A Bundeswettbewerb Mathematik problem and its relation to the Nagel point of a triangle

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Some problems from the German National Mathematics Competition (Bundeswettbewerb Mathematik) are closely connected with Triangle Geometry. While in certain ones, triangles occur explicitely in the problem statement, there are also problems which are not immediately seen to have to do with triangles. An example of the second kind is the **Problem 3 of the Bundeswettbewerb Mathematik 2003**, **1 round**:

In a parallelogram $ABCD,$ points M and N are chosen on the sides AB and BC in a such way that they don't coincide with a vertex, and that the segments AM and NC have equal length. Let Q be the intersection of the segments AN and CM.
To prove that DQ bisects the angle ADC.

This problem is quickly rewritten "from the perspective of triangle ABC":

Let ABC be an arbitrary triangle. The parallel to BC through A meets the parallel to AB through C at D.
Now let M and N be points on the sides AB and $BC,$ which satisfy $AM = CN$.
To prove: The intersection Q of AN and CM lies on the angle bisector of the angle ADC.

The **solution** is not difficult: After the Sine Law in the triangles ADQ and $CDQ,$ we get

$$\frac{\sin\measuredangle ADQ}{\sin\measuredangle CDQ} = \frac{AQ \bullet \sin\measuredangle QAD : DQ}{CQ \bullet \sin\measuredangle QCD : DQ} = \frac{AQ}{CQ} \bullet \frac{\sin\measuredangle QAD}{\sin\measuredangle QCD}.$$

But $\measuredangle QAD = 180° - \measuredangle QNC$ (since $AD \parallel BC$); thus $\sin\measuredangle QAD = \sin\measuredangle QNC,$ and analogously $\sin\measuredangle QCD = \sin\measuredangle QMA,$ and consequently

$$\frac{\sin\measuredangle ADQ}{\sin\measuredangle CDQ} = \frac{AQ}{CQ} \bullet \frac{\sin\measuredangle QNC}{\sin\measuredangle QMA} = \frac{AQ}{\sin\measuredangle QMA} : \frac{CQ}{\sin\measuredangle QNC}.$$

After the Sine Law in the triangles AMQ and $CNQ,$ this transforms to

$$\frac{\sin\measuredangle ADQ}{\sin\measuredangle CDQ} = \frac{AM}{\sin\measuredangle AQM} : \frac{CN}{\sin\measuredangle CQN} = \frac{AM}{CN} \bullet \frac{\sin\measuredangle CQN}{\sin\measuredangle AQM}.$$

But $AM = CN$ and $\measuredangle CQN = \measuredangle AQM$. Thus,

$$\frac{\sin\measuredangle ADQ}{\sin\measuredangle CDQ} = 1 \bullet 1 = 1,$$

i. e. $\sin\measuredangle ADQ = \sin\measuredangle CDQ$. This yields either $\measuredangle ADQ = \measuredangle CDQ$ or $\measuredangle ADQ + \measuredangle CDQ = 180°$. But as $\measuredangle ADQ + \measuredangle CDQ = \measuredangle ADC \neq 180°,$ we must have $\measuredangle ADQ = \measuredangle CDQ$. Thus, the point Q lies on the angle bisector of the angle $ADC,$ what concludes the proof.

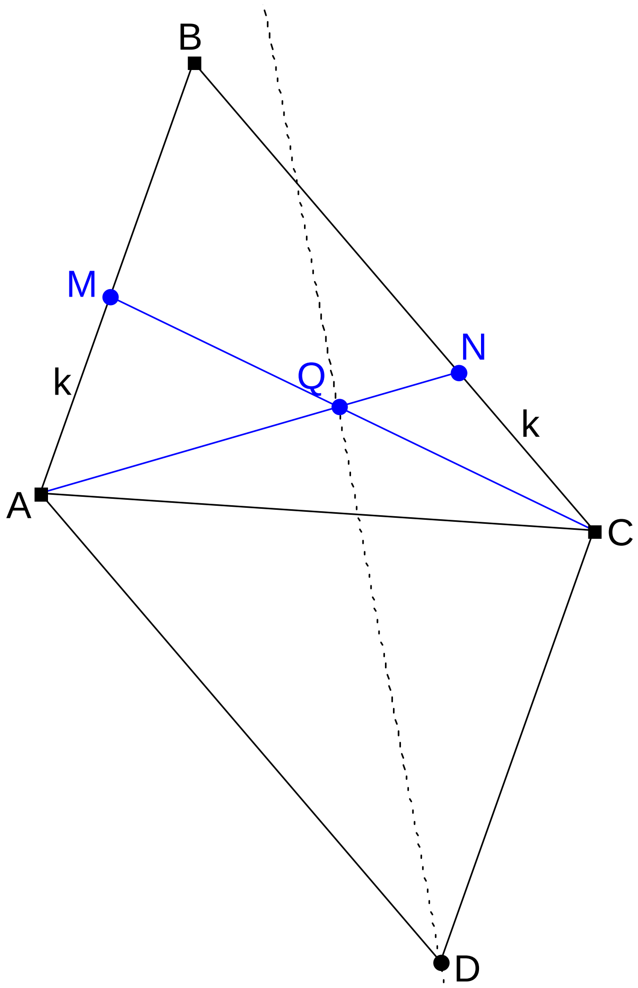

Fig. 1

Our problem faciliates the proof of the following theorem ([2], page 12; [3], page 55):

Nagel theorem. The incenter of a triangle ABC is the Nagel point of the medial triangle of ΔABC.

We begin with some explanations. The medial triangle of a triangle ABC is the triangle from the midpoints of the sides of ΔABC, i. e. from the midpoints of the segments BC, CA and AB. More difficult is the definition of the Nagel point (Fig. 2):

The excircle of triangle ABC which touches the side BC in the interior is called the *a*-**excircle** of triangle ABC. Let this a-excircle touch BC at N; similarly, let the b-excircle touch CA at P and the c-excircle touch AB at M.

Then the lines AN, BP and CM meet at a point, the so-called **Nagel point** of ΔABC.

The proof of the result that the lines AN, BP and CM meet at a point uses the following distances:

$$AM = s - b; \qquad BM = s - a;$$
$$BN = s - c; \qquad CN = s - b;$$
$$CP = s - a; \qquad AP = s - c,$$

where $s = \frac{1}{2}(a + b + c)$ is the halved perimeter of ΔABC. These distances were shown in [2], page 6, in [3], page 29, and in [4], chapter 1 §4.

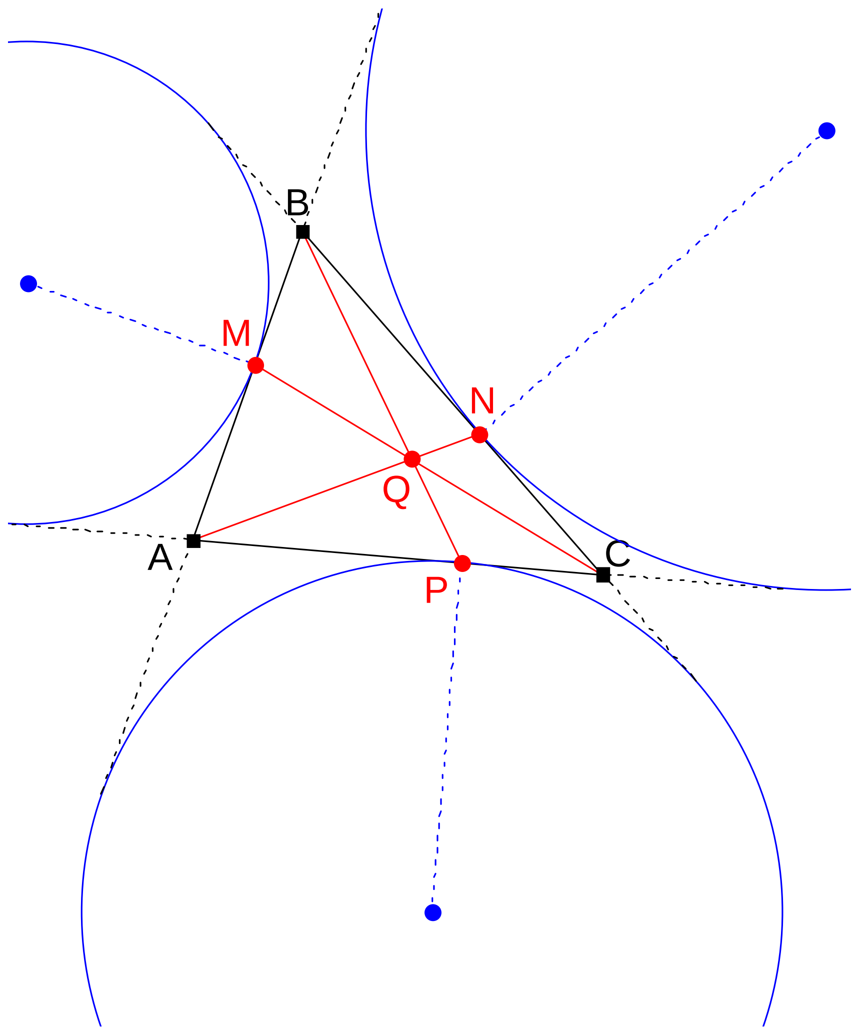

Fig. 2

This gives

$$AM = CN; \qquad CP = BM; \qquad BN = AP. \tag{1}$$

Then,

$$\frac{AM}{BM} \bullet \frac{BN}{CN} \bullet \frac{CP}{AP} = \frac{CN}{BM} \bullet \frac{AP}{CN} \bullet \frac{BM}{AP} = 1,$$

and with directed segments

$$\frac{AM}{MB} \bullet \frac{BN}{NC} \bullet \frac{CP}{PA} = 1.$$

Hence, after the Ceva theorem, the lines AN, BP and CM are concurrent. The existence of the Nagel point is established.

Now we undertake an auxiliary construction:

The parallels to BC through A, to CA through B, and to AB through C enclose a triangle GDE, which is called the **antimedial triangle** of ΔABC (see Fig. 3). Then, $ABCD$ is a parallelogram, and D is the intersection of the parallel to BC through A with the parallel to AB through C. Hence, the point D coincides with the point D from the problem. If Q is the Nagel point of triangle ABC, i. e. the intersection of the lines AN, BP and CM, we have $AM = CN$, and can apply the problem and get: The point Q lies on the angle bisector of the angle ADC.

But since this angle bisector is one of the three angle bisectors of triangle GDE, and since we can analogously prove that *Q* lies on the two other angle bisectors, *Q* is the incenter of triangle *GDE*.

In brief: We have shown that the Nagel point of a triangle is the incenter of the antimedial triangle.

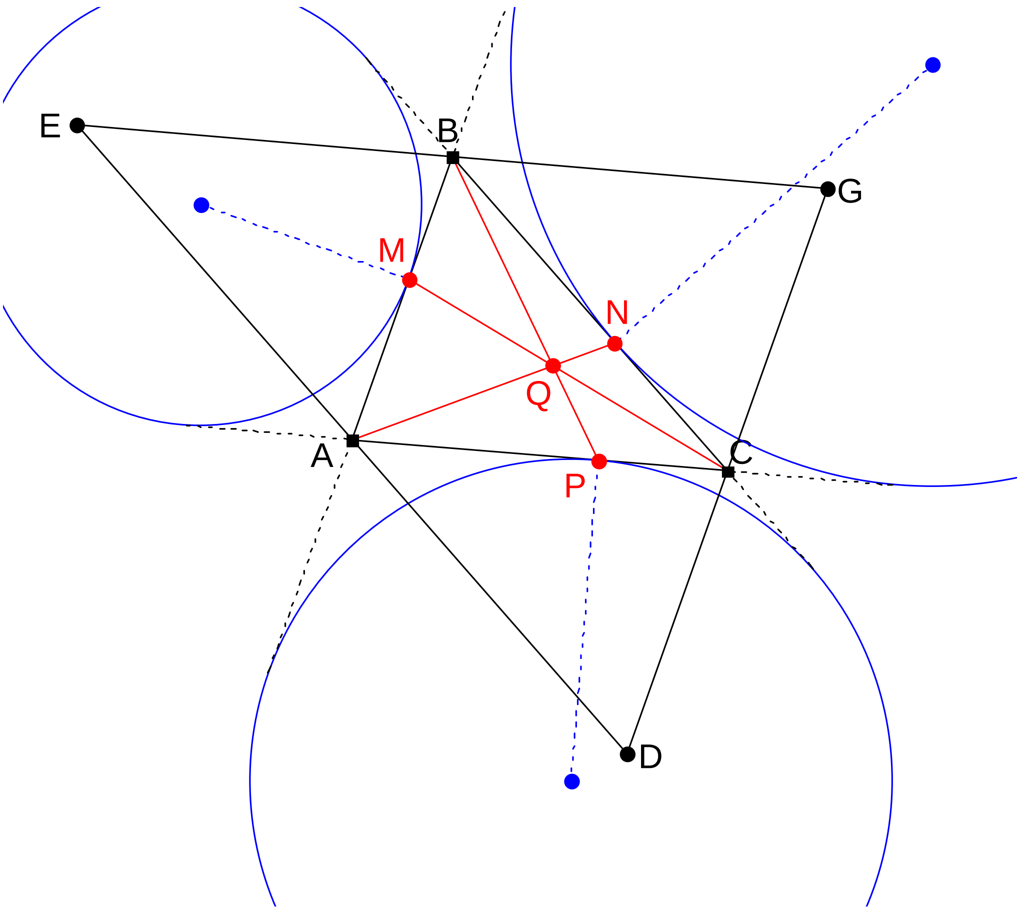

Fig. 3

Now consider a triangle *ABC* and its medial triangle (Fig. 4). Remembering that the sides of the medial triangle are parallel to the respective sides of the original triangle, we see that every triangle is the antimedial triangle of its medial triangle. Hence, the Nagel point of the medial triangle of a triangle Δ*ABC* is the incenter of Δ*ABC*.

This proves the Nagel theorem.

– This derivation of the Nagel theorem is apparently new.

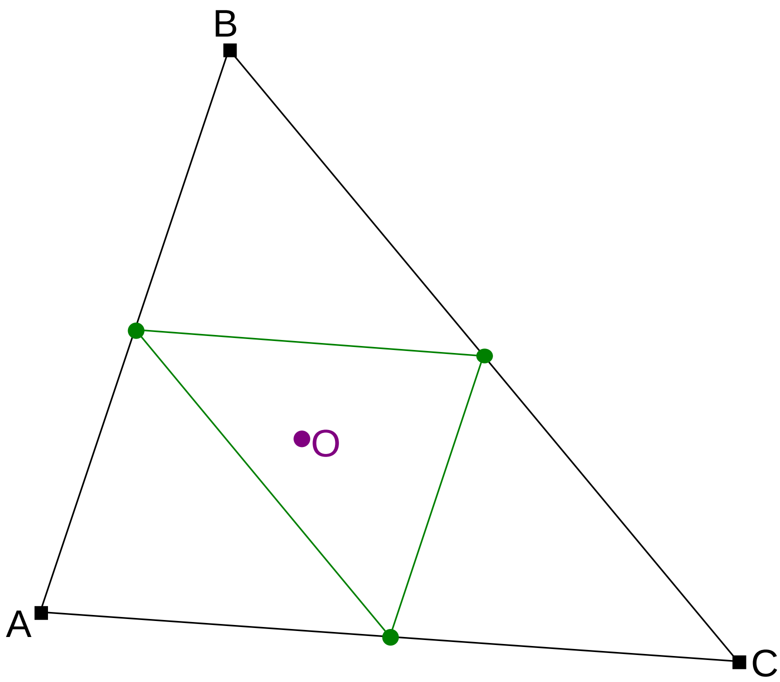

Fig. 4

References

[1] P. Baptist: *Die Entwicklung der neueren Dreiecksgeometrie*, Mannheim-Leipzig-Wien-Zürich 1992.

[2] R. Honsberger: *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, USA 1995.

[3] E. Donath: *Die merkwürdigen Punkte und Linien des ebenen Dreiecks*, Berlin 1976.

[4] H. S. M. Coxeter, S. L. Greitzer: *Zeitlose Geometrie*, Stuttgart 1983.