# From the Complete Quadrilateral to the Droz-Farny Theorem

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## §1. The extended Steiner-Miquel Theorem

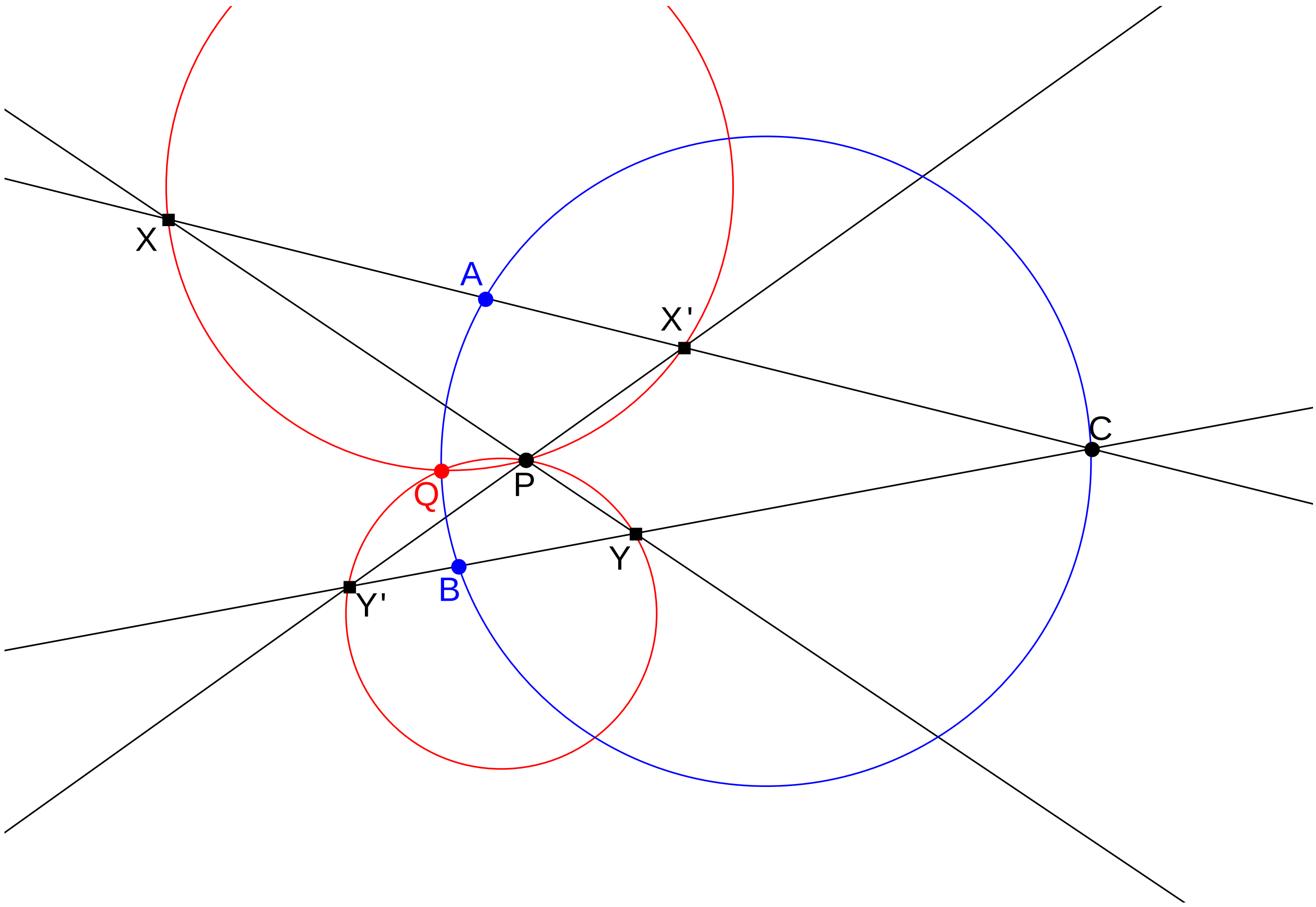


Fig. 1

In the following, the circle through three certain points $P_1,\ P_2,\ P_3$ will be abbreviated as "circle $P_1P_2P_3$".

A phrasing like "the point of intersection of the circles $PXX'$ and $PYY'$ different from $P$" should be understood as follows: If the two circles have two intersections, we take the one different from $P$; if the two circles touch, we take $P$.

Now we begin with our observations.

Regard four points $X,\ X',\ Y,\ Y'$. Let the lines $XX'$ and $YY'$ meet at $C$, and the lines $XY$ and $X'Y'$ meet at $P$.

We shall prove:

**Theorem 1**. Let $k$ be any real number, and $A,\ B$ two points on the segments $XX',\ YY'$, respectively, satisfying

$$\frac{XA}{AX'} = \frac{YB}{BY'} = k$$

(where segments are directed). Then:

**a**) The point of intersection $Q$ of the circles $PXX'$ and $PYY'$ different from $P$ lies on the circle $CAB$. (Fig. 1.)

**b**) The point $Q$ also lies on the circles $CXY$ and $CX'Y'$. (Fig. 2.)

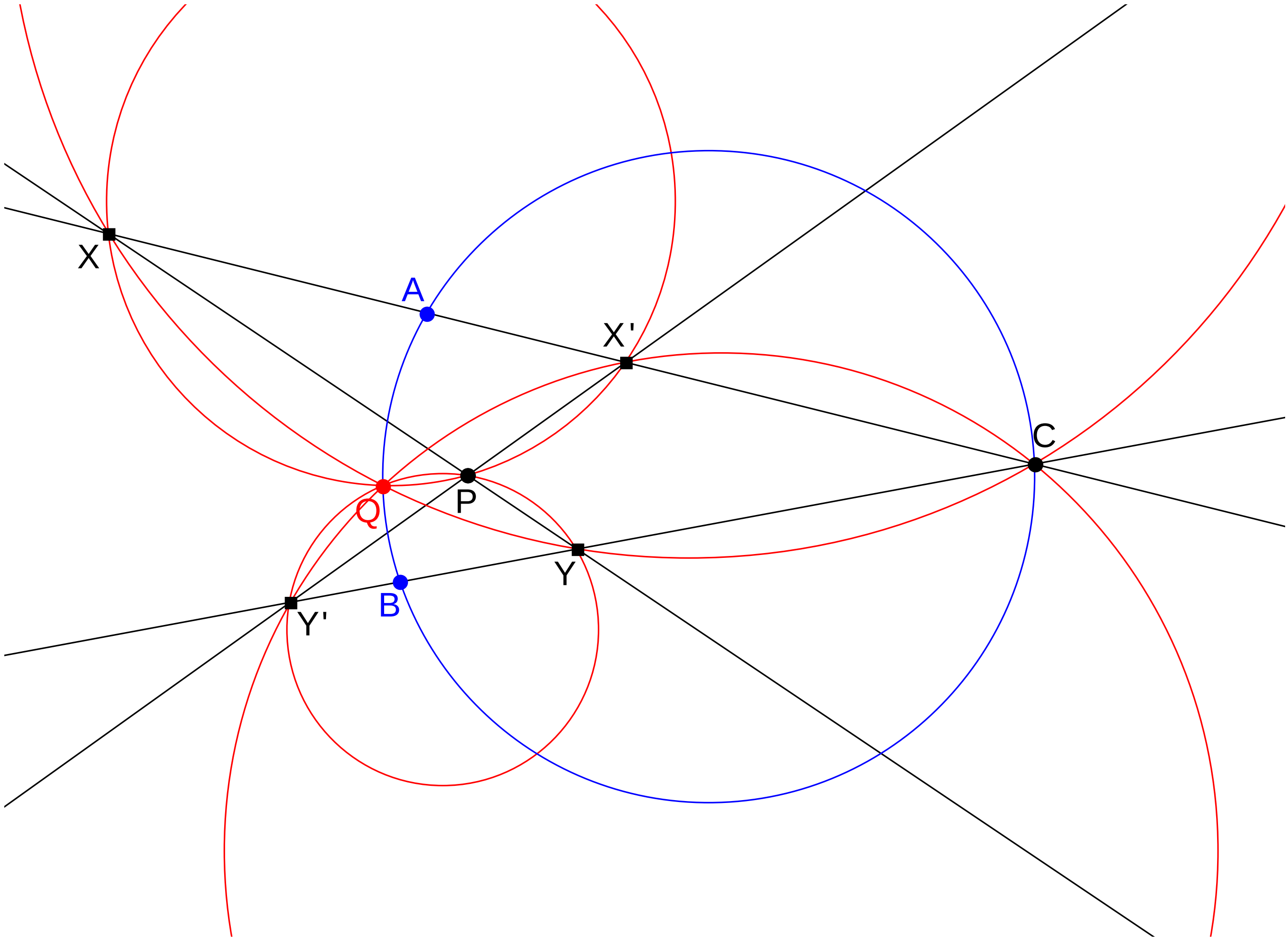


Fig. 2

*Proof of Theorem* 1. See Fig. 3. Since the point $Q$ lies on the circles $PXX'$ and $PYY'$, we have $\measuredangle QYY' = \measuredangle QPY'$ and $\measuredangle QXX' = 180° - \measuredangle QPX'$, hence $\measuredangle QXX' = \measuredangle QPY'$. This gives $\measuredangle QXX' = \measuredangle QYY'$. Similarly, $\measuredangle QX'X = \measuredangle QY'Y$. Thus, the triangles $QXX'$ and $QYY'$ are similar. But from

$$\frac{XA}{AX'} = \frac{YB}{BY'},$$

we see that the points $A$ and $B$ are *corresponding* points in these two triangles; hence, for instance, we get $\measuredangle QAX = \measuredangle QBY$. Consequently, $\measuredangle QAC = 180° - \measuredangle QAX = 180° - \measuredangle QBY = 180° - \measuredangle QBC$. This proves the point $Q$ to lie on the circle $CAB$. Theorem 1 **a**) is verified.

Now vary the parameter $k$. The point $Q$ remains fixed, being defined indepently of $k$. For $k = 0$, we have $A = X$ and $B = Y$; then Theorem 1 **a**) yields that the point $Q$ will lie on the circle $CXY$. For the limiting case $k = \infty$, we have $A = X'$ and $B = Y'$; then Theorem 1 **a**) yields that the point $Q$ will lie on the circle $CX'Y'$. Hence Theorem 1 **b**) is proven.

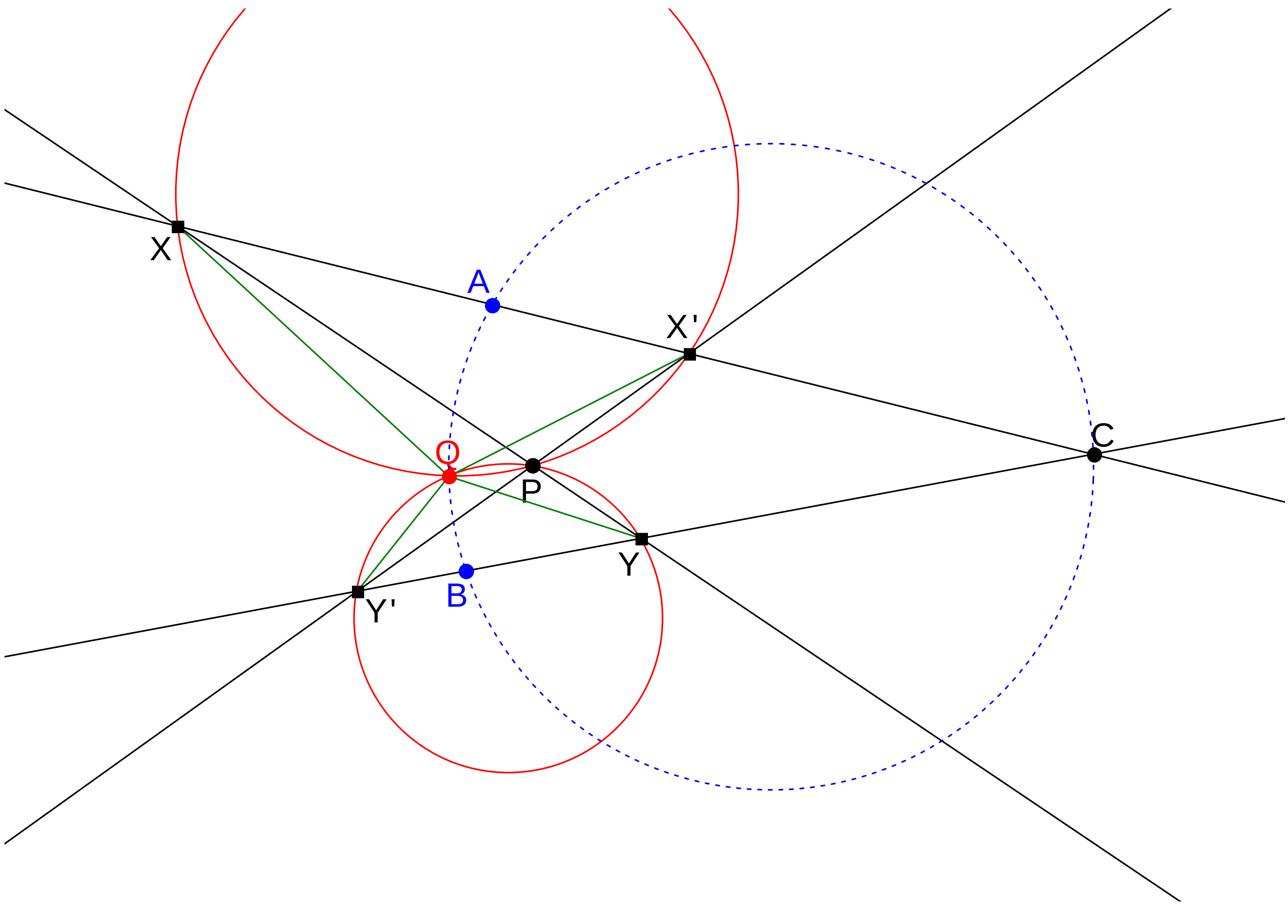


Fig. 3

The elegant proof of Theorem 1 **a**) was given by Nikolaos Dergiades. See Hyacinthos messages #7377 and #7378.

We can rewrite Theorem 1 **b**) thus: The circles $PXX'$, $PYY'$, $CXY$, $CX'Y'$ have a common point. This theorem can be stated in a very nice way:

**Theorem 2**. Four lines form four triangles; the circumcircles of these triangles have a common point.

Or, in a stronger variant: If we take two of the four triangles, then the intersection of the circumcircles of these triangles different from their common vertex lies on the circumcircles of the two other triangles.

In different sources, this result is called Clifford theorem, Steiner theorem, Miquel theorem or Steiner-Miquel theorem, to mention the most customary names only. (Fig. 4.) Sensibly, Theorem 1 can be called **extended Steiner-Miquel theorem**.

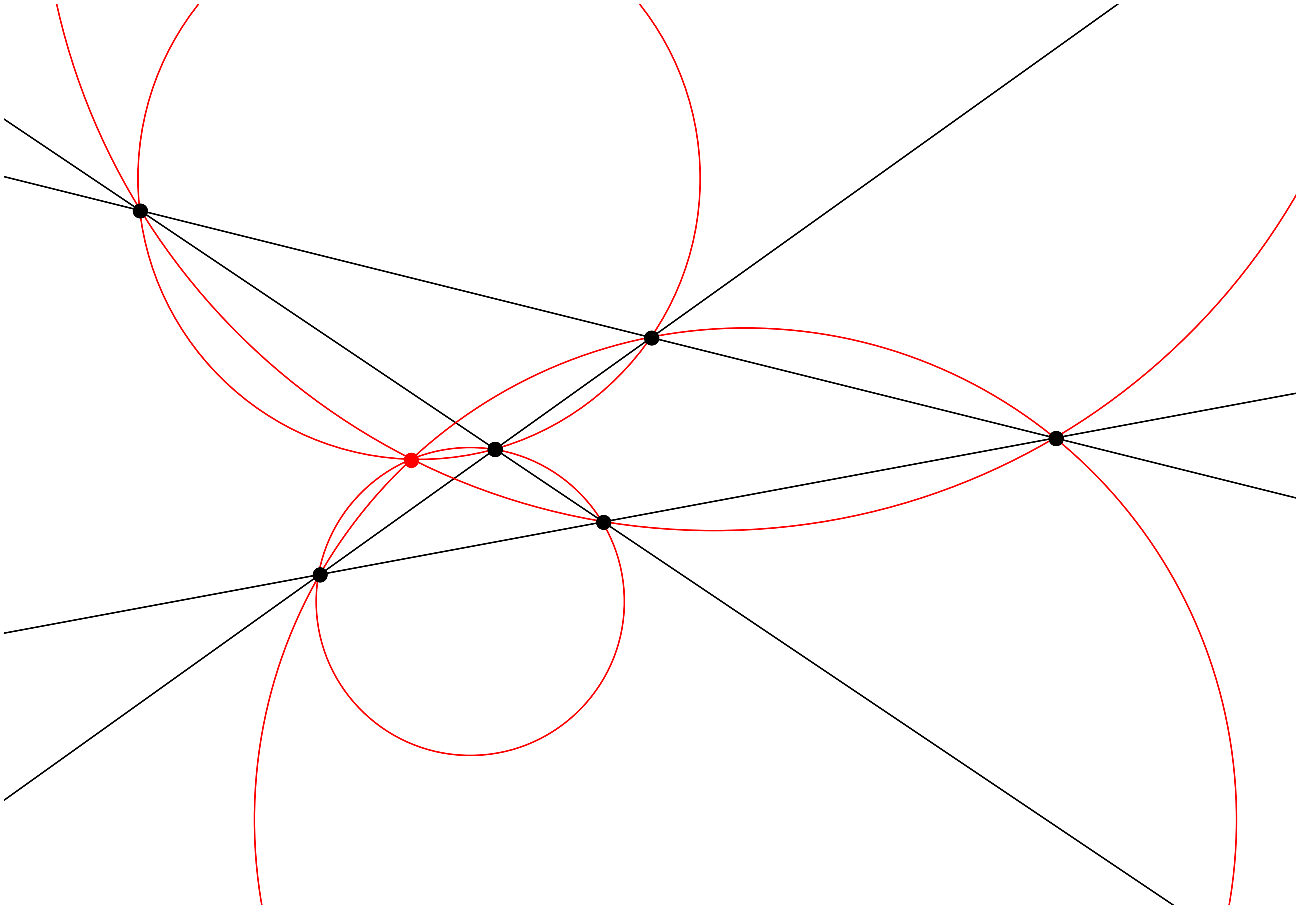

Fig. 4

## §2. An Application to Triangles

Now we are drawing a corrolary from Theorem 1:

**Theorem 3**. Let $\Delta ABC$ be a triangle. A line $g$ meets the sidelines $BC,\ CA,\ AB$ at the points $A',\ B',\ C';$ another line $g'$ meets the sidelines $BC,\ CA,\ AB$ at the points $A'',\ B'',\ C''$. Further, let $P$ be the point of intersection of the lines $g$ and $g'$. Assume that the equations

$$\frac{A'B}{BA''} = \frac{B'A}{AB''}; \qquad \frac{B'C}{CB''} = \frac{C'B}{BC''}; \qquad \frac{C'A}{AC''} = \frac{A'C}{CA''}$$

hold. Then the circles $PA'A'',\ PB'B'',\ PC'C''$ have a point of intersection $Q$ different from $P,$ and this point of intersection lies on the circumcircle of triangle $ABC$.

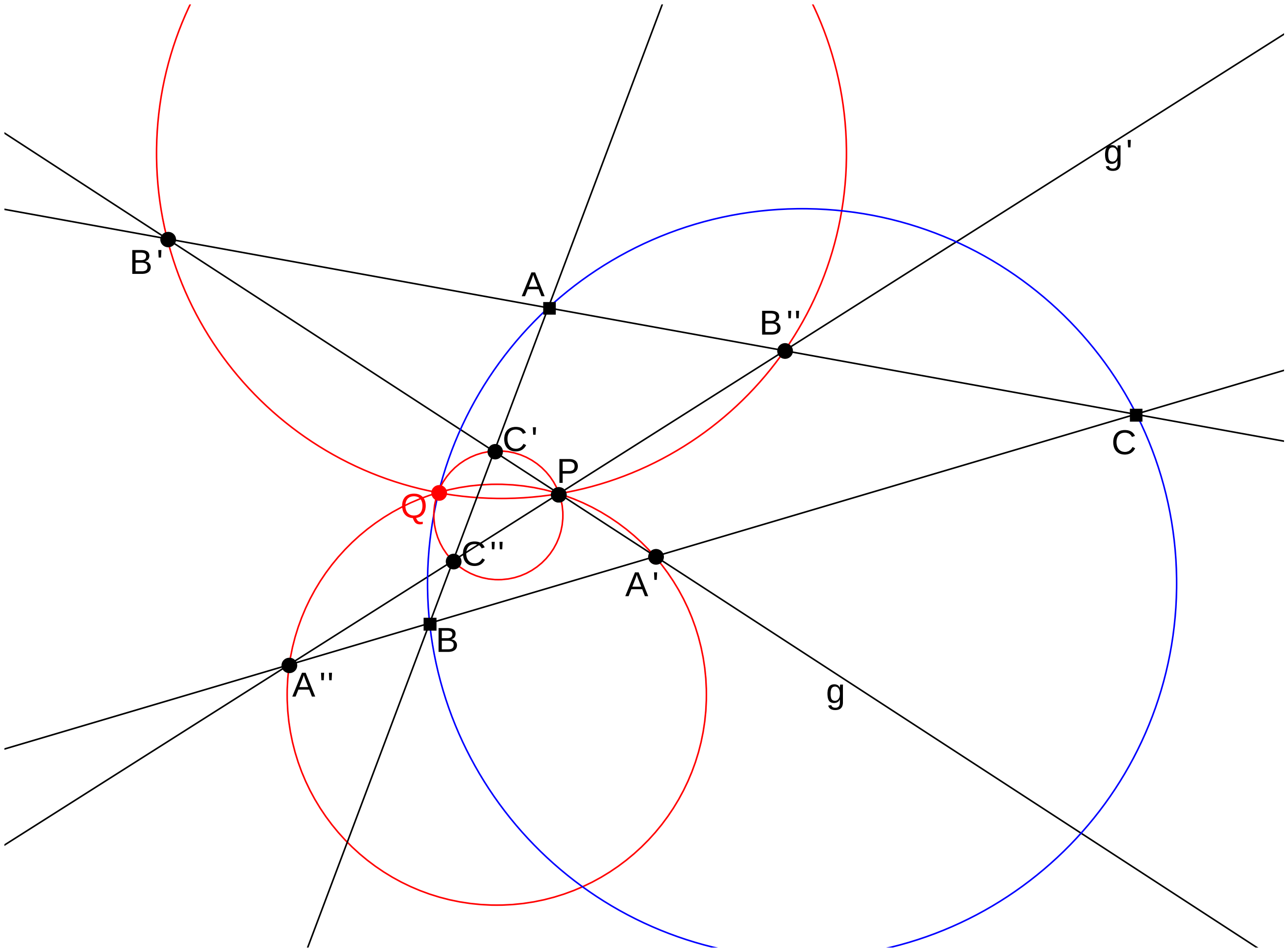


Fig. 5

*Proof*. Apply Theorem 1 to the points $A'$, $A''$, $B'$, $B''$ keeping in view that the points $A$, $B$, $C$, $P$, $A'$, $A''$, $B'$, $B''$ will take the part of the points $A$, $B$, $C$, $P$, $Y$, $Y'$, $X$, $X'$. From

$$\frac{A'B}{BA''} = \frac{B'A}{AB''}$$

it follows that the intersection $Q$ of the circles $PA'A''$ and $PB'B''$ different from $P$ lies on the circle $CAB$ and on the circles $CA'B'$ and $CA''B''$. I. e., the intersection $Q$ of the circles $CA'B'$ and $CAB$ different from $C$ is the intersection of the circles $PA'A''$ and $PB'B''$ different from $P$. Analogously, the intersection $Q'$ of the circles $ABC$ and $AB'C'$ different from $A$ is the intersection of the circles $PB'B''$ and $PC'C''$ different from $P$. But application of Theorem 2 on the lines $BC$, $CA$, $AB$, $g$ shows that the circles $AB'C'$, $BC'A'$, $CA'B'$, $ABC$ have a common point. Thus, the intersection $Q$ of the circles $CA'B'$ and $CAB$ different from $C$ coincides with the intersection $Q'$ of the circles $ABC$ and $AB'C'$ different from $A$. This point $Q = Q'$ is then the intersection of the circles $PA'A''$, $PB'B''$, $PC'C''$ different from $P$. Hence, we can state that the intersection of the circles $PA'A''$, $PB'B''$, $PC'C''$ different from $P$ lies on the circle $ABC$, i. e. on the circumcircle of triangle $ABC$. Theorem 3 is proven.

Now we inquire if all three of the equations

$$\frac{A'B}{BA''} = \frac{B'A}{AB''}; \qquad \frac{B'C}{CB''} = \frac{C'B}{BC''}; \qquad \frac{C'A}{AC''} = \frac{A'C}{CA''}$$

are necessary to ensure that the assertion of Theorem 3 holds. No, it turns out: If one of the three equation holds, the other two follow. In other words, the three equations are equivalent.

*Proof*. We will only show that

the equation $\frac{A'B}{BA''} = \frac{B'A}{AB''}$ entails $\frac{B'C}{CB''} = \frac{C'B}{BC''}$.

Analogous statements will follow by cyclic or symmetric permutation.

We will use directed segments; hereby, we direct the sidelines of triangle $ABC$ in such a way that $BC > 0$, $CA > 0$, $AB > 0$ (and hence $CB < 0$, $AC < 0$, $BA < 0$). We denote $a = BC$, $b = CA$, $c = AB$.

The Menelaos theorem for triangle $CA'B'$ and the collinear points $A$, $C'$, $B$ gives

$$\frac{A'B}{BC} \bullet \frac{CA}{AB'} \bullet \frac{B'C'}{C'A'} = -1.$$

We infer that

$$\frac{B'C'}{C'A'} = -\frac{BC}{A'B} \bullet \frac{AB'}{CA} = -\frac{a}{A'B} \bullet \frac{AB'}{b} = -\frac{AB'}{A'B} \bullet \frac{a}{b} = \frac{B'A}{A'B} \bullet \frac{a}{b}.$$

But analogously,

$$\frac{B''C''}{C''A''} = \frac{B''A}{A''B} \bullet \frac{a}{b}, \qquad \text{hence} \qquad \frac{B''C''}{C''A''} = \frac{AB''}{BA''} \bullet \frac{a}{b}.$$

We resume:

$$\frac{B'C'}{C'A'} = \frac{B'A}{A'B} \bullet \frac{a}{b} \qquad \text{and} \qquad \frac{B''C''}{C''A''} = \frac{AB''}{BA''} \bullet \frac{a}{b}. \tag{1}$$

Analogously,

$$\frac{C'A'}{A'B'} = \frac{C'B}{B'C} \bullet \frac{b}{c} \qquad \text{and} \qquad \frac{C''A''}{A''B''} = \frac{BC''}{CB''} \bullet \frac{b}{c}. \tag{2}$$

But $A'B : BA'' = B'A : AB''$ yields $B'A : A'B = AB'' : BA''$. Substitution in (1) gives

$$\frac{B'C'}{C'A'} = \frac{B''C''}{C''A''}.$$

But for the collinear points $A'$, $B'$, $C'$ and for the collinear points $A''$, $B''$, $C''$, we have

$$\frac{C'A'}{A'B'} = -\frac{C'A'}{B'A'} = -1 : \frac{B'A'}{C'A'} = -1 : \frac{B'C' + C'A'}{C'A'} = -1 : \left(\frac{B'C'}{C'A'} + 1\right) \qquad \text{and}$$
$$\frac{C''A''}{A''B''} = -\frac{C''A''}{B''A''} = -1 : \frac{B''A''}{C''A''} = -1 : \frac{B''C'' + C''A''}{C''A''} = -1 : \left(\frac{B''C''}{C''A''} + 1\right).$$

From

$$\frac{B'C'}{C'A'} = \frac{B''C''}{C''A''}$$

it hence follows

$$\frac{C'A'}{A'B'} = \frac{C''A''}{A''B''}.$$

Substitution in (2) shows

$$\frac{C'B}{B'C} \bullet \frac{b}{c} = \frac{BC''}{CB''} \bullet \frac{b}{c},$$

thus $C'B : B'C = BC'' : CB''$, and hence $B'C : CB'' = C'B : BC''$, qed.

Note that we have incidentally established the relations $B'C' : C'A' = B''C'' : C''A''$ and $C'A' : A'B' = C''A'' : A''B''$. We can express them as a double-ratio:

$B'C' : C'A' : A'B' = B''C'' : C''A'' : A''B''$. It follows:

**Theorem 4**. Under the conditions of Theorem 3, the relation $B'C' : C'A' : A'B' = B''C'' : C''A'' : A''B''$ holds.

## §3. The Droz-Farny Theorem

We conclude with a rather difficult theorem:

**Theorem 5**. Let $g$ and $g'$ be two mutually orthogonal lines through the orthocenter $H$ of a triangle $ABC$. The line $g$ meets the sidelines $BC$, $CA$, $AB$ at the points $A'$, $B'$, $C'$; the line $g'$ meets the sidelines $BC$, $CA$, $AB$ at the points $A''$, $B''$, $C''$.

**a**) The circles with diameters $A'A''$, $B'B''$, $C'C''$ pass through the point $H$.

**b**) These circles have a common point $Q$ different from $H$; this point $Q$ lies on the circumcircle of triangle $ABC$.

**c**) The circles with diameters $A'A''$, $B'B''$, $C'C''$ are coaxal.

**d**) The midpoints of the segments $A'A''$, $B'B''$, $C'C''$ lie on one line.

**e**) The equation $B'C' : C'A' : A'B' = B''C'' : C''A'' : A''B''$ holds.

Part **d**) of this theorem is called **Droz-Farny theorem**, being ascribed to A. Droz-Farny in [1], page 72 without further references. Part **e**) is due to Floor van Lamoen, Hyacinthos message #6144. More about the Droz-Farny configuration can be found in the Hyacinthos messages #6157 by Jean-Pierre Ehrmann and #6245 by me. A very different proof of part **d**) was found by Nick Reingold: see Hyacinthos messages #7383 and #7384.

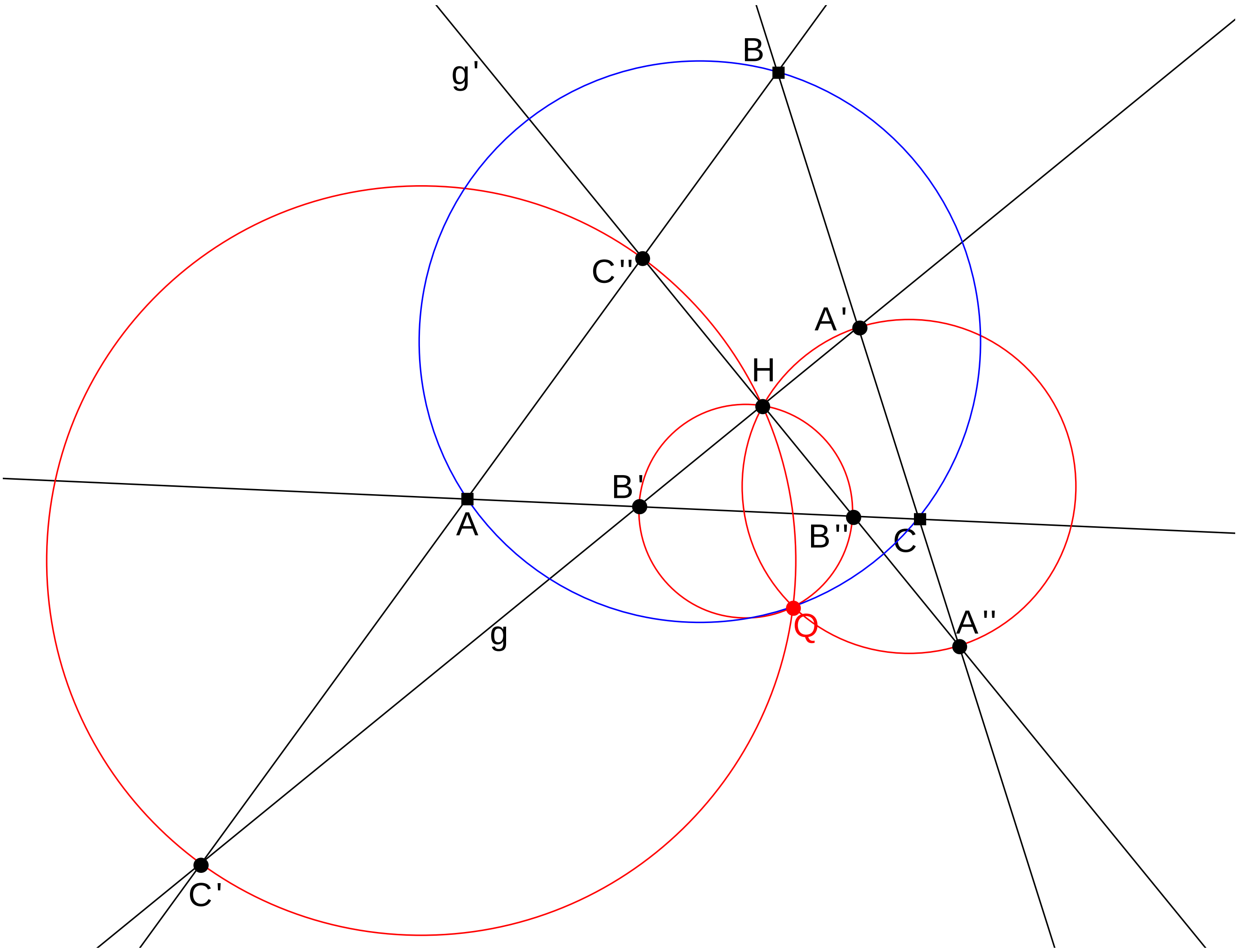

Fig. 6

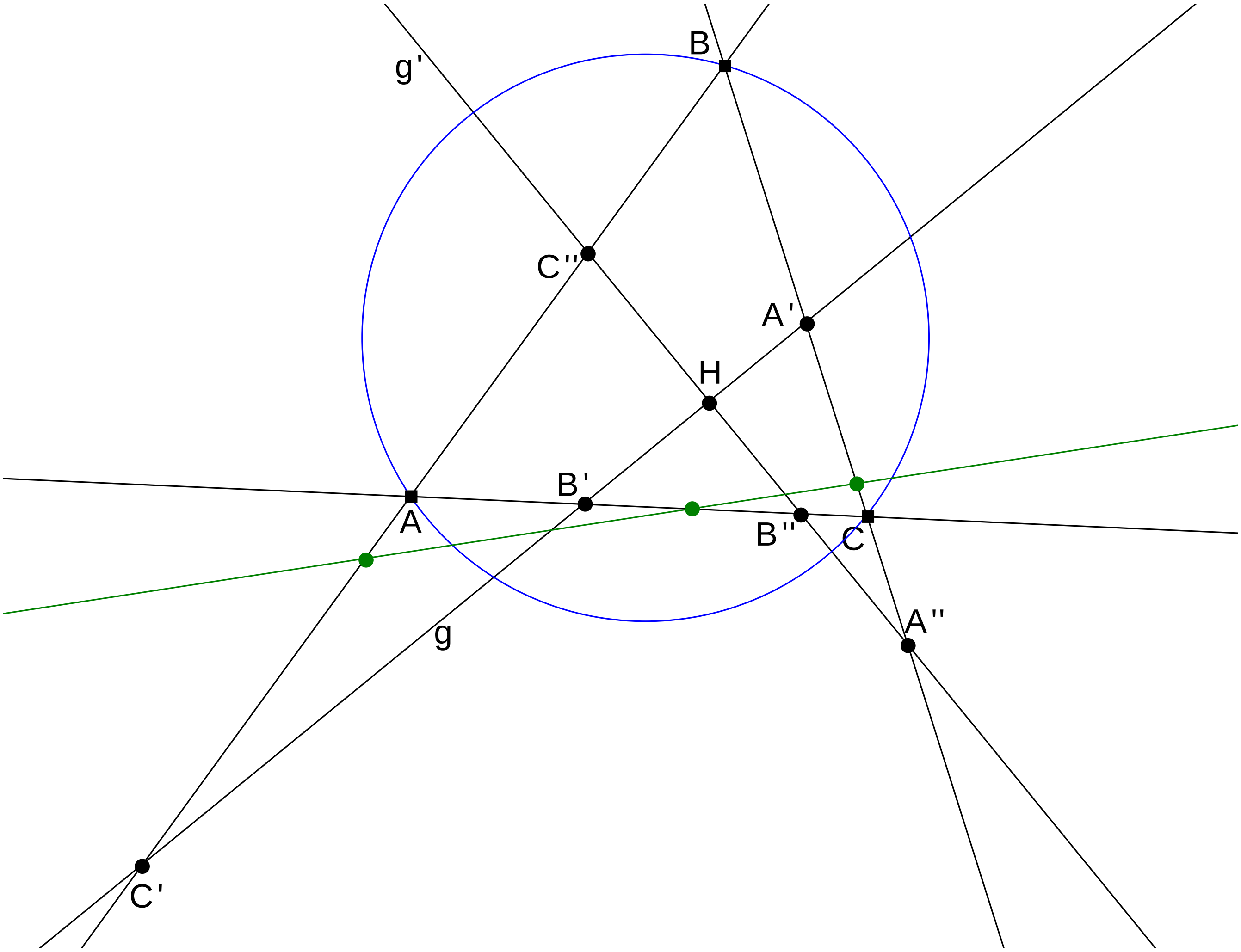


Fig. 7

The following *proof of Theorem* 5 is probably new.

At first, **a**) is virtually trivial: Since the lines $g$ and $g'$ are perpendicular, $\measuredangle A'HA'' = 90°$, $\measuredangle B'HB'' = 90°$ and $\measuredangle C'HC'' = 90°$, and thus the circles with diameters $A'A''$, $B'B''$, $C'C''$ pass through $H$.

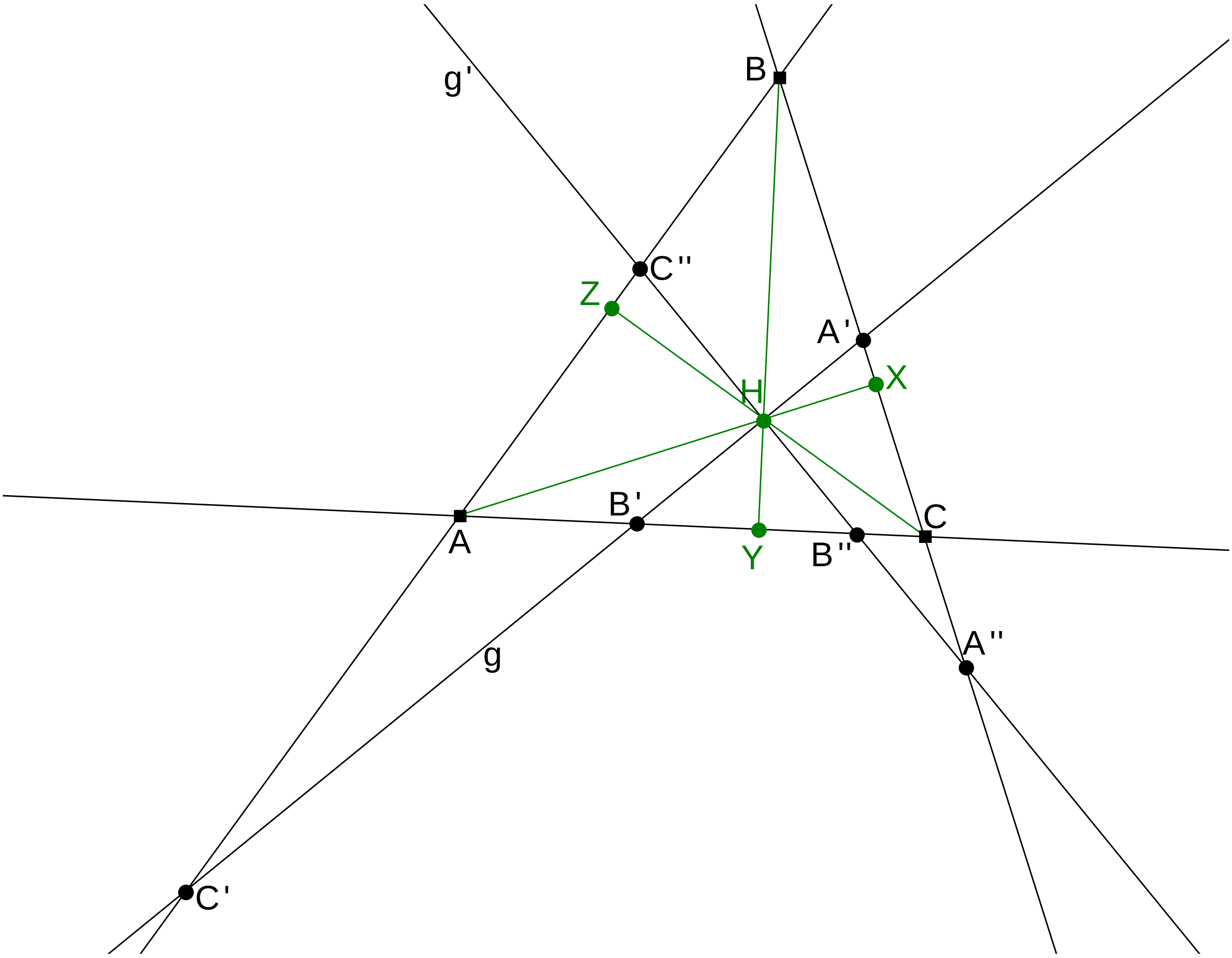


Fig. 8

Now let $AX,\ BY,\ CZ$ be the altitudes of triangle $ABC$. See Fig. 8. After the Sine Law,

$$\frac{A'B}{BA''} = \frac{BH \bullet \sin\measuredangle BHA' \ : \ \sin\measuredangle HA'B}{BH \bullet \sin\measuredangle BHA'' \ : \ \sin\measuredangle HA''B} = \frac{\sin\measuredangle BHA' \ : \ \sin\measuredangle HA'B}{\sin\measuredangle BHA'' \ : \ \sin\measuredangle HA''B} = \frac{\sin\measuredangle BHA' \bullet \sin\measuredangle HA''B}{\sin\measuredangle BHA'' \bullet \sin\measuredangle HA'B}$$

and

$$\frac{B'A}{AB''} = \frac{AH \bullet \sin\measuredangle AHB' \ : \ \sin\measuredangle HB'A}{AH \bullet \sin\measuredangle AHB'' \ : \ \sin\measuredangle HB''A} = \frac{\sin\measuredangle AHB' \ : \ \sin\measuredangle HB'A}{\sin\measuredangle AHB'' \ : \ \sin\measuredangle HB''A} = \frac{\sin\measuredangle AHB' \bullet \sin\measuredangle HB''A}{\sin\measuredangle AHB'' \bullet \sin\measuredangle HB'A}.$$

But $\measuredangle BHA' = \measuredangle YHB'$. In the right-angled triangle $B'YH$, we get $\measuredangle YHB' = 90° - \measuredangle HB'Y$; hence, $\measuredangle BHA' = 90° - \measuredangle HB'Y = 90° - \measuredangle HB'B''$. In the right-angled triangle $B'HB''$, we get $90° - \measuredangle HB'B'' = \measuredangle HB''B'$; hence, $\measuredangle BHA' = \measuredangle HB''B' = \measuredangle HB''A$. So, $\sin\measuredangle BHA' = \sin\measuredangle HB''A$. Similarly, $\sin\measuredangle BHA'' = \sin\measuredangle HB'A$, $\sin\measuredangle AHB' = \sin\measuredangle HA''B$ and $\sin\measuredangle AHB'' = \sin\measuredangle HA'B$. It follows that

$$\frac{\sin\measuredangle BHA' \bullet \sin\measuredangle HA''B}{\sin\measuredangle BHA'' \bullet \sin\measuredangle HA'B} = \frac{\sin\measuredangle AHB' \bullet \sin\measuredangle HB''A}{\sin\measuredangle AHB'' \bullet \sin\measuredangle HB'A},$$

and therefore

$$\frac{A'B}{BA''} = \frac{B'A}{AB''}.$$

In an analogous manner,

$$\frac{B'C}{CB''} = \frac{C'B}{BC''}; \qquad \frac{C'A}{AC''} = \frac{A'C}{CA''}.$$

Now we can apply Theorem 3 (with $P = H$), and infer that the circles $HA'A''$, $HB'B''$, $HC'C''$ have a common point $Q$ different from $H$, and that this common point lies on the circumcircle of triangle $ABC$. But we have already seen that $H$ lies on the circles with diameters $A'A''$, $B'B''$, $C'C''$, so that the circles $HA'A''$, $HB'B''$, $HC'C''$ are simply these circles with diameters $A'A''$, $B'B''$, $C'C''$. Hence we get that the circles with diameters $A'A''$, $B'B''$, $C'C''$ have a common point $Q$ different from $H$, and this common point lies on the circumcircle of triangle $ABC$. This completes the proof of part **b**).

Having two distinct common points, namely $H$ and $Q$, the circles with diameters $A'A''$, $B'B''$, $C'C''$ must be coaxal, what proves part **c**).

Since the center of a circle lies on the perpendicular bisector of any chord, the centers of the circles with diameters $A'A''$, $B'B''$, $C'C''$ lie on the perpendicular bisector of the segment $HQ$ (which is the common chord of the three circles). Now, these centers are naturally the midpoints of segments $A'A''$, $B'B''$, $C'C''$. This establishes part **d**).

Part **e**) follows directly from Theorem 4.

Another proof of part **b**) was provided by Nikolaos Dergiades in Hyacinthos message #7381. That proof uses Theorem 1, too.

## References

[1] Ross Honsberger: *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, USA 1995.