limits_{i\in\mathbb{N}} I_i Y^i$, and thus, there exists a sequence $(p_{k,i})_{i\in\mathbb{N}} \in A^{\mathbb{N}}$ such that $p_k = \sum\limits_{i\in\mathbb{N}} p_{k,i} Y^i$, such that $p_{k,i} \in I_i$ for every $i \in \mathbb{N}$, and such that only finitely many $i \in \mathbb{N}$ satisfy $p_{k,i} \neq 0$. Thus, $P(X) = \sum\limits_{k=0}^{n} p_k X^k$

becomes $P(X) = \sum\limits_{k=0}^{n} \sum\limits_{i\in\mathbb{N}} p_{k,i} Y^i X^k$ (since $p_k = \sum\limits_{i\in\mathbb{N}} p_{k,i} Y^i$). Hence,

$$\begin{aligned}
P(uY) &= \sum_{k=0}^{n} \sum_{i\in\mathbb{N}} p_{k,i} Y^i \underbrace{(uY)^k}_{\substack{=u^k Y^k \\ =Y^k u^k}} = \sum_{k=0}^{n} \sum_{i\in\mathbb{N}} p_{k,i} \underbrace{Y^i Y^k}_{=Y^{i+k}} u^k \\
&= \sum_{k=0}^{n} \sum_{i\in\mathbb{N}} p_{k,i} Y^{i+k} u^k = \sum_{k\in\{0,1,\ldots,n\}} \sum_{i\in\mathbb{N}} p_{k,i} Y^{i+k} u^k \\
&= \sum_{(k,i)\in\{0,1,\ldots,n\}\times\mathbb{N}} p_{k,i} Y^{i+k} u^k = \sum_{\ell\in\mathbb{N}} \sum_{\substack{(k,i)\in\{0,1,\ldots,n\}\times\mathbb{N}; \\ i+k=\ell}} p_{k,i} \underbrace{Y^{i+k}}_{=Y^\ell} u^k \\
&= \sum_{\ell\in\mathbb{N}} \sum_{\substack{(k,i)\in\{0,1,\ldots,n\}\times\mathbb{N}; \\ i+k=\ell}} p_{k,i} Y^\ell u^k = \sum_{\ell\in\mathbb{N}} \sum_{\substack{(k,i)\in\{0,1,\ldots,n\}\times\mathbb{N}; \\ i+k=\ell}} p_{k,i} u^k Y^\ell .
\end{aligned}$$

Hence, $P(uY) = 0$ becomes $\sum\limits_{\ell\in\mathbb{N}} \sum\limits_{\substack{(k,i)\in\{0,1,\ldots,n\}\times\mathbb{N}; \\ i+k=\ell}} p_{k,i} u^k Y^\ell = 0$. In other words, the polynomial $\sum\limits_{\ell\in\mathbb{N}} \underbrace{\sum\limits_{\substack{(k,i)\in\{0,1,\ldots,n\}\times\mathbb{N}; \\ i+k=\ell}} p_{k,i} u^k}_{\in B} Y^\ell \in B[Y]$ equals 0. Hence, its coefficient before Y^n equals 0 as well. But its coefficient before Y^n is $\sum\limits_{\substack{(k,i)\in\{0,1,\ldots,n\}\times\mathbb{N}; \\ i+k=n}} p_{k,i} u^k$. Hence, $\sum\limits_{\substack{(k,i)\in\{0,1,\ldots,n\}\times\mathbb{N}; \\ i+k=n}} p_{k,i} u^k$ equals 0.

Thus,

$$0 = \sum_{\substack{(k,i)\in\{0,1,\ldots,n\}\times\mathbb{N}; \\ i+k=n}} p_{k,i} u^k = \sum_{k\in\{0,1,\ldots,n\}} \sum_{\substack{i\in\mathbb{N}; \\ i+k=n}} p_{k,i} u^k = \sum_{k\in\{0,1,\ldots,n\}} p_{k,n-k} u^k$$

$$\left(\begin{array}{c} \text{since } \{i\in\mathbb{N} \mid i+k=n\} = \{i\in\mathbb{N} \mid i=n-k\} = \{n-k\} \text{ (because } n-k\in\mathbb{N}, \\ \text{since } k\in\{0,1,\ldots,n\}\text{) yields } \sum\limits_{\substack{i\in\mathbb{N}; \\ i+k=n}} p_{k,i} u^k = \sum\limits_{i\in\{n-k\}} p_{k,i} u^k = p_{k,n-k} u^k \end{array} \right).$$

Note that

$$\begin{aligned}
\sum_{i\in\mathbb{N}} p_{n,i} Y^i &= p_n \qquad \left(\text{since } \sum_{i\in\mathbb{N}} p_{k,i} Y^i = p_k \text{ for every } k\in\{0,1,\ldots,n\} \right) \\
&= 1 = 1\cdot Y^0
\end{aligned}$$

in $A[Y]$, and thus the coefficient of the polynomial $\sum\limits_{i\in\mathbb{N}} p_{n,i} Y^i \in A[Y]$ before Y^0 is 1; but the coefficient of the polynomial $\sum\limits_{i\in\mathbb{N}} p_{n,i} Y^i \in A[Y]$ before Y^0 is $p_{n,0}$; hence, $p_{n,0} = 1$.

Define an $(n+1)$-tuple $(a_0, a_1, \ldots, a_n) \in A^{n+1}$ by $(a_k = p_{k,n-k}$ for every $k\in\{0,1,\ldots,n\})$. Then, $a_n = p_{n,n-n} = p_{n,0} = 1$. Besides,

$$\sum_{k=0}^{n} a_k u^k = \sum_{k=0}^{n} p_{k,n-k} u^k = \sum_{k\in\{0,1,\ldots,n\}} p_{k,n-k} u^k = 0.$$

Finally, $a_k = p_{k,n-k} \in I_{n-k}$ (since $p_{k,i} \in I_i$ for every $i \in \mathbb{N}$) for every $k \in \{0, 1, ..., n\}$. In other words, $a_i \in I_{n-i}$ for every $i \in \{0, 1, ..., n\}$.

Altogether, we now know that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i} \text{ for every } i \in \{0, 1, ..., n\}.$$

Thus, by Definition 9, the element u is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$. This proves Lemma $\mathcal{F}$.

Combining Lemmata $\mathcal{E}$ and $\mathcal{F}$, we obtain that u is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$ if and only if uY is n-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$. This proves Theorem 7.

The next theorem is an analogue of Theorem 5 for integrality over ideal semifiltrations:

> **Theorem 8.** Let A and B be two rings such that $A \subseteq B$. Let $(I_\rho)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A.
>
> **(a)** Let $u \in A$. Then, u is 1-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$ if and only if $u \in I_1$.
>
> **(b)** Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$, and that y is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$. Then, $x + y$ is nm-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$.
>
> **(c)** Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$, and that y is n-integral over A. Then, xy is nm-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$.

Proof of Theorem 8. **(a)** In order to verify Theorem 8 **(a)**, we have to prove the following two lemmata:

Lemma $\mathcal{G}$: If u is 1-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$, then $u \in I_1$.

Lemma $\mathcal{H}$: If $u \in I_1$, then u is 1-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$.

Proof of Lemma $\mathcal{G}$: Assume that u is 1-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$. Then, by Definition 9 (applied to $n = 1$), there exists some $(a_0, a_1) \in A^2$ such that

$$\sum_{k=0}^{1} a_k u^k = 0, \qquad a_1 = 1, \qquad \text{and} \qquad a_i \in I_{1-i} \text{ for every } i \in \{0, 1\}.$$

Thus, $a_0 \in I_{1-0}$ (since $a_i \in I_{1-i}$ for every $i \in \{0, 1\}$). Also,

$$0 = \sum_{k=0}^{1} a_k u^k = a_0 \underbrace{u^0}_{=1} + \underbrace{a_1}_{=1} \underbrace{u^1}_{=u} = a_0 + u,$$

so that $u = -\underbrace{a_0}_{\in I_{1-0} = I_1} \in I_1$ (since I_1 is an ideal). This proves Lemma $\mathcal{G}$.

Proof of Lemma $\mathcal{H}$: Assume that $u \in I_1$. Then, $-u \in I_1$ (since I_1 is an ideal). Set $a_0 = -u$ and $a_1 = 1$. Then, $\sum\limits_{k=0}^{1} a_k u^k = \underbrace{a_0}_{=-u} \underbrace{u^0}_{=1} + \underbrace{a_1}_{=1} \underbrace{u^1}_{=u} = -u + u = 0$. Also, $a_i \in I_{1-i}$ for every $i \in \{0, 1\}$ (since $a_0 = -u \in I_1 = I_{1-0}$ and $a_1 = 1 \in A = I_0 = I_{1-1}$). Altogether, we now know that $(a_0, a_1) \in A^2$ and

$$\sum_{k=0}^{1} a_k u^k = 0, \qquad a_1 = 1, \qquad \text{and} \qquad a_i \in I_{1-i} \text{ for every } i \in \{0, 1\}.$$

Thus, by Definition 9 (applied to $n = 1$), the element u is 1-integral over $\left(A, (I_\rho)_{\rho \in \mathbb{N}}\right)$. This proves Lemma $\mathcal{H}$.

Combining Lemmata $\mathcal{G}$ and $\mathcal{H}$, we obtain that u is 1-integral over $\left(A, (I_\rho)_{\rho \in \mathbb{N}}\right)$ if and only if $u \in I_1$. This proves Theorem 8 **(a)**.

(b) Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$. Theorem 7 (applied to x and m instead of u and n) yields that xY is m-integral over $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$ (since x is m-integral over $\left(A, (I_\rho)_{\rho \in \mathbb{N}}\right)$). Also, Theorem 7 (applied to y instead of u) yields that yY is n-integral over $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$ (since y is n-integral over $\left(A, (I_\rho)_{\rho \in \mathbb{N}}\right)$). Hence, Theorem 5 **(b)** (applied to $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$, $B[Y]$, xY and yY instead of A, B, x and y, respectively) yields that $xY + yY$ is nm-integral over $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$. Since $xY + yY = (x + y)Y$, this means that $(x + y)Y$ is nm-integral over $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$. Hence, Theorem 7 (applied to $x + y$ and nm instead of u and n) yields that $x + y$ is nm-integral over $\left(A, (I_\rho)_{\rho \in \mathbb{N}}\right)$. This proves Theorem 8 **(b)**.

(c) First, a trivial observation:

Lemma $\mathcal{I}$: Let A, A' and B' be three rings such that $A \subseteq A' \subseteq B'$. Let $v \in B'$. Let $n \in \mathbb{N}$. If v is n-integral over A, then v is n-integral over A'.

Proof of Lemma $\mathcal{I}$: Assume that v is n-integral over A. Then, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(v) = 0$. Since $A \subseteq A'$, we can identify the polynomial ring $A[X]$ with a subring of the polynomial ring $A'[X]$ (as explained in Definition 7). Thus, $P \in A[X]$ yields $P \in A'[X]$. Hence, there exists a monic polynomial $P \in A'[X]$ with $\deg P = n$ and $P(v) = 0$. Thus, v is n-integral over A'. This proves Lemma $\mathcal{I}$.

Now let us prove Theorem 8 **(c)**.

Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$. Theorem 7 (applied to x and m instead of u and n) yields that xY is m-integral over $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$ (since x is m-integral over $\left(A, (I_\rho)_{\rho \in \mathbb{N}}\right)$). On the other hand, Lemma $\mathcal{I}$ (applied to $A' = A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$, $B' = B[Y]$ and $v = y$) yields that y is n-integral over $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$ (since y is n-integral over A, and $A \subseteq A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right] \subseteq B[Y]$). Hence, Theorem 5 **(c)** (applied to $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$, $B[Y]$ and xY instead of A, B and x, respectively) yields that $xY \cdot y$ is nm-integral over $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$. Since $xY \cdot y = xyY$, this means that xyY is nm-integral over $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. Hence, Theorem 7 (applied to xy and nm instead of u and n) yields that xy is nm-integral over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$. This proves Theorem 8 **(c)**.

The next theorem imitates Theorem 4 for integrality over ideal semifiltrations:

> **Theorem 9.** Let A and B be two rings such that $A \subseteq B$. Let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A.
>
> Let $v \in B$ and $u \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$.
>
> **(a)** Then, $\left(I_\rho A\left[v\right]\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of $A\left[v\right]$. [4]
>
> **(b)** Assume that v is m-integral over A, and that u is n-integral over $\left(A\left[v\right], \left(I_\rho A\left[v\right]\right)_{\rho\in\mathbb{N}}\right)$. Then, u is nm-integral over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$.

Proof of Theorem 9. **(a)** More generally:

Lemma $\mathcal{J}$: Let A and A' be two rings such that $A \subseteq A'$. Let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Then, $\left(I_\rho A'\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A'.

Proof of Lemma $\mathcal{J}$: Since $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A, the set I_ρ is an ideal of A for every $\rho \in \mathbb{N}$, and we have

$$\begin{aligned} I_0 &= A; \\ I_a I_b &\subseteq I_{a+b} \qquad \text{for every } a \in \mathbb{N} \text{ and } b \in \mathbb{N}. \end{aligned}$$

Now, the set $I_\rho A'$ is an ideal of A' for every $\rho \in \mathbb{N}$ (since I_ρ is an ideal of A). Hence, $\left(I_\rho A'\right)_{\rho\in\mathbb{N}}$ is a sequence of ideals of A'. It satisfies

$$\begin{aligned} I_0 A' &= AA' = A'; \\ I_a A' \cdot I_b A' &= I_a I_b A' \subseteq I_{a+b} A' \ \left(\text{since } I_a I_b \subseteq I_{a+b}\right) \qquad \text{for every } a \in \mathbb{N} \text{ and } b \in \mathbb{N}. \end{aligned}$$

Thus, by Definition 6 (applied to A' and $\left(I_\rho A'\right)_{\rho\in\mathbb{N}}$ instead of A and $\left(I_\rho\right)_{\rho\in\mathbb{N}}$), it follows that $\left(I_\rho A'\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A'. This proves Lemma $\mathcal{J}$.

Now let us prove Theorem 9 **(a)**. In fact, Lemma $\mathcal{J}$ (applied to $A' = A\left[v\right]$) yields that $\left(I_\rho A\left[v\right]\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of $A\left[v\right]$. This proves Theorem 9 **(a)**.

(b) First, we will show a simple fact:

Lemma $\mathcal{K}$: Let A, A' and B' be three rings such that $A \subseteq A' \subseteq B'$. Let $v \in B'$. Then, $A' \cdot A\left[v\right] = A'\left[v\right]$.

Proof of Lemma $\mathcal{K}$: We have $\underbrace{A'}_{\subseteq A'\left[v\right]} \cdot \underbrace{A\left[v\right]}_{\substack{\subseteq A'\left[v\right], \\ \text{since } A \subseteq A'}} \subseteq A'\left[v\right] \cdot A'\left[v\right] = A'\left[v\right]$ (since $A'\left[v\right]$ is a ring). On the other hand, let x be an element of $A'\left[v\right]$. Then, there exists some $n \in \mathbb{N}$ and some $\left(a_0, a_1, ..., a_n\right) \in \left(A'\right)^{n+1}$ such that $x = \sum\limits_{k=0}^{n} a_k v^k$. Thus,

$$x = \sum_{k=0}^{n} \underbrace{a_k}_{\in A'} \underbrace{v^k}_{\in A\left[v\right]} \in \sum_{k=0}^{n} A' \cdot A\left[v\right] \subseteq A' \cdot A\left[v\right] \qquad \left(\text{since } A' \cdot A\left[v\right] \text{ is an additive group}\right).$$

[4] Here and in the following, whenever A and B are two rings such that $A \subseteq B$, whenever v is an element of B, and whenever I is an ideal of A, you should read the term $IA\left[v\right]$ as $I\left(A\left[v\right]\right)$, not as $\left(IA\right)\left[v\right]$. For instance, you should read the term $I_\rho A\left[v\right]$ (in Theorem 9 **(a)**) as $I_\rho\left(A\left[v\right]\right)$, not as $\left(I_\rho A\right)\left[v\right]$.

Thus, we have proved that $x \in A' \cdot A[v]$ for every $x \in A'[v]$. Therefore, $A'[v] \subseteq A' \cdot A[v]$. Combined with $A' \cdot A[v] \subseteq A'[v]$, this yields $A' \cdot A[v] = A'[v]$. Hence, we have established Lemma $\mathcal{K}$.

Now let us prove Theorem 9 **(b)**. In fact, consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$. We have $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq A[Y]$, and (as explained in Definition 7) we can identify the polynomial ring $A[Y]$ with a subring of $(A[v])[Y]$ (since $A \subseteq A[v]$). Hence, $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq (A[v])[Y]$. On the other hand, $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right] \subseteq (A[v])[Y]$.

Now, we will show that $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right] = \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[v]$.

In fact, Definition 8 yields

$$\begin{aligned}
(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right] &= \sum_{i\in\mathbb{N}} I_i A[v] \cdot Y^i = \sum_{i\in\mathbb{N}} I_i Y^i \cdot A[v] = A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \cdot A[v] \\
&\qquad \left(\text{since } \sum_{i\in\mathbb{N}} I_i Y^i = A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right) \\
&= \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[v]
\end{aligned}$$

(by Lemma $\mathcal{K}$ (applied to $A' = A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ and $B' = (A[v])[Y]$)).

Note that (as explained in Definition 7) we can identify the polynomial ring $(A[v])[Y]$ with a subring of $B[Y]$ (since $A[v] \subseteq B$). Thus, $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq (A[v])[Y]$ yields $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq B[Y]$.

Besides, Lemma $\mathcal{I}$ (applied to $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$, $B[Y]$ and m instead of A', B' and n) yields that v is m-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ (since v is m-integral over A, and $A \subseteq A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq B[Y]$).

Now, Theorem 7 (applied to $A[v]$ and $(I_\rho A[v])_{\rho\in\mathbb{N}}$ instead of A and $(I_\rho)_{\rho\in\mathbb{N}}$) yields that uY is n-integral over $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right]$ (since u is n-integral over $\left(A[v], (I_\rho A[v])_{\rho\in\mathbb{N}}\right)$). Since $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right] = \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[v]$, this means that uY is n-integral over $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[v]$. Now, Theorem 4 (applied to $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$, $B[Y]$ and uY instead of A, B and u) yields that uY is nm-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ (since v is m-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$, and uY is n-integral over $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[v]$). Thus, Theorem 7 (applied to nm instead of n) yields that u is nm-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$. This proves Theorem 9 **(b)**.

3. Generalizing to two ideal semifiltrations

Theorem 10. Let A be a ring.

(a) Then, $(A)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A.

(b) Let $(I_\rho)_{\rho\in\mathbb{N}}$ and $(J_\rho)_{\rho\in\mathbb{N}}$ be two ideal semifiltrations of A. Then, $(I_\rho J_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A.

Proof of Theorem 10. **(a)** Clearly, $(A)_{\rho\in\mathbb{N}}$ is a sequence of ideals of A. Hence, in order to prove that $(A)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A, it is enough to verify that it satisfies the two conditions

$$\begin{aligned} A &= A; \\ AA &\subseteq A \qquad\qquad \text{for every } a\in\mathbb{N} \text{ and } b\in\mathbb{N}. \end{aligned}$$

But these two conditions are obviously satisfied. Hence, $(A)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A (by Definition 6, applied to $(A)_{\rho\in\mathbb{N}}$ instead of $(I_\rho)_{\rho\in\mathbb{N}}$). This proves Theorem 10 **(a)**.

(b) Since $(I_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A, it is a sequence of ideals of A, and it satisfies the two conditions

$$\begin{aligned} I_0 &= A; \\ I_aI_b &\subseteq I_{a+b} \qquad\qquad \text{for every } a\in\mathbb{N} \text{ and } b\in\mathbb{N} \end{aligned}$$

(by Definition 6). Since $(J_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A, it is a sequence of ideals of A, and it satisfies the two conditions

$$\begin{aligned} J_0 &= A; \\ J_aJ_b &\subseteq J_{a+b} \qquad\qquad \text{for every } a\in\mathbb{N} \text{ and } b\in\mathbb{N} \end{aligned}$$

(by Definition 6, applied to $(J_\rho)_{\rho\in\mathbb{N}}$ instead of $(I_\rho)_{\rho\in\mathbb{N}}$).

Now, $I_\rho J_\rho$ is an ideal of A for every $\rho\in\mathbb{N}$ (since I_ρ and J_ρ are ideals of A for every $\rho\in\mathbb{N}$, and the product of any two ideals of A is an ideal of A). Hence, $(I_\rho J_\rho)_{\rho\in\mathbb{N}}$ is a sequence of ideals of A. Thus, in order to prove that $(I_\rho J_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A, it is enough to verify that it satisfies the two conditions

$$\begin{aligned} I_0J_0 &= A; \\ I_aJ_a\cdot I_bJ_b &\subseteq I_{a+b}J_{a+b} \qquad\qquad \text{for every } a\in\mathbb{N} \text{ and } b\in\mathbb{N}. \end{aligned}$$

But these two conditions are satisfied, since

$$\begin{aligned} &\underbrace{I_0}_{=A}\underbrace{J_0}_{=A} = AA = A; \\ &I_aJ_a\cdot I_bJ_b = \underbrace{I_aI_b}_{\subseteq I_{a+b}}\underbrace{J_aJ_b}_{\subseteq J_{a+b}} \subseteq I_{a+b}J_{a+b} \qquad\qquad \text{for every } a\in\mathbb{N} \text{ and } b\in\mathbb{N}. \end{aligned}$$

Hence, $(I_\rho J_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A (by Definition 6, applied to $(I_\rho J_\rho)_{\rho\in\mathbb{N}}$ instead of $(I_\rho)_{\rho\in\mathbb{N}}$). This proves Theorem 10 **(b)**.

Now let us generalize Theorem 7:

Theorem 11. Let A and B be two rings such that $A\subseteq B$. Let $(I_\rho)_{\rho\in\mathbb{N}}$ and $(J_\rho)_{\rho\in\mathbb{N}}$ be two ideal semifiltrations of A. Let $n\in\mathbb{N}$. Let $u\in B$.

We know that $(I_\rho J_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A (according to Theorem 10 **(b)**).

Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$.

We will abbreviate the ring $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ by $A_{[I]}$.

By Lemma $\mathcal{J}$ (applied to $A_{[I]}$ and $(J_\tau)_{\tau\in\mathbb{N}}$ instead of A' and $(I_\rho)_{\rho\in\mathbb{N}}$), the sequence $\left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}$ is an ideal semifiltration of $A_{[I]}$ (since $A \subseteq A_{[I]}$ and since $(J_\tau)_{\tau\in\mathbb{N}} = (J_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A).

Then, the element u of B is n-integral over $\left(A, (I_\rho J_\rho)_{\rho\in\mathbb{N}}\right)$ if and only if the element uY of the polynomial ring $B[Y]$ is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$.

(Here, $A_{[I]} \subseteq B[Y]$ because $A_{[I]} = A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq A[Y]$ and we consider $A[Y]$ as a subring of $B[Y]$ as explained in Definition 7.)

Proof of Theorem 11. First, note that

$$\sum_{\ell\in\mathbb{N}} I_\ell Y^\ell = \sum_{i\in\mathbb{N}} I_i Y^i \qquad \text{(here we renamed } \ell \text{ as } i \text{ in the sum)}$$
$$= A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] = A_{[I]}.$$

In order to verify Theorem 11, we have to prove the following two lemmata:

Lemma $\mathcal{E}'$: If u is n-integral over $\left(A, (I_\rho J_\rho)_{\rho\in\mathbb{N}}\right)$, then uY is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$.

Lemma $\mathcal{F}'$: If uY is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$, then u is n-integral over $\left(A, (I_\rho J_\rho)_{\rho\in\mathbb{N}}\right)$.

Proof of Lemma $\mathcal{E}'$: Assume that u is n-integral over $\left(A, (I_\rho J_\rho)_{\rho\in\mathbb{N}}\right)$. Then, by Definition 9 (applied to $(I_\rho J_\rho)_{\rho\in\mathbb{N}}$ instead of $(I_\rho)_{\rho\in\mathbb{N}}$), there exists some $(a_0, a_1, ..., a_n) \in A^{n+1}$ such that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i}J_{n-i} \text{ for every } i \in \{0, 1, ..., n\}.$$

Note that $a_k Y^{n-k} \in A_{[I]}$ for every $k \in \{0, 1, ..., n\}$ (because $a_k \in I_{n-k}J_{n-k} \subseteq I_{n-k}$ (since I_{n-k} is an ideal of A) and thus $a_k Y^{n-k} \in I_{n-k}Y^{n-k} \subseteq \sum_{i\in\mathbb{N}} I_i Y^i = A_{[I]}$). Thus, we can define an $(n+1)$-tuple $(b_0, b_1, ..., b_n) \in \left(A_{[I]}\right)^{n+1}$ by $\left(b_k = a_k Y^{n-k} \text{ for every } k \in \{0, 1, ..., n\}\right)$. Then,

$$\sum_{k=0}^{n} b_k \cdot (uY)^k = \sum_{k=0}^{n} a_k Y^{n-k} \cdot (uY)^k = \sum_{k=0}^{n} a_k Y^{n-k} u^k Y^k = \sum_{k=0}^{n} a_k u^k \underbrace{Y^{n-k}Y^k}_{=Y^n} = Y^n \cdot \underbrace{\sum_{k=0}^{n} a_k u^k}_{=0} = 0;$$

$$b_n = \underbrace{a_n}_{=1} \underbrace{Y^{n-n}}_{=Y^0=1} = 1,$$

and

$$b_i = \underbrace{a_i}_{\substack{\in I_{n-i}J_{n-i} \\ =J_{n-i}I_{n-i}}} Y^{n-i} \in J_{n-i} \underbrace{I_{n-i}Y^{n-i}}_{\substack{\subseteq \sum\limits_{\ell\in\mathbb{N}} I_\ell Y^\ell \\ =A_{[I]}}} \subseteq J_{n-i}A_{[I]}$$

for every $i \in \{0, 1, ..., n\}$.

Altogether, we now know that $(b_0, b_1, ..., b_n) \in \left(A_{[I]}\right)^{n+1}$ and

$$\sum_{k=0}^{n} b_k \cdot (uY)^k = 0, \qquad b_n = 1, \qquad \text{and} \qquad b_i \in J_{n-i}A_{[I]} \text{ for every } i \in \{0, 1, ..., n\}.$$

Hence, by Definition 9 (applied to $A_{[I]}$, $B[Y]$, $\left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}$, uY and $(b_0, b_1, ..., b_n)$ instead of A, B, $(I_\rho)_{\rho\in\mathbb{N}}$, u and $(a_0, a_1, ..., a_n)$), the element uY is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$. This proves Lemma $\mathcal{E}'$.

Proof of Lemma $\mathcal{F}'$: Assume that uY is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$. Then, by Definition 9 (applied to $A_{[I]}$, $B[Y]$, $\left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}$, uY and $(p_0, p_1, ..., p_n)$ instead of A, B, $(I_\rho)_{\rho\in\mathbb{N}}$, u and $(a_0, a_1, ..., a_n)$), there exists some $(p_0, p_1, ..., p_n) \in \left(A_{[I]}\right)^{n+1}$ such that

$$\sum_{k=0}^{n} p_k \cdot (uY)^k = 0, \qquad p_n = 1, \qquad \text{and} \qquad p_i \in J_{n-i}A_{[I]} \text{ for every } i \in \{0, 1, ..., n\}.$$

For every $k \in \{0, 1, ..., n\}$, we have

$$\begin{aligned} p_k &\in J_{n-k}A_{[I]} = J_{n-k}\sum_{i\in\mathbb{N}} I_i Y^i \qquad \left(\text{since } A_{[I]} = \sum_{i\in\mathbb{N}} I_i Y^i\right) \\ &= \sum_{i\in\mathbb{N}} J_{n-k} I_i Y^i = \sum_{i\in\mathbb{N}} I_i J_{n-k} Y^i, \end{aligned}$$

and thus, there exists a sequence $(p_{k,i})_{i\in\mathbb{N}} \in A^{\mathbb{N}}$ such that $p_k = \sum\limits_{i\in\mathbb{N}} p_{k,i}Y^i$, such that $p_{k,i} \in I_i J_{n-k}$ for every $i \in \mathbb{N}$, and such that only finitely many $i \in \mathbb{N}$ satisfy $p_{k,i} \neq 0$. Thus,

$$\begin{aligned} \sum_{k=0}^{n} p_k \cdot (uY)^k &= \sum_{k=0}^{n} \sum_{i\in\mathbb{N}} p_{k,i}Y^i \cdot \underbrace{(uY)^k}_{\substack{=u^kY^k \\ =Y^ku^k}} \qquad \left(\text{since } p_k = \sum_{i\in\mathbb{N}} p_{k,i}Y^i\right) \\ &= \sum_{k=0}^{n} \sum_{i\in\mathbb{N}} p_{k,i} \underbrace{Y^i \cdot Y^k}_{=Y^{i+k}} u^k \\ &= \sum_{k=0}^{n} \sum_{i\in\mathbb{N}} p_{k,i}Y^{i+k}u^k = \sum_{k\in\{0,1,...,n\}} \sum_{i\in\mathbb{N}} p_{k,i}Y^{i+k}u^k \\ &= \sum_{(k,i)\in\{0,1,...,n\}\times\mathbb{N}} p_{k,i}Y^{i+k}u^k = \sum_{\ell\in\mathbb{N}} \sum_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N}; \\ i+k=\ell}} p_{k,i} \underbrace{Y^{i+k}}_{=Y^\ell} u^k \\ &= \sum_{\ell\in\mathbb{N}} \sum_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N}; \\ i+k=\ell}} p_{k,i}Y^\ell u^k = \sum_{\ell\in\mathbb{N}} \sum_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N}; \\ i+k=\ell}} p_{k,i}u^kY^\ell. \end{aligned}$$

Hence, $\sum\limits_{k=0}^{n} p_k \cdot (uY)^k = 0$ becomes $\sum\limits_{\ell\in\mathbb{N}} \sum\limits_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N};\\ i+k=\ell}} p_{k,i}u^kY^\ell = 0$. In other words, the polynomial $\sum\limits_{\ell\in\mathbb{N}} \underbrace{\sum\limits_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N};\\ i+k=\ell}} p_{k,i}u^k}_{\in B} Y^\ell \in B[Y]$ equals 0. Hence, its coefficient before Y^n equals 0 as well. But its coefficient before Y^n is $\sum\limits_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N};\\ i+k=n}} p_{k,i}u^k$. Hence, $\sum\limits_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N};\\ i+k=n}} p_{k,i}u^k$ equals 0.

Thus,

$$0 = \sum_{\substack{(k,i)\in\{0,1,...,n\}\times\mathbb{N};\\ i+k=n}} p_{k,i}u^k = \sum_{k\in\{0,1,...,n\}} \sum_{\substack{i\in\mathbb{N};\\ i+k=n}} p_{k,i}u^k = \sum_{k\in\{0,1,...,n\}} p_{k,n-k}u^k$$

$$\left(\begin{array}{c}\text{since } \{i\in\mathbb{N} \mid i+k=n\} = \{i\in\mathbb{N} \mid i=n-k\} = \{n-k\} \text{ (because } n-k\in\mathbb{N}, \\ \text{since } k\in\{0,1,...,n\}\text{) yields } \sum\limits_{\substack{i\in\mathbb{N};\\ i+k=n}} p_{k,i}u^k = \sum\limits_{i\in\{n-k\}} p_{k,i}u^k = p_{k,n-k}u^k\end{array}\right).$$

Note that

$$\sum_{i\in\mathbb{N}} p_{n,i}Y^i = p_n \qquad \left(\text{since } \sum_{i\in\mathbb{N}} p_{k,i}Y^i = p_k \text{ for every } k\in\{0,1,...,n\}\right)$$
$$= 1 = 1\cdot Y^0$$

in $A[Y]$, and thus the coefficient of the polynomial $\sum\limits_{i\in\mathbb{N}} p_{n,i}Y^i \in A[Y]$ before Y^0 is 1; but the coefficient of the polynomial $\sum\limits_{i\in\mathbb{N}} p_{n,i}Y^i \in A[Y]$ before Y^0 is $p_{n,0}$; hence, $p_{n,0} = 1$.

Define an $(n+1)$-tuple $(a_0, a_1, ..., a_n) \in A^{n+1}$ by $(a_k = p_{k,n-k}$ for every $k\in\{0,1,...,n\})$. Then, $a_n = p_{n,n-n} = p_{n,0} = 1$. Besides,

$$\sum_{k=0}^{n} a_k u^k = \sum_{k=0}^{n} p_{k,n-k}u^k = \sum_{k\in\{0,1,...,n\}} p_{k,n-k}u^k = 0.$$

Finally, $a_k = p_{k,n-k} \in I_{n-k}J_{n-k}$ (since $p_{k,i} \in I_iJ_{n-k}$ for every $i\in\mathbb{N}$) for every $k\in\{0,1,...,n\}$. In other words, $a_i \in I_{n-i}J_{n-i}$ for every $i\in\{0,1,...,n\}$.

Altogether, we now know that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i}J_{n-i} \text{ for every } i\in\{0,1,...,n\}.$$

Thus, by Definition 9 (applied to $(I_\rho J_\rho)_{\rho\in\mathbb{N}}$ instead of $(I_\rho)_{\rho\in\mathbb{N}}$), the element u is n-integral over $\left(A, (I_\rho J_\rho)_{\rho\in\mathbb{N}}\right)$. This proves Lemma $\mathcal{F}'$.

Combining Lemmata $\mathcal{E}'$ and $\mathcal{F}'$, we obtain that u is n-integral over $\left(A, (I_\rho J_\rho)_{\rho\in\mathbb{N}}\right)$ if and only if uY is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$. This proves Theorem 11.

For the sake of completeness, we mention the following trivial fact (which shows why Theorem 11 generalizes Theorem 7):

Theorem 12. Let A and B be two rings such that $A \subseteq B$. Let $n \in \mathbb{N}$. Let $u \in B$.

We know that $(A)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A (according to Theorem 10 **(a)**).

Then, the element u of B is n-integral over $\left(A, (A)_{\rho\in\mathbb{N}}\right)$ if and only if u is n-integral over A.

Proof of Theorem 12. In order to verify Theorem 12, we have to prove the following two lemmata:

Lemma $\mathcal{L}$: If u is n-integral over $\left(A, (A)_{\rho\in\mathbb{N}}\right)$, then u is n-integral over A.

Lemma $\mathcal{M}$: If u is n-integral over A, then u is n-integral over $\left(A, (A)_{\rho\in\mathbb{N}}\right)$.

Proof of Lemma $\mathcal{L}$: Assume that u is n-integral over $\left(A, (A)_{\rho\in\mathbb{N}}\right)$. Then, by Definition 9 (applied to $(A)_{\rho\in\mathbb{N}}$ instead of $(I_\rho)_{\rho\in\mathbb{N}}$), there exists some $(a_0, a_1, ..., a_n) \in A^{n+1}$ such that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in A \text{ for every } i \in \{0, 1, ..., n\}.$$

Define a polynomial $P \in A[X]$ by $P(X) = \sum\limits_{k=0}^{n} a_k X^k$. Then, $P(X) = \sum\limits_{k=0}^{n} a_k X^k = \underbrace{a_n}_{=1} X^n + \sum\limits_{k=0}^{n-1} a_k X^k = X^n + \sum\limits_{k=0}^{n-1} a_k X^k$. Hence, the polynomial P is monic, and $\deg P = n$. Besides, $P(u) = 0$ (since $P(X) = \sum\limits_{k=0}^{n} a_k X^k$ yields $P(u) = \sum\limits_{k=0}^{n} a_k u^k = 0$). Thus, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$. Hence, u is n-integral over A. This proves Lemma $\mathcal{L}$.

Proof of Lemma $\mathcal{M}$: Assume that u is n-integral over A. Then, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$. Since $\deg P = n$, there exists some $(n+1)$-tuple $(a_0, a_1, ..., a_n) \in A^{n+1}$ such that $P(X) = \sum\limits_{k=0}^{n} a_k X^k$. Thus, $a_n = 1$ (since P is monic, and $\deg P = n$). Also, $\sum\limits_{k=0}^{n} a_k X^k = P(X)$ yields $\sum\limits_{k=0}^{n} a_k u^k = P(u) = 0$. Altogether, we now know that $(a_0, a_1, ..., a_n) \in A^{n+1}$ and

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in A \text{ for every } i \in \{0, 1, ..., n\}.$$

Hence, by Definition 9 (applied to $(A)_{\rho\in\mathbb{N}}$ instead of $(I_\rho)_{\rho\in\mathbb{N}}$), the element u is n-integral over $\left(A, (A)_{\rho\in\mathbb{N}}\right)$. This proves Lemma $\mathcal{M}$.

Combining Lemmata $\mathcal{L}$ and $\mathcal{M}$, we obtain that u is n-integral over $\left(A, (A)_{\rho\in\mathbb{N}}\right)$ if and only if u is n-integral over A. This proves Theorem 12.

Finally, let us generalize Theorem 8 **(c)**:

Theorem 13. Let A and B be two rings such that $A \subseteq B$. Let $(I_\rho)_{\rho\in\mathbb{N}}$ and $(J_\rho)_{\rho\in\mathbb{N}}$ be two ideal semifiltrations of A.

Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$, and that y is n-integral over $\left(A, (J_\rho)_{\rho\in\mathbb{N}}\right)$. Then, xy is nm-integral over $\left(A, (I_\rho J_\rho)_{\rho\in\mathbb{N}}\right)$.

Proof of Theorem 13. First, a trivial observation:

Lemma $\mathcal{I}'$: Let A, A' and B' be three rings such that $A \subseteq A' \subseteq B'$. Let $(I_\rho)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $v \in B'$. Let $n \in \mathbb{N}$. If v is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$, then v is n-integral over $\left(A', (I_\rho A')_{\rho\in\mathbb{N}}\right)$. (Note that $(I_\rho A')_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A', according to Lemma $\mathcal{J}$.)

Proof of Lemma $\mathcal{I}'$: Assume that v is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$. Then, by Definition 9 (applied to B' and v instead of B and u), there exists some $(a_0, a_1, ..., a_n) \in A^{n+1}$ such that

$$\sum_{k=0}^{n} a_k v^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i} \text{ for every } i \in \{0, 1, ..., n\}.$$

But $(a_0, a_1, ..., a_n) \in A^{n+1}$ yields $(a_0, a_1, ..., a_n) \in (A')^{n+1}$ (since $A \subseteq A'$), and $a_i \in I_{n-i}$ yields $a_i \in I_{n-i}A'$ (since $I_{n-i} \subseteq I_{n-i}A'$) for every $i \in \{0, 1, ..., n\}$. Thus, $(a_0, a_1, ..., a_n) \in (A')^{n+1}$ and

$$\sum_{k=0}^{n} a_k v^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i}A' \text{ for every } i \in \{0, 1, ..., n\}.$$

Hence, by Definition 9 (applied to B', A', $(I_\rho A')_{\rho\in\mathbb{N}}$ and v instead of B, A, $(I_\rho)_{\rho\in\mathbb{N}}$ and u), the element v is n-integral over $\left(A', (I_\rho A')_{\rho\in\mathbb{N}}\right)$. This proves Lemma $\mathcal{I}'$.

Now let us prove Theorem 13.

We have $(J_\rho)_{\rho\in\mathbb{N}} = (J_\tau)_{\tau\in\mathbb{N}}$. Hence, y is n-integral over $\left(A, (J_\tau)_{\tau\in\mathbb{N}}\right)$ (since y is n-integral over $\left(A, (J_\rho)_{\rho\in\mathbb{N}}\right)$).

Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$. We will abbreviate the ring $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ by $A_{[I]}$. We have $A_{[I]} \subseteq B[Y]$, because $A_{[I]} = A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq A[Y]$ and we consider $A[Y]$ as a subring of $B[Y]$ as explained in Definition 7.

Theorem 7 (applied to x and m instead of u and n) yields that xY is m-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ (since x is m-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$). In other words, xY is m-integral over $A_{[I]}$ (since $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] = A_{[I]}$).

On the other hand, Lemma $\mathcal{I}'$ (applied to $A_{[I]}$, $B[Y]$, $(J_\tau)_{\tau\in\mathbb{N}}$ and y instead of A', B', $(I_\rho)_{\rho\in\mathbb{N}}$ and v) yields that y is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$ (since y is n-integral over $\left(A, (J_\tau)_{\tau\in\mathbb{N}}\right)$, and $A \subseteq A_{[I]} \subseteq B[Y]$).

Hence, Theorem 8 **(c)** (applied to $A_{[I]}$, $B[Y]$, $(J_\tau A_{[I]})_{\tau\in\mathbb{N}}$, y, xY, m and n instead of A, B, $(I_\rho)_{\rho\in\mathbb{N}}$, x, y, n and m respectively) yields that $y\cdot xY$ is mn-integral over $\left(A_{[I]}, (J_\tau A_{[I]})_{\tau\in\mathbb{N}}\right)$ (since y is n-integral over $\left(A_{[I]}, (J_\tau A_{[I]})_{\tau\in\mathbb{N}}\right)$, and xY is m-integral over $A_{[I]}$). Since $y\cdot xY = xyY$ and $mn = nm$, this means that xyY is nm-integral over $\left(A_{[I]}, (J_\tau A_{[I]})_{\tau\in\mathbb{N}}\right)$. Hence, Theorem 11 (applied to xy and nm instead of u and n) yields that xy is nm-integral over $\left(A, (I_\rho J_\rho)_{\rho\in\mathbb{N}}\right)$. This proves Theorem 13.

4. Accelerating ideal semifiltrations

We start this section with an obvious observation:

Theorem 14. Let A be a ring. Let $(I_\rho)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $\lambda\in\mathbb{N}$. Then, $(I_{\lambda\rho})_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A.

Proof of Theorem 14. Since $(I_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A, it is a sequence of ideals of A, and it satisfies the two conditions

$$I_0 = A;$$
$$I_a I_b \subseteq I_{a+b} \qquad \text{for every } a\in\mathbb{N} \text{ and } b\in\mathbb{N}$$

(by Definition 6).

Now, $I_{\lambda\rho}$ is an ideal of A for every $\rho\in\mathbb{N}$ (since $(I_\rho)_{\rho\in\mathbb{N}}$ is a sequence of ideals of A). Hence, $(I_{\lambda\rho})_{\rho\in\mathbb{N}}$ is a sequence of ideals of A. Thus, in order to prove that $(I_{\lambda\rho})_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A, it is enough to verify that it satisfies the two conditions

$$I_{\lambda\cdot 0} = A;$$
$$I_{\lambda a} I_{\lambda b} \subseteq I_{\lambda(a+b)} \qquad \text{for every } a\in\mathbb{N} \text{ and } b\in\mathbb{N}.$$

But these two conditions are satisfied, since

$$I_{\lambda\cdot 0} = I_0 = A;$$
$$\begin{aligned} I_{\lambda a} I_{\lambda b} &\subseteq I_{\lambda a+\lambda b} && \left(\text{since } (I_\rho)_{\rho\in\mathbb{N}} \text{ is an ideal semifiltration of } A\right) \\ &= I_{\lambda(a+b)} && \text{for every } a\in\mathbb{N} \text{ and } b\in\mathbb{N}. \end{aligned}$$

Hence, $(I_{\lambda\rho})_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A (by Definition 6, applied to $(I_{\lambda\rho})_{\rho\in\mathbb{N}}$ instead of $(I_\rho)_{\rho\in\mathbb{N}}$). This proves Theorem 14.

I refer to $(I_{\lambda\rho})_{\rho\in\mathbb{N}}$ as the λ-*acceleration* of the ideal semifiltration $(I_\rho)_{\rho\in\mathbb{N}}$.

Now, Theorem 11, itself a generalization of Theorem 7, is going to be generalized once more:

Theorem 15. Let A and B be two rings such that $A\subseteq B$. Let $(I_\rho)_{\rho\in\mathbb{N}}$ and $(J_\rho)_{\rho\in\mathbb{N}}$ be two ideal semifiltrations of A. Let $n\in\mathbb{N}$. Let $u\in B$. Let $\lambda\in\mathbb{N}$.

We know that $(I_{\lambda\rho})_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A (according to Theorem 14).

Hence, $\left(I_{\lambda\rho}J_\rho\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A (according to Theorem 10 **(b)**, applied to $\left(I_{\lambda\rho}\right)_{\rho\in\mathbb{N}}$ instead of $\left(I_\rho\right)_{\rho\in\mathbb{N}}$).

Consider the polynomial ring $A\left[Y\right]$ and its A-subalgebra $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$.

We will abbreviate the ring $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ by $A_{[I]}$.

By Lemma $\mathcal{J}$ (applied to $A_{[I]}$ and $\left(J_\tau\right)_{\tau\in\mathbb{N}}$ instead of A' and $\left(I_\rho\right)_{\rho\in\mathbb{N}}$), the sequence $\left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}$ is an ideal semifiltration of $A_{[I]}$ (since $A \subseteq A_{[I]}$ and since $\left(J_\tau\right)_{\tau\in\mathbb{N}} = \left(J_\rho\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A).

Then, the element u of B is n-integral over $\left(A, \left(I_{\lambda\rho}J_\rho\right)_{\rho\in\mathbb{N}}\right)$ if and only if the element uY^λ of the polynomial ring $B\left[Y\right]$ is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$. (Here, $A_{[I]} \subseteq B\left[Y\right]$ because $A_{[I]} = A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right] \subseteq A\left[Y\right]$ and we consider $A\left[Y\right]$ as a subring of $B\left[Y\right]$ as explained in Definition 7.)

Proof of Theorem 15. First, note that

$$\sum_{\ell\in\mathbb{N}} I_\ell Y^\ell = \sum_{i\in\mathbb{N}} I_i Y^i \qquad \text{(here we renamed } \ell \text{ as } i \text{ in the sum)}$$
$$= A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right] = A_{[I]}.$$

In order to verify Theorem 15, we have to prove the following two lemmata:

Lemma $\mathcal{E}''$: If u is n-integral over $\left(A, \left(I_{\lambda\rho}J_\rho\right)_{\rho\in\mathbb{N}}\right)$, then uY^λ is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$.

Lemma $\mathcal{F}''$: If uY^λ is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$, then u is n-integral over $\left(A, \left(I_{\lambda\rho}J_\rho\right)_{\rho\in\mathbb{N}}\right)$.

Proof of Lemma $\mathcal{E}''$: Assume that u is n-integral over $\left(A, \left(I_{\lambda\rho}J_\rho\right)_{\rho\in\mathbb{N}}\right)$. Then, by Definition 9 (applied to $\left(I_{\lambda\rho}J_\rho\right)_{\rho\in\mathbb{N}}$ instead of $\left(I_\rho\right)_{\rho\in\mathbb{N}}$), there exists some $\left(a_0, a_1, ..., a_n\right) \in A^{n+1}$ such that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{\lambda(n-i)}J_{n-i} \text{ for every } i \in \left\{0, 1, ..., n\right\}.$$

Note that $a_k Y^{\lambda(n-k)} \in A_{[I]}$ for every $k \in \left\{0, 1, ..., n\right\}$ (because $a_k \in I_{\lambda(n-k)}J_{n-k} \subseteq I_{\lambda(n-k)}$ (since $I_{\lambda(n-k)}$ is an ideal of A) and thus $a_k Y^{\lambda(n-k)} \in I_{\lambda(n-k)} Y^{\lambda(n-k)} \subseteq \sum\limits_{i\in\mathbb{N}} I_i Y^i = A_{[I]}$). Thus, we can define an $(n+1)$-tuple $\left(b_0, b_1, ..., b_n\right) \in \left(A_{[I]}\right)^{n+1}$ by

$$\left(b_k = a_k Y^{\lambda(n-k)} \text{ for every } k \in \left\{0, 1, ..., n\right\}\right).$$

Then,

$$\sum_{k=0}^{n} b_k \cdot \left(uY^{\lambda}\right)^k = \sum_{k=0}^{n} a_k Y^{\lambda(n-k)} \cdot \underbrace{\left(uY^{\lambda}\right)^k}_{\substack{=u^k\left(Y^{\lambda}\right)^k \\ =u^kY^{\lambda k}}} = \sum_{k=0}^{n} a_k Y^{\lambda(n-k)} u^k Y^{\lambda k} = \sum_{k=0}^{n} a_k u^k \underbrace{Y^{\lambda(n-k)} Y^{\lambda k}}_{\substack{=Y^{\lambda(n-k)+\lambda k} \\ =Y^{\lambda n}}} = Y^{\lambda n} \cdot \underbrace{\sum_{k=0}^{n} a_k u^k}_{=0} = 0;$$

$$b_n = \underbrace{a_n}_{=1} \underbrace{Y^{\lambda(n-n)}}_{=Y^{\lambda\cdot 0}=Y^0=1} = 1,$$

and

$$b_i = \underbrace{a_i}_{\substack{\in I_{\lambda(n-i)}J_{n-i} \\ =J_{n-i}I_{\lambda(n-i)}}} Y^{\lambda(n-i)} \in J_{n-i} \underbrace{I_{\lambda(n-i)} Y^{\lambda(n-i)}}_{\substack{\subseteq \sum_{\ell\in\mathbb{N}} I_\ell Y^\ell \\ =A_{[I]}}} \subseteq J_{n-i} A_{[I]}$$

for every $i \in \{0, 1, ..., n\}$.

Altogether, we now know that $(b_0, b_1, ..., b_n) \in \left(A_{[I]}\right)^{n+1}$ and

$$\sum_{k=0}^{n} b_k \cdot \left(uY^{\lambda}\right)^k = 0, \qquad b_n = 1, \qquad \text{and} \qquad b_i \in J_{n-i}A_{[I]} \text{ for every } i \in \{0, 1, ..., n\}.$$

Hence, by Definition 9 (applied to $A_{[I]}$, $B[Y]$, $\left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}$, uY^{λ} and $(b_0, b_1, ..., b_n)$ instead of A, B, $(I_\rho)_{\rho\in\mathbb{N}}$, u and $(a_0, a_1, ..., a_n)$), the element uY^{λ} is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$. This proves Lemma $\mathcal{E}''$.

Proof of Lemma $\mathcal{F}''$: Assume that uY^{λ} is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$. Then, by Definition 9 (applied to $A_{[I]}$, $B[Y]$, $\left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}$, uY^{λ} and $(p_0, p_1, ..., p_n)$ instead of A, B, $(I_\rho)_{\rho\in\mathbb{N}}$, u and $(a_0, a_1, ..., a_n)$), there exists some $(p_0, p_1, ..., p_n) \in \left(A_{[I]}\right)^{n+1}$ such that

$$\sum_{k=0}^{n} p_k \cdot \left(uY^{\lambda}\right)^k = 0, \qquad p_n = 1, \qquad \text{and} \qquad p_i \in J_{n-i}A_{[I]} \text{ for every } i \in \{0, 1, ..., n\}.$$

For every $k \in \{0, 1, ..., n\}$, we have

$$p_k \in J_{n-k}A_{[I]} = J_{n-k} \sum_{i\in\mathbb{N}} I_i Y^i \qquad \left(\text{since } A_{[I]} = \sum_{i\in\mathbb{N}} I_i Y^i\right)$$
$$= \sum_{i\in\mathbb{N}} J_{n-k} I_i Y^i = \sum_{i\in\mathbb{N}} I_i J_{n-k} Y^i,$$

and thus, there exists a sequence $(p_{k,i})_{i\in\mathbb{N}} \in A^{\mathbb{N}}$ such that $p_k = \sum_{i\in\mathbb{N}} p_{k,i} Y^i$, such that $p_{k,i} \in I_i J_{n-k}$ for every $i \in \mathbb{N}$, and such that only finitely many $i \in \mathbb{N}$ satisfy $p_{k,i} \neq 0$.

Thus,

$$\sum_{k=0}^{n} p_k \cdot \left(uY^{\lambda}\right)^k = \sum_{k=0}^{n} \sum_{i\in\mathbb{N}} p_{k,i} Y^i \cdot \underbrace{\left(uY^{\lambda}\right)^k}_{\substack{=u^k\left(Y^{\lambda}\right)^k \\ =u^kY^{\lambda k} \\ =Y^{\lambda k}u^k}} \qquad \left(\text{since } p_k = \sum_{i\in\mathbb{N}} p_{k,i} Y^i\right)$$

$$= \sum_{k=0}^{n} \sum_{i\in\mathbb{N}} p_{k,i} \underbrace{Y^i \cdot Y^{\lambda k}}_{=Y^{i+\lambda k}} u^k$$

$$= \sum_{k=0}^{n} \sum_{i\in\mathbb{N}} p_{k,i} Y^{i+\lambda k} u^k = \sum_{k\in\{0,1,\ldots,n\}} \sum_{i\in\mathbb{N}} p_{k,i} Y^{i+\lambda k} u^k$$

$$= \sum_{(k,i)\in\{0,1,\ldots,n\}\times\mathbb{N}} p_{k,i} Y^{i+\lambda k} u^k = \sum_{\ell\in\mathbb{N}} \sum_{\substack{(k,i)\in\{0,1,\ldots,n\}\times\mathbb{N}; \\ i+\lambda k=\ell}} p_{k,i} \underbrace{Y^{i+\lambda k}}_{=Y^{\ell}} u^k$$

$$= \sum_{\ell\in\mathbb{N}} \sum_{\substack{(k,i)\in\{0,1,\ldots,n\}\times\mathbb{N}; \\ i+\lambda k=\ell}} p_{k,i} Y^{\ell} u^k = \sum_{\ell\in\mathbb{N}} \sum_{\substack{(k,i)\in\{0,1,\ldots,n\}\times\mathbb{N}; \\ i+\lambda k=\ell}} p_{k,i} u^k Y^{\ell}.$$

Hence, $\sum\limits_{k=0}^{n} p_k \cdot \left(uY^{\lambda}\right)^k = 0$ becomes $\sum\limits_{\ell\in\mathbb{N}} \sum\limits_{\substack{(k,i)\in\{0,1,\ldots,n\}\times\mathbb{N}; \\ i+\lambda k=\ell}} p_{k,i} u^k Y^{\ell} = 0$. In other words, the polynomial $\sum\limits_{\ell\in\mathbb{N}} \underbrace{\sum\limits_{\substack{(k,i)\in\{0,1,\ldots,n\}\times\mathbb{N}; \\ i+\lambda k=\ell}} p_{k,i} u^k}_{\in B} Y^{\ell} \in B[Y]$ equals 0. Hence, its coefficient before $Y^{\lambda n}$ equals 0 as well. But its coefficient before $Y^{\lambda n}$ is $\sum\limits_{\substack{(k,i)\in\{0,1,\ldots,n\}\times\mathbb{N}; \\ i+\lambda k=\lambda n}} p_{k,i} u^k$. Hence, $\sum\limits_{\substack{(k,i)\in\{0,1,\ldots,n\}\times\mathbb{N}; \\ i+\lambda k=\lambda n}} p_{k,i} u^k$ equals 0.

Thus,

$$0 = \sum_{\substack{(k,i)\in\{0,1,\ldots,n\}\times\mathbb{N}; \\ i+\lambda k=\lambda n}} p_{k,i} u^k = \sum_{k\in\{0,1,\ldots,n\}} \sum_{\substack{i\in\mathbb{N}; \\ i+\lambda k=\lambda n}} p_{k,i} u^k = \sum_{k\in\{0,1,\ldots,n\}} p_{k,\lambda(n-k)} u^k$$

$$\left(\begin{array}{c} \text{since } \{i\in\mathbb{N} \mid i+\lambda k=\lambda n\} = \{i\in\mathbb{N} \mid i=\lambda n-\lambda k\} \\ = \{i\in\mathbb{N} \mid i=\lambda(n-k)\} = \{\lambda(n-k)\} \text{ (because } \lambda(n-k)\in\mathbb{N}, \\ \text{since } k\in\{0,1,\ldots,n\} \text{ yields } n-k\in\mathbb{N} \text{ and we have } \lambda\in\mathbb{N}) \\ \text{yields } \sum\limits_{\substack{i\in\mathbb{N}; \\ i+\lambda k=\lambda n}} p_{k,i}u^k = \sum\limits_{i\in\{\lambda(n-k)\}} p_{k,i}u^k = p_{k,\lambda(n-k)}u^k \end{array}\right).$$

Note that

$$\sum_{i\in\mathbb{N}} p_{n,i} Y^i = p_n \qquad \left(\text{since } \sum_{i\in\mathbb{N}} p_{k,i} Y^i = p_k \text{ for every } k\in\{0,1,\ldots,n\}\right)$$

$$= 1 = 1 \cdot Y^0$$

in $A[Y]$, and thus the coefficient of the polynomial $\sum\limits_{i\in\mathbb{N}} p_{n,i} Y^i \in A[Y]$ before Y^0 is 1; but the coefficient of the polynomial $\sum\limits_{i\in\mathbb{N}} p_{n,i} Y^i \in A[Y]$ before Y^0 is $p_{n,0}$; hence, $p_{n,0} = 1$.

Define an $(n+1)$-tuple $(a_0, a_1, ..., a_n) \in A^{n+1}$ by $\left(a_k = p_{k,\lambda(n-k)} \text{ for every } k \in \{0, 1, ..., n\}\right)$. Then, $a_n = p_{n,\lambda(n-n)} = p_{n,\lambda\cdot 0} = p_{n,0} = 1$. Besides,

$$\sum_{k=0}^{n} a_k u^k = \sum_{k=0}^{n} p_{k,\lambda(n-k)} u^k = \sum_{k\in\{0,1,...,n\}} p_{k,\lambda(n-k)} u^k = 0.$$

Finally, $a_k = p_{k,\lambda(n-k)} \in I_{\lambda(n-k)} J_{n-k}$ (since $p_{k,i} \in I_i J_{n-k}$ for every $i \in \mathbb{N}$) for every $k \in \{0, 1, ..., n\}$. In other words, $a_i \in I_{\lambda(n-i)} J_{n-i}$ for every $i \in \{0, 1, ..., n\}$.

Altogether, we now know that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{\lambda(n-i)} J_{n-i} \text{ for every } i \in \{0, 1, ..., n\}.$$

Thus, by Definition 9 (applied to $(I_{\lambda\rho} J_\rho)_{\rho\in\mathbb{N}}$ instead of $(I_\rho)_{\rho\in\mathbb{N}}$), the element u is n-integral over $\left(A, (I_{\lambda\rho} J_\rho)_{\rho\in\mathbb{N}}\right)$. This proves Lemma $\mathcal{F}''$.

Combining Lemmata $\mathcal{E}''$ and $\mathcal{F}''$, we obtain that u is n-integral over $\left(A, (I_{\lambda\rho} J_\rho)_{\rho\in\mathbb{N}}\right)$ if and only if uY^λ is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$. This proves Theorem 15.

A particular case of Theorem 15:

Theorem 16. Let A and B be two rings such that $A \subseteq B$. Let $(I_\rho)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $n \in \mathbb{N}$. Let $u \in B$. Let $\lambda \in \mathbb{N}$.

We know that $(I_{\lambda\rho})_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A (according to Theorem 14).

Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ defined in Definition 8.

Then, the element u of B is n-integral over $\left(A, (I_{\lambda\rho})_{\rho\in\mathbb{N}}\right)$ if and only if the element uY^λ of the polynomial ring $B[Y]$ is n-integral over the ring $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$. (Here, $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq B[Y]$ because $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq A[Y]$ and we consider $A[Y]$ as a subring of $B[Y]$ as explained in Definition 7).

Proof of Theorem 16. Theorem 10 **(a)** states that $(A)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A.

We will abbreviate the ring $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ by $A_{[I]}$.

We have the following five equivalences:

- The element u of B is n-integral over $\left(A, (I_{\lambda\rho})_{\rho\in\mathbb{N}}\right)$ if and only if the element u of B is n-integral over $\left(A, (I_{\lambda\rho} A)_{\rho\in\mathbb{N}}\right)$ (since $I_{\lambda\rho} = I_{\lambda\rho} A$).

- The element u of B is n-integral over $\left(A, (I_{\lambda\rho} A)_{\rho\in\mathbb{N}}\right)$ if and only if the element uY^λ of the polynomial ring $B[Y]$ is n-integral over $\left(A_{[I]}, \left(AA_{[I]}\right)_{\tau\in\mathbb{N}}\right)$ (according to Theorem 15, applied to $(A)_{\rho\in\mathbb{N}}$ instead of $(J_\rho)_{\rho\in\mathbb{N}}$).

- The element uY^{λ} of the polynomial ring $B[Y]$ is n-integral over $\left(A_{[I]}, \left(AA_{[I]}\right)_{\tau\in\mathbb{N}}\right)$ if and only if the element uY^{λ} of the polynomial ring $B[Y]$ is n-integral over $\left(A_{[I]}, \left(A_{[I]}\right)_{\rho\in\mathbb{N}}\right)$ (since $\left(\underbrace{AA_{[I]}}_{=A_{[I]}}\right)_{\tau\in\mathbb{N}} = \left(A_{[I]}\right)_{\tau\in\mathbb{N}} = \left(A_{[I]}\right)_{\rho\in\mathbb{N}}$).

- The element uY^{λ} of the polynomial ring $B[Y]$ is n-integral over $\left(A_{[I]}, \left(A_{[I]}\right)_{\rho\in\mathbb{N}}\right)$ if and only if the element uY^{λ} of the polynomial ring $B[Y]$ is n-integral over $A_{[I]}$ (by Theorem 12, applied to $A_{[I]}$, $B[Y]$ and uY^{λ} instead of A, B and u).

- The element uY^{λ} of the polynomial ring $B[Y]$ is n-integral over $A_{[I]}$ if and only if the element uY^{λ} of the polynomial ring $B[Y]$ is n-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ (since $A_{[I]} = A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$).

Combining these five equivalences, we obtain that the element u of B is n-integral over $\left(A, (I_{\lambda\rho})_{\rho\in\mathbb{N}}\right)$ if and only if the element uY^{λ} of the polynomial ring $B[Y]$ is n-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$. This proves Theorem 16.

Finally we can generalize even Theorem 2:

Theorem 17. Let A and B be two rings such that $A \subseteq B$. Let $(I_\rho)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $n \in \mathbb{N}$. Let $v \in B$. Let a_0, a_1, ..., a_n be $n+1$ elements of A such that $\sum\limits_{i=0}^{n} a_i v^i = 0$ and $a_i \in I_{n-i}$ for every $i \in \{0, 1, ..., n\}$.

Let $k \in \{0, 1, ..., n\}$. We know that $\left(I_{(n-k)\rho}\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A (according to Theorem 14, applied to $\lambda = n-k$).

Then, $\sum\limits_{i=0}^{n-k} a_{i+k} v^i$ is n-integral over $\left(A, \left(I_{(n-k)\rho}\right)_{\rho\in\mathbb{N}}\right)$.

Proof of Theorem 17. Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ defined in Definition 8. We have $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq B[Y]$, because $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq A[Y]$ and we consider $A[Y]$ as a subring of $B[Y]$ as explained in Definition 7.

As usual, note that

$$\sum_{\ell\in\mathbb{N}} I_\ell Y^\ell = \sum_{i\in\mathbb{N}} I_i Y^i \qquad \text{(here we renamed } \ell \text{ as } i \text{ in the sum)}$$
$$= A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right].$$

In the ring $B[Y]$, we have

$$\sum_{i=0}^{n} a_i Y^{n-i} \underbrace{(vY)^i}_{=v^iY^i=Y^iv^i} = \sum_{i=0}^{n} a_i \underbrace{Y^{n-i}Y^i}_{=Y^n} v^i = Y^n \underbrace{\sum_{i=0}^{n} a_i v^i}_{=0} = 0.$$

Besides, $a_i Y^{n-i} \in A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ for every $i \in \{0,1,...,n\}$ (since $\underbrace{a_i}_{\in I_{n-i}} Y^{n-i} \in I_{n-i}Y^{n-i} \subseteq \sum_{\ell\in\mathbb{N}} I_\ell Y^\ell = A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$). Hence, Theorem 2 (applied to $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$, $B[Y]$, vY and $a_i Y^{n-i}$ instead of A, B, v and a_i) yields that $\sum\limits_{i=0}^{n-k} a_{i+k} Y^{n-(i+k)} (vY)^i$ is n-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$. Since

$$\sum_{i=0}^{n-k} a_{i+k} Y^{n-(i+k)} \underbrace{(vY)^i}_{=v^iY^i=Y^iv^i} = \sum_{i=0}^{n-k} a_{i+k} \underbrace{Y^{n-(i+k)}Y^i}_{=Y^{(n-(i+k))+i}=Y^{n-k}} v^i = \sum_{i=0}^{n-k} a_{i+k} v^i \cdot Y^{n-k},$$

this means that $\sum\limits_{i=0}^{n-k} a_{i+k} v^i \cdot Y^{n-k}$ is n-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$.

But Theorem 16 (applied to $u = \sum\limits_{i=0}^{n-k} a_{i+k} v^i$ and $\lambda = n-k$) yields that $\sum\limits_{i=0}^{n-k} a_{i+k} v^i$ is n-integral over $\left(A, \left(I_{(n-k)\rho}\right)_{\rho\in\mathbb{N}}\right)$ if and only if $\sum\limits_{i=0}^{n-k} a_{i+k} v^i \cdot Y^{n-k}$ is n-integral over the ring $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$. Since we know that $\sum\limits_{i=0}^{n-k} a_{i+k} v^i \cdot Y^{n-k}$ is n-integral over the ring $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$, this yields that $\sum\limits_{i=0}^{n-k} a_{i+k} v^i$ is n-integral over $\left(A, \left(I_{(n-k)\rho}\right)_{\rho\in\mathbb{N}}\right)$. This proves Theorem 17.

5. Generalizing a lemma by Lombardi

Now, we are going to generalize Theorem 2 from [3] (which is the main result of [3])[5]. First, a very technical lemma:

Lemma 18. Let A and B be two rings such that $A \subseteq B$. Let $x \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Let $u \in B$. Let $\mu \in \mathbb{N}$ and $\nu \in \mathbb{N}$. Assume that

$$u^n \in \left\langle u^0, u^1, ..., u^{n-1}\right\rangle_A \cdot \left\langle x^0, x^1, ..., x^\nu\right\rangle_A \tag{4}$$

and that

$$u^m x^\mu \in \left\langle u^0, u^1, ..., u^{m-1}\right\rangle_A \cdot \left\langle x^0, x^1, ..., x^\mu\right\rangle_A + \left\langle u^0, u^1, ..., u^m\right\rangle_A \cdot \left\langle x^0, x^1, ..., x^{\mu-1}\right\rangle_A. \tag{5}$$

Then, u is $(n\mu + m\nu)$-integral over A.

Before we prove this lemma, we recall a basic mathematical principle:

Principle of strong induction (form #1). Let $\mathfrak{A}(i)$ be an assertion for every $i \in \mathbb{N}$. If

every $I \in \mathbb{N}$ satisfying $(\mathfrak{A}(i)$ for every $i \in \mathbb{N}$ such that $i < I)$ satisfies $\mathfrak{A}(I)$,

then

every $i \in \mathbb{N}$ satisfies $\mathfrak{A}(i)$.

---

[5] *Caveat:* The notion "integral over (A, J) " defined in [3] has nothing to do with *our* notion "n-integral over $(A, (I_n)_{n\in\mathbb{N}})$ ".

By renaming i, I and $\mathfrak{A}$ as j, J and $\mathfrak{B}$, respectively, we can rewrite this principle as follows:

> **Principle of strong induction (form #2).** Let $\mathfrak{B}(j)$ be an assertion for every $j \in \mathbb{N}$. If
>
> every $J \in \mathbb{N}$ satisfying $(\mathfrak{B}(j)$ for every $j \in \mathbb{N}$ such that $j < J)$ satisfies $\mathfrak{B}(J)$,
>
> then
>
> $$\text{every } j \in \mathbb{N} \text{ satisfies } \mathfrak{B}(j).$$

Proof of Lemma 18. Let

$$S = (\{0,1,...,n-1\} \times \{0,1,...,\mu-1\}) \cup (\{0,1,...,m-1\} \times \{\mu,\mu+1,...,\mu+\nu-1\}).$$

Then, $(0,0) \in S$ [6]. Besides, $|S| = n\mu + m\nu$ [7]. Also,

$$j < \mu + \nu \text{ for every } (i,j) \in S \qquad (6)$$

[8].

---

[6]since

$$\begin{aligned}(0,0) &\in \{0,1,...,n-1\} \times \{0,1,...,\mu-1\} \\ &\subseteq (\{0,1,...,n-1\} \times \{0,1,...,\mu-1\}) \cup (\{0,1,...,m-1\} \times \{\mu,\mu+1,...,\mu+\nu-1\}) = S\end{aligned}$$

[7]since

$$\begin{aligned}&(\{0,1,...,n-1\} \times \{0,1,...,\mu-1\}) \cap (\{0,1,...,m-1\} \times \{\mu,\mu+1,...,\mu+\nu-1\}) \\ &= (\{0,1,...,n-1\} \cap \{0,1,...,m-1\}) \times \underbrace{(\{0,1,...,\mu-1\} \cap \{\mu,\mu+1,...,\mu+\nu-1\})}_{=\varnothing} \\ &= (\{0,1,...,n-1\} \cap \{0,1,...,m-1\}) \times \varnothing = \varnothing\end{aligned}$$

yields

$$\begin{aligned}&|(\{0,1,...,n-1\} \times \{0,1,...,\mu-1\}) \cup (\{0,1,...,m-1\} \times \{\mu,\mu+1,...,\mu+\nu-1\})| \\ &= \underbrace{|\{0,1,...,n-1\} \times \{0,1,...,\mu-1\}|}_{=|\{0,1,...,n-1\}|\cdot|\{0,1,...,\mu-1\}|} + \underbrace{|\{0,1,...,m-1\} \times \{\mu,\mu+1,...,\mu+\nu-1\}|}_{=|\{0,1,...,m-1\}|\cdot|\{\mu,\mu+1,...,\mu+\nu-1\}|} \\ &= \underbrace{|\{0,1,...,n-1\}|}_{=n} \cdot \underbrace{|\{0,1,...,\mu-1\}|}_{=\mu} + \underbrace{|\{0,1,...,m-1\}|}_{=m} \cdot \underbrace{|\{\mu,\mu+1,...,\mu+\nu-1\}|}_{=\nu} = n\mu + m\nu,\end{aligned}$$

so that

$$\begin{aligned}|S| &= |(\{0,1,...,n-1\} \times \{0,1,...,\mu-1\}) \cup (\{0,1,...,m-1\} \times \{\mu,\mu+1,...,\mu+\nu-1\})| \\ &= n\mu + m\nu\end{aligned}$$

[8]In fact,

$$\begin{aligned}S &= \left(\{0,1,...,n-1\} \times \underbrace{\{0,1,...,\mu-1\}}_{\substack{\subseteq\{0,1,...,\mu+\nu-1\}, \\ \text{since } \mu-1\leq\mu+\nu-1}}\right) \cup \left(\{0,1,...,m-1\} \times \underbrace{\{\mu,\mu+1,...,\mu+\nu-1\}}_{\substack{\subseteq\{0,1,...,\mu+\nu-1\}, \\ \text{since } \mu\geq 0}}\right) \\ &\subseteq (\{0,1,...,n-1\} \times \{0,1,...,\mu+\nu-1\}) \cup (\{0,1,...,m-1\} \times \{0,1,...,\mu+\nu-1\}) \\ &= (\{0,1,...,n-1\} \cup \{0,1,...,m-1\}) \times \{0,1,...,\mu+\nu-1\}.\end{aligned}$$

Let U be the A-submodule $\langle u^i x^j \mid (i,j) \in S \rangle_A$ of B. Then, U is an $(n\mu + m\nu)$-generated A-module (since $|S| = n\mu + m\nu$). Besides, clearly,

$$u^i x^j \in U \text{ for every } (i,j) \in S \tag{7}$$

(since $U = \langle u^i x^j \mid (i,j) \in S \rangle_A$). Thus, $u^0 x^0 \in U$ (by (7), applied to $(i,j) = (0,0)$), since $(0,0) \in S$. Since $\underbrace{u^0}_{=1} \underbrace{x^0}_{=1} = 1$, this becomes $1 \in U$.

Now, we will show that

$$\text{every } i \in \mathbb{N} \text{ and } j \in \mathbb{N} \text{ satisfying } j < \mu + \nu \text{ satisfy } u^i x^j \in U. \tag{8}$$

Proof of (8). For every $i \in \mathbb{N}$, define an assertion $\mathfrak{A}(i)$ by

$$\mathfrak{A}(i) = \left(\text{every } j \in \mathbb{N} \text{ satisfies } \left(\text{if } j < \mu + \nu, \text{ then } u^i x^j \in U\right)\right).$$

Let us now show that

$$\text{every } I \in \mathbb{N} \text{ satisfying } (\mathfrak{A}(i) \text{ for every } i \in \mathbb{N} \text{ such that } i < I) \text{ satisfies } \mathfrak{A}(I). \tag{9}$$

Proof of (9). Let $I \in \mathbb{N}$ be such that

$$(\mathfrak{A}(i) \text{ for every } i \in \mathbb{N} \text{ such that } i < I). \tag{10}$$

We must prove that $\mathfrak{A}(I)$ holds.

For every $j \in \mathbb{N}$, define an assertion $\mathfrak{B}(j)$ by

$$\mathfrak{B}(j) = \left(\text{if } j < \mu + \nu, \text{ then } u^I x^j \in U\right).$$

Let us now show that

$$\text{every } J \in \mathbb{N} \text{ satisfying } (\mathfrak{B}(j) \text{ for every } j \in \mathbb{N} \text{ such that } j < J) \text{ satisfies } \mathfrak{B}(J). \tag{11}$$

Proof of (11). Let $J \in \mathbb{N}$ be such that

$$(\mathfrak{B}(j) \text{ for every } j \in \mathbb{N} \text{ such that } j < J). \tag{12}$$

We must prove that $\mathfrak{B}(J)$ holds.

Assume that $J < \mu + \nu$. Then,

$$u^I x^j \in U \text{ for every } j \in \mathbb{N} \text{ such that } j < J \tag{13}$$

(since for every $j \in \mathbb{N}$ such that $j < J$, the assertion $\mathfrak{B}(j)$ holds (due to (12)), i. e., the assertion $\left(\text{if } j < \mu + \nu, \text{ then } u^I x^j \in U\right)$ holds, which yields $u^I x^j \in U$ (since $j < J$ and $J < \mu + \nu$ yield $j < \mu + \nu$)). Now,

$$\begin{aligned}
&\left\langle u^I \right\rangle_A \cdot \underbrace{\left\langle x^0, x^1, ..., x^{J-1} \right\rangle_A}_{=\langle x^j \mid j \in \{0,1,...,J-1\}\rangle_A} \\
&= \left\langle u^I \right\rangle_A \cdot \left\langle x^j \mid j \in \{0,1,...,J-1\} \right\rangle_A = \left\langle u^I x^j \mid j \in \{0,1,...,J-1\} \right\rangle_A \\
&= \left\{ \sum_{j \in \{0,1,...,J-1\}} a_j u^I x^j \mid (a_j)_{j \in \{0,1,...,J-1\}} \in A^{\{0,1,...,J-1\}} \right\} \subseteq U,
\end{aligned} \tag{14}$$

Hence, for every $(i,j) \in S$, we have $j \in \{0,1,...,\mu+\nu-1\}$ and thus $j < \mu + \nu$.

since $\sum\limits_{j\in\{0,1,...,J-1\}} a_j u^I x^j \in U$ for every $(a_j)_{j\in\{0,1,...,J-1\}} \in A^{\{0,1,...,J-1\}}$ (since $u^I x^j \in U$ for every $j \in \{0,1,...,J-1\}$ (by (13), since $j < J$), and since U is an A-module). Also,

$$u^i x^j \in U \text{ for every } i \in \mathbb{N} \text{ and } j \in \mathbb{N} \text{ such that } i < I \text{ and } j < \mu+\nu \qquad (15)$$

(since for every $i \in \mathbb{N}$ and $j \in \mathbb{N}$ such that $i < I$ and $j < \mu + \nu$, the assertion (if $j < \mu+\nu$, then $u^i x^j \in U$) holds (because (10) and $i < I$ yield $\mathfrak{A}(i)$), and thus $u^i x^j \in U$ (since $j < \mu+\nu$)). Now,

$$\begin{aligned}
&\underbrace{\left\langle u^0, u^1, ..., u^{I-1}\right\rangle_A}_{=\langle u^i \ \mid\ i\in\{0,1,...,I-1\}\rangle_A} \cdot \underbrace{\left\langle x^0, x^1, ..., x^{\mu+\nu-1}\right\rangle_A}_{=\langle x^j \ \mid\ j\in\{0,1,...,\mu+\nu-1\}\rangle_A} \\
&= \left\langle u^i \ \mid\ i \in \{0,1,...,I-1\}\right\rangle_A \cdot \left\langle x^j \ \mid\ j \in \{0,1,...,\mu+\nu-1\}\right\rangle_A \\
&= \left\langle u^i x^j \ \mid\ (i,j) \in \{0,1,...,I-1\} \times \{0,1,...,\mu+\nu-1\}\right\rangle_A \\
&= \left\{ \sum_{(i,j)\in\{0,1,...,I-1\}\times\{0,1,...,\mu+\nu-1\}} a_{i,j} u^i x^j \ \mid\ (a_{i,j})_{(i,j)\in\{0,1,...,I-1\}\times\{0,1,...,\mu+\nu-1\}} \in A^{\{0,1,...,I-1\}\times\{0,1,...,\mu+\nu-1\}} \right\} \\
&\subseteq U, \qquad (16)
\end{aligned}$$

because $\sum\limits_{(i,j)\in\{0,1,...,I-1\}\times\{0,1,...,\mu+\nu-1\}} a_{i,j} u^i x^j \in U$ for every $(a_{i,j})_{(i,j)\in\{0,1,...,I-1\}\times\{0,1,...,\mu+\nu-1\}} \in A^{\{0,1,...,I-1\}\times\{0,1,...,\mu+\nu-1\}}$ (since $u^i x^j \in U$ for every $(i,j) \in \{0,1,...,I-1\}\times\{0,1,...,\mu+\nu-1\}$ (by (15), since $i < I$ (because $i \in \{0,1,...,I-1\}$) and $j < \mu+\nu$ (because $j \in \{0,1,...,\mu+\nu-1\}$)), and since U is an A-module).

Note that $J < \mu+\nu$ yields $J \leq \mu+\nu-1$ (since J and $\mu+\nu$ are integers).

Trivially,

$$(I \geq m \ \wedge\ J \geq \mu) \ \vee\ (I < m \ \wedge\ J \geq \mu) \ \vee\ (I \geq n \ \wedge\ J < \mu) \ \vee\ (I < n \ \wedge\ J < \mu)$$

[9]. Hence, one of the following four cases must hold:

Case 1: We have $I \geq m \ \wedge\ J \geq \mu$.

Case 2: We have $I < m \ \wedge\ J \geq \mu$.

Case 3: We have $I \geq n \ \wedge\ J < \mu$.

Case 4: We have $I < n \ \wedge\ J < \mu$.

---

[9] since

$$\begin{aligned}
&\underbrace{(I \geq m \ \wedge\ J \geq \mu) \ \vee\ (I < m \ \wedge\ J \geq \mu)}_{\substack{= (I\geq m \ \vee\ I<m) \ \wedge\ (J\geq\mu) \\ = (J\geq\mu) \text{ (since } (I\geq m \ \vee\ I<m) \text{ is true)}}} \vee \underbrace{(I \geq n \ \wedge\ J < \mu) \ \vee\ (I < n \ \wedge\ J < \mu)}_{\substack{= (I\geq n \ \vee\ I<n) \ \wedge\ (J<\mu) \\ = (J<\mu) \text{ (since } (I\geq n \ \vee\ I<n) \text{ is true)}}} \\
&= (J \geq \mu) \ \vee\ (J < \mu) \ = \ \text{true}
\end{aligned}$$

In Case 1, we have $I - m \geq 0$ (since $I \geq m$) and $J - \mu \geq 0$ (since $J \geq \mu$), thus

$$\begin{aligned}
& \underbrace{u^I}_{=u^{I-m}u^m} \underbrace{x^J}_{=x^\mu x^{J-\mu}} \\
&= u^{I-m} \underbrace{u^m x^\mu}_{\substack{\in \left\langle u^0,u^1,...,u^{m-1}\right\rangle_A \cdot \left\langle x^0,x^1,...,x^\mu\right\rangle_A + \left\langle u^0,u^1,...,u^m\right\rangle_A \cdot \left\langle x^0,x^1,...,x^{\mu-1}\right\rangle_A \\ \text{(by (5))}}} x^{J-\mu} \\
&\in u^{I-m} \left(\left\langle u^0, u^1, ..., u^{m-1} \right\rangle_A \cdot \left\langle x^0, x^1, ..., x^\mu \right\rangle_A + \left\langle u^0, u^1, ..., u^m \right\rangle_A \cdot \left\langle x^0, x^1, ..., x^{\mu-1} \right\rangle_A \right) x^{J-\mu} \\
&= \underbrace{u^{I-m} \left\langle u^0, u^1, ..., u^{m-1} \right\rangle_A}_{\substack{=\left\langle u^{I-m}u^0, u^{I-m}u^1, ..., u^{I-m}u^{m-1}\right\rangle_A \\ =\left\langle u^{(I-m)+0}, u^{(I-m)+1}, ..., u^{(I-m)+(m-1)}\right\rangle_A \\ =\left\langle u^{I-m}, u^{I-m+1}, ..., u^{I-1}\right\rangle_A \\ \subseteq \left\langle u^0,u^1,...,u^{I-1}\right\rangle_A \text{ (since} \\ \{I-m, I-m+1, ..., I-1\} \subseteq \{0,1,...,I-1\}, \\ \text{since } I-m \geq 0)}} \cdot \underbrace{\left\langle x^0, x^1, ..., x^\mu \right\rangle_A x^{J-\mu}}_{\substack{=\left\langle x^0x^{J-\mu}, x^1x^{J-\mu}, ..., x^\mu x^{J-\mu}\right\rangle_A \\ =\left\langle x^{0+(J-\mu)}, x^{1+(J-\mu)}, ..., x^{\mu+(J-\mu)}\right\rangle_A \\ =\left\langle x^{J-\mu}, x^{J-\mu+1}, ..., x^J\right\rangle_A \\ \subseteq \left\langle x^0,x^1,...,x^{\mu+\nu-1}\right\rangle_A \text{ (since} \\ \{J-\mu, J-\mu+1, ..., J\} \subseteq \{0,1,...,\mu+\nu-1\}, \\ \text{since } J-\mu \geq 0 \text{ and } J \leq \mu+\nu-1)}} \\
&\quad + \underbrace{u^{I-m} \left\langle u^0, u^1, ..., u^m \right\rangle_A}_{\substack{=\left\langle u^{I-m}u^0, u^{I-m}u^1, ..., u^{I-m}u^m\right\rangle_A \\ =\left\langle u^{(I-m)+0}, u^{(I-m)+1}, ..., u^{(I-m)+m}\right\rangle_A \\ =\left\langle u^{I-m}, u^{I-m+1}, ..., u^I\right\rangle_A \\ \subseteq \left\langle u^0,u^1,...,u^I\right\rangle_A \text{ (since} \\ \{I-m, I-m+1, ..., I\} \subseteq \{0,1,...,I\}, \\ \text{since } I-m \geq 0)}} \cdot \underbrace{\left\langle x^0, x^1, ..., x^{\mu-1} \right\rangle_A x^{J-\mu}}_{\substack{=\left\langle x^0x^{J-\mu}, x^1x^{J-\mu}, ..., x^{\mu-1}x^{J-\mu}\right\rangle_A \\ =\left\langle x^{0+(J-\mu)}, x^{1+(J-\mu)}, ..., x^{(\mu-1)+(J-\mu)}\right\rangle_A \\ =\left\langle x^{J-\mu}, x^{J-\mu+1}, ..., x^{J-1}\right\rangle_A \\ \subseteq \left\langle x^0,x^1,...,x^{J-1}\right\rangle_A \text{ (since} \\ \{J-\mu, J-\mu+1, ..., J-1\} \subseteq \{0,1,...,J-1\}, \\ \text{since } J-\mu \geq 0)}} \\
&\subseteq \underbrace{\left\langle u^0, u^1, ..., u^{I-1} \right\rangle_A \cdot \left\langle x^0, x^1, ..., x^{\mu+\nu-1} \right\rangle_A}_{\subseteq U \text{ by (16)}} + \underbrace{\left\langle u^0, u^1, ..., u^I \right\rangle_A}_{=\left\langle u^0,u^1,...,u^{I-1}\right\rangle_A + \left\langle u^I\right\rangle_A} \cdot \left\langle x^0, x^1, ..., x^{J-1} \right\rangle_A \\
&\subseteq U + \underbrace{\left(\left\langle u^0, u^1, ..., u^{I-1} \right\rangle_A + \left\langle u^I \right\rangle_A \right) \cdot \left\langle x^0, x^1, ..., x^{J-1} \right\rangle_A}_{=\left\langle u^0,u^1,...,u^{I-1}\right\rangle_A \cdot \left\langle x^0,x^1,...,x^{J-1}\right\rangle_A + \left\langle u^I\right\rangle_A \cdot \left\langle x^0,x^1,...,x^{J-1}\right\rangle_A} \\
&= U + \left\langle u^0, u^1, ..., u^{I-1} \right\rangle_A \cdot \underbrace{\left\langle x^0, x^1, ..., x^{J-1} \right\rangle_A}_{\substack{\subseteq \left\langle x^0,x^1,...,x^{\mu+\nu-1}\right\rangle_A \text{ (since} \\ \{0,1,...,J-1\} \subseteq \{0,1,...,\mu+\nu-1\}, \\ \text{since } J-1 \leq J \leq \mu+\nu-1)}} + \left\langle u^I \right\rangle_A \cdot \left\langle x^0, x^1, ..., x^{J-1} \right\rangle_A \\
&\subseteq U + \underbrace{\left\langle u^0, u^1, ..., u^{I-1} \right\rangle_A \cdot \left\langle x^0, x^1, ..., x^{\mu+\nu-1} \right\rangle_A}_{\subseteq U \text{ by (16)}} + \underbrace{\left\langle u^I \right\rangle_A \cdot \left\langle x^0, x^1, ..., x^{J-1} \right\rangle_A}_{\subseteq U \text{ by (14)}} \\
&\subseteq U + U + U \subseteq U \qquad \text{(since } U \text{ is an } A\text{-module)}.
\end{aligned}$$

Thus, we have proved that $u^I x^J \in U$ holds in Case 1.

In Case 2, we have $I \in \{0, 1, ..., m-1\}$ (since $I < m$ and $I \in \mathbb{N}$) and $J \in \{\mu, \mu+1, ..., \mu+\nu-1\}$ (since $J \geq \mu$ and $J < \mu+\nu$), thus

$$\begin{aligned}
(I, J) &\in \{0, 1, ..., m-1\} \times \{\mu, \mu+1, ..., \mu+\nu-1\} \\
&\subseteq (\{0, 1, ..., n-1\} \times \{0, 1, ..., \mu-1\}) \cup (\{0, 1, ..., m-1\} \times \{\mu, \mu+1, ..., \mu+\nu-1\}) = S,
\end{aligned}$$

so that $u^I x^J \in U$ (by (7), applied to I and J instead of i and j). Thus, we have proved that $u^I x^J \in U$ holds in Case 2.

In Case 3, we have $I - n \geq 0$ (since $I \geq n$) and $J + \nu \leq \mu + \nu - 1$ (since $J < \mu$

yields $J+\nu<\mu+\nu$, and since $J+\nu$ and $\mu+\nu$ are integers), thus

$$\begin{aligned}
&\underbrace{u^I}_{=u^{I-n}u^n} x^J \\
&= u^{I-n} \underbrace{u^n}_{\substack{\in \left\langle u^0,u^1,...,u^{n-1}\right\rangle_A \cdot \left\langle x^0,x^1,...,x^\nu\right\rangle_A \\ \text{(by (4))}}} x^J \in \underbrace{u^{I-n}\left\langle u^0,u^1,...,u^{n-1}\right\rangle_A}_{\substack{=\left\langle u^{I-n}u^0,u^{I-n}u^1,...,u^{I-n}u^{n-1}\right\rangle_A \\ =\left\langle u^{(I-n)+0},u^{(I-n)+1},...,u^{(I-n)+(n-1)}\right\rangle_A \\ =\left\langle u^{I-n},u^{I-n+1},...,u^{I-1}\right\rangle_A \\ \subseteq\left\langle u^0,u^1,...,u^{I-1}\right\rangle_A \text{ (since} \\ \{I-n,I-n+1,...,I-1\}\subseteq\{0,1,...,I-1\}, \\ \text{since } I-n\geq 0)}} \cdot \underbrace{\left\langle x^0,x^1,...,x^\nu\right\rangle_A x^J}_{\substack{=\left\langle x^0x^J,x^1x^J,...,x^\nu x^J\right\rangle_A \\ =\left\langle x^{0+J},x^{1+J},...,x^{\nu+J}\right\rangle_A \\ =\left\langle x^J,x^{J+1},...,x^{J+\nu}\right\rangle_A \\ \subseteq\left\langle x^0,x^1,...,x^{\mu+\nu-1}\right\rangle_A \text{ (since} \\ \{J,J+1,...,J+\nu\}\subseteq\{0,1,...,\mu+\nu-1\}, \\ \text{since } J\geq 0 \text{ and } J+\nu\leq\mu+\nu-1)}} \\
&\subseteq \left\langle u^0,u^1,...,u^{I-1}\right\rangle_A \cdot \left\langle x^0,x^1,...,x^{\mu+\nu-1}\right\rangle_A \subseteq U \qquad \text{(by (16))}.
\end{aligned}$$

Thus, we have proved that $u^I x^J \in U$ holds in Case 3.

In Case 4, we have $I \in \{0,1,...,n-1\}$ (since $I<n$ and $I\in\mathbb{N}$) and $J \in \{0,1,...,\mu-1\}$ (since $J<\mu$ and $J\in\mathbb{N}$), thus

$$\begin{aligned}
&(I,J) \in \{0,1,...,n-1\} \times \{0,1,...,\mu-1\} \\
&\qquad \subseteq (\{0,1,...,n-1\} \times \{0,1,...,\mu-1\}) \cup (\{0,1,...,m-1\} \times \{\mu,\mu+1,...,\mu+\nu-1\}) = S,
\end{aligned}$$

so that $u^I x^J \in U$ (by (7), applied to I and J instead of i and j). Thus, we have proved that $u^I x^J \in U$ holds in Case 4.

Therefore, we have proved that $u^I x^J \in U$ holds in each of the four cases 1, 2, 3 and 4. Hence, $u^I x^J \in U$ always holds.

Hence, we have proved that if $J<\mu+\nu$, then $u^I x^J \in U$. In other words, we have proved the assertion $\mathfrak{B}(J)$ (because $\mathfrak{B}(J) = \left(\text{if } J<\mu+\nu\text{, then } u^I x^J \in U\right)$).

Thus, we have proved (11). Hence, the Principle of strong induction (form #2) yields that

$$\text{every } j\in\mathbb{N} \text{ satisfies } \mathfrak{B}(j).$$

In other words,

$$\text{every } j\in\mathbb{N} \text{ satisfies } \left(\text{if } j<\mu+\nu\text{, then } u^I x^j \in U\right).$$

Thus, the assertion $\mathfrak{A}(I)$ holds (because $\mathfrak{A}(I) = \left(\text{every } j\in\mathbb{N} \text{ satisfies } \left(\text{if } j<\mu+\nu\text{, then } u^I x^j \in U\right)\right)$).

Thus, we have proved (9). Hence, the Principle of strong induction (form #1) yields that

$$\text{every } i\in\mathbb{N} \text{ satisfies } \mathfrak{A}(i).$$

In other words,

$$\text{every } i\in\mathbb{N} \text{ satisfies } \left(\text{every } j\in\mathbb{N} \text{ satisfies } \left(\text{if } j<\mu+\nu\text{, then } u^i x^j \in U\right)\right)$$

(since $\mathfrak{A}(i) = (\text{every } j\in\mathbb{N} \text{ satisfies } (\text{if } j<\mu+\nu\text{, then } u^i x^j \in U))$). This is equivalent to (8). Thus, (8) is proven.

Now,

$$u\cdot u^i x^j \in U \text{ for every } (i,j)\in S, \tag{17}$$

because $\underbrace{u\cdot u^i}_{=u^{i+1}} x^j = u^{i+1}x^j \in U$ (by (8) (applied to $i+1$ instead of i), since $j<\mu+\nu$ (by (6))).

Now,

$$uU = u\left\langle u^i x^j \mid (i,j)\in S\right\rangle_A = \left\langle u\cdot u^i x^j \mid (i,j)\in S\right\rangle_A$$
$$= \left\{\sum_{(i,j)\in S} a_{i,j} u\cdot u^i x^j \mid (a_{i,j})_{(i,j)\in S}\in A^S\right\}\subseteq U,$$

because $\sum\limits_{(i,j)\in S} a_{i,j} u\cdot u^i x^j \in U$ for every $(a_{i,j})_{(i,j)\in S}\in A^S$ (since $\sum\limits_{(i,j)\in S} a_{i,j} \underbrace{u\cdot u^i x^j}_{\in U \text{ by (17)}} \in U$, because U is an A-module).

Altogether, U is an $(n\mu+m\nu)$-generated A-submodule of B such that $1\in U$ and $uU\subseteq U$. Thus, $u\in B$ satisfies Assertion $\mathcal{C}$ of Theorem 1 with n replaced by $n\mu+m\nu$. Hence, $u\in B$ satisfies the four equivalent assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of Theorem 1 with n replaced by $n\mu+m\nu$. Consequently, u is $(n\mu+m\nu)$-integral over A. This proves Lemma 18.

We record a weaker variant of Lemma 18:

Lemma 19. Let A and B be two rings such that $A\subseteq B$. Let $x\in B$ and $y\in B$ be such that $xy\in A$. Let $m\in\mathbb{N}$ and $n\in\mathbb{N}$. Let $u\in B$. Let $\mu\in\mathbb{N}$ and $\nu\in\mathbb{N}$. Assume that

$$u^n \in \left\langle u^0,u^1,...,u^{n-1}\right\rangle_A\cdot\left\langle x^0,x^1,...,x^\nu\right\rangle_A \tag{18}$$

and that

$$u^m \in \left\langle u^0,u^1,...,u^{m-1}\right\rangle_A\cdot\left\langle y^0,y^1,...,y^\mu\right\rangle_A+\left\langle u^0,u^1,...,u^{m}\right\rangle_A\cdot\left\langle y^1,y^2,...,y^\mu\right\rangle_A. \tag{19}$$

Then, u is $(n\mu+m\nu)$-integral over A.

Proof of Lemma 19. For every $i\in\{0,1,...,\mu\}$, we have $\mu\geq i$ and thus $\mu-i\geq 0$, so that

$$y^i \underbrace{x^\mu}_{=x^{\mu-i}x^i} = y^i x^{\mu-i}x^i = \underbrace{x^i y^i}_{\substack{=(xy)^i\in A,\\ \text{since } xy\in A}} x^{\mu-i}\in\left\langle x^{\mu-i}\right\rangle_A \tag{20}$$
$$\subseteq \left\langle x^0,x^1,...,x^\mu\right\rangle_A \tag{21}$$

(since $\{\mu-i\}\subseteq\{0,1,...,\mu\}$, because $\mu-i\in\{0,1,...,\mu\}$, since $i\in\{0,1,...,\mu\}$). Now,

$$\underbrace{\left\langle y^0,y^1,...,y^\mu\right\rangle_A}_{=\langle y^i \mid i\in\{0,1,...,\mu\}\rangle_A} x^\mu = \left\langle y^i \mid i\in\{0,1,...,\mu\}\right\rangle_A x^\mu = \left\langle y^i x^\mu \mid i\in\{0,1,...,\mu\}\right\rangle_A$$
$$= \left\{\sum_{i\in\{0,1,...,\mu\}} a_i y^i x^\mu \mid (a_i)_{i\in\{0,1,...,\mu\}}\in A^{\{0,1,...,\mu\}}\right\}\subseteq\left\langle x^0,x^1,...,x^\mu\right\rangle_A, \tag{22}$$

since $\sum\limits_{i\in\{0,1,...,\mu\}} a_i y^i x^\mu \in \langle x^0,x^1,...,x^\mu\rangle_A$ for every $(a_i)_{i\in\{0,1,...,\mu\}}\in A^{\{0,1,...,\mu\}}$ (since $\sum\limits_{i\in\{0,1,...,\mu\}} a_i \underbrace{y^i x^\mu}_{\substack{\in\left\langle x^0,x^1,...,x^\mu\right\rangle_A\\ \text{by (21)}}} \in$ $\langle x^0,x^1,...,x^\mu\rangle_A$, because $\langle x^0,x^1,...,x^\mu\rangle_A$ is an A-module).

Besides, for every $i \in \{1, 2, ..., \mu\}$, we have

$$\begin{aligned} y^i x^\mu &\in \left\langle x^{\mu-i} \right\rangle_A \qquad \text{(by (20), since } i \in \{1, 2, ..., \mu\} \text{ yields } i \in \{0, 1, ..., \mu\}\text{)} \\ &\subseteq \left\langle x^0, x^1, ..., x^{\mu-1} \right\rangle_A \end{aligned} \tag{23}$$

(since $\{\mu - i\} \subseteq \{0, 1, ..., \mu - 1\}$, because $\mu - i \in \{0, 1, ..., \mu - 1\}$, since $i \in \{1, 2, ..., \mu\}$). Now,

$$\begin{aligned} &\underbrace{\left\langle y^1, y^2, ..., y^\mu \right\rangle_A}_{=\langle y^i \ \mid \ i \in \{1,2,...,\mu\}\rangle_A} x^\mu = \left\langle y^i \ \mid \ i \in \{1, 2, ..., \mu\} \right\rangle_A x^\mu = \left\langle y^i x^\mu \ \mid \ i \in \{1, 2, ..., \mu\} \right\rangle_A \\ &= \left\{ \sum_{i \in \{1,2,...,\mu\}} a_i y^i x^\mu \ \mid \ (a_i)_{i \in \{1,2,...,\mu\}} \in A^{\{1,2,...,\mu\}} \right\} \subseteq \left\langle x^0, x^1, ..., x^{\mu-1} \right\rangle_A, \end{aligned} \tag{24}$$

since $\sum\limits_{i \in \{1,2,...,\mu\}} a_i y^i x^\mu \in \langle x^0, x^1, ..., x^{\mu-1} \rangle_A$ for every $(a_i)_{i \in \{1,2,...,\mu\}} \in A^{\{1,2,...,\mu\}}$ (since $\sum\limits_{i \in \{1,2,...,\mu\}} a_i \underbrace{y^i x^\mu}_{\substack{\in \left\langle x^0, x^1, ..., x^{\mu-1} \right\rangle_A \\ \text{by (23)}}} \in \langle x^0, x^1, ..., x^{\mu-1} \rangle_A$, because $\langle x^0, x^1, ..., x^{\mu-1} \rangle_A$ is an A-module).

Now, (19) yields

$$\begin{aligned} u^m x^\mu &\in \left(\left\langle u^0, u^1, ..., u^{m-1} \right\rangle_A \cdot \left\langle y^0, y^1, ..., y^\mu \right\rangle_A + \left\langle u^0, u^1, ..., u^m \right\rangle_A \cdot \left\langle y^1, y^2, ..., y^\mu \right\rangle_A \right) x^\mu \\ &= \left\langle u^0, u^1, ..., u^{m-1} \right\rangle_A \cdot \underbrace{\left\langle y^0, y^1, ..., y^\mu \right\rangle_A x^\mu}_{\substack{\subseteq \left\langle x^0, x^1, ..., x^\mu \right\rangle_A \\ \text{(by (22))}}} + \left\langle u^0, u^1, ..., u^m \right\rangle_A \cdot \underbrace{\left\langle y^1, y^2, ..., y^\mu \right\rangle_A x^\mu}_{\substack{\subseteq \left\langle x^0, x^1, ..., x^{\mu-1} \right\rangle_A \\ \text{(by (24))}}} \\ &\subseteq \left\langle u^0, u^1, ..., u^{m-1} \right\rangle_A \cdot \left\langle x^0, x^1, ..., x^\mu \right\rangle_A + \left\langle u^0, u^1, ..., u^m \right\rangle_A \cdot \left\langle x^0, x^1, ..., x^{\mu-1} \right\rangle_A. \end{aligned}$$

In other words, (5) holds. Also, (4) holds (because (18) holds, and because (4) is the same as (18)). Thus, Lemma 18 yields that u is $(n\mu + m\nu)$-integral over A. This proves Lemma 19.

Something trivial now:

Lemma 20. Let A and B be two rings such that $A \subseteq B$. Let $x \in B$. Let $n \in \mathbb{N}$. Let $u \in B$. Assume that u is n-integral over $A[x]$. Then, there exists some $\nu \in \mathbb{N}$ such that

$$u^n \in \left\langle u^0, u^1, ..., u^{n-1} \right\rangle_A \cdot \left\langle x^0, x^1, ..., x^\nu \right\rangle_A.$$

Proof of Lemma 20. There exists a monic polynomial $P \in (A[x])[X]$ with $\deg P = n$ and $P(u) = 0$ (since u is n-integral over $A[x]$). Since $P \in (A[x])[X]$ is a monic polynomial with $\deg P = n$, there exist elements α_0, α_1, ..., α_{n-1} of $A[x]$ such that $P(X) = X^n + \sum\limits_{i=0}^{n-1} \alpha_i X^i$. Thus, $P(u) = u^n + \sum\limits_{i=0}^{n-1} \alpha_i u^i$, so that $P(u) = 0$ becomes $u^n + \sum\limits_{i=0}^{n-1} \alpha_i u^i = 0$. Hence, $u^n = -\sum\limits_{i=0}^{n-1} \alpha_i u^i$.

For every $i \in \{0, 1, ..., n-1\}$, we have $\alpha_i \in A[x]$, and thus there exist some $\nu_i \in \mathbb{N}$ and some $(\beta_{i,0}, \beta_{i,1}, ..., \beta_{i,\nu_i}) \in A^{\nu_i+1}$ such that $\alpha_i = \sum\limits_{k=0}^{\nu_i} \beta_{i,k} x^k$. Hence, $\alpha_i \in \langle x^0, x^1, ..., x^{\nu_i} \rangle_A$ for every $i \in \{0, 1, ..., n-1\}$.

Let $\nu = \max\{\nu_0, \nu_1, ..., \nu_{n-1}\}$. Then, for every $i \in \{0, 1, ..., n-1\}$, we have $\nu_i \leq \nu$, hence $\{0, 1, ..., \nu_i\} \subseteq \{0, 1, ..., \nu\}$, thus $\langle x^0, x^1, ..., x^{\nu_i}\rangle_A \subseteq \langle x^0, x^1, ..., x^{\nu}\rangle_A$, and thus $\alpha_i \in \langle x^0, x^1, ..., x^{\nu}\rangle_A$ (since $\alpha_i \in \langle x^0, x^1, ..., x^{\nu_i}\rangle_A$). Therefore,

$$\begin{aligned} u^n &= -\sum_{i=0}^{n-1} \alpha_i u^i = -\sum_{i=0}^{n-1} \underbrace{u^i}_{\in \langle u^0, u^1, ..., u^{n-1}\rangle_A} \underbrace{\alpha_i}_{\in \langle x^0, x^1, ..., x^{\nu}\rangle_A} \\ &\in -\sum_{i=0}^{n-1} \left\langle u^0, u^1, ..., u^{n-1}\right\rangle_A \cdot \left\langle x^0, x^1, ..., x^{\nu}\right\rangle_A \subseteq \left\langle u^0, u^1, ..., u^{n-1}\right\rangle_A \cdot \left\langle x^0, x^1, ..., x^{\nu}\right\rangle_A \end{aligned}$$

(since $\langle u^0, u^1, ..., u^{n-1}\rangle_A \cdot \langle x^0, x^1, ..., x^{\nu}\rangle_A$ is an A-module). This proves Lemma 20.

A consequence of Lemmata 19 and 20 is the following theorem:

> **Theorem 21.** Let A and B be two rings such that $A \subseteq B$. Let $x \in B$ and $y \in B$ be such that $xy \in A$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Let $u \in B$. Assume that u is n-integral over $A[x]$, and that u is m-integral over $A[y]$. Then, there exists some $\lambda \in \mathbb{N}$ such that u is λ-integral over A.

Proof of Theorem 21. Since u is n-integral over $A[x]$, Lemma 20 yields that there exists some $\nu \in \mathbb{N}$ such that

$$u^n \in \left\langle u^0, u^1, ..., u^{n-1}\right\rangle_A \cdot \left\langle x^0, x^1, ..., x^{\nu}\right\rangle_A.$$

In other words, (18) holds.

Since u is m-integral over $A[y]$, Lemma 20 (with x, n and ν replaced by y, m and μ) yields that there exists some $\mu \in \mathbb{N}$ such that

$$u^m \in \left\langle u^0, u^1, ..., u^{m-1}\right\rangle_A \cdot \left\langle y^0, y^1, ..., y^{\mu}\right\rangle_A.$$

Hence,

$$u^m \in \left\langle u^0, u^1, ..., u^{m-1}\right\rangle_A \cdot \left\langle y^0, y^1, ..., y^{\mu}\right\rangle_A + \left\langle u^0, u^1, ..., u^{m}\right\rangle_A \cdot \left\langle y^1, y^2, ..., y^{\mu}\right\rangle_A$$

(because

$$\begin{aligned} &\left\langle u^0, u^1, ..., u^{m-1}\right\rangle_A \cdot \left\langle y^0, y^1, ..., y^{\mu}\right\rangle_A \\ &\subseteq \left\langle u^0, u^1, ..., u^{m-1}\right\rangle_A \cdot \left\langle y^0, y^1, ..., y^{\mu}\right\rangle_A + \left\langle u^0, u^1, ..., u^{m}\right\rangle_A \cdot \left\langle y^1, y^2, ..., y^{\mu}\right\rangle_A \end{aligned}$$

). In other words, (19) holds.

Since both (18) and (19) hold, Lemma 19 yields that u is $(n\mu + m\nu)$-integral over A. Thus, there exists some $\lambda \in \mathbb{N}$ such that u is λ-integral over A (namely, $\lambda = n\mu + m\nu$). This proves Theorem 21.

We record a generalization of Theorem 21 (which will turn out to be easily seen equivalent to Theorem 21):

> **Theorem 22.** Let A and B be two rings such that $A \subseteq B$. Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Let $u \in B$. Assume that u is n-integral over $A[x]$, and that u is m-integral over $A[y]$. Then, there exists some $\lambda \in \mathbb{N}$ such that u is λ-integral over $A[xy]$.

Proof of Theorem 22. Obviously, $A \subseteq A[xy]$ yields $A[x] \subseteq (A[xy])[x]$ and $A[y] \subseteq (A[xy])[y]$.

Since u is n-integral over $A[x]$, Lemma $\mathcal{I}$ (applied to B, $(A[xy])[x]$, $A[x]$ and u instead of B', A', A and v) yields that u is n-integral over $(A[xy])[x]$.

Since u is m-integral over $A[y]$, Lemma $\mathcal{I}$ (applied to B, $(A[xy])[y]$, $A[y]$, m and u instead of B', A', A, n and v) yields that u is m-integral over $(A[xy])[y]$.

Now, Theorem 21 (applied to $A[xy]$ instead of A) yields that there exists some $\lambda \in \mathbb{N}$ such that u is λ-integral over $A[xy]$ (because $xy \in A[xy]$, because u is n-integral over $(A[xy])[x]$, and because u is m-integral over $(A[xy])[y]$). This proves Theorem 22.

Theorem 22 has a "relative version":

> **Theorem 23.** Let A and B be two rings such that $A \subseteq B$. Let $(I_\rho)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $x \in B$ and $y \in B$.
>
> **(a)** Then, $(I_\rho A[x])_{\rho\in\mathbb{N}}$ is an ideal semifiltration of $A[x]$. Besides, $(I_\rho A[y])_{\rho\in\mathbb{N}}$ is an ideal semifiltration of $A[y]$. Besides, $(I_\rho A[xy])_{\rho\in\mathbb{N}}$ is an ideal semifiltration of $A[xy]$.
>
> **(b)** Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Let $u \in B$. Assume that u is n-integral over $\left(A[x], (I_\rho A[x])_{\rho\in\mathbb{N}}\right)$, and that u is m-integral over $\left(A[y], (I_\rho A[y])_{\rho\in\mathbb{N}}\right)$. Then, there exists some $\lambda \in \mathbb{N}$ such that u is λ-integral over $\left(A[xy], (I_\rho A[xy])_{\rho\in\mathbb{N}}\right)$.

Proof of Theorem 23. **(a)** Since $(I_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A, Lemma $\mathcal{J}$ (applied to $A[x]$ instead of A') yields that $(I_\rho A[x])_{\rho\in\mathbb{N}}$ is an ideal semifiltration of $A[x]$.

Since $(I_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A, Lemma $\mathcal{J}$ (applied to $A[y]$ instead of A') yields that $(I_\rho A[y])_{\rho\in\mathbb{N}}$ is an ideal semifiltration of $A[y]$.

Since $(I_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A, Lemma $\mathcal{J}$ (applied to $A[xy]$ instead of A') yields that $(I_\rho A[xy])_{\rho\in\mathbb{N}}$ is an ideal semifiltration of $A[xy]$.

Thus, Theorem 23 **(a)** is proven.

(b) We formulate a lemma:

Lemma $\mathcal{N}$: Let A, A' and B be three rings such that $A \subseteq A' \subseteq B$. Let $v \in B$. Let $(I_\rho)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$. We have $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq A[Y]$, and (as explained in Definition 7) we can identify the polynomial ring $A[Y]$ with a subring of $(A[v])[Y]$ (since $A \subseteq A[v]$). Hence, $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] \subseteq (A[v])[Y]$. On the other hand, $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right] \subseteq (A[v])[Y]$.

(a) We have

$$(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right] = \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[v].$$

(b) Let $u \in B$. Let $n \in \mathbb{N}$. Then, the element u of B is n-integral over $\left(A[v], (I_\rho A[v])_{\rho\in\mathbb{N}}\right)$ if and only if the element uY of the polynomial ring $B[Y]$ is n-integral over the ring $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[v]$.

Proof of Lemma $\mathcal{N}$: **(a)** We have proven Lemma $\mathcal{N}$ **(a)** during the proof of Theorem 9 **(b)**.

(b) Theorem 7 (applied to $A[v]$ and $(I_\rho A[v])_{\rho\in\mathbb{N}}$ instead of A and $(I_\rho)_{\rho\in\mathbb{N}}$) yields that the element u of B is n-integral over $\left(A[v], (I_\rho A[v])_{\rho\in\mathbb{N}}\right)$ if and only if the element uY of the polynomial ring $B[Y]$ is n-integral over the ring $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right]$. In other words, the element u of B is n-integral over $\left(A[v], (I_\rho A[v])_{\rho\in\mathbb{N}}\right)$ if and only if the element uY of the polynomial ring $B[Y]$ is n-integral over the ring $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[v]$ (because Lemma $\mathcal{N}$ **(a)** yields $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right] = \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[v]$). This proves Lemma $\mathcal{N}$ **(b)**.

Now, let us prove Theorem 23 **(b)**. In fact, for every $v \in B$, we can consider the polynomial ring $(A[v])[Y]$ and its $A[v]$-subalgebra $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right]$. We have $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right] \subseteq (A[v])[Y]$, and (as explained in Definition 7) we can identify the polynomial ring $(A[v])[Y]$ with a subring of $B[Y]$ (since $A[v] \subseteq B$). Hence, $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right] \subseteq B[Y]$.

Lemma $\mathcal{N}$ **(b)** (applied to x instead of v) yields that the element u of B is n-integral over $\left(A[x], (I_\rho A[x])_{\rho\in\mathbb{N}}\right)$ if and only if the element uY of the polynomial ring $B[Y]$ is n-integral over the ring $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[x]$. But since the element u of B is n-integral over $\left(A[x], (I_\rho A[x])_{\rho\in\mathbb{N}}\right)$, this yields that the element uY of the polynomial ring $B[Y]$ is n-integral over the ring $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[x]$.

Lemma $\mathcal{N}$ **(b)** (applied to y and m instead of v and n) yields that the element u of B is m-integral over $\left(A[y], (I_\rho A[y])_{\rho\in\mathbb{N}}\right)$ if and only if the element uY of the polynomial ring $B[Y]$ is m-integral over the ring $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[y]$. But since the element u of B is m-integral over $\left(A[y], (I_\rho A[y])_{\rho\in\mathbb{N}}\right)$, this yields that the element uY of the polynomial ring $B[Y]$ is m-integral over the ring $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[y]$.

Since uY is n-integral over the ring $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[x]$, and since uY is m-integral over the ring $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[y]$, Theorem 22 (applied to $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$, $B[Y]$ and uY instead of A, B and u) yields that there exists some $\lambda \in \mathbb{N}$ such that uY is λ-integral over $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[xy]$.

Lemma $\mathcal{N}$ **(b)** (applied to xy and λ instead of v and n) yields that the element u of B is λ-integral over $\left(A[xy], (I_\rho A[xy])_{\rho\in\mathbb{N}}\right)$ if and only if the element uY of the polynomial ring $B[Y]$ is λ-integral over the ring $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[xy]$. But since the element uY of the polynomial ring $B[Y]$ is λ-integral over the ring $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[xy]$, this yields that the element u of B is λ-integral over $\left(A[xy], (I_\rho A[xy])_{\rho\in\mathbb{N}}\right)$. Thus, Theorem 23 **(b)** is proven.

We notice that Corollary 3 can be derived from Lemma 18:

Second proof of Corollary 3. Let $n = 1$. Let $m = 1$. We have

$$u^n \in \left\langle u^0, u^1, ..., u^{n-1} \right\rangle_A \cdot \left\langle v^0, v^1, ..., v^\alpha \right\rangle_A$$

[10] and

$$u^m v^\beta \in \left\langle u^0, u^1, ..., u^{m-1} \right\rangle_A \cdot \left\langle v^0, v^1, ..., v^\beta \right\rangle_A + \left\langle u^0, u^1, ..., u^m \right\rangle_A \cdot \left\langle v^0, v^1, ..., v^{\beta-1} \right\rangle_A$$

[11]. Thus, Lemma 18 (applied to v, β and α instead of x, μ and ν) yields that u is $(n\beta + m\alpha)$-integral over A. This means that u is $(\alpha + \beta)$-integral over A (because $n\beta + m\alpha = 1\beta + 1\alpha = \beta + \alpha = \alpha + \beta$). This proves Corollary 3 once again.

In how far does this all generalize Theorem 2 from [3]? Actually, Theorem 2 from [3] can be easily reduced to the case when $J = 0$ (by passing from the ring A to its localization A_{1+J}), and in this case it easily follows from Lemma 18.

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[10] because

$$u^n = u^1 = u = \sum_{i=0}^{\alpha} \underbrace{s_i}_{\in A} v^i \in \left\langle v^0, v^1, ..., v^\alpha \right\rangle_A = A \cdot \left\langle v^0, v^1, ..., v^\alpha \right\rangle_A$$
$$= \left\langle u^0, u^1, ..., u^{n-1} \right\rangle_A \cdot \left\langle v^0, v^1, ..., v^\alpha \right\rangle_A$$

(since $A = \langle 1 \rangle_A = \left\langle u^0 \right\rangle_A = \left\langle u^0, u^1, ..., u^{n-1} \right\rangle_A$, as $n = 1$)

[11] because

$$\underbrace{u^m}_{=u^1=u} v^\beta = uv^\beta = \sum_{i=0}^{\beta} t_i v^{\beta-i} = \sum_{i=0}^{\beta} t_{\beta-i} v^{\beta-(\beta-i)} \qquad \text{(here we substituted } \beta - i \text{ for } i \text{ in the sum)}$$
$$= \sum_{i=0}^{\beta} \underbrace{t_{\beta-i}}_{\in A} v^i \in \left\langle v^0, v^1, ..., v^\beta \right\rangle_A = A \cdot \left\langle v^0, v^1, ..., v^\beta \right\rangle_A$$
$$= \left\langle u^0, u^1, ..., u^{m-1} \right\rangle_A \cdot \left\langle v^0, v^1, ..., v^\beta \right\rangle_A$$

(since $A = \langle 1 \rangle_A = \left\langle u^0 \right\rangle_A = \left\langle u^0, u^1, ..., u^{m-1} \right\rangle_A$, as $m = 1$) and

$$\left\langle u^0, u^1, ..., u^{m-1} \right\rangle_A \cdot \left\langle v^0, v^1, ..., v^\beta \right\rangle_A$$
$$\subseteq \left\langle u^0, u^1, ..., u^{m-1} \right\rangle_A \cdot \left\langle v^0, v^1, ..., v^\beta \right\rangle_A + \left\langle u^0, u^1, ..., u^m \right\rangle_A \cdot \left\langle v^0, v^1, ..., v^{\beta-1} \right\rangle_A$$