For every $k \in \mathbb{N}$, for every $k$-tuple $v \in \{0, 1\}^k$ and every $i \in \{1, 2, ..., k\}$, let us denote by $v_i$ the $i$-th component of the $k$-tuple $v$ (remember that $v$ is an element of $\{0, 1\}^k$, that is, a $k$-tuple of elements of $\{0, 1\}$). Then, every $v \in \{0, 1\}^k$ satisfies $v = (v_1, v_2, ..., v_k)$.

Let $n > 1$ be an integer. For any $n$-tuple $a \in \{0, 1\}^n$, we define two integers $F(a)$ and $G(a)$ by

$F(a) = |\{i \in \{1, 2, ..., n-1\} \mid a_i = a_{i+1} = 0\}|$;

$G(a) = |\{i \in \{1, 2, ..., n-1\} \mid a_i = a_{i+1} = 1\}|$.

Set

$T = \{a \in \{0, 1\}^n \mid F(a) = G(a)\}$.

Prove that

$|T| = \begin{cases} 2 \binom{n-2}{(n-2)/2}, & \text{if } n \text{ is even;} \\ 2 \binom{n-2}{(n-1)/2}, & \text{if } n \text{ is odd} \end{cases}$.

**Example.** For instance, if $n = 4$, then

$T = \{(0, 0, 1, 1), (0, 1, 0, 1), (1, 0, 1, 0), (1, 1, 0, 0)\}$, so that $|T| = 4$.

**Solution by Darij Grinberg.**

First, we are going to introduce some notations:

- For any assertion $U$, we denote by $[U]$ the Boolean value of the assertion $U$ (that is, $[U] = \begin{cases} 1, & \text{if } U \text{ is true;} \\ 0, & \text{if } U \text{ is false}. \end{cases}$).

It is then clear that if $B$ is a set, and $U(a)$ is an assertion for every element $a$ of $B$, then

$\sum_{a \in B} [U(a)] = |\{a \in B \mid U(a) \text{ is true}\}|$.

Also, if $U_1, U_2, ..., U_m$ are $m$ assertions, then $[U_1 \text{ and } U_2 \text{ and } ... \text{ and } U_m] = \prod_{j=1}^{m} [U_j]$.

\[\sum_{a \in B} [U(a)] = \sum_{a \in B} \begin{cases} 1, & \text{if } U(a) \text{ is true;} \\ 0, & \text{if } U(a) \text{ is false} \end{cases} = \sum_{a \in B \text{ such that } U(a) \text{ is true}} 1 + \sum_{a \in B \text{ such that } U(a) \text{ is false}} 0 = \sum_{a \in B \text{ such that } U(a) \text{ is true}} 1 = |\{a \in B \mid U(a) \text{ is true}\}|.\]
• If \( n \) is an integer and \( k \) is a real, then we define the binomial coefficient \( \binom{n}{k} \) by

\[
\binom{n}{k} = \begin{cases} 
\frac{n(n-1)\ldots(n-k+1)}{k!}, & \text{if } k \in \mathbb{N}; \\
0, & \text{otherwise}
\end{cases}
\]

where \( \mathbb{N} \) denotes the set \( \{0, 1, 2, \ldots\} \). This definition agrees with the standard definition of \( \binom{n}{k} \) in the case when \( k \in \mathbb{Z} \).

A consequence of this definition is that if \( S \) is a finite set and \( k \) is a real, then

\[
\binom{|S|}{k} = |\{ A \in \mathcal{P}(S) \mid |A| = k\}|
\]

2.

A simple fact:

**Lemma 1.** Every integer \( \eta > 0 \) satisfies

\[
\binom{2\eta}{\eta} = 2 \binom{2\eta - 1}{\eta - 1}.
\]

3

Notice that

\[
[\alpha = 1] = \alpha \quad \text{for every } \alpha \in \{0, 1\},
\]

since

\[
\alpha = \begin{cases} 
1, & \text{if } \alpha = 1; \\
0, & \text{if } \alpha \neq 1
\end{cases} \quad \text{(since } \alpha \in \{0, 1\}\text{)}
\]

\[
= \begin{cases} 
1, & \text{if } \alpha = 1 \text{ is true}; \\
0, & \text{if } \alpha = 1 \text{ is false } = [\alpha = 1].
\end{cases}
\]

2In fact, if \( k \in \mathbb{N} \), then this is well-known, and otherwise \( \binom{|S|}{k} = |\{ A \in \mathcal{P}(S) \mid |A| = k\}| \) follows

from \( \binom{|S|}{k} = 0 \) and

\[
\left| \left\{ A \in \mathcal{P}(S) \mid |A| = k \right\} \right| = |\emptyset| = 0
\]

3Proof. We have

\[
\binom{2\eta}{\eta} = \frac{(2\eta)!}{\eta! \cdot (2\eta - \eta)!} = \frac{2\eta \cdot (2\eta - 1)!}{(\eta \cdot (\eta - 1))! \cdot (2\eta - \eta)!} = \frac{2\eta \cdot (2\eta - 1)!}{(\eta - 1)! \cdot (2\eta - \eta)!} = 2 \cdot \frac{(2\eta - 1)!}{(\eta - 1)! \cdot ((2\eta - 1) - (\eta - 1))!} = \frac{2\eta - 1}{\eta - 1}.
\]

qed.
Also, \[ \alpha = 0 \] for every \( \alpha \in \{0, 1\} \), \hfill (2)

since \[ \alpha = \begin{cases} 0, & \text{if } \alpha = 0; \\ 1, & \text{if } \alpha \neq 0 \end{cases} \quad \text{(since } \alpha \in \{0, 1\} \text{)} \]

\[ = \begin{cases} 1 - 1, & \text{if } \alpha = 0; \\ 1 - 0, & \text{if } \alpha \neq 0 \end{cases} = 1 - \begin{cases} 1, & \text{if } \alpha = 0 \text{ is true}; \\ 0, & \text{if } \alpha = 0 \text{ is false}; \end{cases} = 1 - [\alpha = 0]. \]

Now, let us solve the problem. Define a map \( f : \{0, 1\}^n \to \mathcal{P} (\{1, 2, ..., n\}) \) by

\[ f (a) = \{i \in \{1, 2, ..., n\} \mid a_i = 1\} \quad \text{for any n-tuple } a \in \{0, 1\}^n. \]

Define a map \( g : \mathcal{P} (\{1, 2, ..., n\}) \to \{0, 1\}^n \) by

\[ g (A) = ([1 \in A], [2 \in A], ..., [n \in A]) \quad \text{for any } A \in \mathcal{P} (\{1, 2, ..., n\}). \]

Then, for any n-tuple \( a \in \{0, 1\}^n \), we have

\[ (g \circ f) (a) = g (f (a)) = ([1 \in f (a)], [2 \in f (a)], ..., [n \in f (a)]) = ([a_1 = 1], [a_2 = 1], ..., [a_n = 1]) \]

(since \([i \in f (a)] = [a_i = 1]\) for any \(i \in \{1, 2, ..., n\}\) by the definition of \(f\))

\[ = (a_1, a_2, ..., a_n) \quad \text{(since } [a_i = 1] = a_i \text{ for any } i \in \{1, 2, ..., n\} \text{ by (1), since } a_i \in \{0, 1\}) \]

\[ = a. \]

Thus, \( g \circ f = \text{id} \). On the other hand, for any \( A \in \mathcal{P} (\{1, 2, ..., n\}) \), we have \( A \subseteq \{1, 2, ..., n\}\) and thus

\[ (f \circ g) (A) = f (g (A)) = \{i \in \{1, 2, ..., n\} \mid (g (A))_i = 1\} = \{i \in \{1, 2, ..., n\} \mid i \in A = 1\} \]

\[ \quad \text{(since } g (A) = ([1 \in A], [2 \in A], ..., [n \in A]) \text{ and thus } (g (A))_i = [i \in A] \text{ for every } i \in \{1, 2, ..., n\}) \]

\[ = \{i \in \{1, 2, ..., n\} \mid i \in A \} = A \cap \{1, 2, ..., n\} = A \quad \text{(since } A \subseteq \{1, 2, ..., n\}). \]

Therefore, \( f \circ g = \text{id} \).

Since \( g \circ f = \text{id} \) and \( f \circ g = \text{id} \), the maps \( f \) and \( g \) are mutually inverse. Hence, the map \( f : \{0, 1\}^n \to \mathcal{P} (\{1, 2, ..., n\}) \) is a bijection.

For any n-tuple \( a \in \{0, 1\}^n \), we have

\[ |f (a)| = |\{i \in \{1, 2, ..., n\} \mid a_i = 1\}| = \sum_{i \in \{1, 2, ..., n\}} [a_i = 1] = \sum_{i \in \{1, 2, ..., n\}} a_i. \]

Thus, for any real \( k \), we have

\[ \left| \left\{ a \in \{0, 1\}^n \mid \sum_{i \in \{1, 2, ..., n\}} a_i = k \right\} \right| = \left| \{a \in \{0, 1\}^n \mid |f (a)| = k\} \right| = \left| f^{-1} (\{A \in \mathcal{P} (\{1, 2, ..., n\}) \mid |A| = k\}) \right| \]

(since \( f \) is a bijection)

\[ = \binom{n}{k}. \]

3
Applying this to \( n - 2 \) instead of \( n \), we obtain
\[
\left\{ a \in \{0, 1\}^{n-2} \mid \sum_{i \in \{1,2,\ldots,n-2\}} a_i = k \right\} = \binom{n-2}{k}.
\] (3)

Now, for any \( n \)-tuple \( a \in \{0, 1\}^n \), we have
\[
F(a) = \left| \left\{ i \in \{1, 2, \ldots, n-1\} \mid a_i = a_{i+1} = 1 \right\} \right| = \sum_{i \in \{1,2,\ldots,n-1\}} [a_i = a_{i+1} = 0]
\]
\[
= \sum_{i \in \{1,2,\ldots,n-1\}} [a_i = 0 \text{ and } a_{i+1} = 0] = \sum_{i \in \{1,2,\ldots,n-1\}} [a_i = 0] [a_{i+1} = 0] = \sum_{i \in \{1,2,\ldots,n-1\}} (1 - a_i) (1 - a_{i+1})
\]
\[
\left( \text{since } [a_i = 0] = 1 - a_i \text{ (by (2), applied to } \alpha = a_i \in \{0,1\} \text{)} \right)
\]
\[
\text{and } [a_{i+1} = 0] = 1 - a_{i+1} \text{ (by (2), applied to } \alpha = a_{i+1} \in \{0,1\} \text{)}
\]
\[
G(a) = \left| \left\{ i \in \{1, 2, \ldots, n-1\} \mid a_i = a_{i+1} = 1 \right\} \right| = \sum_{i \in \{1,2,\ldots,n-1\}} [a_i = a_{i+1} = 1]
\]
\[
= \sum_{i \in \{1,2,\ldots,n-1\}} [a_i = 1 \text{ and } a_{i+1} = 1] = \sum_{i \in \{1,2,\ldots,n-1\}} [a_i = 1] [a_{i+1} = 1] = \sum_{i \in \{1,2,\ldots,n-1\}} a_i a_{i+1}
\]
\[
\left( \text{since } [a_i = 1] = a_i \text{ (by (1), applied to } \alpha = a_i \in \{0,1\} \text{)} \right)
\]
\[
\text{and } [a_{i+1} = 1] = a_{i+1} \text{ (by (1), applied to } \alpha = a_{i+1} \in \{0,1\} \text{)}
\]
so that
\[
F(a) - G(a) = \sum_{i \in \{1,2,\ldots,n-1\}} (1 - a_i) (1 - a_{i+1}) - \sum_{i \in \{1,2,\ldots,n-1\}} a_i a_{i+1}
\]
\[
= \sum_{i \in \{1,2,\ldots,n-1\}} \left( \frac{(1 - a_i) (1 - a_{i+1}) a_i a_{i+1}}{1-a_i-a_{i+1}+a_i a_{i+1}} \right) = \sum_{i \in \{1,2,\ldots,n-1\}} (1 - a_i - a_{i+1})
\]
\[
= \sum_{i \in \{1,2,\ldots,n-1\}} \left( 1 - \sum_{i \in \{1,2,\ldots,n-1\}} a_i \right) \sum_{i \in \{1,2,\ldots,n-1\}} a_i - \sum_{i \in \{1,2,\ldots,n-1\}} a_i
\]
\[
= (n-1) - \left( a_1 + \sum_{i \in \{2,3,\ldots,n-1\}} a_i \right) - \left( \sum_{i \in \{2,3,\ldots,n-1\}} a_i + a_n \right)
\]
\[
= (n-1) - a_1 - a_n - 2 \sum_{i \in \{2,3,\ldots,n-1\}} a_i.
\] (4)

Define a map \( u : \{0, 1\}^n \rightarrow \{0, 1\}^{n-2} \) by
\[
u(a) = (a_2, a_3, \ldots, a_{n-1}) \quad \text{for any } n\text{-tuple } a \in \{0, 1\}^n.
\]
Then, \( (u(a))_i = a_{i+1} \) for any \( n\)-tuple \( a \in \{0, 1\}^n \) and any \( i \in \{1, 2, \ldots, n-2\} \).

Define a map \( v : \{0, 1\}^n \rightarrow \{0, 1\} \times \{0, 1\}^{n-2} \times \{0, 1\} \) by
\[
v(a) = (a_1, u(a), a_n) \quad \text{for any } n\text{-tuple } a \in \{0, 1\}^n.
\]

4
Define a map \( w: \{0, 1\} \times \{0, 1\}^{n-2} \times \{0, 1\} \to \{0, 1\}^n \) by
\[
w(x, b, y) = (x, b_1, b_2, ..., b_{n-2}, y)
\]
for any triple \((x, b, y) \in \{0, 1\} \times \{0, 1\}^{n-2} \times \{0, 1\}\).

Then, any \(n\)-tuple \(a \in \{0, 1\}^n\) satisfies
\[
(u(a))_i = a_{i+1} \text{ for any } i \in \{1, 2, ..., n-2\},
\]
and thus
\[
(w \circ v)(a) = w(v(a)) = w(a, u(a), a_n)
\]
\[
= \begin{cases} a_1, (u(a))_1, (u(a))_2, \ldots, (u(a))_{n-2}, a_n \\ = a_{i+1} \text{ by (5)} = a_{i+1} \text{ by (5)} = a_{(n-2)+1} \text{ by (5)} \end{cases}
\]
\[
= (a_1, a_{i+1}, a_{i+1}, \ldots, a_{(n-2)+1}, a_n) = (a_1, a_2, a_3, \ldots, a_{n-1}, a_n) = a,
\]
so that \(w \circ v = \text{id}\). Besides, any triple \((x, b, y) \in \{0, 1\} \times \{0, 1\}^{n-2} \times \{0, 1\}\) satisfies
\[
w(x, b, y) = (x, b_1, b_2, ..., b_{n-2}, y), \quad \text{so that}
\]
\[
(w(x, b, y))_{i+1} = b_i \text{ for any } i \in \{1, 2, ..., n-2\},
\]
and thus
\[
u(w(x, b, y)) = ((w(x, b, y))_2, (w(x, b, y))_3, \ldots, (w(x, b, y))_{n-1})
\]
\[
= \begin{cases} (w(x, b, y))_{i+1}, (w(x, b, y))_{2+1}, \ldots, (w(x, b, y))_{(n-2)+1} \\ = b_1 \text{ by (6)} = b_2 \text{ by (6)} = b_{n-2} \text{ by (6)} \end{cases}
\]
\[
= (b_1, b_2, \ldots, b_{n-2}) = b,
\]
so that
\[
(v \circ w)(x, b, y) = v(w(x, b, y)) = ((w(x, b, y))_1, u(w(x, b, y)), (w(x, b, y))_n)
\]
\[
= (x, b_1, b_2, \ldots, b_{n-2}, y) \quad \text{since} \quad (w(x, b, y))_1 = (x, b_1, b_2, \ldots, b_{n-2}, y)_1 = x,
\]
\[
\quad \quad \quad \quad \quad u(w(x, b, y)) = b \text{ and } (w(x, b, y))_n = (x, b_1, b_2, \ldots, b_{n-2}, y)_n = y,
\]
and thus \(v \circ w = \text{id}\).

Since \(v \circ w = \text{id}\) and \(w \circ v = \text{id}\), the maps \(v\) and \(w\) are mutually inverse. Hence, the map \(v\) is a bijection.
Now, for any $x \in \{0, 1\}$ and $y \in \{0, 1\}$, we have

$$\{a \in T \mid (a_1, a_n) = (x, y)\}$$

$$= T \cap \{a \in \{0, 1\}^n \mid (a_1, a_n) = (x, y)\}$$

$$= \{a \in \{0, 1\}^n \mid F(a) = G(a)\} \cap \{a \in \{0, 1\}^n \mid (a_1, a_n) = (x, y)\}$$

$$= \{a \in \{0, 1\}^n \mid F(a) = G(a) \text{ and } (a_1, a_n) = (x, y)\}$$

$$= \{a \in \{0, 1\}^n \mid F(a) = G(a) \text{ and } a_1 = x \text{ and } a_n = y\}$$

$$= \{a \in \{0, 1\}^n \mid (n - 1) - a_1 - a_n - 2 \sum_{i \in \{2, 3, \ldots, n-1\}} a_i = 0 \text{ and } a_1 = x \text{ and } a_n = y\}$$

(by (4))

$$= \{a \in \{0, 1\}^n \mid 2 \sum_{i \in \{2, 3, \ldots, n-1\}} a_i = (n - 1) - a_1 - a_n \text{ and } a_1 = x \text{ and } a_n = y\}$$

$$= \{a \in \{0, 1\}^n \mid 2 \sum_{i \in \{2, 3, \ldots, n-1\}} a_i = (n - 1) - x - y \text{ and } a_1 = x \text{ and } a_n = y\}$$

$$= \{a \in \{0, 1\}^n \mid 2 \sum_{i \in \{2, 3, \ldots, n-1\}} (u(a))_{i-1} = (n - 1) - x - y \text{ and } a_1 = x \text{ and } a_n = y\}$$

(because $u(a) = (a_2, a_3, \ldots, a_{n-1})$, so that $a_i = (u(a))_{i-1}$ for any $i \in \{2, 3, \ldots, n-1\}$)

$$= \{a \in \{0, 1\}^n \mid 2 \sum_{i \in \{1, 2, \ldots, n-2\}} (u(a))_i = (n - 1) - x - y \text{ and } a_1 = x \text{ and } a_n = y\}$$

$$= \{a \in \{0, 1\}^n \mid 2 \sum_{i \in \{2, 3, \ldots, n-1\}} (u(a))_{i-1} = \sum_{i \in \{1, 2, \ldots, n-2\}} (u(a))_i\}$$
\[
= \left\{ a \in \{0,1\}^n \mid \sum_{i \in \{1,2,\ldots,n-2\}} (u(a))_i = ((n-1) - x - y) / 2 \text{ and } a_1 = x \text{ and } a_n = y \right\}
\]

\[
= \left\{ a \in \{0,1\}^n \mid u(a) \in \left\{ b \in \{0,1\}^{n-2} \mid \sum_{i \in \{1,2,\ldots,n-2\}} b_i = ((n-1) - x - y) / 2 \right\} \right. \quad \text{and } a_1 = x \text{ and } a_n = y
\]

\[
\quad \text{since } \sum_{i \in \{1,2,\ldots,n-2\}} (u(a))_i = ((n-1) - x - y) / 2 \text{ is equivalent to }
\]

\[
\left. u(a) \in \left\{ b \in \{0,1\}^{n-2} \mid \sum_{i \in \{1,2,\ldots,n-2\}} b_i = ((n-1) - x - y) / 2 \right\} \right). \quad \text{(by (3), applied to } k = ((n-1) - x - y) / 2) \text{.}
\]

Since \(v\) is a bijection, this yields

\[
\left| \left\{ a \in T \mid (a_1, a_n) = (x, y) \right\} \right|
\]

\[
= \left| \left\{ x \times \left\{ b \in \{0,1\}^{n-2} \mid \sum_{i \in \{1,2,\ldots,n-2\}} b_i = ((n-1) - x - y) / 2 \right\} \times \{y\} \right\} \right|
\]

\[
= \left| \left\{ x \right\} \cdot \left| \left\{ b \in \{0,1\}^{n-2} \mid \sum_{i \in \{1,2,\ldots,n-2\}} b_i = ((n-1) - x - y) / 2 \right\} \right| \cdot \left| \{y\} \right| \right|_{=1}
\]

\[
= \left| \left\{ b \in \{0,1\}^{n-2} \mid \sum_{i \in \{1,2,\ldots,n-2\}} b_i = ((n-1) - x - y) / 2 \right\} \right|
\]

\[
= \left| \left\{ a \in \{0,1\}^{n-2} \mid \sum_{i \in \{1,2,\ldots,n-2\}} a_i = ((n-1) - x - y) / 2 \right\} \right|
\]

\[
= \binom{n-2}{((n-1) - x - y) / 2} \quad \text{(by (3), applied to } k = ((n-1) - x - y) / 2) \text{.}
\]

(7)
Now,
\[ T = \{ a \in T \mid (a_1, a_n) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\} \} \]

(\text{because any } a \in T \text{ satisfies } a \in \{0, 1\}^n, \text{ thus } a_1 \in \{0, 1\} \text{ and } a_n \in \{0, 1\},)

and hence \( (a_1, a_n) \in \{0, 1\} \times \{0, 1\} = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \)

\[ = \{ a \in T \mid (a_1, a_n) = (0, 0) \text{ or } (a_1, a_n) = (0, 1) \text{ or } (a_1, a_n) = (1, 0) \text{ or } (a_1, a_n) = (1, 1) \} \]

\[ = \{ a \in T \mid (a_1, a_n) = (0, 0) \} \cup \{ a \in T \mid (a_1, a_n) = (0, 1) \} \]

\[ \cup \{ a \in T \mid (a_1, a_n) = (1, 0) \} \cup \{ a \in T \mid (a_1, a_n) = (1, 1) \} . \]

Since the four sets
\[ \{ a \in T \mid (a_1, a_n) = (0, 0) \}, \quad \{ a \in T \mid (a_1, a_n) = (0, 1) \}, \]
\[ \{ a \in T \mid (a_1, a_n) = (1, 0) \}, \quad \{ a \in T \mid (a_1, a_n) = (1, 1) \} \]

are pairwise disjoint (because every element \( a \) of \( T \) lies in at most one of these four sets, since every element \( a \) of \( T \) satisfies at most one of the four equations

\( (a_1, a_n) = (0, 0), \quad (a_1, a_n) = (0, 1), \quad (a_1, a_n) = (1, 0), \quad (a_1, a_n) = (1, 1) \)

), this yields

\[ |T| = \left| \{ a \in T \mid (a_1, a_n) = (0, 0) \} \right| + \left| \{ a \in T \mid (a_1, a_n) = (0, 1) \} \right| \]

\[ = \frac{n - 2}{((n - 1) - 0 - 0)/2} \quad \text{(by (7), applied to } x=0 \text{ and } y=0) \]

\[ + \left| \{ a \in T \mid (a_1, a_n) = (1, 0) \} \right| + \left| \{ a \in T \mid (a_1, a_n) = (1, 1) \} \right| \]

\[ = \frac{n - 2}{((n - 1) - 1 - 0)/2} \quad \text{(by (7), applied to } x=1 \text{ and } y=0) \]

\[ = \frac{n - 2}{((n - 1) - 0 - 0)/2} + \frac{n - 2}{((n - 1) - 0 - 1)/2} + \frac{n - 2}{((n - 1) - 1 - 0)/2} + \frac{n - 2}{((n - 1) - 1 - 1)/2} \]

\[ = \binom{n - 2}{n - 1} + \binom{n - 2}{n - 2} + \binom{n - 2}{(n - 2)/2} + \binom{n - 2}{(n - 3)/2} \]

\[ = \binom{n - 2}{n - 1} + 2\binom{n - 2}{(n - 2)/2} + \binom{n - 2}{(n - 3)/2} . \]

(8)

Now, if \( n \) is odd, then \( n - 2 \) is odd, so that \( (n - 2)/2 \notin \mathbb{N} \), and thus \( \binom{n - 2}{(n - 2)/2} = \)

8
0, so that

\[ |T| = \left( \frac{n-2}{(n-1)/2} \right) + 2 \left( \frac{n-2}{(n-2)/2} \right) + \left( \frac{n-2}{(n-3)/2} \right) \tag{by (8)} \]

\[ = \left( \frac{n-2}{(n-1)/2} \right) + 2 \left( \frac{n-2}{(n-3)/2} \right) = \left( \frac{n-2}{(n-1)/2} \right) + \left( \frac{n-2}{(n-1)/2 - 1} \right) \]

(by the recurrence of the binomial coefficients, since \( (n-1)/2 \in \mathbb{Z} \) (since \( n-1 \) is even, since \( n \) is odd) and \( (n-1)/2 > 0 \) (since \( n > 1 \))

\[ = \left( \frac{n-1}{(n-1)/2} \right) \quad \text{(9)} \]

\[ = \left( \frac{2(n-1)/2}{(n-1)/2} \right) \quad \text{(since } n-1 = 2(n-1)/2) \]

\[ = 2 \left( \frac{2(n-1)/2 - 1}{(n-1)/2 - 1} \right) \quad \text{(by Lemma 1, applied to } \eta = (n-1)/2, \text{ since } (n-1)/2 \text{ is an integer and } (n-1)/2 > 0) \]

\[ = 2 \left( \frac{n-2}{(n-3)/2} \right) \quad \text{(since } 2(n-1)/2 - 1 = (n-1) - 1 = n-2 \text{ and } (n-1)/2 - 1 = (n-3)/2) \]

\[ = 2 \left( \frac{n-2}{\lfloor (n-2)/2 \rfloor} \right) \tag{10} \]

(since \( (n-3)/2 = \lfloor (n-2)/2 \rfloor \)). On the other hand, if \( n \) is even, then \( n-1 \) and \( n-3 \) are odd, so that \( (n-1)/2 \notin \mathbb{N} \) and \( (n-3)/2 \notin \mathbb{N} \), and thus

\[ |T| = \left( \frac{n-2}{(n-1)/2} \right) + 2 \left( \frac{n-2}{(n-2)/2} \right) + \left( \frac{n-2}{(n-3)/2} \right) \tag{by (8)} \]

\[ = 2 \left( \frac{n-2}{(n-2)/2} \right) \quad \text{=0, since } (n-1)/2 \notin \mathbb{N} \]

\[ = 2 \left( \frac{n-2}{\lfloor (n-2)/2 \rfloor} \right) \quad \text{(11)} \]

\[ = 2 \left( \frac{n-2}{\lfloor (n-2)/2 \rfloor} \right) \quad \text{(12)} \]

(since \( (n-2)/2 = \frac{n/2 - 1}{\in \mathbb{Z} \text{ (since } n \text{ is even)} \) yields \( (n-2)/2 = \lfloor (n-2)/2 \rfloor \).

Combining (11) and (9), we obtain

\[ |T| = \begin{cases} 
2 \left( \frac{n-2}{(n-2)/2} \right), & \text{if } n \text{ is even;} \\
\left( \frac{n-1}{(n-1)/2} \right), & \text{if } n \text{ is odd} 
\end{cases} \]

\[ ^4 \text{This is because } (n-3)/2 = (n-1)/2 - 1 \in \mathbb{Z} \text{ and } (n-2)/2 = (n-3)/2 + 1/2 \in \mathbb{Z} \text{ and } (n-2)/2 \in [(n-3)/2], (n-3)/2 + 1]. \]
for any $n > 1$. On the other hand, combining (12) and (10), we obtain

$$|T| = \begin{cases} 
2 \left( \frac{n - 2}{\lfloor (n - 2)/2 \rfloor} \right), & \text{if } n \text{ is even;} \\
2 \left( \frac{n - 2}{\lceil (n - 2)/2 \rceil} \right), & \text{if } n \text{ is odd}
\end{cases}$$

for any $n > 1$. This solves the problem.