American Mathematical Monthly  
Problem 11453 by Richard Stanley, generalized

Let $\Delta$ be a finite set of finite sets. Assume that
every set $F \in \Delta$ and every subset $G$ of $F$ satisfy $G \in \Delta$. \hfill (1)

Let $k$ be a nonnegative integer. Assume that
for every set $G \in \Delta$ satisfying $|G| \leq k$, we have $2^{k+1-|G|} \cdot \sum_{F \in \Delta; F \supseteq G} (-1)^{|F|}$. \hfill (2)

Prove that $2^{k+1} \mid |\Delta|$.

Solution by Darij Grinberg.

Remark about notation: Let us denote by $\mathbb{N}$ the set \{0, 1, 2, ...\} (and not the set \{1, 2, 3, ...\}, as some authors do).

We start with a simple lemma:

Lemma 1. Let $G$ be a finite set. Then,
$$\sum_{F \in \mathcal{P}(G)} (-2)^{|F|} = (-1)^{|G|}.$$  \hfill (3)

Proof of Lemma 1. For every $k \in \mathbb{N}$, we have
$$|\{F \in \mathcal{P}(G) \mid |F| = k\}| = \binom{|G|}{k}. \hfill (3)$$

Now,
$$\sum_{F \in \mathcal{P}(G)} (-2)^{|F|} = \sum_{k \in \mathbb{N}} \sum_{F \in \mathcal{P}(G); |F| = k} (-2)^{|F|} = \sum_{k \in \mathbb{N}} \sum_{F \in \mathcal{P}(G); |F| = k} (-2)^k = \sum_{k \in \mathbb{N}} \underbrace{|\{F \in \mathcal{P}(G) \mid |F| = k\}| \cdot (-2)^k}_{=|\{F \in \mathcal{P}(G) \mid |F| = k\}| \cdot (-2)^k}$$
$$= \sum_{k \in \mathbb{N}} \binom{|G|}{k} \cdot (-2)^k = \sum_{k \in \mathbb{N}} \binom{|G|}{k} \cdot (-2)^k$$
$$\left(\text{here we replaced the } \sum_{k \in \mathbb{N}} \text{ sign by an } \sum_{k \in \mathbb{N}; k \leq |G|} \text{ sign, since the addends for } k > |G| \text{ are zero (as } \binom{|G|}{k} = 0 \text{ for } k > |G|)\right)$$
and
\[
\left(\frac{-1}{-(2)+1}\right)^{|G|} = ((-2) + 1)^{|G|} = \sum_{k=0}^{|G|} \binom{|G|}{k} \cdot (-2)^k \cdot \frac{|G|}{k} = 1
\]
(by the binomial formula)

Thus,
\[
\sum_{F \in P(G)} (-2)^{|F|} = (-1)^{|G|}.
\]

This completes the proof of Lemma 1.

Now let us solve the problem: Let \( G \in \Delta \). Then, \( \{F \in \Delta \mid G \supseteq F\} \subseteq P(G) \) and \( P(G) \subseteq \{F \in \Delta \mid G \supseteq F\} \). Hence, \( \{F \in \Delta \mid G \supseteq F\} = P(G) \).

Now,
\[
\sum_{G \in \Delta} (-2)^{|G|} \sum_{F \in \Delta; F \supseteq G} (-1)^{|F|}
\]

\[
= \sum_{F \in \Delta} (-2)^{|F|} \sum_{G \in \Delta; G \supseteq F} (-1)^{|G|}
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= \sum_{G \in \Delta} \sum_{F \in \Delta; G \supseteq F} (-2)^{|F|} (-1)^{|G|} = \sum_{G \in \Delta} \sum_{F \in \Delta; G \supseteq F} (-2)^{|F|} (-1)^{|G|} = \sum_{G \in \Delta} (-1)^{|G|} \sum_{F \in \Delta; G \supseteq F} (-2)^{|F|}
\]

\[
= \sum_{G \in \Delta} (-1)^{|G|} \sum_{F \in P(G)} (-2)^{|F|}
\]

\[
\text{since } \{F \in \Delta \mid G \supseteq F\} = P(G) \text{ yields } \sum_{F \in \Delta; G \supseteq F} (-2)^{|F|} = \sum_{F \in P(G)} (-2)^{|F|}
\]

\[
= \sum_{G \in \Delta} (-1)^{|G|} (-1)^{|G|} = (-1)^{|G|} \quad \text{since Lemma 1 yields } \sum_{F \in P(G)} (-2)^{|F|} = (-1)^{|G|}
\]

\[
= \sum_{G \in \Delta} 1 = |\Delta| \cdot 1 = |\Delta|.
\]

\[
\text{In fact, let } U \subseteq \{F \in \Delta \mid G \supseteq F\}. \text{ Then, } G \supseteq U, \text{ thus } U \subseteq G, \text{ which means that } U \in P(G). \text{ Hence, we have proven that } U \in P(G) \text{ for every } U \subseteq \{F \in \Delta \mid G \supseteq F\}. \text{ Thus, } \{F \in \Delta \mid G \supseteq F\} \subseteq P(G).
\]

\[
\text{In fact, let } U \in P(G). \text{ Then, } U \subseteq G, \text{ thus } U \in \Delta \text{ (by (1), applied to } G \text{ and } U \text{ instead of } F \text{ and } G) \text{ and } G \supseteq U, \text{ so that } U \in \{F \in \Delta \mid G \supseteq F\}. \text{ Hence, we have proven that } U \in \{F \in \Delta \mid G \supseteq F\} \text{ for every } U \in P(G). \text{ Thus, } P(G) \subseteq \{F \in \Delta \mid G \supseteq F\}.
\]
But for every $G \in \Delta$, we have
\[2^{k+1} | (-2)^{|G|} \sum_{F \in \Delta: F \supseteq G} (-1)^{|F|}\] (5)

Thus,
\[2^{k+1} | \sum_{G \in \Delta} (-2)^{|G|} \sum_{F \in \Delta: F \supseteq G} (-1)^{|F|}.\]

This becomes $2^{k+1} | |\Delta|$ (due to (4)). Thus, the problem is solved.

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3In fact, since $|G| \in \mathbb{N}$, we have either $|G| \leq k$ or $|G| \geq k + 1$. In both of these cases, (5) holds, since:

- if $|G| \leq k$, then $2^{k+1-|G|} | \sum_{F \in \Delta: F \supseteq G} (-1)^{|F|}$ (by (2)) and $2^{|G|} | (-2)^{|G|}$ (since $(-2)^{|G|} = (-1)^{|G|} 2^{|G|}$)

  lead to $2^{|G|} 2^{k+1-|G|} | (-2)^{|G|} \sum_{F \in \Delta: F \supseteq G} (-1)^{|F|}$, what becomes $2^{k+1} | (-2)^{|G|} \sum_{F \in \Delta: F \supseteq G} (-1)^{|F|}$ (since $2^{|G|} 2^{k+1-|G|} = 2^{|G|} + (k+1-|G|) = 2^{k+1}$), so that (5) holds;

- if $|G| \geq k + 1$, then $2^{k+1} | 2^{|G|}$ and thus $2^{k+1} | (-1)^{|G|} 2^{|G|} = (-2)^{|G|}$, so that (5) holds.

Hence, (5) always holds. This completes the proof of (5).