Circumscribed quadrilaterals revisited / Darij Grinberg
(updated version, 5 October 2012)

The aim of this note is to prove some new properties of circumscribed quadrilaterals and give new proofs to classical ones.[1]

We start with some trivialities (Fig. 1).

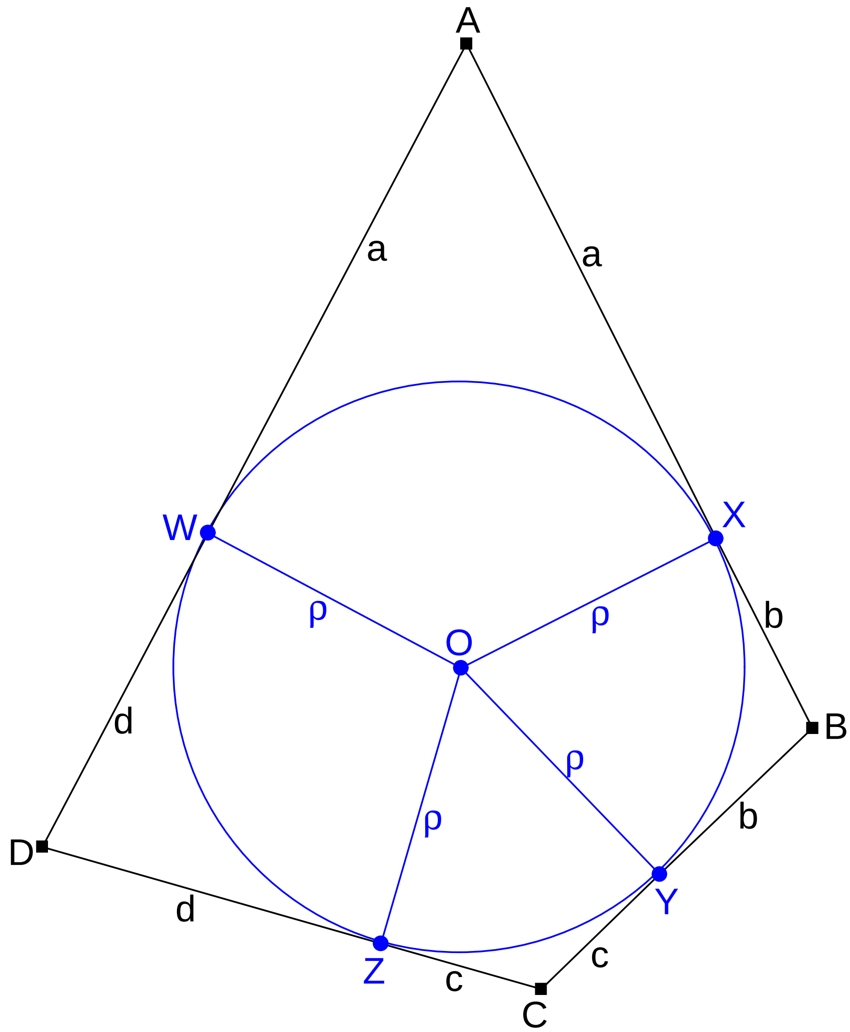

Fig. 1

Let $ABCD$ be a circumscribed quadrilateral, that is, a quadrilateral which has an incircle. Let this incircle have the center O and the radius ρ and touch its sides AB, BC, CD, DA at the points X, Y, Z, W, respectively. Then, for some very obvious reasons, $OX \perp AB$, $OY \perp BC$, $OZ \perp CD$, $OW \perp DA$ and $OX = OY = OZ = OW = \rho$. Moreover, $AW = AX$, $BX = BY$, $CY = CZ$, $DZ = DW$, since the two tangents from a point to a circle are equal in length. We denote

$$a = AW = AX; \qquad b = BX = BY; \qquad c = CY = CZ; \qquad d = DZ = DW.$$

(Thus, we denote by a, b, c, d not, as usual, the sidelengths of the quadrilateral $ABCD$, but the segments $AW = AX$, $BX = BY$, $CY = CZ$, $DZ = DW$.)

Then, the sidelengths of quadrilateral $ABCD$ are

$$AB = AX + BX = a + b; \qquad BC = BY + CY = b + c;$$
$$CD = CZ + DZ = c + d; \qquad DA = DW + AW = d + a.$$

[1] I am grateful to George Baloglou for correcting a mistake in Theorem 13.

Hence,

$$AB + CD = (a + b) + (c + d) = (b + c) + (d + a) = BC + DA.$$

Thus we have shown the maybe most famous fact about circumscribed quadrilaterals:

Theorem 1. If $ABCD$ is a circumscribed quadrilateral[2], then $AB + CD = BC + DA$.

In words: In a circumscribed quadrilateral, the sums of the lengths of opposite sides are equal.

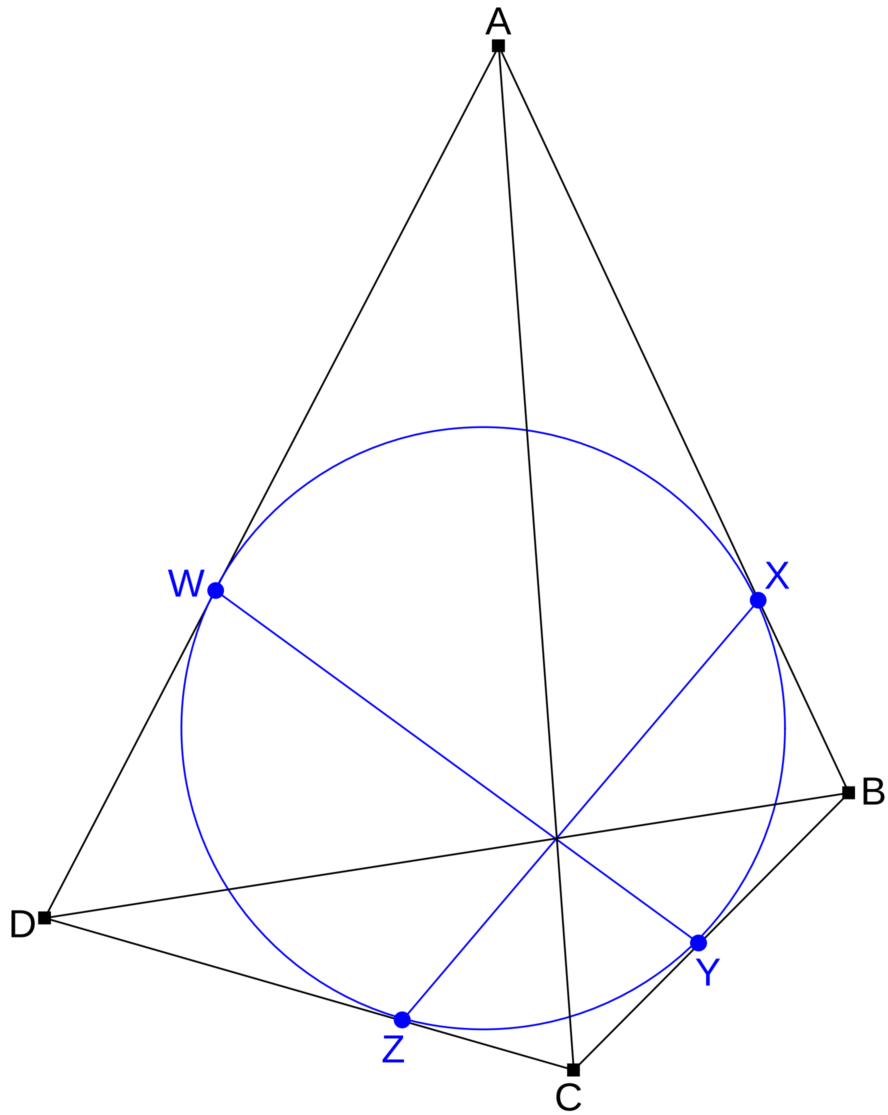

Fig. 2

Now, let's get serious and turn to the first nontrivial result about circumscribed quadrilaterals (Fig. 2):

Theorem 2. The four lines AC, BD, XZ, YW concur at one point.

This theorem is still rather well-known; it is problem 105 in [1] and also appears in [6], [8] and [10]. Here we give two proofs of this theorem.

[2]In the following, we assume in every theorem that $ABCD$ is a circumscribed quadrilateral (and the notations are those introduced before).

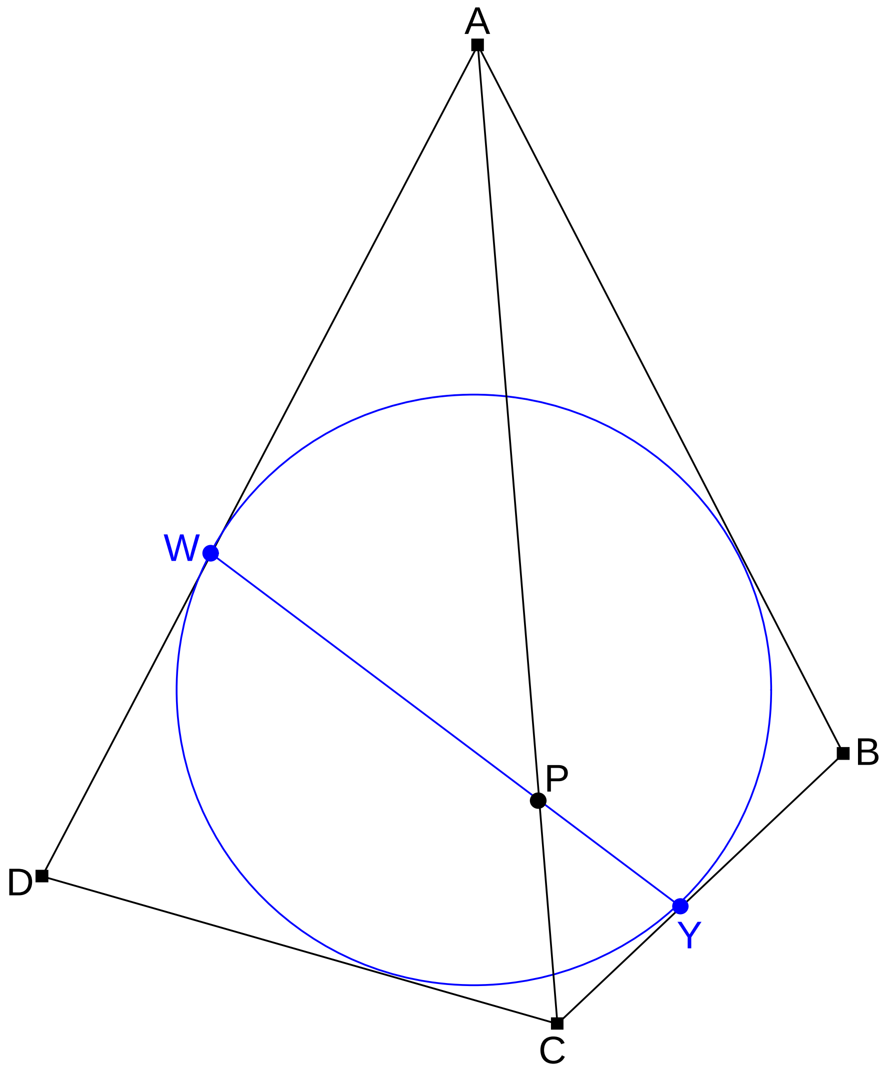

Fig. 3

First proof of Theorem 2. (See Fig. 3.) Let P be the point of intersection of the lines AC and YW.

The lines BC and DA touch the incircle of the quadrilateral $ABCD$ at the points Y and W. Hence, by the tangent-chordal angle theorem, both angles $\measuredangle CYW$ and $\measuredangle DWY$ are equal to the chordal angle of the chord YW in the incircle of the quadrilateral $ABCD$. Thus, $\measuredangle CYW = \measuredangle DWY$. In other words, $\measuredangle CYP = 180^\circ - \measuredangle AWP$. Thus, $\sin\measuredangle CYP = \sin\measuredangle AWP$. But after the sine law in triangle AWP, we have $AP = AW \cdot \dfrac{\sin\measuredangle AWP}{\sin\measuredangle APW}$, and after the sine law in triangle CYP, we have $CP = CY \cdot \dfrac{\sin\measuredangle CYP}{\sin\measuredangle CPY}$. Thus,

$$\frac{AP}{CP} = \frac{AW \cdot \dfrac{\sin\measuredangle AWP}{\sin\measuredangle APW}}{CY \cdot \dfrac{\sin\measuredangle CYP}{\sin\measuredangle CPY}} = \frac{AW \cdot \dfrac{\sin\measuredangle AWP}{\sin\measuredangle APW}}{CY \cdot \dfrac{\sin\measuredangle AWP}{\sin\measuredangle APW}} = \frac{AW}{CY} = \frac{a}{c}.$$

Now, let P' be the point of intersection of the lines AC and XZ. Then, we similarly find $\dfrac{AP'}{CP'} = \dfrac{a}{c}$. Thus, $\dfrac{AP}{CP} = \dfrac{AP'}{CP'}$. This means that the points P and P' divide the segment AC in the same ratio; hence, these points P and P' coincide. Since the point P is the point of intersection of the lines AC and YW, and the point P' is the point of intersection of the lines AC and XZ, it thus follows that the lines AC, XZ and YW

concur at one point. Similarly, we can verify that the lines BD, XZ and YW concur at one point. Hence, all four lines AC, BD, XZ and YW concur at one point, and Theorem 2 is proven.

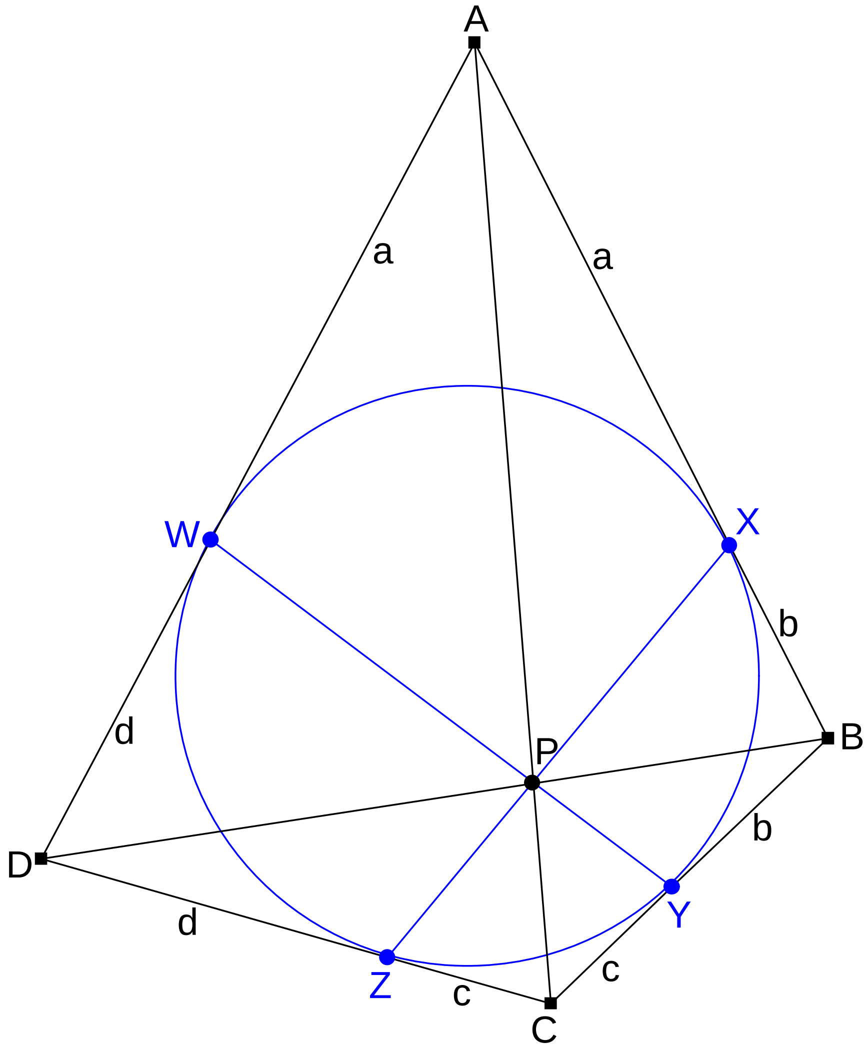

Fig. 4

This proof of Theorem 2 has a nice consequence (Fig. 4): The point of intersection of the four lines AC, BD, XZ, YW must obviously coincide with the point of intersection P of the lines AC and YW defined in the above proof of Theorem 2. Now, we have shown that this point P satisfies $\frac{AP}{CP} = \frac{a}{c}$. Similarly, $\frac{BP}{DP} = \frac{b}{d}$. Thus, we get:

Theorem 3. If P is the point of intersection of the lines AC, BD, XZ, YW, then $\frac{AP}{CP} = \frac{a}{c}$ and $\frac{BP}{DP} = \frac{b}{d}$.

Note that this result appeared in [7] and [8].

Second proof of Theorem 2. We will show that the lines AC, BD and XZ concur. Then, analogously we can show that the lines AC, BD and YW concur, and thus it will follow that all four lines AC, BD, XZ and YW concur, thus proving Theorem 2.

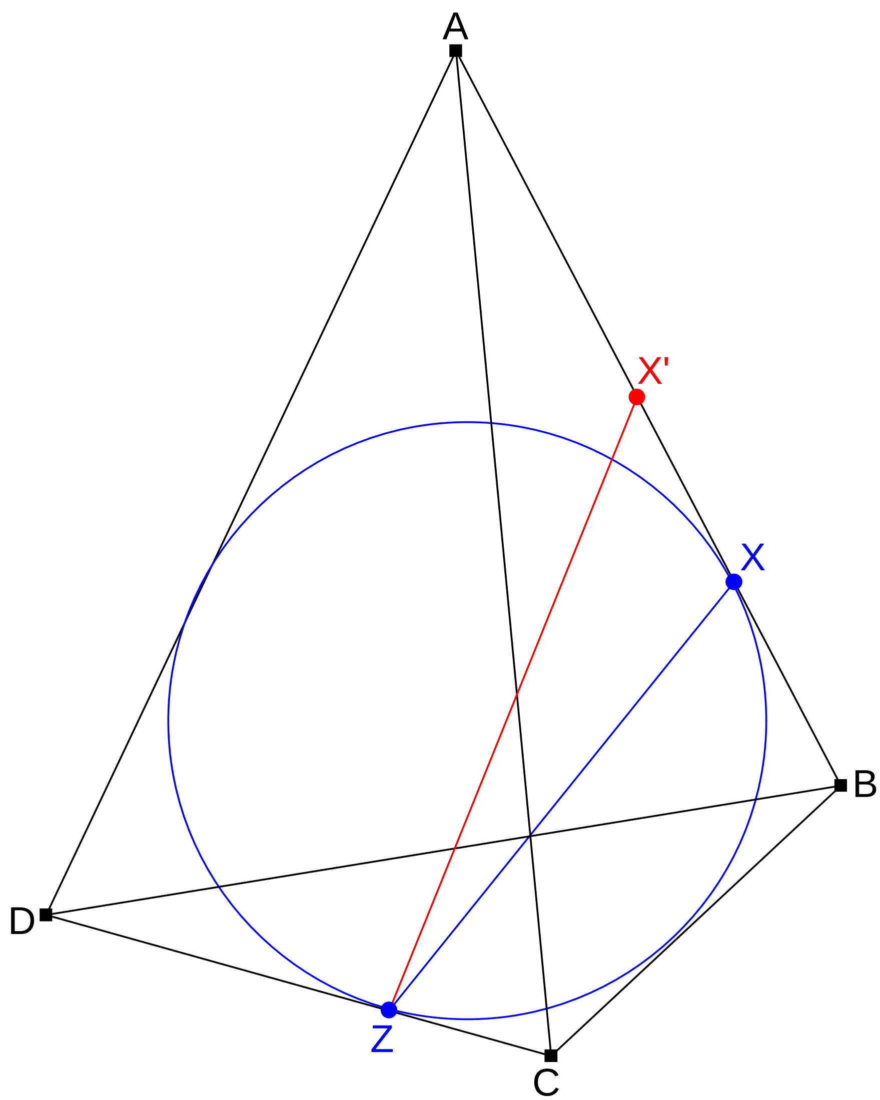

Fig. 5

(See Fig. 5.) Now, in order to show that the lines AC, BD and XZ concur, it appears reasonable to apply the Brianchon theorem in a limiting case. However, one has to be careful doing this. Here is how one should *not* proceed:

"Consider the degenerate hexagon $AXBCZD$ (degenerate, since its adjacent sides AX and XB lie on one line, and its adjacent sides CZ and ZD lie on one line). This hexagon is obviously circumscribed, since all of its sides AX, XB, BC, CZ, ZD, DA touch one circle (namely, the incircle of the quadrilateral $ABCD$). Hence, the main diagonals AC, XZ and BD of this hexagon concur, and the proof is complete."

The mistake - to be more precise, the gap - in this argumentation becomes clear if one applies it to the hexagon $AX'BCZD$, where X' is an arbitrary point on the line AB. This hexagon, too, appears to be circumscribed, since all of its sides AX', $X'B$, BC, CZ, ZD, DA touch one circle (namely, the incircle of the quadrilateral $ABCD$) - if they are extended to lines (but this should not be a problem, since we are talking about projective theorems, and thus arrangement shouldn't matter). Thus, by the Brianchon theorem, it seems to follow that the lines AC, $X'Z$ and BD concur - but this is nonsense for every point X' different from X.

So where is the mistake? The trick is: A geometrical theorem can be used in a degenerate case if either its proof still functions in this case, or one can deduce the degenerate case from the generic case by a limiting argument. Our application of the Brianchon theorem to the hexagon $AX'BCZD$ did not match any of these two conditions; thus, it was not legitimate. Hence, there is no wonder the resulting assertion was wrong.

However, one can rescue the above proof of Theorem 2. In order to do this, one must find an argument that shows why the Brianchon theorem can be applied to the degenerate hexagon $AXBCZD$, but not to the degenerate hexagon $AX'BCZD$ with $X' \neq X$.

In order to find such an argument, let's recall how the Brianchon theorem is derived from the Pascal theorem using the polar transformation.

The Pascal theorem states: If six points A_1, B_1, C_1, D_1, E_1, F_1 lie on one circle, then the points of intersection $A_1B_1 \cap D_1E_1$, $B_1C_1 \cap E_1F_1$ and $C_1D_1 \cap F_1A_1$ are collinear; hereby, if two "adjacent" points - i. e., for instance, the points A_1 and B_1 - coincide, then the line A_1B_1 has to be interpreted as the tangent to the circle at the point A_1, and not as an arbitrary line through the point A_1.

After the polar transformation, this becomes: If six lines a_1, b_1, c_1, d_1, e_1, f_1 touch a circle, then the lines $(a_1 \cap b_1) * (d_1 \cap e_1)$, $(b_1 \cap c_1) * (e_1 \cap f_1)$ and $(c_1 \cap d_1) * (f_1 \cap a_1)$ are concurrent[3]; hereby, if two "adjacent" lines - i. e., for instance, the lines a_1 and b_1 - coincide, then the point of intersection $a_1 \cap b_1$ has to be interpreted as the point of tangency of the line a_1 with the circle, and not as an arbitrary point on the line a_1.

In other words: The hexagon formed by the lines a_1, b_1, c_1, d_1, e_1, f_1 may be degenerated, but if two adjacent sides lie on one line, then the vertex where these sides meet must be the point of tangency of this line with the circle, and not just an arbitrary point on this line.

This is fulfilled for the degenerate hexagon $AXBCZD$ [4], but not for the degenerate hexagon $AX'BCZD$ with $X' \neq X$. Thus, the above argumentation for the hexagon $AXBCZD$ is correct - thus Theorem 2 is proven -, but the same argumentation for the hexagon $AX'BCZD$ is wrong.

Now, we head over to a less classical result, one noted by myself in 2003 (Fig. 6):

Theorem 4. Let the perpendicular to the line AB at the point A meet the line BO at a point M. Let the perpendicular to the line AD at the point A meet the line DO at a point N. Then, $MN \perp AC$.

[3] Hereby, we use the abbreviation $G * H$ for the line joining two points G and H.

[4] The adjacent sides AX and XB of this hexagon lie on one line - and the vertex where they meet, namely the vertex X, is indeed the point of tangency of this line with the circle. The same holds for the adjacent sides CZ and ZD.

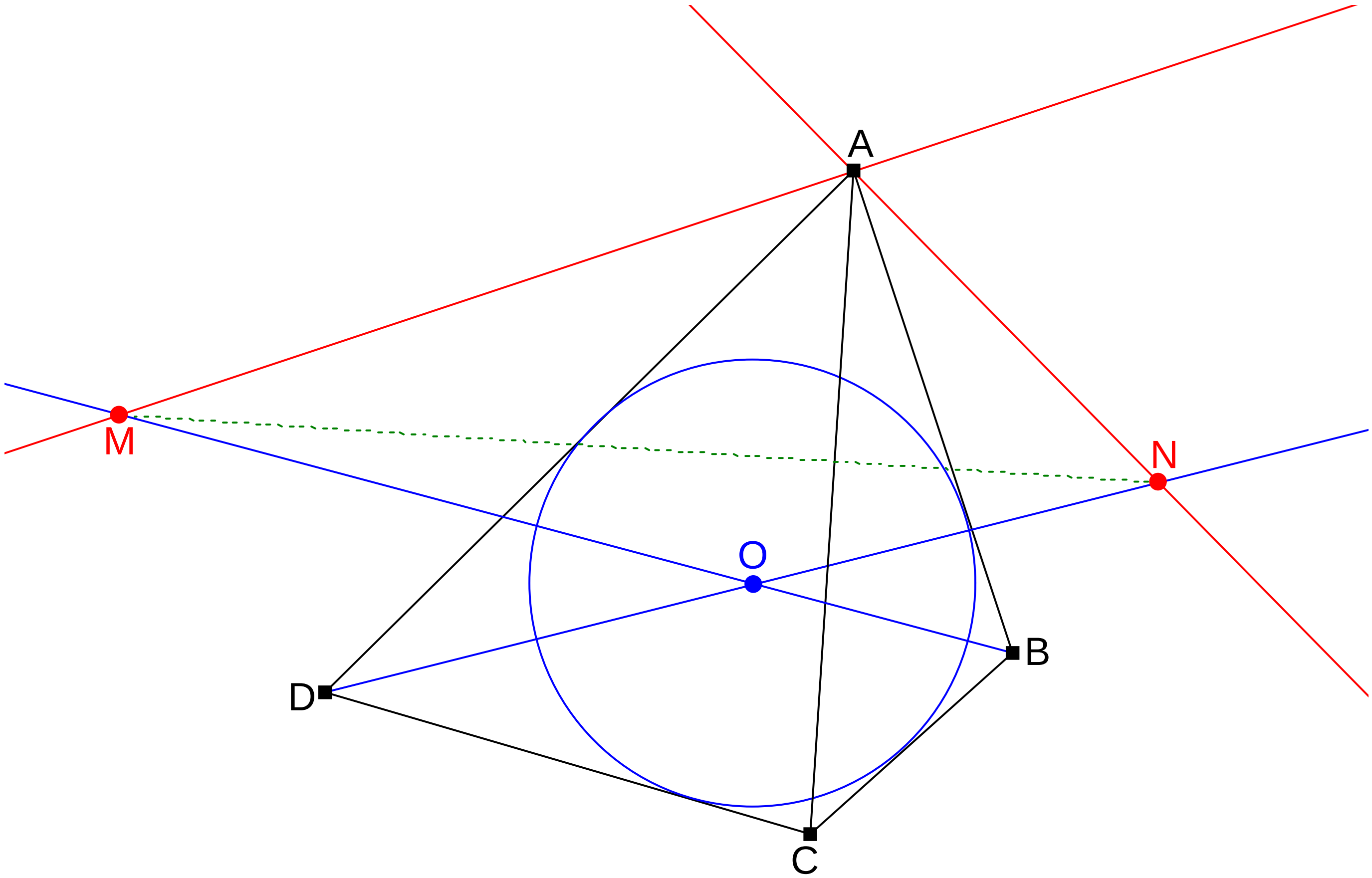

Fig. 6

In [4], this theorem appears as Theorem 1 and receives two proofs. Here is a different *proof of Theorem 4:*

(See Fig. 7.) Let L_b and L_d be the orthogonal projections of the points B and D on the line AC. Then, the lines BL_b and DL_d, both being perpendicular to AC, must be parallel to each other, and thus Thales yields $\frac{BL_b}{DL_d} = \frac{BP}{DP}$. But according to Theorem 3, we have $\frac{BP}{DP} = \frac{b}{d}$. Thus $\frac{BL_b}{DL_d} = \frac{b}{d}$, or, equivalently, $\frac{BL_b}{b} = \frac{DL_d}{d}$.

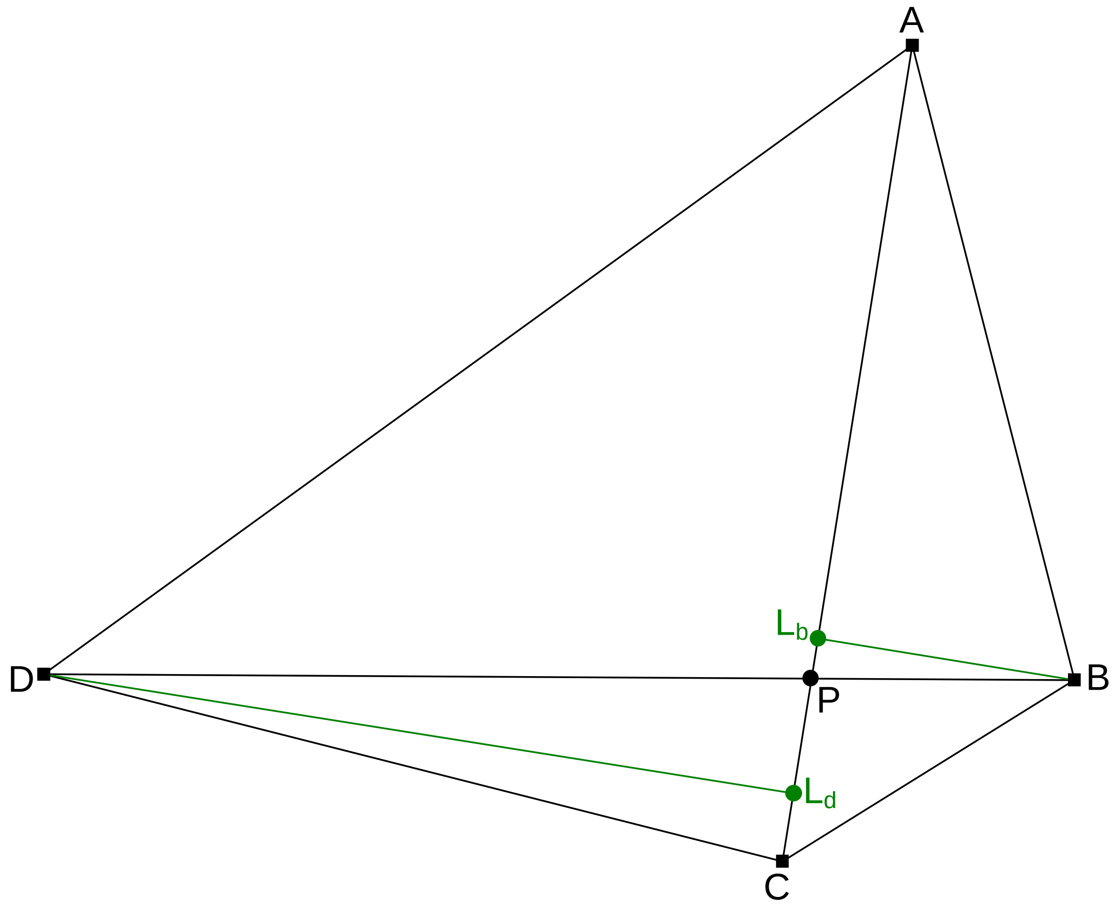

Fig. 7

(See Fig. 8.) Let R be the orthogonal projection of the point M on the line AC. Then, $\measuredangle ARM = 90^\circ$. Compared with $\measuredangle BL_bA = 90^\circ$, this yields $\measuredangle ARM = \measuredangle BL_bA$. On the other hand, $\measuredangle MAB = 90^\circ$, so that $\measuredangle MAR = \measuredangle MAB - \measuredangle L_bAB = 90^\circ - \measuredangle L_bAB$. But in the right-angled triangle AL_bB, we have $\measuredangle ABL_b = 90^\circ - \measuredangle L_bAB$. Hence, $\measuredangle MAR = \measuredangle ABL_b$. From $\measuredangle ARM = \measuredangle BL_bA$ and $\measuredangle MAR = \measuredangle ABL_b$, we see that the triangles ARM and BL_bA are similar; thus, $\dfrac{AR}{BL_b} = \dfrac{AM}{AB}$.

On the other hand, the point M lies on the line BO, and from $AM \perp AB$ and $OX \perp AB$ it follows that $AM \parallel OX$. Hence, by Thales, $\dfrac{AM}{AB} = \dfrac{OX}{BX}$. Thus, we obtain

$$\frac{AR}{BL_b} = \frac{AM}{AB} = \frac{OX}{BX} = \frac{\rho}{b}, \qquad \text{so that} \qquad AR = BL_b \cdot \frac{\rho}{b} = \rho \cdot \frac{BL_b}{b}.$$

Similarly, we can denote by R' the orthogonal projection of the point N on the line AC, and show that $AR' = \rho \cdot \dfrac{DL_d}{d}$. Since $\dfrac{BL_b}{b} = \dfrac{DL_d}{d}$, we thus get $AR = AR'$. Since the points R and R' both lie on the segment AC, this yields that these points R and R' coincide. Now, since the point R is the orthogonal projection of the point M on the line AC, we have $MR \perp AC$, so that the point M lies on the perpendicular to the line AC at the point R. Similarly, the point N lies on the perpendicular to the line AC at the point R'. But since $R = R'$, these two perpendiculars coincide, and thus the points M and N lie on one and the same perpendicular to the line AC. This means $MN \perp AC$, and Theorem 4 is proven.

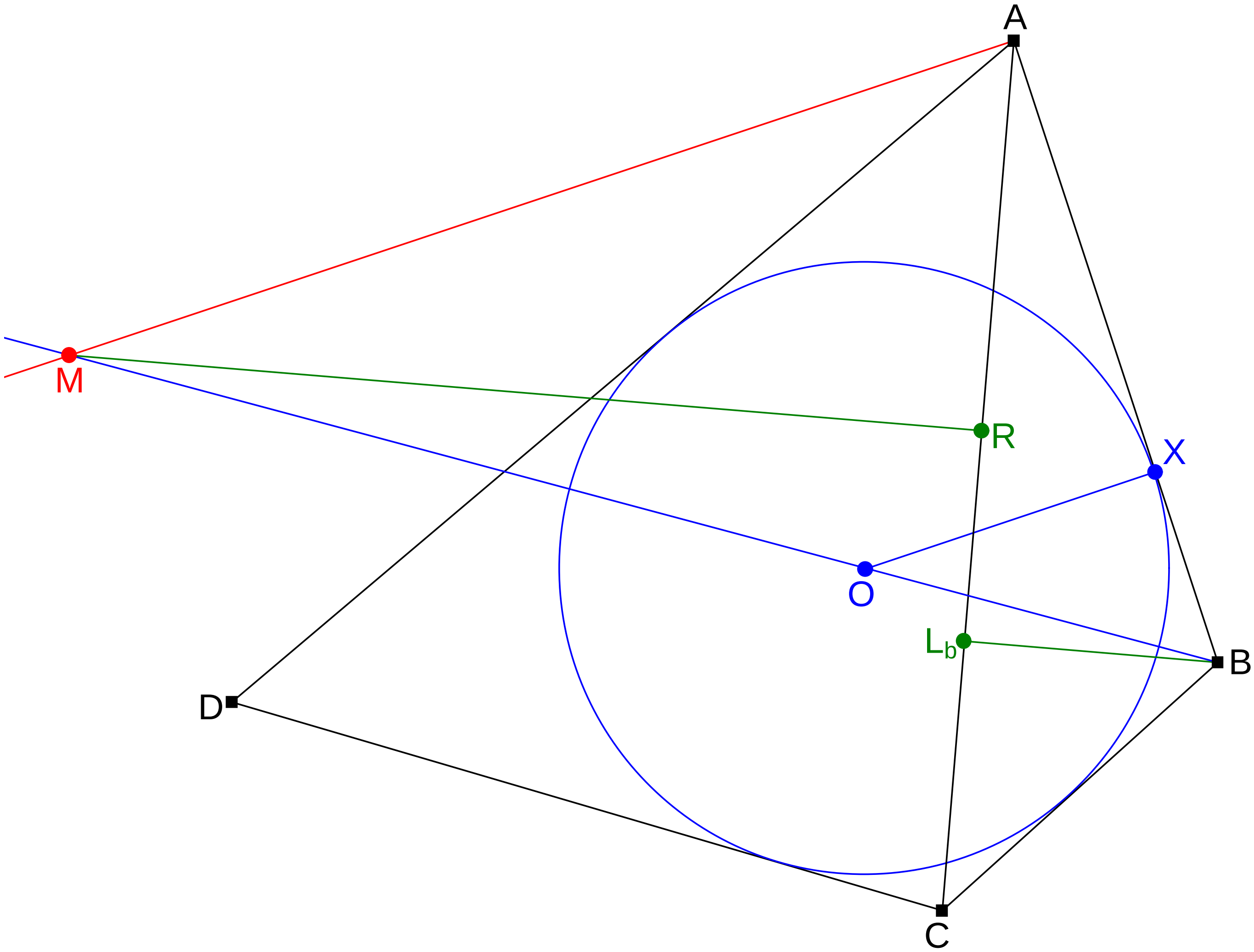

Fig. 8

In [2], Jean-Pierre Ehrmann showed an alternate approach to Theorem 4 with the help of hyperbola properties. A corollary of this approach is the following fact:

Theorem 5. Denote the distances from the points B and D to the line MN by m and n, respectively. Then, $\dfrac{m}{AB} = \dfrac{n}{AD}$.

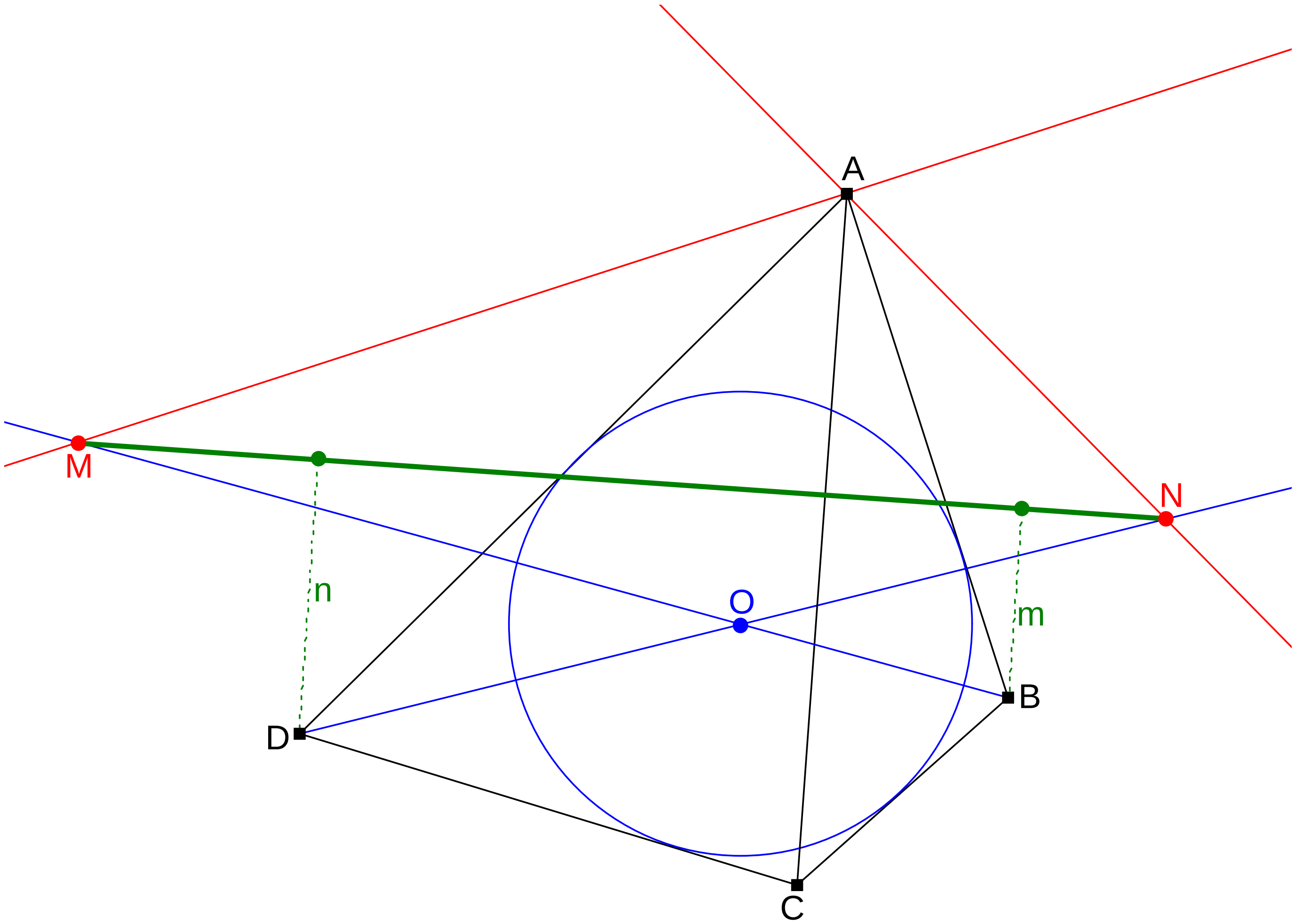

Fig. 9

Here is an elementary *proof of Theorem 5.* First, we focus on the points X, Y, Z, W. We will use directed segments; in the following, the directed distance between two points P_1 and P_2 will be denoted by $\overline{P_1P_2}$ (as opposed to the non-directed distance, which we will continue to write as P_1P_2). Also, we direct the lines AB, BC, CD, DA in such a way that the directed segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ are positive (and thus the segments $\overline{BA}$, $\overline{CB}$, $\overline{DC}$, $\overline{AD}$ are negative). Then,

$$a = AW = AX; \qquad b = BX = BY; \qquad c = CY = CZ; \qquad d = DZ = DW$$

becomes

$$a = \overline{WA} = \overline{AX}; \qquad b = \overline{XB} = \overline{BY}; \qquad c = \overline{YC} = \overline{CZ}; \qquad d = \overline{ZD} = \overline{DW}.$$

(See Fig. 10.) Now, let T be the point on the line AC satisfying $\frac{\overline{AT}}{\overline{TC}} = -\frac{a}{c}$. Then, $\frac{\overline{TC}}{\overline{AT}} = -\frac{c}{a}$, what rewrites as $\frac{\overline{CT}}{\overline{TA}} = -\frac{c}{a}$. Hence,

$$\frac{\overline{AX}}{\overline{XB}} \cdot \frac{\overline{BY}}{\overline{YC}} \cdot \frac{\overline{CT}}{\overline{TA}} = \frac{a}{b} \cdot \frac{b}{c} \cdot \left(-\frac{c}{a}\right) = -1.$$

By the Menelaos theorem, applied to the triangle ABC and the points X, Y, T on its sides AB, BC, CA, this yields that the points X, Y, T are collinear. In other words, the point T lies on the line XY. As the definition of the point T is symmetric in B and D, we can similarly show that this point T lies on the line ZW.

Note that we have thus shown an interesting side-result: Our point T lies on the lines AC, XY and ZW and divides the segment AC in the ratio $\frac{\overline{AT}}{\overline{TC}} = -\frac{a}{c}$. Comparing this with $\frac{\overline{AP}}{\overline{PC}} = \frac{a}{c}$ (this is just the equation $\frac{AP}{CP} = \frac{a}{c}$ from Theorem 3, after being rewritten with directed segments), we see that $\frac{\overline{AT}}{\overline{TC}} = -\frac{\overline{AP}}{\overline{PC}}$, so that the point T is the harmonic conjugate of the point P with respect to the segment AC. Thus, we have shown:

Theorem 6. The lines AC, XY, ZW concur at one point T. This point T divides the segment AC in the ratio $\frac{\overline{AT}}{\overline{TC}} = -\frac{a}{c}$ and is the harmonic conjugate of the point P with respect to the segment AC.

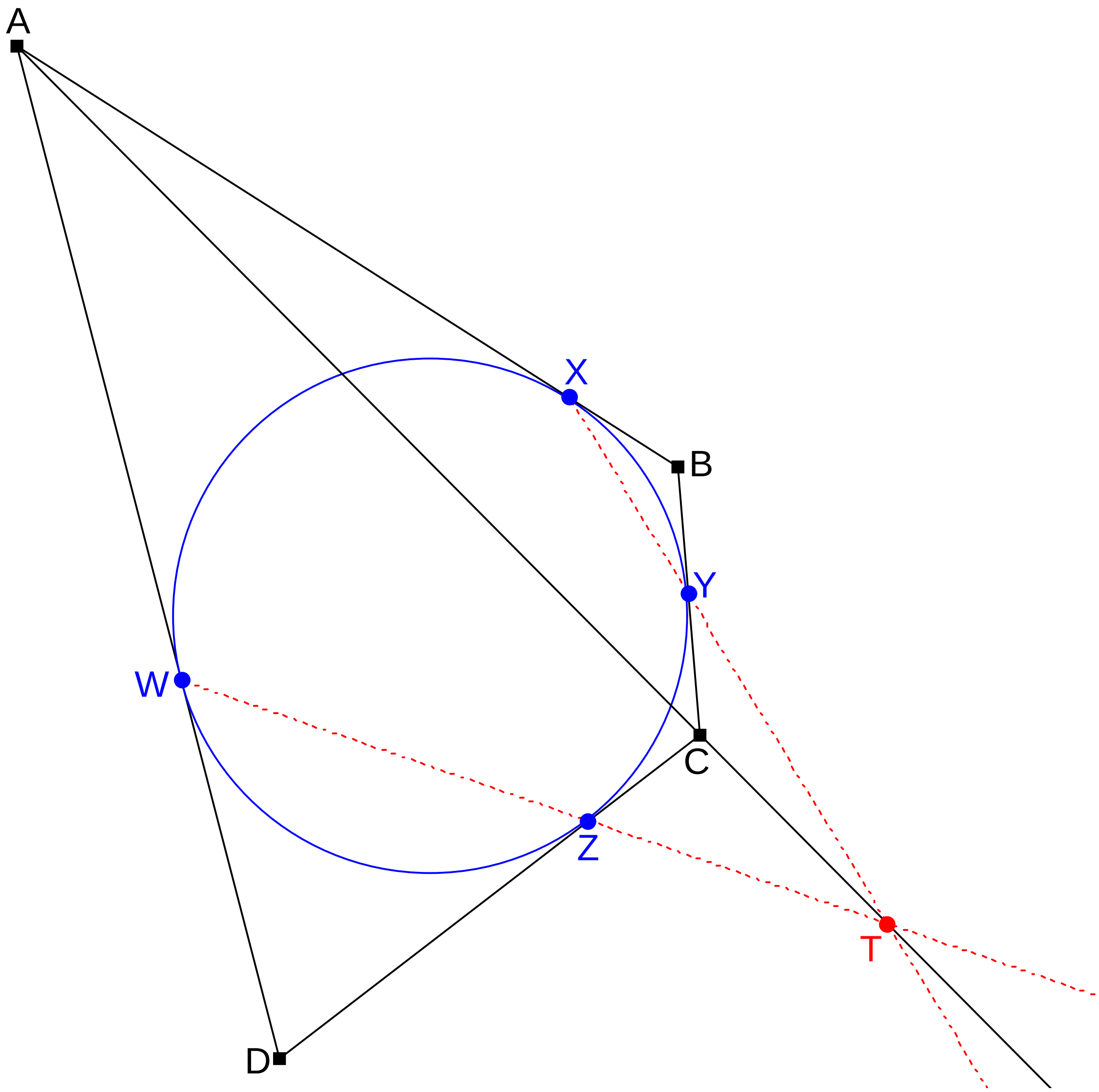

Fig. 10

(See Fig. 11.) Now, let M' be the orthogonal projection of the point B on the line MN. Then, the distance m from the point B to the line MN equals to the segment

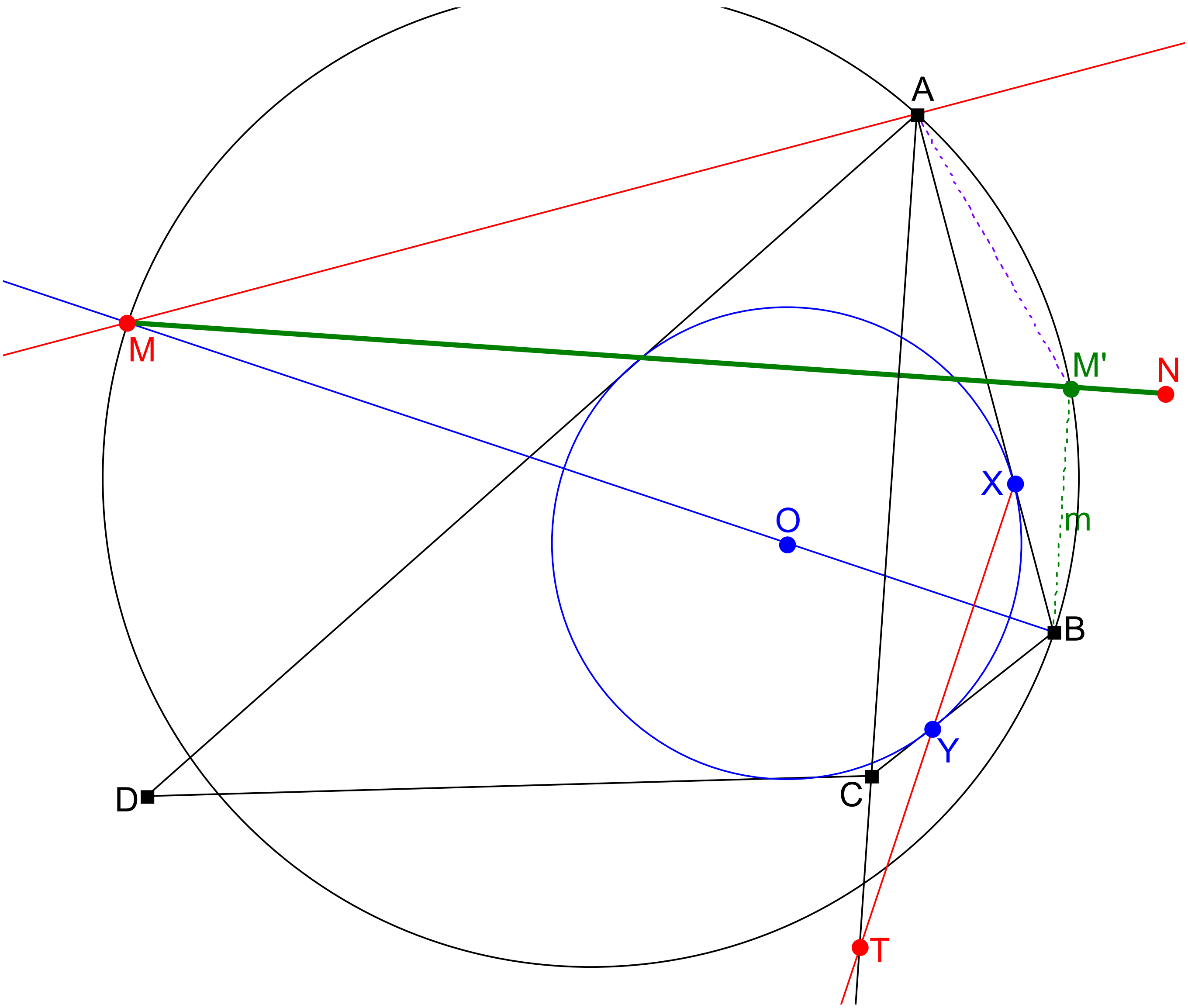

Fig. 11

BM'; so we have $m = BM'$.

On the other hand, $BM' \perp MN$, combined with $MN \perp AC$, yields $BM' \parallel AC$, so that $\measuredangle M'BA = \measuredangle XAT$.

Since $\measuredangle MM'B = 90^\circ$ and $\measuredangle MAB = 90^\circ$, the points M' and A lie on the circle with diameter MB. Thus, the quadrilateral $AM'BM$ is cyclic, so that $\measuredangle BM'A = 180^\circ - \measuredangle AMB$. On the other hand, in the right-angled triangle AMB, we have $\measuredangle AMB = 90^\circ - \measuredangle ABM$. But since the point M lies on the line BO, i. e. on the angle bisector of the angle ABC (since the point O is the incenter of the quadrilateral $ABCD$), we have $\measuredangle ABM = \frac{\measuredangle ABC}{2}$. Finally, since $BX = BY$, the triangle XBY is isosceles, so that its base angle $\measuredangle BXY$ equals

$$\measuredangle BXY = \frac{180^\circ - \measuredangle XBY}{2} = 90^\circ - \frac{\measuredangle XBY}{2} = 90^\circ - \frac{\measuredangle ABC}{2}.$$

Thus,

$$\begin{aligned}\measuredangle BM'A &= 180^\circ - \measuredangle AMB = 180^\circ - (90^\circ - \measuredangle ABM) = 90^\circ + \measuredangle ABM = 90^\circ + \frac{\measuredangle ABC}{2} \\ &= 180^\circ - \left(90^\circ - \frac{\measuredangle ABC}{2}\right) = 180^\circ - \measuredangle BXY = \measuredangle AXT.\end{aligned}$$

Since $\measuredangle M'BA = \measuredangle XAT$ and $\measuredangle BM'A = \measuredangle AXT$, the triangles $BM'A$ and AXT are similar. Thus, $\frac{BM'}{AB} = \frac{AX}{TA}$. Since $m = BM'$ and $a = AX$, we therefore have $\frac{m}{AB} = \frac{a}{TA}$. Similarly, $\frac{n}{AD} = \frac{a}{TA}$. Hence, $\frac{m}{AB} = \frac{n}{AD}$, what proves Theorem 5.

In the remaining part of the article, we will consider some metric identities at the circumscribed quadrilateral (Fig. 12).

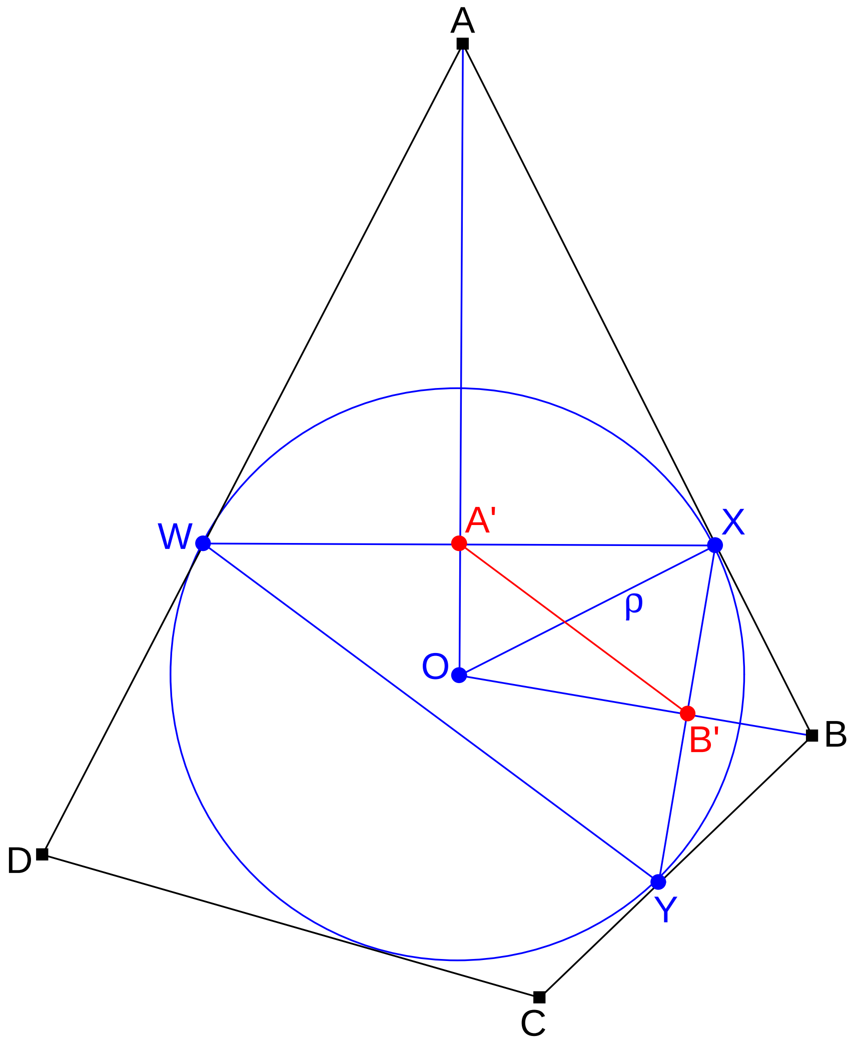

Fig. 12

The points X and Y, being the points of tangency of the incircle of the quadrilateral $ABCD$ with its sides AB and BC, are symmetric to each other with respect to the angle bisector BO of the angle ABC. Hence, the segment XY is perpendicular to the line BO and is bisected by this line. So the midpoint B' of the segment XY lies on the line BO. Similarly, the midpoint A' of the segment WX lies on the line AO.

Now, from $XY \perp BO$ we see that $\measuredangle XB'O = 90^\circ$, while from $OX \perp AB$ we have $\measuredangle BXO = 90^\circ$. Thus, $\measuredangle XB'O = \measuredangle BXO$. Also, trivially, $\measuredangle XOB' = \measuredangle BOX$. Thus, the triangles $XB'O$ and BXO are similar, so that $\frac{OB'}{OX} = \frac{OX}{OB}$, and thus $OB \cdot OB' = OX^2 = \rho^2$.

Similarly, $OA \cdot OA' = \rho^2$. Hence, $OB \cdot OB' = OA \cdot OA'$, so that $\frac{OB}{OA} = \frac{OA'}{OB'}$. Together with $\measuredangle BOA = \measuredangle A'OB'$, this yields the similarity of triangles BOA and $A'OB'$. Consequently,

$$\frac{A'B'}{AB} = \frac{OA'}{OB}, \qquad \text{thus} \qquad A'B' = AB \cdot \frac{OA'}{OB} = AB \cdot \frac{OA \cdot OA'}{OA \cdot OB} = AB \cdot \frac{\rho^2}{OA \cdot OB}.$$

Now, the points A' and B' are the midpoints of the sides WX and XY of triangle

WXY; thus, $A'B' = \frac{YW}{2}$. Hence, $AB \cdot \frac{\rho^2}{OA \cdot OB} = \frac{YW}{2}$. Consequently,

$$AB = \frac{YW}{2} \cdot \frac{OA \cdot OB}{\rho^2}.$$

Similar relations must obviously hold for BC, CD and DA. We summarize:

Theorem 7. We have

$$AB = \frac{YW}{2} \cdot \frac{OA \cdot OB}{\rho^2}; \qquad BC = \frac{XZ}{2} \cdot \frac{OB \cdot OC}{\rho^2};$$

$$CD = \frac{YW}{2} \cdot \frac{OC \cdot OD}{\rho^2}; \qquad DA = \frac{XZ}{2} \cdot \frac{OD \cdot OA}{\rho^2}.$$

(See Fig. 13.)

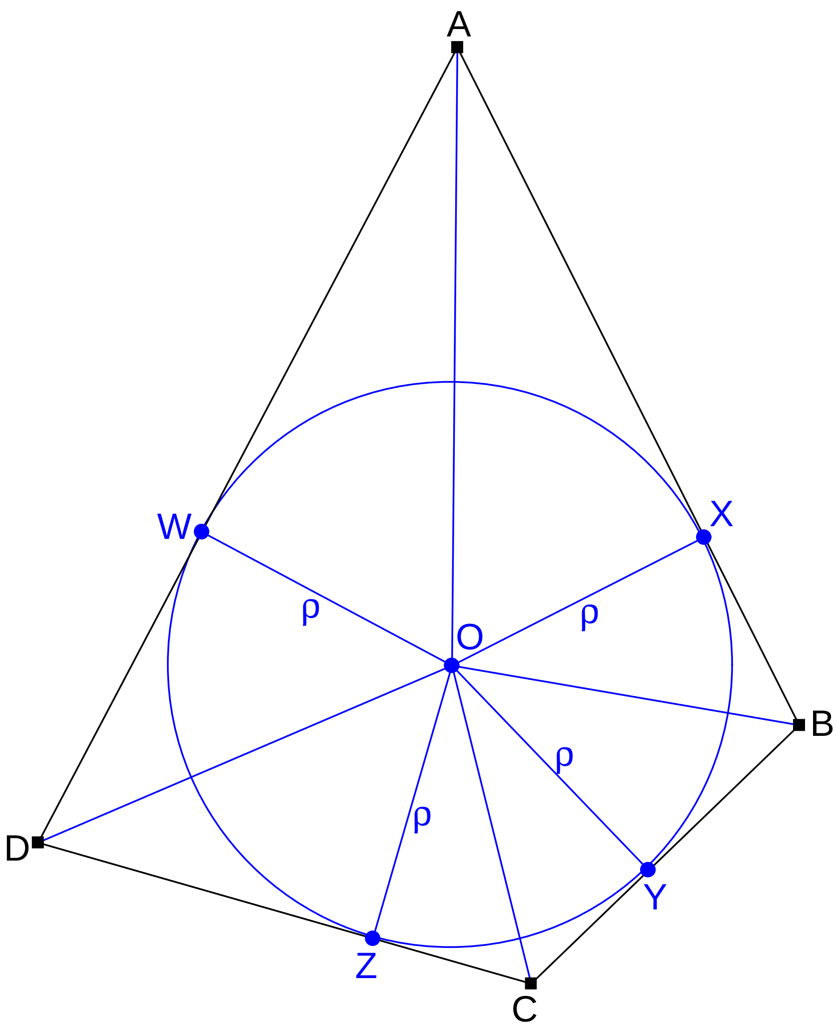

Fig. 13

These equations can be used for deriving some other formulas. For instance, $AB = \frac{YW}{2} \cdot \frac{OA \cdot OB}{\rho^2}$ transforms into

$$OA \cdot OB = \rho^2 \cdot AB : \frac{YW}{2} = \frac{2\rho^2 \cdot AB}{YW}.$$

Similarly,

$$OC \cdot OD = \frac{2\rho^2 \cdot CD}{YW}.$$

Thus,

$$\frac{OA \cdot OB}{OC \cdot OD} = \frac{\left(\dfrac{2\rho^2 \cdot AB}{YW}\right)}{\left(\dfrac{2\rho^2 \cdot CD}{YW}\right)} = \frac{AB}{CD}.$$

Similarly, $\dfrac{OB \cdot OC}{OD \cdot OA} = \dfrac{BC}{DA}$. So we have shown:

Theorem 8. We have

$$\frac{AB}{CD} = \frac{OA \cdot OB}{OC \cdot OD}; \qquad \frac{BC}{DA} = \frac{OB \cdot OC}{OD \cdot OA}.$$

Proving these equations was a 10th grade problem in the 4th round of the 14th DeMO (East German mathematical olympiad) 1974/75. This theorem entails

$$\frac{AB \cdot BC}{CD \cdot DA} = \frac{AB}{CD} \cdot \frac{BC}{DA} = \frac{OA \cdot OB}{OC \cdot OD} \cdot \frac{OB \cdot OC}{OD \cdot OA} = \frac{OB^2}{OD^2},$$

or, equivalently,

$$\frac{OB^2}{AB \cdot BC} = \frac{OD^2}{CD \cdot DA}.$$

Similarly, $\dfrac{OA^2}{DA \cdot AB} = \dfrac{OC^2}{BC \cdot CD}$. Thus we arrive at:

Theorem 9. We have

$$\frac{OB^2}{AB \cdot BC} = \frac{OD^2}{CD \cdot DA}; \qquad \frac{OA^2}{DA \cdot AB} = \frac{OC^2}{BC \cdot CD}.$$

This also appears with proof in [5].

Now we show a harder identity given in the China IMO TST 2003 ([8]):

Theorem 10. We have

$$OA \cdot OC + OB \cdot OD = \sqrt{AB \cdot BC \cdot CD \cdot DA}.$$

Proof of Theorem 10. (See Fig. 14.) Let X' and Z' be the antipodes of the points X and Z on the incircle of the quadrilateral $ABCD$ [5], or, in other words, the reflections of the points X and Z with respect to the center O of this incircle. Then, the segment XX' is a diameter of the incircle of the quadrilateral $ABCD$, and thus $\measuredangle XYX' = 90°$, so that $YX' \perp XY$. On the other hand, $XY \perp BO$. Hence, $YX' \parallel BO$, so that $\measuredangle XX'Y = \measuredangle BOX$. Together with $\measuredangle XYX' = \measuredangle BXO$ (since $\measuredangle XYX' = 90°$ and $\measuredangle BXO = 90°$) this entails that the triangles $XX'Y$ and BOX are similar; consequently, $\dfrac{X'Y}{X'X} = \dfrac{OX}{OB}$, so that $X'Y = X'X \cdot \dfrac{OX}{OB}$. Now, $X'X = 2 \cdot OX$ (since the point X' is the reflection of X in O), and thus

$$X'Y = 2 \cdot OX \cdot \frac{OX}{OB} = \frac{2 \cdot OX^2}{OB} = \frac{2\rho^2}{OB}.$$

[5]The *antipode* of a point P on a circle k is defined as the point P' on the circle k such that the segment PP' is a diameter of k.

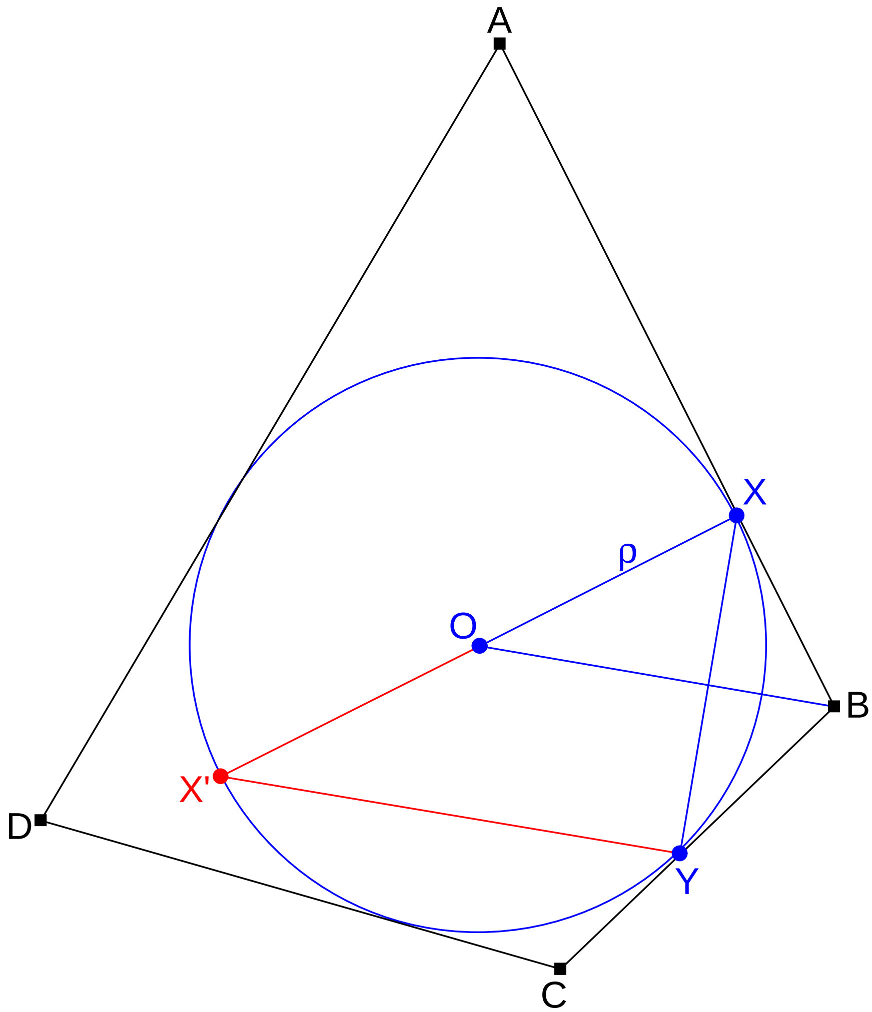

Fig. 14

Similarly,

$$Z'Y = \frac{2\rho^2}{OC}; \qquad Z'W = \frac{2\rho^2}{OD}; \qquad X'W = \frac{2\rho^2}{OA}.$$

Finally, $X'Z' = XZ$, since the points X' and Z' are the reflections of the points X and Z in the point O, and reflections preserve distances.

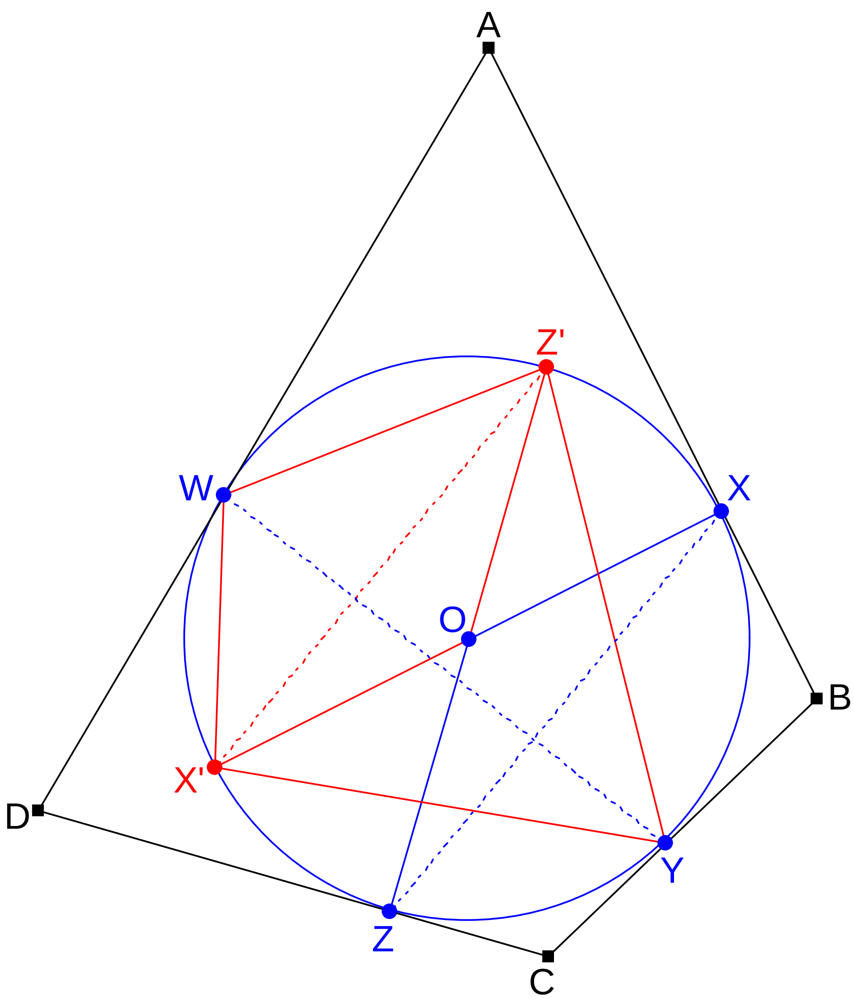

Fig. 15

(See Fig. 15.) Now, the points X', Y, Z', W all lie on the incircle of the quadrilateral $ABCD$; thus, the quadrilateral $X'YZ'W$ is cyclic, so that, after the Ptolemy theorem,

$$X'Y \cdot Z'W + X'W \cdot Z'Y = X'Z' \cdot YW.$$

According to the above formulas, this becomes

$$\begin{aligned} \frac{2\rho^2}{OB} \cdot \frac{2\rho^2}{OD} + \frac{2\rho^2}{OA} \cdot \frac{2\rho^2}{OC} &= XZ \cdot YW, \qquad \text{i. e.} \\ 4\rho^4 \cdot \left(\frac{1}{OB \cdot OD} + \frac{1}{OA \cdot OC} \right) &= XZ \cdot YW, \qquad \text{i. e.} \\ 4\rho^4 \cdot \frac{OA \cdot OC + OB \cdot OD}{OA \cdot OB \cdot OC \cdot OD} &= XZ \cdot YW. \end{aligned}$$

Hence,

$$OA \cdot OC + OB \cdot OD = \frac{XZ \cdot YW \cdot OA \cdot OB \cdot OC \cdot OD}{4\rho^4}. \qquad (1)$$

But Theorem 7 yields

$$\begin{aligned} & AB \cdot BC \cdot CD \cdot DA \\ = & \left(\frac{YW}{2} \cdot \frac{OA \cdot OB}{\rho^2} \right) \cdot \left(\frac{XZ}{2} \cdot \frac{OB \cdot OC}{\rho^2} \right) \cdot \left(\frac{YW}{2} \cdot \frac{OC \cdot OD}{\rho^2} \right) \cdot \left(\frac{XZ}{2} \cdot \frac{OD \cdot OA}{\rho^2} \right) \\ = & \left(\frac{XZ \cdot YW \cdot OA \cdot OB \cdot OC \cdot OD}{4\rho^4} \right)^2, \end{aligned}$$

so that

$$\frac{XZ \cdot YW \cdot OA \cdot OB \cdot OC \cdot OD}{4\rho^4} = \sqrt{AB \cdot BC \cdot CD \cdot DA}.$$

Hence, (1) becomes

$$OA \cdot OC + OB \cdot OD = \sqrt{AB \cdot BC \cdot CD \cdot DA},$$

and Theorem 10 is proven.

For our further observations, we denote by $|P_1P_2...P_n|$ the (non-directed) area of an arbitrary polygon $P_1P_2...P_n$.

Now, we denote the interior angles of the quadrilateral $ABCD$ by

$$\alpha = \measuredangle DAB; \qquad \beta = \measuredangle ABC; \qquad \gamma = \measuredangle BCD; \qquad \delta = \measuredangle CDA.$$

Then, we are going to show:

Theorem 11. We have

$$OA \cdot OC = \frac{(a+c)\cdot\rho}{\sin\dfrac{\alpha+\gamma}{2}}; \qquad OB \cdot OD = \frac{(b+d)\cdot\rho}{\sin\dfrac{\beta+\delta}{2}};$$

$$\frac{OA \cdot OC}{OB \cdot OD} = \frac{a+c}{b+d}; \qquad OA \cdot OC + OB \cdot OD = \frac{(a+b+c+d)\cdot\rho}{\sin\dfrac{\alpha+\gamma}{2}}.$$

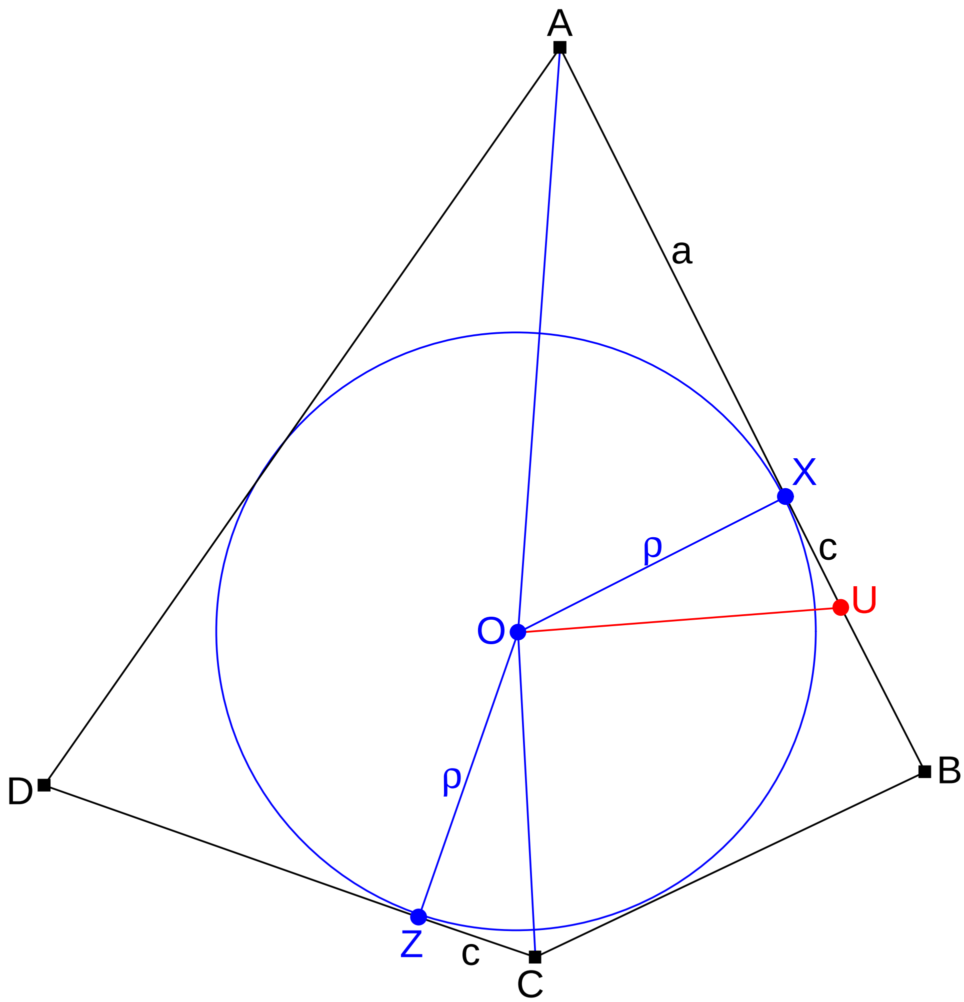

Fig. 16

Proof of Theorem 11. (See Fig. 16.) Let U be the point on the ray XB satisfying $UX = c$. Comparing this with $c = CZ$, we get $UX = CZ$. Furthermore, $\measuredangle OXU = 90^\circ = \measuredangle OZC$ and $OX = OZ$. Thus, the triangles OXU and OZC are congruent, so that $OU = OC$ and $\measuredangle XOU = \measuredangle ZOC$.

Since the point O, being the incenter of the quadrilateral $ABCD$, lies on the angle bisector of its angle DAB, we have $\measuredangle XAO = \frac{\measuredangle DAB}{2} = \frac{\alpha}{2}$; in the right-angled triangle AXO, we thus obtain $\measuredangle XOA = 90^\circ - \measuredangle XAO = 90^\circ - \frac{\alpha}{2}$. Similarly, $\measuredangle ZOC = 90^\circ - \frac{\gamma}{2}$; since $\measuredangle XOU = \measuredangle ZOC$, this becomes $\measuredangle XOU = 90^\circ - \frac{\gamma}{2}$. Hence, $\measuredangle AOU = \measuredangle XOA + \measuredangle XOU = \left(90^\circ - \frac{\alpha}{2}\right) + \left(90^\circ - \frac{\gamma}{2}\right) = 180^\circ - \frac{\alpha+\gamma}{2}$, so that $\sin \measuredangle AOU = \sin \frac{\alpha+\gamma}{2}$.

From $AX = a$ and $UX = c$, we conclude that $AU = AX + UX = a + c$.

Now, the area of a triangle equals half of the product of two of its sides and the sine of the angle between them; applying this to triangle AOU, we get $|AOU| = \frac{1}{2} \cdot OA \cdot OU \cdot \sin \measuredangle AOU$; since $OU = OC$ and $\sin \measuredangle AOU = \sin \frac{\alpha+\gamma}{2}$, this becomes $|AOU| = \frac{1}{2} \cdot OA \cdot OC \cdot \sin \frac{\alpha+\gamma}{2}$.

On the other hand, the area of a triangle equals half of the product of a side with the respective altitude; applied to the triangle AOU (in which OX is the altitude to the side AU), this yields $|AOU| = \frac{1}{2} \cdot AU \cdot OX$; since $AU = a + c$ and $OX = \rho$, this rewrites as $|AOU| = \frac{1}{2} \cdot (a+c) \cdot \rho$.

Comparing the equations $|AOU| = \frac{1}{2} \cdot OA \cdot OC \cdot \sin \frac{\alpha+\gamma}{2}$ and $|AOU| = \frac{1}{2} \cdot (a+c) \cdot \rho$, we see that $OA \cdot OC \cdot \sin \frac{\alpha+\gamma}{2} = (a+c) \cdot \rho$, and thus

$$OA \cdot OC = \frac{(a+c) \cdot \rho}{\sin \frac{\alpha+\gamma}{2}}.$$

Similarly,

$$OB \cdot OD = \frac{(b+d) \cdot \rho}{\sin \frac{\beta+\delta}{2}}.$$

Now, by the sum of angles in the quadrilateral $ABCD$, we have $\alpha + \beta + \gamma + \delta = 360^\circ$, so that $\frac{\alpha+\gamma}{2} + \frac{\beta+\delta}{2} = \frac{\alpha+\beta+\gamma+\delta}{2} = \frac{360^\circ}{2} = 180^\circ$, and thus $\sin \frac{\beta+\delta}{2} = \sin \frac{\alpha+\gamma}{2}$. Hence, the equation

$$OB \cdot OD = \frac{(b+d) \cdot \rho}{\sin \frac{\beta+\delta}{2}} \qquad \text{becomes} \qquad OB \cdot OD = \frac{(b+d) \cdot \rho}{\sin \frac{\alpha+\gamma}{2}}.$$

Thus,

$$\frac{OA \cdot OC}{OB \cdot OD} = \frac{\left(\dfrac{(a+c) \cdot \rho}{\sin \dfrac{\alpha+\gamma}{2}}\right)}{\left(\dfrac{(b+d) \cdot \rho}{\sin \dfrac{\alpha+\gamma}{2}}\right)} = \frac{a+c}{b+d}$$

and

$$OA \cdot OC + OB \cdot OD = \frac{(a+c) \cdot \rho}{\sin \dfrac{\alpha+\gamma}{2}} + \frac{(b+d) \cdot \rho}{\sin \dfrac{\alpha+\gamma}{2}} = \frac{(a+b+c+d) \cdot \rho}{\sin \dfrac{\alpha+\gamma}{2}}.$$

Therefore, Theorem 11 is proven.

Now, Theorem 11 asserts

$$OA \cdot OC + OB \cdot OD = \frac{(a+b+c+d) \cdot \rho}{\sin \dfrac{\alpha+\gamma}{2}},$$

while Theorem 10 states that

$$OA \cdot OC + OB \cdot OD = \sqrt{AB \cdot BC \cdot CD \cdot DA}.$$

Hence,

$$\frac{(a+b+c+d) \cdot \rho}{\sin \dfrac{\alpha+\gamma}{2}} = \sqrt{AB \cdot BC \cdot CD \cdot DA},$$

so that

$$(a+b+c+d) \cdot \rho = \sqrt{AB \cdot BC \cdot CD \cdot DA} \cdot \sin \frac{\alpha+\gamma}{2}.$$

(See Fig. 13.) Now, the area of a right-angled triangle equals half of the product of its two catets; for the right-angled triangle AWO, this yields $|AWO| = \frac{1}{2} \cdot AW \cdot OW = \frac{1}{2} \cdot a \cdot \rho$. Similarly, $|AXO| = \frac{1}{2} \cdot a \cdot \rho$, and thus $|AWOX| = |AWO| + |AXO| = \frac{1}{2} \cdot a \cdot \rho + \frac{1}{2} \cdot a \cdot \rho = a \cdot \rho$. Similarly, $|BXOY| = b \cdot \rho$, $|CYOZ| = c \cdot \rho$ and $|DZOW| = d \cdot \rho$. Hence,

$$\begin{aligned} |ABCD| &= |AWOX| + |BXOY| + |CYOZ| + |DZOW| = a \cdot \rho + b \cdot \rho + c \cdot \rho + d \cdot \rho \\ &= (a+b+c+d) \cdot \rho = \sqrt{AB \cdot BC \cdot CD \cdot DA} \cdot \sin \frac{\alpha+\gamma}{2}. \end{aligned}$$

Thus, we conclude:

Theorem 12. The area $|ABCD|$ of a circumscribed quadrilateral $ABCD$ equals

$$|ABCD| = \sqrt{AB \cdot BC \cdot CD \cdot DA} \cdot \sin \frac{\alpha+\gamma}{2}.$$

This is not an unknown formula, however it is usually derived from the generalized Brahmagupta formula for the area of an arbitrary quadrilateral ([9]) which, in turn, is proven by a long trigonometric calculation. Here we gave a rather long, yet synthetic proof of Theorem 12.

Next, we are going to prove a result due to A. Zaslavsky, M. Isaev and D. Tsvetov which was given in the final (fifth) round of the Allrussian Mathematical Olympiad 2005 as problem 7 for class 11 ([11]):

Theorem 13. The incenter O of a circumscribed quadrilateral $ABCD$ coincides with the centroid of the quadrilateral $ABCD$ if and only if either the quadrilateral $ABCD$ is a rhombus or $OA \cdot OC = OB \cdot OD$. (See Fig. 17.)

Hereby, the *centroid* of the quadrilateral $ABCD$ is defined as follows:

Let E, F, G, H be the midpoints of the sides AB, BC, CD, DA of the quadrilateral $ABCD$. Then, according to the Varignon theorem, the quadrilateral $EFGH$ is a parallelogram, so that its two diagonals EG and FH bisect each other. In other words, the segments EG and FH have a common midpoint. This midpoint is called the **centroid** of the quadrilateral $ABCD$.

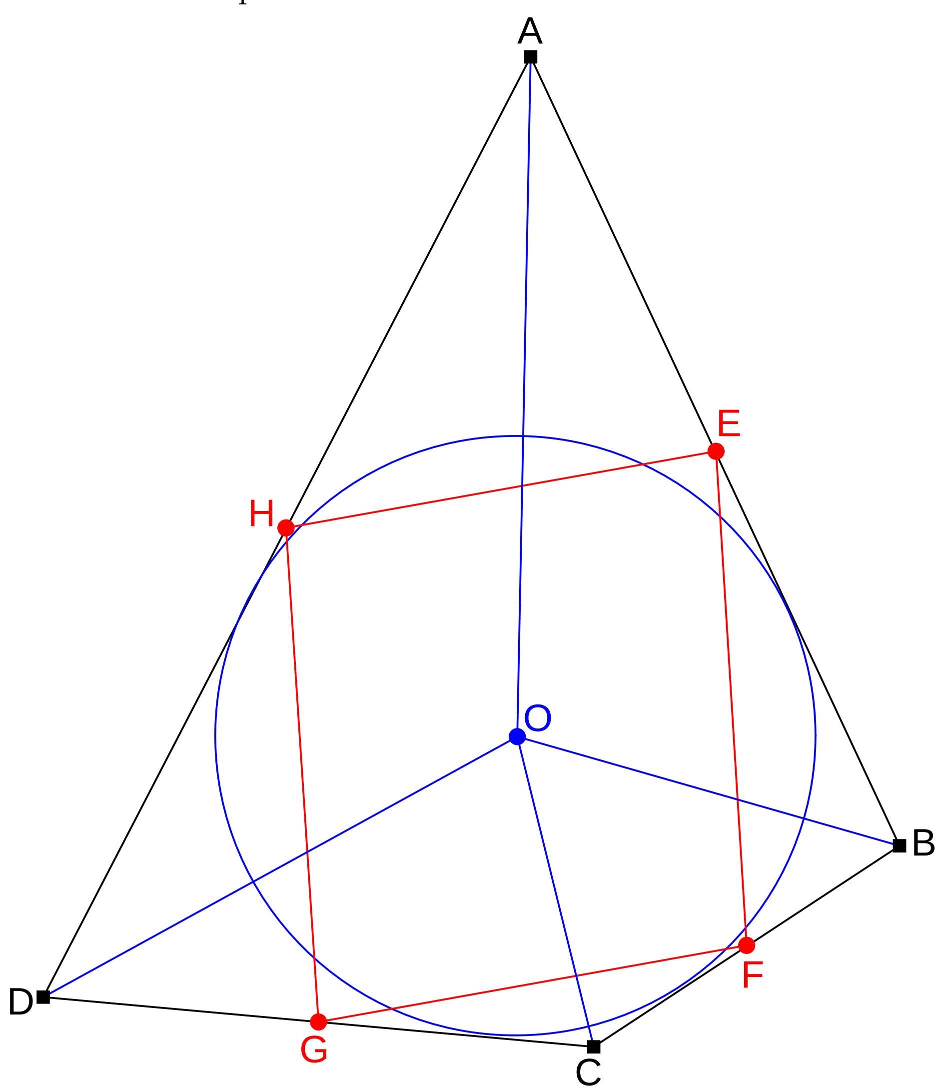

Fig. 17

Now, let's *prove Theorem 13.* In order to do this, we have to verify two assertions:

Assertion 1. If the point O is the centroid of the quadrilateral $ABCD$, then either the quadrilateral $ABCD$ is a rhombus or $OA \cdot OC = OB \cdot OD$.

Assertion 2. If either the quadrilateral $ABCD$ is a rhombus or $OA \cdot OC = OB \cdot OD$, then the point O is the centroid of the quadrilateral $ABCD$.

Before we establish any of these assertions, we start with a few observations holding for every circumscribed quadrilateral $ABCD$ (Fig. 18):

Since the point E is the midpoint of the segment AB, we have $AE = \frac{AB}{2} = \frac{a+b}{2}$, and thus

$$EX = |AX - AE| = \left| a - \frac{a+b}{2} \right| = \left| \frac{a-b}{2} \right| = \frac{|a-b|}{2}.$$

Similarly, $GZ = \frac{|c-d|}{2}$.

Also, note that the triangles EOX and GOZ are right-angled at their vertices X and Z, since $\measuredangle OXE = 90^\circ$ and $\measuredangle OZG = 90^\circ$.

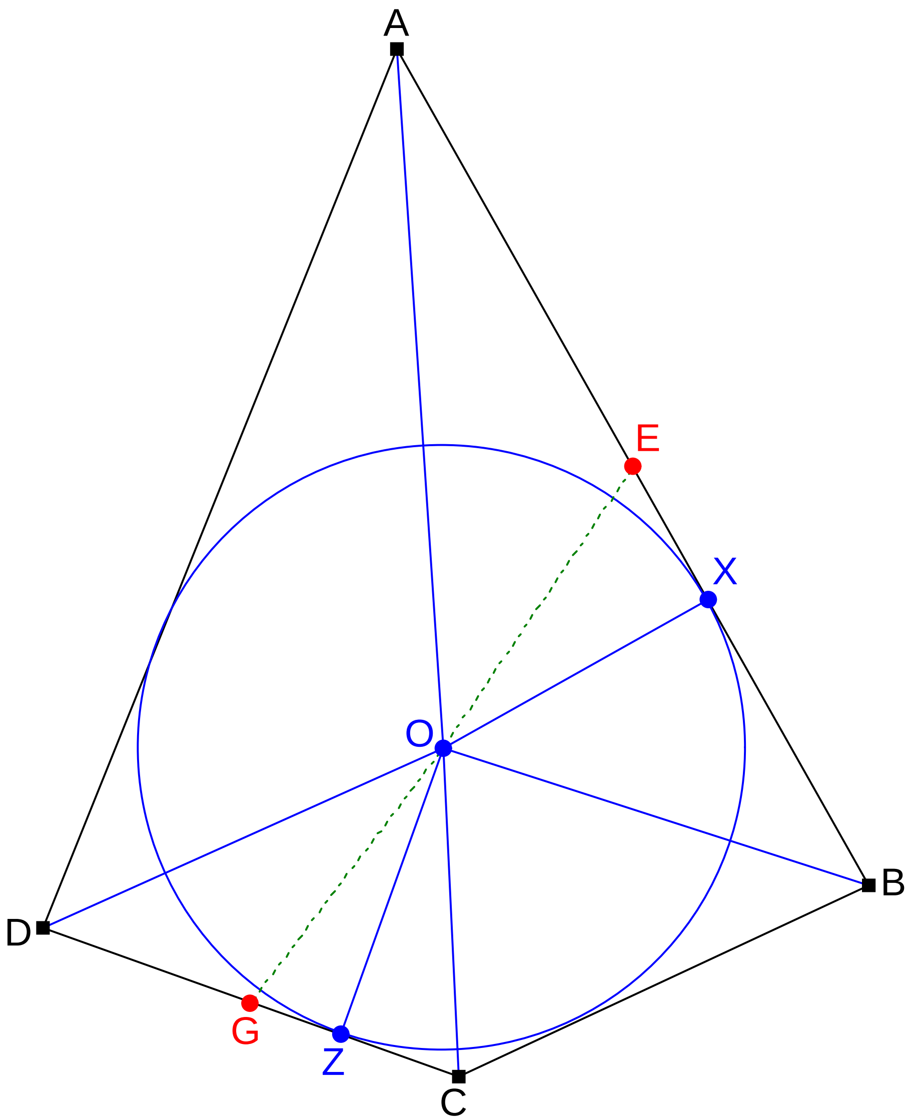

Fig. 18

Now, we are going to establish Assertions 1 and 2.

Proof of Assertion 1. We distinguish between two cases:

Case 1: We have $a + c \neq b + d$.

Case 2: We have $a + c = b + d$.

Let us first consider Case 1. The point O is the centroid of the quadrilateral $ABCD$, i. e. the midpoint of the segment EG. Thus, $OE = OG$. Also, $OX = OZ$.

Hence, the two right-angled triangles EOX and GOZ have the hypotenuse and one catet in common; thus, they are congruent, and we conclude that $EX = GZ$. Since $EX = \frac{|a-b|}{2}$ and $GZ = \frac{|c-d|}{2}$, this yields $|a-b| = |c-d|$. Thus, either $a-b = c-d$, or $a-b = d-c$. Now, $a-b = d-c$ would lead to $a+c = b+d$, what is impossible since we have $a+c \neq b+d$ (because we are in Case 1). Hence, it remains only the possibility $a-b = c-d$, that is, $a+d = b+c$. Similarly to $a-b = c-d$, we can prove that $a-d = c-b$, and thus $2a = (a+d)+(a-d) = (b+c)+(c-b) = 2c$. In other words, $a = c$. Similarly, $b = d$. Hence, opposite sides of the quadrilateral $ABCD$ are equal; this means that the quadrilateral $ABCD$ is a parallelogram, and since it is circumscribed, it must be a rhombus (in fact, among all parallelograms, only rhombi are circumscribed). Thus, we have shown that the quadrilateral $ABCD$ is a rhombus in Case 1.

Now, let us consider Case 2. In this case, $a+c = b+d$. As we have $\frac{OA \cdot OC}{OB \cdot OD} = \frac{a+c}{b+d}$ from Theorem 11, this yields $OA \cdot OC = OB \cdot OD$. Thus, $OA \cdot OC = OB \cdot OD$ holds in Case 2.

Hence, we have shown that the quadrilateral $ABCD$ is a rhombus in Case 1, and that $OA \cdot OC = OB \cdot OD$ in Case 2. Since these cases cover all possibilities, we conclude that either the quadrilateral $ABCD$ is a rhombus or $OA \cdot OC = OB \cdot OD$. Assertion 1 is proven.

Proof of Assertion 2. Assume that either the quadrilateral $ABCD$ is a rhombus or $OA \cdot OC = OB \cdot OD$. We can WLOG assume that $OA \cdot OC = OB \cdot OD$ (because the case when the quadrilateral $ABCD$ is a rhombus is trivial for symmetry reasons).

From Theorem 11, we have $\frac{OA \cdot OC}{OB \cdot OD} = \frac{a+c}{b+d}$, so that $OA \cdot OC = OB \cdot OD$ immediately yields $a+c = b+d$. Hence, $a-b = d-c$, and thus $EX = \frac{|a-b|}{2} = \frac{|d-c|}{2} = \frac{|c-d|}{2} = GZ$. Furthermore, $OX = OZ$. Thus, the two right-angled triangles EOX and GOZ have the same catets; hence, they are congruent, and it follows that $OE = OG$. So the point O lies on the perpendicular bisector of the segment EG. Similarly, the point O lies on the perpendicular bisector of the segment FH.

Since the circumscribed quadrilateral $ABCD$ is convex, and E, F, G, H are the midpoints of its sides, the lines EG and FH cannot be parallel. Thus, the perpendicular bisectors of the segments EG and FH are not parallel as well; therefore, they have one and only one common point. This common point is obviously the centroid of the quadrilateral $ABCD$ (since this centroid is the common midpoint of the segments EG and FH and thus lies on their perpendicular bisectors).

But as we have shown that the point O lies on the perpendicular bisectors of the segments EG and FH, the point O must be this common point. Hence, the point O is the centroid of the quadrilateral $ABCD$. Assertion 2 is shown, and the proof of Theorem 13 is complete.

Now we return to the case of an arbitrary circumscribed quadrilateral $ABCD$. We prove an identity formulated by Pengshi in [12]:

Theorem 14. The radius ρ of the incircle of the circumscribed quadrilateral $ABCD$ satisfies

$$\rho^2 = \frac{bcd + cda + dab + abc}{a+b+c+d}.$$

Our proof of this theorem will only slightly differ from Anipoh's in [12]; the key is the following lemma:

Theorem 15. Let x, y, z, w be four angles such that $x+y+z+w=180^\circ$. Then,

$$\begin{aligned} & \tan x+\tan y+\tan z+\tan w \\ = & \tan y\cdot\tan z\cdot\tan w+\tan z\cdot\tan w\cdot\tan x+\tan w\cdot\tan x\cdot\tan y+\tan x\cdot\tan y\cdot\tan z. \end{aligned}$$

Proof of Theorem 15. From $x+y+z+w=180^\circ$ it follows that $x+y=180^\circ-(z+w)$, so that $\tan(x+y)=\tan(180^\circ-(z+w))=-\tan(z+w)$ and thus $\tan(x+y)+\tan(z+w)=0$. But the addition formulas for the tan function yield $\tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}$ and $\tan(z+w)=\dfrac{\tan z+\tan w}{1-\tan z\tan w}$; hence, $\tan(x+y)+\tan(z+w)=0$ becomes $\dfrac{\tan x+\tan y}{1-\tan x\tan y}+\dfrac{\tan z+\tan w}{1-\tan z\tan w}=0$. Multiplication by $(1-\tan x\tan y)(1-\tan z\tan w)$ yields

$$(\tan x+\tan y)(1-\tan z\tan w)+(\tan z+\tan w)(1-\tan x\tan y)=0,$$

thus

$$\begin{aligned} &(\tan x+\tan y-\tan z\tan w\tan x-\tan y\tan z\tan w) \\ &\qquad +(\tan z+\tan w-\tan x\tan y\tan z-\tan w\tan x\tan y)=0, \end{aligned}$$

thus

$$\tan x+\tan y+\tan z+\tan w=\tan y\tan z\tan w+\tan z\tan w\tan x+\tan w\tan x\tan y+\tan x\tan y\tan z.$$

This proves Theorem 15.

Now we come to the *proof of Theorem 14:* With the notations α, β, γ, δ for the angles of the quadrilateral $ABCD$, we have

$$\alpha+\beta+\gamma+\delta=\measuredangle DAB+\measuredangle ABC+\measuredangle BCD+\measuredangle CDA=360^\circ$$

(by the sum of angles in the quadrilateral $ABCD$). Now set $x=\dfrac{\alpha}{2}$, $y=\dfrac{\beta}{2}$, $z=\dfrac{\gamma}{2}$, $w=\dfrac{\delta}{2}$. Then,

$$x+y+z+w=\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}=\frac{\alpha+\beta+\gamma+\delta}{2}=\frac{360^\circ}{2}=180^\circ.$$

Thus, Theorem 15 yields

$$\begin{aligned} & \tan x+\tan y+\tan z+\tan w \\ = & \tan y\cdot\tan z\cdot\tan w+\tan z\cdot\tan w\cdot\tan x+\tan w\cdot\tan x\cdot\tan y+\tan x\cdot\tan y\cdot\tan z. \end{aligned} \tag{2}$$

(See Fig. 16.) During the proof of Theorem 11, we have shown that $\measuredangle XAO=\dfrac{\alpha}{2}$. Since $OX\perp AB$, the triangle AXO is right-angled at X. Hence, $OX=AX\cdot\tan\measuredangle XAO$, so that $\rho=a\cdot\tan x$ (since $OX=\rho$, $AX=a$ and $\measuredangle XAO=\dfrac{\alpha}{2}=x$). Thus $\tan x=\dfrac{\rho}{a}$; similarly, $\tan y=\dfrac{\rho}{b}$, $\tan z=\dfrac{\rho}{c}$, and $\tan w=\dfrac{\rho}{d}$. Thus, (2) becomes

$$\frac{\rho}{a}+\frac{\rho}{b}+\frac{\rho}{c}+\frac{\rho}{d}=\frac{\rho}{b}\cdot\frac{\rho}{c}\cdot\frac{\rho}{d}+\frac{\rho}{c}\cdot\frac{\rho}{d}\cdot\frac{\rho}{a}+\frac{\rho}{d}\cdot\frac{\rho}{a}\cdot\frac{\rho}{b}+\frac{\rho}{a}\cdot\frac{\rho}{b}\cdot\frac{\rho}{c}.$$

Multiplication by $abcd$ yields

$$\rho bcd + \rho cda + \rho dab + \rho abc = \rho^3 a + \rho^3 b + \rho^3 c + \rho^3 d.$$

In other words,

$$\begin{aligned} \rho\left(bcd + cda + dab + abc\right) &= \rho^3\left(a+b+c+d\right), \qquad \text{so that} \\ \rho^2 &= \frac{bcd + cda + dab + abc}{a+b+c+d}, \end{aligned}$$

what proves Theorem 14.

A notation: If P is a point, and g is a line, we denote by $\operatorname{dist}(P;\ g)$ the (non-directed) distance from the point P to the line g. We will often use the following fact:

Area-distance relation: For any three points U, V, W we have $|UVW| = \frac{1}{2} \cdot VW \cdot \operatorname{dist}(U;\ VW)$.

This fact is just a restatement of the fact that the area of a triangle equals $\frac{1}{2} \cdot$ sidelength $\cdot$ corresponding altitude (since in triangle UVW, the altitude from U to VW is $\operatorname{dist}(U;\ VW)$).

Now comes an easy corollary of Theorem 3 (Fig. 4):

Theorem 16. We have

$$\frac{|APB|}{ab} = \frac{|BPC|}{bc} = \frac{|CPD|}{cd} = \frac{|DPA|}{da}. \tag{3}$$

Proof of Theorem 16. By the area-distance relation, $|BAP| = \frac{1}{2} \cdot AP \cdot \operatorname{dist}(B;\ AP)$ and $|BCP| = \frac{1}{2} \cdot CP \cdot \operatorname{dist}(B;\ CP)$, so that

$$\frac{|APB|}{|BPC|} = \frac{|BAP|}{|BCP|} = \frac{\frac{1}{2} \cdot AP \cdot \operatorname{dist}(B;\ AP)}{\frac{1}{2} \cdot CP \cdot \operatorname{dist}(B;\ CP)} = \frac{AP}{CP} \cdot \frac{\operatorname{dist}(B;\ AP)}{\operatorname{dist}(B;\ CP)}.$$

Now, $\frac{\operatorname{dist}(B;\ AP)}{\operatorname{dist}(B;\ CP)} = 1$ (since $\operatorname{dist}(B;\ AP) = \operatorname{dist}(B;\ CP)$, because AP and CP are the same line), and $\frac{AP}{CP} = \frac{a}{c}$ by Theorem 3. Hence, we get $\frac{|APB|}{|BPC|} = \frac{a}{c} \cdot 1 = \frac{a}{c} = \frac{ab}{bc}$, so that $\frac{|APB|}{ab} = \frac{|BPC|}{bc}$. Similarly, $\frac{|BPC|}{bc} = \frac{|CPD|}{cd}$ and $\frac{|CPD|}{cd} = \frac{|DPA|}{da}$. This proves Theorem 16.

Now we shall show a result by A. Zaslavsky from [13] (see also [14]) (Fig. 19):

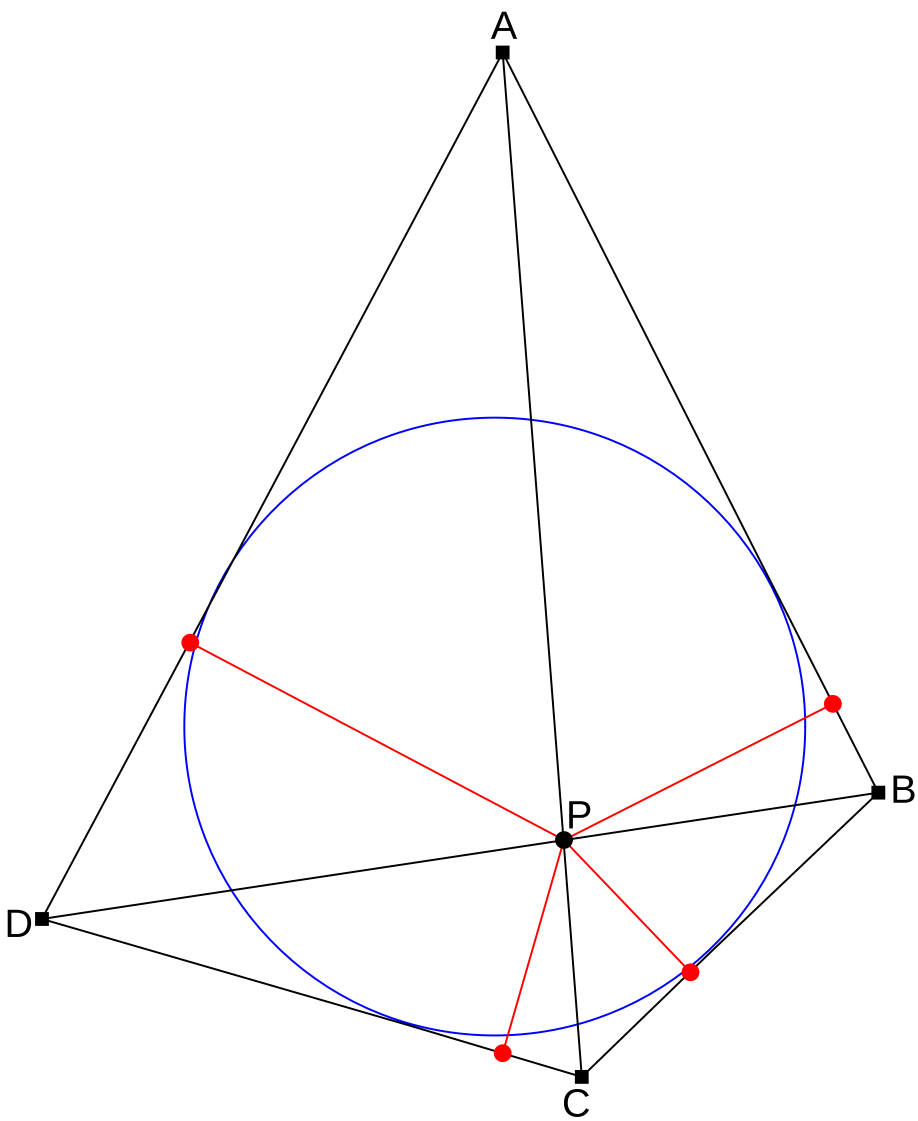

Fig. 19

Theorem 17. We have

$$\frac{1}{\operatorname{dist}(P;\ AB)}+\frac{1}{\operatorname{dist}(P;\ CD)}=\frac{1}{\operatorname{dist}(P;\ BC)}+\frac{1}{\operatorname{dist}(P;\ DA)}.$$

Proof of Theorem 17. Due to the equation (3), we can define

$$\lambda=\frac{|APB|}{ab}=\frac{|BPC|}{bc}=\frac{|CPD|}{cd}=\frac{|DPA|}{da}.$$

Then, $|APB|=\lambda ab$.

By the area-distance relation, $|PAB|=\frac{1}{2}\cdot AB\cdot\operatorname{dist}(P;\ AB)$, so that

$$\operatorname{dist}(P;\ AB)=\frac{2\cdot|PAB|}{AB}=\frac{2\cdot|APB|}{AB}=\frac{2\cdot\lambda ab}{a+b}\qquad(\text{as } |APB|=\lambda ab \text{ and } AB=a+b),$$

and thus

$$\frac{1}{\operatorname{dist}(P;\ AB)}=1\Big/\frac{2\cdot\lambda ab}{a+b}=\frac{a+b}{2\cdot\lambda ab}=\frac{1}{2\lambda}\cdot\frac{a+b}{ab}=\frac{1}{2\lambda}\left(\frac{1}{a}+\frac{1}{b}\right).$$

Similarly, $\dfrac{1}{\text{dist}\,(P;\ CD)} = \dfrac{1}{2\lambda}\left(\dfrac{1}{c} + \dfrac{1}{d}\right)$, so that

$$\frac{1}{\text{dist}\,(P;\ AB)} + \frac{1}{\text{dist}\,(P;\ CD)} = \frac{1}{2\lambda}\left(\frac{1}{a} + \frac{1}{b}\right) + \frac{1}{2\lambda}\left(\frac{1}{c} + \frac{1}{d}\right) = \frac{1}{2\lambda}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right).$$

Similarly,

$$\frac{1}{\text{dist}\,(P;\ BC)} + \frac{1}{\text{dist}\,(P;\ DA)} = \frac{1}{2\lambda}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right).$$

Thus,

$$\frac{1}{\text{dist}\,(P;\ AB)} + \frac{1}{\text{dist}\,(P;\ CD)} = \frac{1}{\text{dist}\,(P;\ BC)} + \frac{1}{\text{dist}\,(P;\ DA)},$$

and Theorem 17 is proven.

Next comes a result whose part **a)** appeared in [15] (with a different proof) (Fig. 20):

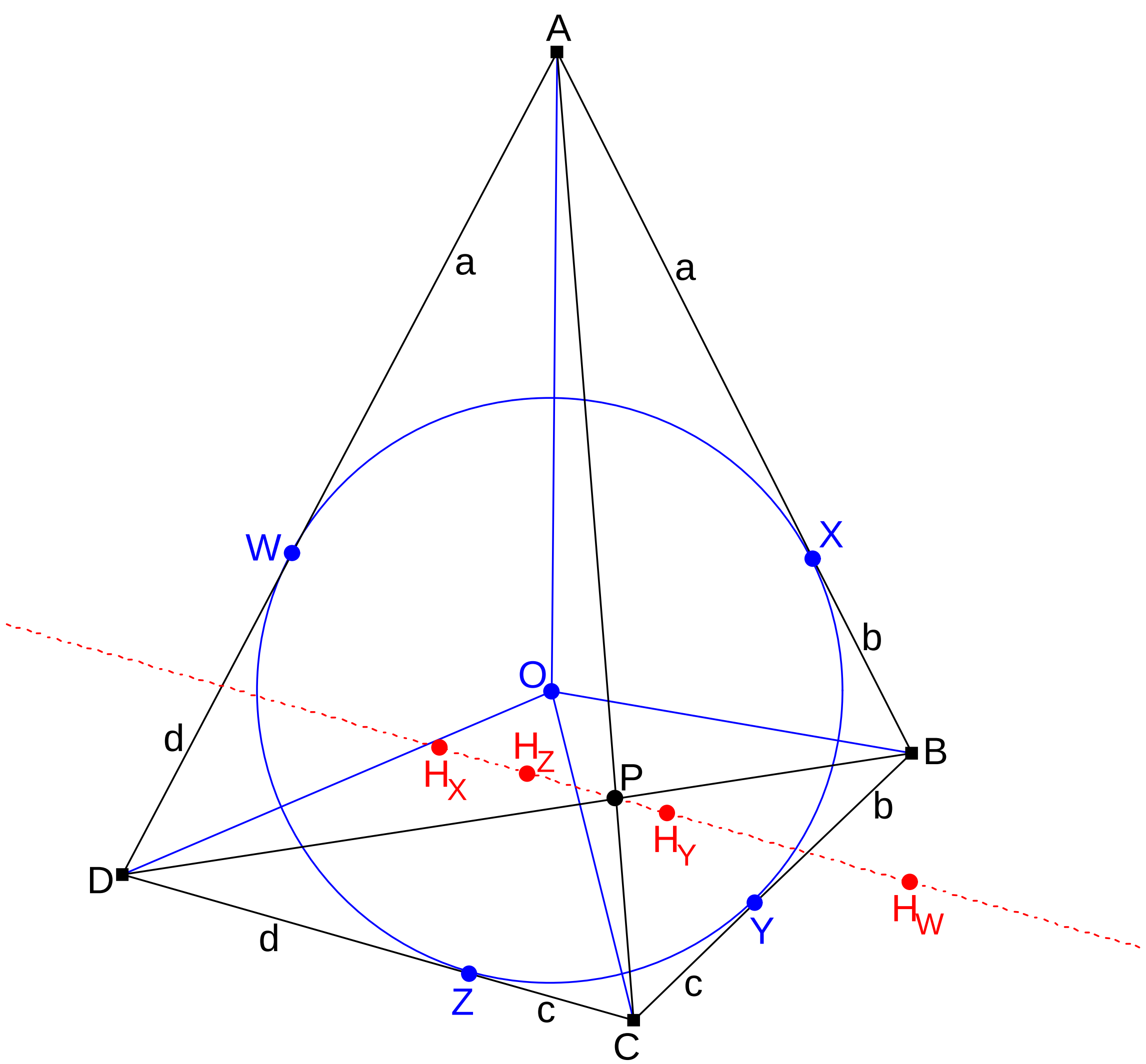

Fig. 20

Theorem 18. Let H_X, H_Y, H_Z, H_W be the orthocenters of triangles AOB, BOC, COD, DOA.

a) The points P, H_X, H_Y, H_Z, H_W are collinear.

b) Using directed segments, we have

$$-\frac{\overline{PH_X}}{ab} = \frac{\overline{PH_Y}}{bc} = -\frac{\overline{PH_Z}}{cd} = \frac{\overline{PH_W}}{da}.$$

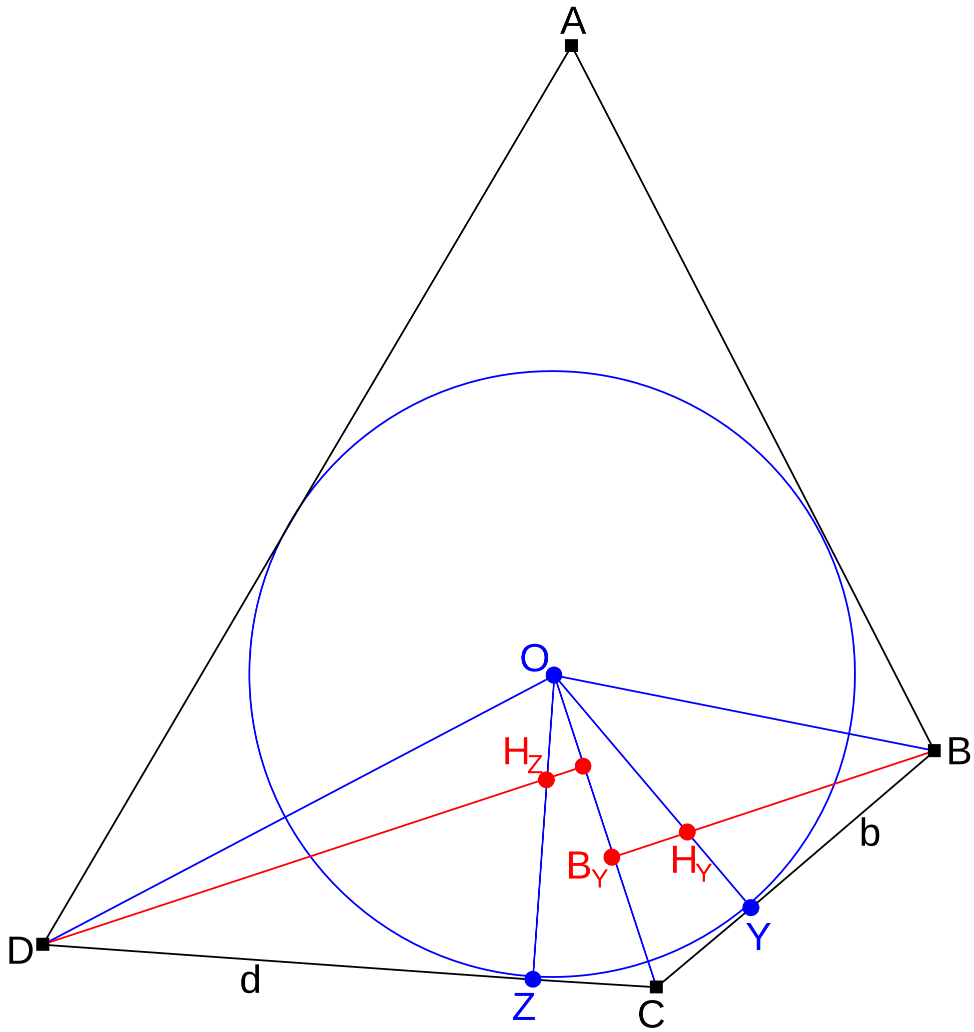

Fig. 21

Proof of Theorem 18. (See Fig. 21.) Let B_Y be the foot of the altitude of triangle BOC issuing from B. Then, the lines BB_Y and OY are two altitudes of triangle BOC (for BB_Y, this is clear, and for OY it follows from $OY \perp BC$), and thus intersect at the orthocenter H_Y of this triangle. Hence, $\measuredangle BYH_Y = 90^\circ$ and

$$\begin{aligned} \measuredangle YBH_Y &= \measuredangle CBB_Y = 90^\circ - \measuredangle BCB_Y \qquad \text{(in the right-angled triangle } BB_YC) \\ &= 90^\circ - \measuredangle BCO. \end{aligned}$$

Thus we have shown that $\measuredangle BYH_Y = 90^\circ$ and $\measuredangle YBH_Y = 90^\circ - \measuredangle BCO$. Similarly, $\measuredangle DZH_Z = 90^\circ$ and $\measuredangle ZDH_Z = 90^\circ - \measuredangle DCO$.

The point O, being the incenter of the quadrilateral $ABCD$, lies on the angle bisector of the angle BCD. Thus, $\measuredangle BCO = \measuredangle DCO$.

From $\measuredangle BYH_Y = 90^\circ = \measuredangle DZH_Z$ and $\measuredangle YBH_Y = 90^\circ - \measuredangle BCO = 90^\circ - \measuredangle DCO = \measuredangle ZDH_Z$, it follows that triangles BYH_Y and DZH_Z are similar. Therefore, $\frac{BH_Y}{DH_Z} = \frac{BY}{DZ}$. Since $BY = b$ and $DZ = d$, this becomes $\frac{BH_Y}{DH_Z} = \frac{b}{d}$.

The line BH_Y is the line BB_Y; thus, $BB_Y \perp CO$ yields $BH_Y \perp CO$. Similarly, $DH_Z \perp CO$. Consequently, $BH_Y \parallel DH_Z$.

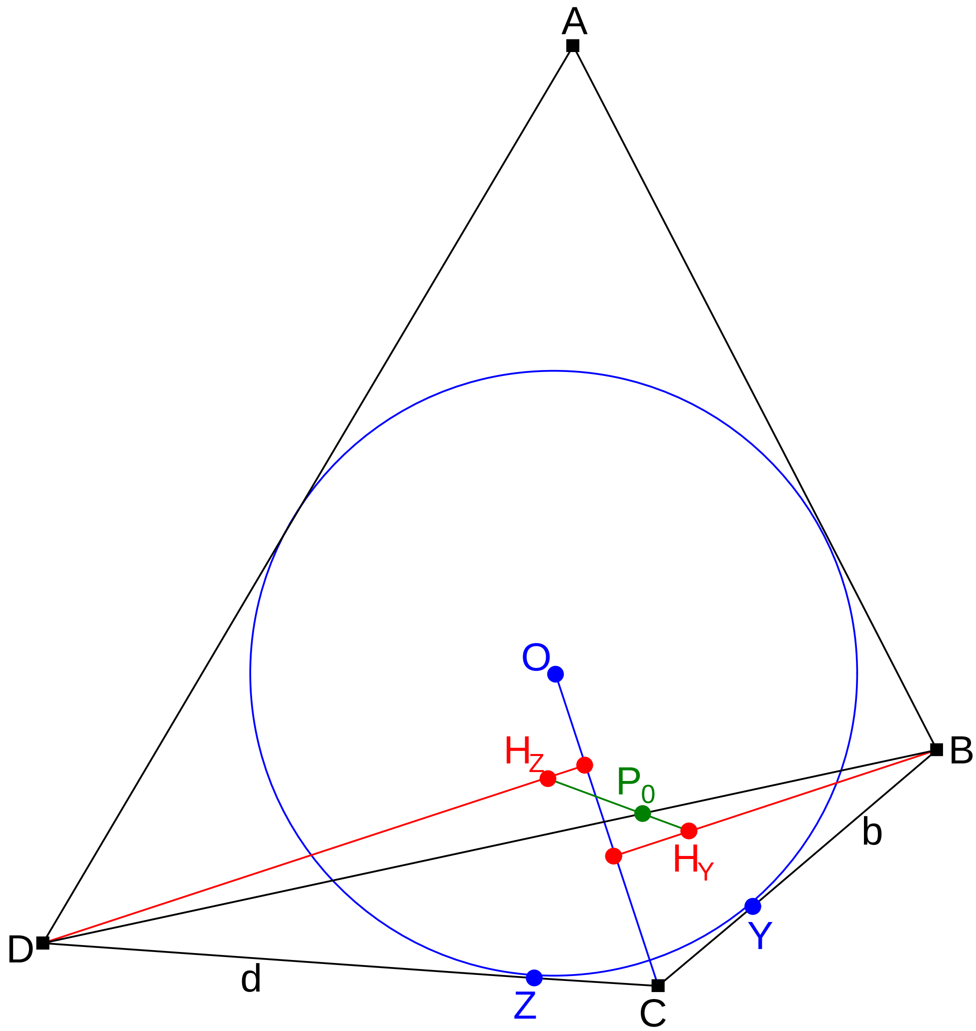

Fig. 22

(See Fig. 22.) Now, denote by P_0 the point of intersection of the lines H_YH_Z and BD. Since $BH_Y \parallel DH_Z$, the Thales theorem yields $\frac{BP_0}{DP_0} = \frac{BH_Y}{DH_Z}$. Since $\frac{BH_Y}{DH_Z} = \frac{b}{d}$, this becomes $\frac{BP_0}{DP_0} = \frac{b}{d}$. But Theorem 3 asserts $\frac{BP}{DP} = \frac{b}{d}$. Thus, $\frac{BP_0}{DP_0} = \frac{BP}{DP}$. Hence, the points P_0 and P divide the segment BD in the same ratio (both internally, as one can see by arrangement considerations[6]). Hence, these points P_0 and P must coincide. Thus, $P_0 \in H_YH_Z$ yields $P \in H_YH_Z$. Hence, the lines PH_Y and PH_Z coincide. Similarly, the lines PH_Z and PH_W coincide, and the lines PH_W and PH_X coincide. Thus, all four lines PH_X, PH_Y, PH_Z, PH_W coincide, i. e., the points P, H_X, H_Y, H_Z, H_W are collinear. Theorem 18 **a)** is proven.

Because of $BH_Y \parallel DH_Z$, the Thales theorem implies $\frac{P_0H_Y}{P_0H_Z} = \frac{BH_Y}{DH_Z}$. As we saw above, $P_0 = P$, so this becomes $\frac{PH_Y}{PH_Z} = \frac{BH_Y}{DH_Z}$. Together with $\frac{BH_Y}{DH_Z} = \frac{b}{d}$, this yields

[6] One could also avoid arrangement considerations by working consequently with directed segments, but this would require more theory.

$\frac{PH_Y}{PH_Z} = \frac{b}{d}$. With directed segments, this transforms into $\frac{\overline{PH_Y}}{\overline{PH_Z}} = -\frac{b}{d}$ (as arrangement considerations show that the directed ratio $\frac{\overline{PH_Y}}{\overline{PH_Z}}$ is negative). Thus, $d \cdot \overline{PH_Y} = -b \cdot \overline{PH_Z}$, so that $\frac{\overline{PH_Y}}{b} = -\frac{\overline{PH_Z}}{d}$. Dividing by c yields $\frac{\overline{PH_Y}}{bc} = -\frac{\overline{PH_Z}}{cd}$. Similarly, $\frac{\overline{PH_W}}{da} = -\frac{\overline{PH_Z}}{cd}$ and $\frac{\overline{PH_W}}{da} = -\frac{\overline{PH_X}}{ab}$. Thus, $-\frac{\overline{PH_X}}{ab} = \frac{\overline{PH_Y}}{bc} = -\frac{\overline{PH_Z}}{cd} = \frac{\overline{PH_W}}{da}$, and Theorem 18 **b)** is proven. This completes the proof of Theorem 18.

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