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Problem 11424 by Emeric Deutsch (edited)

For every $k \in \mathbb{N}$, for every k-tuple $v \in \{0,1\}^k$ and every $i \in \{1,2,...,k\}$, let us denote by v_i the i-th component of the k-tuple v (remember that v is an element of $\{0,1\}^k$, that is, a k-tuple of elements of $\{0,1\}$). Then, every $v \in \{0,1\}^k$ satisfies $v = (v_1, v_2, ..., v_k)$.

Let $n > 1$ be an integer. For any n-tuple $a \in \{0,1\}^n$, we define two integers $F(a)$ and $G(a)$ by

$$F(a) = |\{i \in \{1,2,...,n-1\} \mid a_i = a_{i+1} = 0\}|;$$
$$G(a) = |\{i \in \{1,2,...,n-1\} \mid a_i = a_{i+1} = 1\}|.$$

Set

$$T = \{a \in \{0,1\}^n \mid F(a) = G(a)\}.$$

Prove that

$$|T| = \begin{cases} 2\dbinom{n-2}{(n-2)/2}, & \text{if } n \text{ is even;} \\ \dbinom{n-1}{(n-1)/2}, & \text{if } n \text{ is odd} \end{cases} = 2\dbinom{n-2}{\lfloor (n-2)/2 \rfloor}.$$

Example. For instance, if $n = 4$, then

$$T = \{(0,0,1,1), (0,1,0,1), (1,0,1,0), (1,1,0,0)\}, \qquad \text{so that } |T| = 4.$$

Solution by Darij Grinberg.
First, we are going to introduce some notations:

- For any assertion $\mathcal{U}$, we denote by $[\mathcal{U}]$ the Boolean value of the assertion $\mathcal{U}$ (that is, $[\mathcal{U}] = \begin{cases} 1, \text{ if } \mathcal{U} \text{ is true;} \\ 0, \text{ if } \mathcal{U} \text{ is false} \end{cases}$).
  It is then clear that if B is a set, and $\mathcal{U}(a)$ is an assertion for every element a of B, then
  $$\sum_{a \in B} [\mathcal{U}(a)] = |\{a \in B \mid \mathcal{U}(a) \text{ is true}\}|.$$
  [1]
  Also, if $\mathcal{U}_1$, $\mathcal{U}_2$, ..., $\mathcal{U}_m$ are m assertions, then $[\mathcal{U}_1 \text{ and } \mathcal{U}_2 \text{ and } ... \text{ and } \mathcal{U}_m] = \prod_{j=1}^{m} [\mathcal{U}_j]$.

[1] In fact,

$$\sum_{a \in B} [\mathcal{U}(a)] = \sum_{a \in B} \begin{cases} 1, \text{ if } \mathcal{U}(a) \text{ is true;} \\ 0, \text{ if } \mathcal{U}(a) \text{ is false} \end{cases} = \sum_{\substack{a \in B \\ \text{such that} \\ \mathcal{U}(a) \text{ is true}}} 1 + \underbrace{\sum_{\substack{a \in B \\ \text{such that} \\ \mathcal{U}(a) \text{ is false}}} 0}_{=0} = \sum_{\substack{a \in B \\ \text{such that} \\ \mathcal{U}(a) \text{ is true}}} 1$$
$$= |\{a \in B \mid \mathcal{U}(a) \text{ is true}\}|.$$

- If n is an integer and k is a real, then we define the binomial coefficient $\binom{n}{k}$ by

$$\binom{n}{k} = \begin{cases} \dfrac{n(n-1)\ldots(n-k+1)}{k!}, & \text{if } k \in \mathbb{N}; \\ 0, & \text{otherwise} \end{cases},$$

where $\mathbb{N}$ denotes the set $\{0, 1, 2, \ldots\}$. This definition agrees with the standard definition of $\binom{n}{k}$ in the case when $k \in \mathbb{Z}$.

A consequence of this definition is that if S is a finite set and k is a real, then

$$\binom{|S|}{k} = |\{A \in \mathcal{P}(S) \mid |A| = k\}|$$

[2].

A simple fact:

Lemma 1. Every integer $\eta > 0$ satisfies

$$\binom{2\eta}{\eta} = 2\binom{2\eta-1}{\eta-1}.$$

[3]

Notice that

$$[\alpha = 1] = \alpha \qquad \text{for every } \alpha \in \{0, 1\}, \tag{1}$$

since

$$\begin{aligned} \alpha &= \begin{cases} 1, & \text{if } \alpha = 1; \\ 0, & \text{if } \alpha \neq 1 \end{cases} \qquad (\text{since } \alpha \in \{0, 1\}) \\ &= \begin{cases} 1, & \text{if } \alpha = 1 \text{ is true}; \\ 0, & \text{if } \alpha = 1 \text{ is false} \end{cases} = [\alpha = 1]. \end{aligned}$$

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[2] In fact, if $k \in \mathbb{N}$, then this is well-known, and otherwise $\binom{|S|}{k} = |\{A \in \mathcal{P}(S) \mid |A| = k\}|$ follows from $\binom{|S|}{k} = 0$ and $\left|\left\{A \in \mathcal{P}(S) \mid \underbrace{|A| = k}_{\substack{\text{impossible, since} \\ k \notin \mathbb{N}}}\right\}\right| = |\varnothing| = 0$

[3] *Proof.* We have

$$\begin{aligned} \binom{2\eta}{\eta} &= \frac{(2\eta)!}{\eta! \cdot (2\eta-\eta)!} = \frac{2\eta \cdot (2\eta-1)!}{(\eta \cdot (\eta-1)!) \cdot (2\eta-\eta)!} \qquad (\text{since } (2\eta)! = 2\eta \cdot (2\eta-1)! \text{ and } \eta! = \eta \cdot (\eta-1)!) \\ &= \underbrace{\frac{2\eta}{\eta}}_{=2} \cdot \underbrace{\frac{(2\eta-1)!}{(\eta-1)! \cdot (2\eta-\eta)!}}_{\substack{= \dfrac{(2\eta-1)!}{(\eta-1)! \cdot ((2\eta-1)-(\eta-1))!} \\ (\text{since } 2\eta-\eta=(2\eta-1)-(\eta-1))}} = 2 \cdot \underbrace{\frac{(2\eta-1)!}{(\eta-1)! \cdot ((2\eta-1)-(\eta-1))!}}_{=\binom{2\eta-1}{\eta-1}} \\ &= 2\binom{2\eta-1}{\eta-1}, \end{aligned}$$

qed.

Also,

$$[\alpha = 0] = 1 - \alpha \qquad \text{for every } \alpha \in \{0,1\}, \tag{2}$$

since

$$\alpha = \begin{cases} 0, & \text{if } \alpha = 0; \\ 1, & \text{if } \alpha \neq 0 \end{cases} \qquad (\text{since } \alpha \in \{0,1\})$$

$$= \begin{cases} 1-1, & \text{if } \alpha = 0; \\ 1-0, & \text{if } \alpha \neq 0 \end{cases} = 1 - \begin{cases} 1, & \text{if } \alpha = 0; \\ 0, & \text{if } \alpha \neq 0 \end{cases} = 1 - \begin{cases} 1, & \text{if } \alpha = 0 \text{ is true}; \\ 0, & \text{if } \alpha = 0 \text{ is false}; \end{cases} = 1 - [\alpha = 0].$$

Now, let us solve the problem.

Define a map $f : \{0,1\}^n \to \mathcal{P}(\{1,2,...,n\})$ by

$$f(a) = \{i \in \{1,2,...,n\} \mid a_i = 1\} \qquad \text{for any } n\text{-tuple } a \in \{0,1\}^n.$$

Define a map $g : \mathcal{P}(\{1,2,...,n\}) \to \{0,1\}^n$ by

$$g(A) = ([1 \in A], [2 \in A], ..., [n \in A]) \qquad \text{for any } A \in \mathcal{P}(\{1,2,...,n\}).$$

Then, for any n-tuple $a \in \{0,1\}^n$, we have

$$\begin{aligned}(g \circ f)(a) &= g(f(a)) = ([1 \in f(a)], [2 \in f(a)], ..., [n \in f(a)]) \\ &= ([a_1 = 1], [a_2 = 1], ..., [a_n = 1]) \\ &\qquad (\text{since } [i \in f(a)] = [a_i = 1] \text{ for any } i \in \{1,2,...,n\} \text{ by the definition of } f) \\ &= (a_1, a_2, ..., a_n) \qquad (\text{since } [a_i = 1] = a_i \text{ for any } i \in \{1,2,...,n\} \text{ by (1), since } a_i \in \{0,1\}) \\ &= a.\end{aligned}$$

Thus, $g \circ f = \mathrm{id}$. On the other hand, for any $A \in \mathcal{P}(\{1,2,...,n\})$, we have $A \subseteq \{1,2,...,n\}$ and thus

$$\begin{aligned}(f \circ g)(A) &= f(g(A)) = \{i \in \{1,2,...,n\} \mid (g(A))_i = 1\} \\ &= \{i \in \{1,2,...,n\} \mid [i \in A] = 1\} \\ &\qquad \left(\begin{array}{c}\text{since } g(A) = ([1 \in A], [2 \in A], ..., [n \in A]) \\ \text{and thus } (g(A))_i = [i \in A] \text{ for every } i \in \{1,2,...,n\}\end{array}\right) \\ &= \{i \in \{1,2,...,n\} \mid i \in A\} = A \cap \{1,2,...,n\} = A \qquad (\text{since } A \subseteq \{1,2,...,n\}).\end{aligned}$$

Therefore, $f \circ g = \mathrm{id}$.

Since $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$, the maps f and g are mutually inverse. Hence, the map $f : \{0,1\}^n \to \mathcal{P}(\{1,2,...,n\})$ is a bijection.

For any n-tuple $a \in \{0,1\}^n$, we have

$$|f(a)| = |\{i \in \{1,2,...,n\} \mid a_i = 1\}| = \sum_{i \in \{1,2,...,n\}} \underbrace{[a_i = 1]}_{\substack{= a_i \text{ by (1),} \\ \text{since } a_i \in \{0,1\}}} = \sum_{i \in \{1,2,...,n\}} a_i.$$

Thus, for any real k, we have

$$\begin{aligned}&\left|\left\{a \in \{0,1\}^n \mid \sum_{i \in \{1,2,...,n\}} a_i = k\right\}\right| = |\{a \in \{0,1\}^n \mid |f(a)| = k\}| \\ &= \left|f^{-1}(\{A \in \mathcal{P}(\{1,2,...,n\}) \mid |A| = k\})\right| = |\{A \in \mathcal{P}(\{1,2,...,n\}) \mid |A| = k\}| \\ &\qquad (\text{since } f \text{ is a bijection}) \\ &= \binom{|\{1,2,...,n\}|}{k} = \binom{n}{k}.\end{aligned}$$

Applying this to $n-2$ instead of n, we obtain

$$\left|\left\{a\in\{0,1\}^{n-2}\mid\sum_{i\in\{1,2,...,n-2\}}a_i=k\right\}\right|=\binom{n-2}{k}. \tag{3}$$

Now, for any n-tuple $a\in\{0,1\}^n$, we have

$$\begin{aligned}
F(a)&=|\{i\in\{1,2,...,n-1\}\mid a_i=a_{i+1}=0\}|=\sum_{i\in\{1,2,...,n-1\}}[a_i=a_{i+1}=0]\\
&=\sum_{i\in\{1,2,...,n-1\}}[a_i=0\text{ and }a_{i+1}=0]=\sum_{i\in\{1,2,...,n-1\}}[a_i=0][a_{i+1}=0]=\sum_{i\in\{1,2,...,n-1\}}(1-a_i)(1-a_{i+1})\\
&\qquad\left(\begin{array}{c}\text{since }[a_i=0]=1-a_i\text{ (by (2), applied to }\alpha=a_i\in\{0,1\}\text{)}\\ \text{and }[a_{i+1}=0]=1-a_{i+1}\text{ (by (2), applied to }\alpha=a_{i+1}\in\{0,1\}\text{)}\end{array}\right);\\
G(a)&=|\{i\in\{1,2,...,n-1\}\mid a_i=a_{i+1}=1\}|=\sum_{i\in\{1,2,...,n-1\}}[a_i=a_{i+1}=1]\\
&=\sum_{i\in\{1,2,...,n-1\}}[a_i=1\text{ and }a_{i+1}=1]=\sum_{i\in\{1,2,...,n-1\}}[a_i=1][a_{i+1}=1]=\sum_{i\in\{1,2,...,n-1\}}a_ia_{i+1}\\
&\qquad\left(\begin{array}{c}\text{since }[a_i=1]=a_i\text{ (by (1), applied to }\alpha=a_i\in\{0,1\}\text{)}\\ \text{and }[a_{i+1}=1]=a_{i+1}\text{ (by (1), applied to }\alpha=a_{i+1}\in\{0,1\}\text{)}\end{array}\right),
\end{aligned}$$

so that

$$\begin{aligned}
F(a)-G(a)&=\sum_{i\in\{1,2,...,n-1\}}(1-a_i)(1-a_{i+1})-\sum_{i\in\{1,2,...,n-1\}}a_ia_{i+1}\\
&=\sum_{i\in\{1,2,...,n-1\}}\left(\underbrace{(1-a_i)(1-a_{i+1})}_{=1-a_i-a_{i+1}+a_ia_{i+1}}-a_ia_{i+1}\right)=\sum_{i\in\{1,2,...,n-1\}}(1-a_i-a_{i+1})\\
&=\underbrace{\sum_{i\in\{1,2,...,n-1\}}1}_{=(n-1)\cdot1=n-1}-\underbrace{\sum_{i\in\{1,2,...,n-1\}}a_i}_{=a_1+\sum\limits_{i\in\{2,3,...,n-1\}}a_i}-\underbrace{\sum_{i\in\{1,2,...,n-1\}}a_{i+1}}_{\substack{=\sum\limits_{i\in\{2,3,...,n\}}a_i\\=\sum\limits_{i\in\{2,3,...,n-1\}}a_i+a_n}}\\
&=(n-1)-\left(a_1+\sum_{i\in\{2,3,...,n-1\}}a_i\right)-\left(\sum_{i\in\{2,3,...,n-1\}}a_i+a_n\right)\\
&=(n-1)-a_1-a_n-2\sum_{i\in\{2,3,...,n-1\}}a_i.
\end{aligned} \tag{4}$$

Define a map $u:\{0,1\}^n\to\{0,1\}^{n-2}$ by

$$u(a)=(a_2,a_3,...,a_{n-1})\qquad\text{for any }n\text{-tuple }a\in\{0,1\}^n.$$

Then, $(u(a))_i=a_{i+1}$ for any n-tuple $a\in\{0,1\}^n$ and any $i\in\{1,2,...,n-2\}$.

Define a map $v:\{0,1\}^n\to\{0,1\}\times\{0,1\}^{n-2}\times\{0,1\}$ by

$$v(a)=(a_1,u(a),a_n)\qquad\text{for any }n\text{-tuple }a\in\{0,1\}^n.$$

Define a map $w : \{0,1\} \times \{0,1\}^{n-2} \times \{0,1\} \to \{0,1\}^n$ by

$$w(x,b,y) = (x, b_1, b_2, ..., b_{n-2}, y) \qquad \text{for any triple } (x,b,y) \in \{0,1\} \times \{0,1\}^{n-2} \times \{0,1\}.$$

Then, any n-tuple $a \in \{0,1\}^n$ satisfies

$$(u(a))_i = a_{i+1} \text{ for any } i \in \{1,2,...,n-2\}, \tag{5}$$

and thus

$$\begin{aligned}
(w \circ v)(a) &= w(v(a)) = w(a_1, u(a), a_n) \\
&= \left(a_1, \underbrace{(u(a))_1}_{=a_{1+1} \text{ by (5)}}, \underbrace{(u(a))_2}_{=a_{2+1} \text{ by (5)}}, ..., \underbrace{(u(a))_{n-2}}_{=a_{(n-2)+1} \text{ by (5)}}, a_n \right) \\
&= \left(a_1, a_{1+1}, a_{2+1}, ..., a_{(n-2)+1}, a_n\right) = (a_1, a_2, a_3, ..., a_{n-1}, a_n) = a,
\end{aligned}$$

so that $w \circ v = \mathrm{id}$. Besides, any triple $(x,b,y) \in \{0,1\} \times \{0,1\}^{n-2} \times \{0,1\}$ satisfies

$$\begin{aligned}
w(x,b,y) &= (x, b_1, b_2, ..., b_{n-2}, y), \qquad \text{so that} \\
(w(x,b,y))_{i+1} &= b_i \qquad \text{for any } i \in \{1,2,...,n-2\},
\end{aligned} \tag{6}$$

and thus

$$\begin{aligned}
u(w(x,b,y)) &= \left((w(x,b,y))_2, (w(x,b,y))_3, ..., (w(x,b,y))_{n-1}\right) \\
&= \left(\underbrace{(w(x,b,y))_{1+1}}_{=b_1 \text{ by (6)}}, \underbrace{(w(x,b,y))_{2+1}}_{=b_2 \text{ by (6)}}, ..., \underbrace{(w(x,b,y))_{(n-2)+1}}_{=b_{n-2} \text{ by (6)}} \right) \\
&= (b_1, b_2, ..., b_{n-2}) = b,
\end{aligned}$$

so that

$$\begin{aligned}
(v \circ w)(x,b,y) &= v(w(x,b,y)) = ((w(x,b,y))_1, u(w(x,b,y)), (w(x,b,y))_n) \\
&= (x,b,y) \\
&\qquad \left(\begin{array}{c} \text{since } (w(x,b,y))_1 = (x, b_1, b_2, ..., b_{n-2}, y)_1 = x, \\ u(w(x,b,y)) = b \text{ and } (w(x,b,y))_n = (x, b_1, b_2, ..., b_{n-2}, y)_n = y \end{array} \right),
\end{aligned}$$

and thus $v \circ w = \mathrm{id}$.

Since $v \circ w = \mathrm{id}$ and $w \circ v = \mathrm{id}$, the maps v and w are mutually inverse. Hence, the map v is a bijection.

Now, for any $x \in \{0,1\}$ and $y \in \{0,1\}$, we have

$$\begin{aligned}
&\{a \in T \mid (a_1, a_n) = (x, y)\} \\
&= T \cap \{a \in \{0,1\}^n \mid (a_1, a_n) = (x, y)\} \\
&= \{a \in \{0,1\}^n \mid F(a) = G(a)\} \cap \{a \in \{0,1\}^n \mid (a_1, a_n) = (x, y)\} \\
&= \{a \in \{0,1\}^n \mid F(a) = G(a) \text{ and } (a_1, a_n) = (x, y)\} \\
&= \{a \in \{0,1\}^n \mid F(a) = G(a) \text{ and } a_1 = x \text{ and } a_n = y\} \\
&= \{a \in \{0,1\}^n \mid F(a) - G(a) = 0 \text{ and } a_1 = x \text{ and } a_n = y\} \\
&= \left\{ a \in \{0,1\}^n \mid (n-1) - a_1 - a_n - 2 \sum_{i \in \{2,3,\ldots,n-1\}} a_i = 0 \text{ and } a_1 = x \text{ and } a_n = y \right\} \\
&\qquad \text{(by (4))} \\
&= \left\{ a \in \{0,1\}^n \mid 2 \sum_{i \in \{2,3,\ldots,n-1\}} a_i = (n-1) - a_1 - a_n \text{ and } a_1 = x \text{ and } a_n = y \right\} \\
&= \left\{ a \in \{0,1\}^n \mid 2 \sum_{i \in \{2,3,\ldots,n-1\}} a_i = (n-1) - x - y \text{ and } a_1 = x \text{ and } a_n = y \right\} \\
&= \left\{ a \in \{0,1\}^n \mid 2 \sum_{i \in \{2,3,\ldots,n-1\}} (u(a))_{i-1} = (n-1) - x - y \text{ and } a_1 = x \text{ and } a_n = y \right\} \\
&\qquad \left(\text{because } u(a) = (a_2, a_3, \ldots, a_{n-1}), \text{ so that } a_i = (u(a))_{i-1} \text{ for any } i \in \{2,3,\ldots,n-1\}\right) \\
&= \left\{ a \in \{0,1\}^n \mid 2 \sum_{i \in \{1,2,\ldots,n-2\}} (u(a))_i = (n-1) - x - y \text{ and } a_1 = x \text{ and } a_n = y \right\} \\
&\qquad \left(\text{since } \sum_{i \in \{2,3,\ldots,n-1\}} (u(a))_{i-1} = \sum_{i \in \{1,2,\ldots,n-2\}} (u(a))_i \right)
\end{aligned}$$

$$= \left\{ a \in \{0,1\}^n \mid \sum_{i \in \{1,2,\ldots,n-2\}} (u(a))_i = ((n-1)-x-y)/2 \text{ and } a_1 = x \text{ and } a_n = y \right\}$$

$$= \left\{ a \in \{0,1\}^n \mid u(a) \in \left\{ b \in \{0,1\}^{n-2} \mid \sum_{i \in \{1,2,\ldots,n-2\}} b_i = ((n-1)-x-y)/2 \right\} \text{ and } a_1 = x \text{ and } a_n = y \right\}$$

$$\left(\begin{array}{c} \text{since } \sum_{i \in \{1,2,\ldots,n-2\}} (u(a))_i = ((n-1)-x-y)/2 \text{ is equivalent to} \\ u(a) \in \left\{ b \in \{0,1\}^{n-2} \mid \sum_{i \in \{1,2,\ldots,n-2\}} b_i = ((n-1)-x-y)/2 \right\} \end{array} \right)$$

$$= \left\{ a \in \{0,1\}^n \mid u(a) \in \left\{ b \in \{0,1\}^{n-2} \mid \sum_{i \in \{1,2,\ldots,n-2\}} b_i = ((n-1)-x-y)/2 \right\} \text{ and } a_1 \in \{x\} \text{ and } a_n \in \{y\} \right\}$$

$$= \left\{ a \in \{0,1\}^n \mid (a_1, u(a), a_n) \in \{x\} \times \left\{ b \in \{0,1\}^{n-2} \mid \sum_{i \in \{1,2,\ldots,n-2\}} b_i = ((n-1)-x-y)/2 \right\} \times \{y\} \right\}$$

$$= \left\{ a \in \{0,1\}^n \mid v(a) \in \{x\} \times \left\{ b \in \{0,1\}^{n-2} \mid \sum_{i \in \{1,2,\ldots,n-2\}} b_i = ((n-1)-x-y)/2 \right\} \times \{y\} \right\}$$

$$= v^{-1} \left(\{x\} \times \left\{ b \in \{0,1\}^{n-2} \mid \sum_{i \in \{1,2,\ldots,n-2\}} b_i = ((n-1)-x-y)/2 \right\} \times \{y\} \right).$$

Since v is a bijection, this yields

$$\begin{aligned}
&|\{a \in T \mid (a_1, a_n) = (x,y)\}| \\
&= \left| \{x\} \times \left\{ b \in \{0,1\}^{n-2} \mid \sum_{i \in \{1,2,\ldots,n-2\}} b_i = ((n-1)-x-y)/2 \right\} \times \{y\} \right| \\
&= \underbrace{|\{x\}|}_{=1} \cdot \left| \left\{ b \in \{0,1\}^{n-2} \mid \sum_{i \in \{1,2,\ldots,n-2\}} b_i = ((n-1)-x-y)/2 \right\} \right| \cdot \underbrace{|\{y\}|}_{=1} \\
&= \left| \left\{ b \in \{0,1\}^{n-2} \mid \sum_{i \in \{1,2,\ldots,n-2\}} b_i = ((n-1)-x-y)/2 \right\} \right| \\
&= \left| \left\{ a \in \{0,1\}^{n-2} \mid \sum_{i \in \{1,2,\ldots,n-2\}} a_i = ((n-1)-x-y)/2 \right\} \right| \\
&= \binom{n-2}{((n-1)-x-y)/2} \qquad \text{(by (3), applied to } k = ((n-1)-x-y)/2\text{)}.
\end{aligned} \tag{7}$$

Now,

$$\begin{aligned}
T &= \{a \in T\} = \{a \in T \mid (a_1, a_n) \in \{(0,0), (0,1), (1,0), (1,1)\}\} \\
&\qquad \left(\begin{array}{c}\text{because any } a \in T \text{ satisfies } a \in \{0,1\}^n, \text{ thus } a_1 \in \{0,1\} \text{ and } a_n \in \{0,1\}, \\ \text{and hence } (a_1, a_n) \in \{0,1\} \times \{0,1\} = \{(0,0), (0,1), (1,0), (1,1)\}\end{array}\right) \\
&= \{a \in T \mid (a_1, a_n) = (0,0) \text{ or } (a_1, a_n) = (0,1) \text{ or } (a_1, a_n) = (1,0) \text{ or } (a_1, a_n) = (1,1)\} \\
&= \{a \in T \mid (a_1, a_n) = (0,0)\} \cup \{a \in T \mid (a_1, a_n) = (0,1)\} \\
&\quad \cup \{a \in T \mid (a_1, a_n) = (1,0)\} \cup \{a \in T \mid (a_1, a_n) = (1,1)\}.
\end{aligned}$$

Since the four sets

$$\begin{aligned}
&\{a \in T \mid (a_1, a_n) = (0,0)\}, \qquad && \{a \in T \mid (a_1, a_n) = (0,1)\}, \\
&\{a \in T \mid (a_1, a_n) = (1,0)\}, \qquad && \{a \in T \mid (a_1, a_n) = (1,1)\}
\end{aligned}$$

are pairwise disjoint (because every element a of T lies in at most one of these four sets, since every element a of T satisfies at most one of the four equations

$$(a_1, a_n) = (0,0), \qquad (a_1, a_n) = (0,1), \qquad (a_1, a_n) = (1,0), \qquad (a_1, a_n) = (1,1)$$

), this yields

$$\begin{aligned}
|T| &= \underbrace{|\{a \in T \mid (a_1, a_n) = (0,0)\}|}_{\substack{= \binom{n-2}{((n-1)-0-0)/2} \\ \text{(by (7), applied to } x=0 \text{ and } y=0)}} + \underbrace{|\{a \in T \mid (a_1, a_n) = (0,1)\}|}_{\substack{= \binom{n-2}{((n-1)-0-1)/2} \\ \text{(by (7), applied to } x=0 \text{ and } y=1)}} \\
&\quad + \underbrace{|\{a \in T \mid (a_1, a_n) = (1,0)\}|}_{\substack{= \binom{n-2}{((n-1)-1-0)/2} \\ \text{(by (7), applied to } x=1 \text{ and } y=0)}} + \underbrace{|\{a \in T \mid (a_1, a_n) = (1,1)\}|}_{\substack{= \binom{n-2}{((n-1)-1-1)/2} \\ \text{(by (7), applied to } x=1 \text{ and } y=1)}} \\
&= \binom{n-2}{((n-1)-0-0)/2} + \binom{n-2}{((n-1)-0-1)/2} + \binom{n-2}{((n-1)-1-0)/2} + \binom{n-2}{((n-1)-1-1)/2} \\
&= \binom{n-2}{(n-1)/2} + \binom{n-2}{(n-2)/2} + \binom{n-2}{(n-2)/2} + \binom{n-2}{(n-3)/2} \\
&= \binom{n-2}{(n-1)/2} + 2\binom{n-2}{(n-2)/2} + \binom{n-2}{(n-3)/2}. \qquad (8)
\end{aligned}$$

Now, if n is odd, then $n-2$ is odd, so that $(n-2)/2 \notin \mathbb{N}$, and thus $\binom{n-2}{(n-2)/2} =$

0, so that

$$
\begin{aligned}
|T| &= \binom{n-2}{(n-1)/2} + 2\underbrace{\binom{n-2}{(n-2)/2}}_{=0} + \binom{n-2}{(n-3)/2} \qquad \text{(by (8))} \\
&= \binom{n-2}{(n-1)/2} + \binom{n-2}{(n-3)/2} = \binom{n-2}{(n-1)/2} + \binom{n-2}{(n-1)/2-1} \\
&= \binom{(n-2)+1}{(n-1)/2} \qquad \left(\begin{array}{c}\text{by the recurrence of the binomial coefficients, since} \\ (n-1)/2 \in \mathbb{Z} \text{ (since } n-1 \text{ is even, since } n \text{ is odd)} \\ \text{and } (n-1)/2 > 0 \text{ (since } n > 1)\end{array}\right) \\
&= \binom{n-1}{(n-1)/2} \qquad (9) \\
&= \binom{2(n-1)/2}{(n-1)/2} \qquad (\text{since } n-1 = 2(n-1)/2) \\
&= 2\binom{2(n-1)/2-1}{(n-1)/2-1} \qquad \left(\begin{array}{c}\text{by Lemma 1, applied to } \eta = (n-1)/2, \text{ since } (n-1)/2 \\ \text{is an integer and } (n-1)/2 > 0\end{array}\right) \\
&= 2\binom{n-2}{(n-3)/2} \qquad \left(\begin{array}{c}\text{since } 2(n-1)/2-1 = (n-1)-1 = n-2 \text{ and} \\ (n-1)/2-1 = (n-3)/2\end{array}\right) \\
&= 2\binom{n-2}{\lfloor (n-2)/2 \rfloor} \qquad (10)
\end{aligned}
$$

(since $(n-3)/2 = \lfloor (n-2)/2 \rfloor$ [4]). On the other hand, if n is even, then $n-1$ and $n-3$ are odd, so that $(n-1)/2 \notin \mathbb{N}$ and $(n-3)/2 \notin \mathbb{N}$, and thus

$$
\begin{aligned}
|T| &= \underbrace{\binom{n-2}{(n-1)/2}}_{=0,\text{ since } (n-1)/2 \notin \mathbb{N}} + 2\binom{n-2}{(n-2)/2} + \underbrace{\binom{n-2}{(n-3)/2}}_{=0,\text{ since } (n-3)/2 \notin \mathbb{N}} \qquad \text{(by (8))} \\
&= 2\binom{n-2}{(n-2)/2} \qquad (11) \\
&= 2\binom{n-2}{\lfloor (n-2)/2 \rfloor} \qquad (12)
\end{aligned}
$$

(since $(n-2)/2 = \underbrace{n/2}_{\in\mathbb{Z} \text{ (since } n \text{ is even)}} - 1 \in \mathbb{Z}$ yields $(n-2)/2 = \lfloor (n-2)/2 \rfloor$).

Combining (11) and (9), we obtain

$$
|T| = \begin{cases} 2\binom{n-2}{(n-2)/2}, & \text{if } n \text{ is even;} \\ \binom{n-1}{(n-1)/2}, & \text{if } n \text{ is odd} \end{cases}
$$

[4] This is because $(n-3)/2 = \underbrace{(n-1)/2}_{\in\mathbb{Z}} - 1 \in \mathbb{Z}$ and $(n-2)/2 = (n-3)/2 + 1/2 \in [(n-3)/2, (n-3)/2+1[$.

for any $n > 1$. On the other hand, combining (12) and (10), we obtain

$$|T| = \begin{cases} 2\binom{n-2}{\lfloor (n-2)/2 \rfloor}, & \text{if } n \text{ is even;} \\ 2\binom{n-2}{\lfloor (n-2)/2 \rfloor}, & \text{if } n \text{ is odd} \end{cases} = 2\binom{n-2}{\lfloor (n-2)/2 \rfloor}$$

for any $n > 1$. This solves the problem.