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Problem 11453 by Richard Stanley, generalized

Let Δ be a finite set of finite sets. Assume that

$$\text{every set } F \in \Delta \text{ and every subset } G \text{ of } F \text{ satisfy } G \in \Delta. \quad (1)$$

Let k be a nonnegative integer. Assume that

$$\text{for every set } G \in \Delta \text{ satisfying } |G| \leq k, \text{ we have } 2^{k+1-|G|} \mid \sum_{\substack{F \in \Delta; \\ F \supseteq G}} (-1)^{|F|}. \quad (2)$$

Prove that $2^{k+1} \mid |\Delta|$.

Solution by Darij Grinberg.

Remark about notation: Let us denote by \mathbb{N} the set $\{0, 1, 2, \dots\}$ (and not the set $\{1, 2, 3, \dots\}$, as some authors do).

We start with a simple lemma:

Lemma 1. Let G be a finite set. Then,

$$\sum_{F \in \mathcal{P}(G)} (-2)^{|F|} = (-1)^{|G|}.$$

Proof of Lemma 1. For every $k \in \mathbb{N}$, we have

$$|\{F \in \mathcal{P}(G) \mid |F| = k\}| = \binom{|G|}{k}. \quad (3)$$

Now,

$$\begin{aligned} \sum_{F \in \mathcal{P}(G)} (-2)^{|F|} &= \sum_{k \in \mathbb{N}} \sum_{\substack{F \in \mathcal{P}(G); \\ |F|=k}} (-2)^{|F|} = \sum_{k \in \mathbb{N}} \sum_{\substack{F \in \mathcal{P}(G); \\ |F|=k}} (-2)^k = \sum_{k \in \mathbb{N}} \underbrace{\sum_{\substack{F \in \mathcal{P}(G); \\ |F|=k}} 1}_{=|\{F \in \mathcal{P}(G) \mid |F|=k\}| \cdot 1} \cdot (-2)^k \\ &= \sum_{k \in \mathbb{N}} |\{F \in \mathcal{P}(G) \mid |F|=k\}| \cdot (-2)^k = \sum_{k \in \mathbb{N}} \binom{|G|}{k} \cdot (-2)^k \\ & \quad \text{(by (3))} \end{aligned}$$

$$= \sum_{\substack{k \in \mathbb{N}; \\ k \leq |G|}} \binom{|G|}{k} \cdot (-2)^k$$

\left(\begin{array}{l} \text{here we replaced the } \sum_{k \in \mathbb{N}} \text{ sign by an } \sum_{\substack{k \in \mathbb{N}; \\ k \leq |G|}} \text{ sign, since the addends for} \\ k > |G| \text{ are zero (as } \binom{|G|}{k} = 0 \text{ for } k > |G|) \end{array} \right)

and

$$\begin{aligned}
\binom{-1}{=(-2)+1}^{|G|} &= ((-2) + 1)^{|G|} = \sum_{k=0}^{|G|} \binom{|G|}{k} \cdot (-2)^k \cdot \underbrace{1^{|G|-k}}_{=1} && \text{(by the binomial formula)} \\
&= \sum_{k=0}^{|G|} \binom{|G|}{k} \cdot (-2)^k = \sum_{\substack{k \in \mathbb{N}; \\ k \leq |G|}} \binom{|G|}{k} \cdot (-2)^k.
\end{aligned}$$

Thus,

$$\sum_{F \in \mathcal{P}(G)} (-2)^{|F|} = (-1)^{|G|}.$$

This completes the proof of Lemma 1.

Now let us solve the problem: Let $G \in \Delta$. Then, $\{F \in \Delta \mid G \supseteq F\} \subseteq \mathcal{P}(G)$ ¹ and $\mathcal{P}(G) \subseteq \{F \in \Delta \mid G \supseteq F\}$ ². Hence, $\{F \in \Delta \mid G \supseteq F\} = \mathcal{P}(G)$.

Now,

$$\begin{aligned}
&\sum_{G \in \Delta} (-2)^{|G|} \sum_{\substack{F \in \Delta; \\ F \supseteq G}} (-1)^{|F|} \\
&= \sum_{F \in \Delta} (-2)^{|F|} \sum_{\substack{G \in \Delta; \\ G \supseteq F}} (-1)^{|G|} && \text{(here we renamed } G \text{ and } F \text{ as } F \text{ and } G \text{ in the sums)} \\
&= \sum_{F \in \Delta} \sum_{\substack{G \in \Delta; \\ G \supseteq F}} (-2)^{|F|} (-1)^{|G|} = \sum_{G \in \Delta} \sum_{\substack{F \in \Delta; \\ G \supseteq F}} (-2)^{|F|} (-1)^{|G|} = \sum_{G \in \Delta} (-1)^{|G|} \sum_{\substack{F \in \Delta; \\ G \supseteq F}} (-2)^{|F|} \\
&= \sum_{G \in \Delta} \sum_{\substack{F \in \Delta; \\ G \supseteq F}} (-1)^{|G|} (-2)^{|F|} \\
&= \sum_{G \in \Delta} (-1)^{|G|} \sum_{F \in \mathcal{P}(G)} (-2)^{|F|} \\
&\quad \left(\text{since } \{F \in \Delta \mid G \supseteq F\} = \mathcal{P}(G) \text{ yields } \sum_{\substack{F \in \Delta; \\ G \supseteq F}} (-2)^{|F|} = \sum_{F \in \mathcal{P}(G)} (-2)^{|F|} \right) \\
&= \sum_{G \in \Delta} \underbrace{(-1)^{|G|} (-1)^{|G|}}_{=(-1) \cdot (-1)^{|G|=1 \cdot |G|=1} = 1} \left(\text{since Lemma 1 yields } \sum_{F \in \mathcal{P}(G)} (-2)^{|F|} = (-1)^{|G|} \right) \\
&= \sum_{G \in \Delta} 1 = |\Delta| \cdot 1 = |\Delta|. && (4)
\end{aligned}$$

¹In fact, let $U \in \{F \in \Delta \mid G \supseteq F\}$. Then, $G \supseteq U$, thus $U \subseteq G$, which means that $U \in \mathcal{P}(G)$. Hence, we have proven that $U \in \mathcal{P}(G)$ for every $U \in \{F \in \Delta \mid G \supseteq F\}$. Thus, $\{F \in \Delta \mid G \supseteq F\} \subseteq \mathcal{P}(G)$.

²In fact, let $U \in \mathcal{P}(G)$. Then, $U \subseteq G$, thus $U \in \Delta$ (by (1), applied to G and U instead of F and G) and $G \supseteq U$, so that $U \in \{F \in \Delta \mid G \supseteq F\}$. Hence, we have proven that $U \in \{F \in \Delta \mid G \supseteq F\}$ for every $U \in \mathcal{P}(G)$. Thus, $\mathcal{P}(G) \subseteq \{F \in \Delta \mid G \supseteq F\}$.

But for every $G \in \Delta$, we have

$$2^{k+1} \mid (-2)^{|G|} \sum_{\substack{F \in \Delta; \\ F \supseteq G}} (-1)^{|F|} \quad (5)$$

³ Thus,

$$2^{k+1} \mid \sum_{G \in \Delta} (-2)^{|G|} \sum_{\substack{F \in \Delta; \\ F \supseteq G}} (-1)^{|F|}.$$

This becomes $2^{k+1} \mid |\Delta|$ (due to (4)). Thus, the problem is solved.

³In fact, since $|G| \in \mathbb{N}$, we have either $|G| \leq k$ or $|G| \geq k+1$. In both of these cases, (5) holds, since:

- if $|G| \leq k$, then $2^{k+1-|G|} \mid \sum_{\substack{F \in \Delta; \\ F \supseteq G}} (-1)^{|F|}$ (by (2)) and $2^{|G|} \mid (-2)^{|G|}$ (since $(-2)^{|G|} = (-1)^{|G|} 2^{|G|}$) lead to $2^{|G|} 2^{k+1-|G|} \mid (-2)^{|G|} \sum_{\substack{F \in \Delta; \\ F \supseteq G}} (-1)^{|F|}$, what becomes $2^{k+1} \mid (-2)^{|G|} \sum_{\substack{F \in \Delta; \\ F \supseteq G}} (-1)^{|F|}$ (since $2^{|G|} 2^{k+1-|G|} = 2^{|G|+(k+1-|G|)} = 2^{k+1}$), so that (5) holds;
- if $|G| \geq k+1$, then $2^{k+1} \mid 2^{|G|}$ and thus $2^{k+1} \mid (-1)^{|G|} 2^{|G|} = (-2)^{|G|}$, so that (5) holds.

Hence, (5) always holds. This completes the proof of (5).