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Problem 11403 by Yaming Yu (edited)

For every integer $n \geq 0$, define a polynomial $f_n \in \mathbb{Q}[x]$ by

$$f_n(x) = \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j).$$

Find $\deg f_n$ for every $n > 1$.

Solution by Darij Grinberg.

First, we will show:

Lemma 1. For every integer $n > 1$, we have $f_n(x) = (n-1)(f_{n-1}(x) + x f_{n-2}(x))$.

Proof of Lemma 1. Every integer $i \in \{0, 1, ..., n-1\}$ satisfies

$$i \cdot \binom{n-1}{i} = (n-1) \cdot \binom{n-2}{i-1}. \tag{1}$$

[*Proof of (1):* Let $i \in \{0, 1, ..., n-1\}$. We must prove (1). If $i = 0$, then (1) follows immediately by comparing $\underbrace{i}_{=0} \cdot \binom{n-1}{i} = 0$ with $(n-1) \cdot \underbrace{\binom{n-2}{i-1}}_{\substack{=0 \\ (\text{since } i-1<0 \ (\text{since } i=0<1))}} = 0$.

Thus, for the rest of this proof of (1), we can WLOG assume that $i \neq 0$. Assume this. Hence, $i \geq 1$ (since $i \neq 0$ and $i \in \{0, 1, ..., n-1\}$), so that $i - 1 \geq 0$. Now,

$$i \cdot \underbrace{\binom{n-1}{i}}_{=\frac{(n-1)!}{i!((n-1)-i)!}} = i \cdot \frac{(n-1)!}{i!((n-1)-i)!} = i \cdot \frac{(n-1) \cdot (n-2)!}{((i-1)! \cdot i) \cdot ((n-2)-(i-1))!}$$

$$(\text{since } (n-1)! = (n-1) \cdot (n-2)!, \ i! = (i-1)! \cdot i \text{ and } (n-1)-i = (n-2)-(i-1))$$

$$= (n-1) \cdot \underbrace{\frac{(n-2)!}{(i-1)! \cdot ((n-2)-(i-1))!}}_{=\binom{n-2}{i-1}} = (n-1) \cdot \binom{n-2}{i-1}.$$

This proves (1).]

Now,

$$f_n(x) = \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j) = \sum_{i=0}^{n} \left(\binom{n-1}{i-1} + \binom{n-1}{i} \right) (-x)^{n-i} \prod_{j=0}^{i-1} (x+j)$$

$$\left(\text{as } \binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i} \text{ by the recurrence of the binomial coefficients} \right)$$

$$= \sum_{i=0}^{n} \binom{n-1}{i-1} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j) + \sum_{i=0}^{n} \binom{n-1}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j)$$

$$= \sum_{i=1}^{n} \binom{n-1}{i-1} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j) + \sum_{i=0}^{n} \binom{n-1}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j)$$

$$\left(\begin{array}{c} \text{here we replaced the first } \sum\limits_{i=0}^{n} \text{ sign by an } \sum\limits_{i=1}^{n} \text{ sign,} \\ \text{since the addend for } i=0 \text{ is zero} \\ \left(\text{as } \binom{n-1}{i-1} = \binom{n-1}{-1} = 0 \text{ for } i=0\right) \end{array} \right)$$

$$= \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i-1} \prod_{j=0}^{i} (x+j) + \sum_{i=0}^{n} \binom{n-1}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j)$$

$$\text{(here we substituted } i+1 \text{ for } i \text{ in the first sum)}$$

$$= \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i-1} \prod_{j=0}^{i} (x+j) + \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j)$$

$$\left(\begin{array}{c} \text{here we replaced the } \sum\limits_{i=0}^{n} \text{ sign by an } \sum\limits_{i=0}^{n-1} \text{ sign, since the addend for } i=n \text{ is zero} \\ \left(\text{as } \binom{n-1}{i} = \binom{n-1}{n} = 0 \text{ for } i=n\right) \end{array} \right)$$

$$= \sum_{i=0}^{n-1} \binom{n-1}{i} \left((-x)^{n-i-1} \prod_{j=0}^{i} (x+j) + (-x)^{n-i} \prod_{j=0}^{i-1} (x+j) \right)$$

$$= \sum_{i=0}^{n-1} \binom{n-1}{i} \left((-x)^{n-i-1} \prod_{j=0}^{i} (x+j) + (-x)^{n-i-1} (-x) \prod_{j=0}^{i-1} (x+j) \right)$$

$$= \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i-1} \left(\prod_{j=0}^{i} (x+j) + (-x) \prod_{j=0}^{i-1} (x+j) \right)$$
$$= \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i-1} \left((x+i) \prod_{j=0}^{i-1} (x+j) + (-x) \prod_{j=0}^{i-1} (x+j) \right)$$
$$= \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i-1} \underbrace{((x+i)+(-x))}_{=i} \prod_{j=0}^{i-1} (x+j) = \sum_{i=0}^{n-1} \left(i \cdot \binom{n-1}{i} \right) (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j)$$
$$= \sum_{i=0}^{n-1} \left((n-1) \cdot \binom{n-2}{i-1} \right) (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j) \qquad \text{(by (1))}$$
$$= (n-1) \cdot \sum_{i=0}^{n-1} \binom{n-2}{i-1} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j) . \tag{2}$$

But

$$f_{n-1}(x) = \sum_{i=0}^{n-1} \underbrace{\binom{n-1}{i}}_{\substack{= \binom{n-2}{i} + \binom{n-2}{i-1} \\ \text{(by the recurrence of the binomial coefficients)}}} \underbrace{(-x)^{(n-1)-i}}_{=(-x)^{n-i-1}} \prod_{j=0}^{i-1} (x+j)$$
$$= \sum_{i=0}^{n-1} \left(\binom{n-2}{i} + \binom{n-2}{i-1} \right) (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j)$$
$$= \sum_{i=0}^{n-1} \binom{n-2}{i} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j) + \sum_{i=0}^{n-1} \binom{n-2}{i-1} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j)$$

and

$$f_{n-2}(x) = \sum_{i=0}^{n-2} \binom{n-2}{i} (-x)^{(n-2)-i} \prod_{j=0}^{i-1} (x+j) ,$$

yielding

$$x f_{n-2}(x) = x \sum_{i=0}^{n-2} \binom{n-2}{i} (-x)^{(n-2)-i} \prod_{j=0}^{i-1} (x+j) = -(-x) \sum_{i=0}^{n-2} \binom{n-2}{i} (-x)^{(n-2)-i} \prod_{j=0}^{i-1} (x+j)$$
$$= - \sum_{i=0}^{n-2} \binom{n-2}{i} \underbrace{(-x)(-x)^{(n-2)-i}}_{=(-x)^{(n-2)-i+1} = (-x)^{n-i-1}} \prod_{j=0}^{i-1} (x+j)$$
$$= - \sum_{i=0}^{n-2} \binom{n-2}{i} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j) = - \sum_{i=0}^{n-1} \binom{n-2}{i} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j)$$
$$\left(\begin{array}{c} \text{here we replaced the } \sum\limits_{i=0}^{n-2} \text{ sign by an } \sum\limits_{i=0}^{n-1} \text{ sign, since the addend} \\ \text{for } i = n-1 \text{ is zero (as } \binom{n-2}{i} = \binom{n-2}{n-1} = 0 \text{ for } i = n-1) \end{array} \right),$$

so that

$$\begin{aligned}
& f_{n-1}(x) + x f_{n-2}(x) \\
&= \left(\sum_{i=0}^{n-1} \binom{n-2}{i} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j) + \sum_{i=0}^{n-1} \binom{n-2}{i-1} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j) \right) \\
&+ \left(- \sum_{i=0}^{n-1} \binom{n-2}{i} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j) \right) \\
&= \sum_{i=0}^{n-1} \binom{n-2}{i-1} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j),
\end{aligned}$$

and thus (2) becomes $f_n(x) = (n-1) \cdot (f_{n-1}(x) + x f_{n-2}(x))$. This proves Lemma 1.

Next, we introduce a *notation*: For any polynomial $p \in \mathbb{Q}[x]$, and for any integer $k \geq 0$, we denote by $\operatorname{coeff}(p, k)$ the coefficient of p before x^k. Then, every polynomial $p \in \mathbb{Q}[x]$ satisfies $p(x) = \sum_{k \geq 0} \operatorname{coeff}(p, k) \cdot x^k$.

Now, Lemma 1 yields:[1]

Corollary 2. For every integer $n > 1$, we have $\deg f_n \leq \max\{\deg f_{n-1}, 1 + \deg f_{n-2}\}$ and $\operatorname{coeff}(f_n, s) = (n-1)(\operatorname{coeff}(f_{n-1}, s) + \operatorname{coeff}(f_{n-2}, s-1))$ for every positive integer s.

Proof of Corollary 2. Theorem 1 yields $f_n(x) = (n-1)(f_{n-1}(x) + x f_{n-2}(x))$. Thus,

$$\begin{aligned}
\deg f_n &= \deg(f_n(x)) = \deg\left(\underbrace{(n-1)}_{\substack{\text{is a nonzero} \\ \text{constant}}} (f_{n-1}(x) + x f_{n-2}(x)) \right) = \deg(f_{n-1}(x) + x f_{n-2}(x)) \\
&\leq \max\{\deg(f_{n-1}(x)), \deg(x f_{n-2}(x))\} = \max\{\deg(f_{n-1}(x)), 1 + \deg(f_{n-2}(x))\} \\
&= \max\{\deg f_{n-1}, 1 + \deg f_{n-2}\}
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{coeff}(f_n, s) &= \operatorname{coeff}(f_n(x), s) = \operatorname{coeff}((n-1)(f_{n-1}(x) + x f_{n-2}(x)), s) \\
&= (n-1)(\operatorname{coeff}(f_{n-1}(x), s) + \operatorname{coeff}(x f_{n-2}(x), s)) \\
&= (n-1)(\operatorname{coeff}(f_{n-1}(x), s) + \operatorname{coeff}(f_{n-2}(x), s-1)) \\
&= (n-1)(\operatorname{coeff}(f_{n-1}, s) + \operatorname{coeff}(f_{n-2}, s-1)),
\end{aligned}$$

and Corollary 2 is proven.

Next, we notice that

$$f_0(x) = \sum_{i=0}^{0} \binom{0}{i} (-x)^{0-i} \prod_{j=0}^{i-1} (x+j) = \underbrace{\binom{0}{0}}_{=1} \underbrace{(-x)^{0-0}}_{=(-x)^0=1} \underbrace{\prod_{j=0}^{-1} (x+j)}_{=1} = 1$$

[1]Here and in the following, we are using the convention that the degree of the zero polynomial is $-\infty$.

and

$$\begin{aligned}
f_1(x) &= \sum_{i=0}^{1} \binom{1}{i} (-x)^{1-i} \prod_{j=0}^{i-1} (x+j) \\
&= \underbrace{\binom{1}{0}}_{=1} \underbrace{(-x)^{1-0}}_{=(-x)^1=-x} \underbrace{\prod_{j=0}^{-1} (x+j)}_{=1} + \underbrace{\binom{1}{1}}_{=1} \underbrace{(-x)^{1-1}}_{=(-x)^0=1} \underbrace{\prod_{j=0}^{0} (x+j)}_{=x+0=x} \\
&= (-x) + x = 0.
\end{aligned}$$

Thus, Lemma 1 (applied to $n = 2$) yields

$$f_2(x) = (2-1)(f_{2-1}(x) + x f_{2-2}(x)) = 1 \left(\underbrace{f_1(x)}_{=0} + x \underbrace{f_0(x)}_{=1} \right) = 1(0+x) = 1x = x.$$

Also, Lemma 1 (applied to $n = 3$) yields

$$f_3(x) = (3-1)(f_{3-1}(x) + x f_{3-2}(x)) = 2 \left(\underbrace{f_2(x)}_{=x} + x \underbrace{f_1(x)}_{=0} \right) = 2(x+0) = 2x.$$

Now, our main result:

Theorem 3. For any positive integer u, we have $\deg f_{2u} = \deg f_{2u+1} = u$, $\operatorname{coeff}(f_{2u}, u) > 0$ and $\operatorname{coeff}(f_{2u+1}, u) > 0$.

Proof of Theorem 3. We will show Theorem 3 by induction over u:

Induction base. For $u = 1$, we have $f_{2u}(x) = f_{2\cdot 1}(x) = f_2(x) = x$, thus $\deg f_{2u} = 1 = u$ and $\operatorname{coeff}(f_{2u}, u) = \operatorname{coeff}(f_{2u}, 1) = 1 > 0$. Besides, for $u = 1$, we have $f_{2u+1}(x) = f_{2\cdot 1+1}(x) = f_3(x) = 2x$, thus $\deg f_{2u+1} = 1 = u$ and $\operatorname{coeff}(f_{2u+1}, u) = \operatorname{coeff}(f_{2u+1}, 1) = 2 > 0$. Altogether, we have thus shown that the relations $\deg f_{2u} = \deg f_{2u+1} = u$, $\operatorname{coeff}(f_{2u}, u) > 0$ and $\operatorname{coeff}(f_{2u+1}, u) > 0$ hold for $u = 1$. In other words, Theorem 3 is proven for $u = 1$. This completes the induction base.

Induction step. Let $k \geq 2$ be an integer. Assume that Theorem 3 holds for $u = k-1$. We want to prove that Theorem 3 holds for $u = k$ as well.

Since Theorem 3 holds for $u = k-1$, we have $\deg f_{2(k-1)} = \deg f_{2(k-1)+1} = k-1$, $\operatorname{coeff}\left(f_{2(k-1)}, k-1\right) > 0$ and $\operatorname{coeff}\left(f_{2(k-1)+1}, k-1\right) > 0$.

Now, Corollary 2 (applied to $n = 2k$ and $s = k$) yields

$$\begin{aligned}
\deg f_{2k} &\leq \max\{\deg f_{2k-1}, 1 + \deg f_{2k-2}\} = \max\left\{\deg f_{2(k-1)+1}, 1 + \deg f_{2(k-1)}\right\} \\
&= \max\{k-1, 1+(k-1)\} = \max\{k-1, k\} = k
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{coeff}(f_{2k}, k) &= (2k-1)(\operatorname{coeff}(f_{2k-1}, k) + \operatorname{coeff}(f_{2k-2}, k-1)) \\
&= (2k-1) \left(\underbrace{\operatorname{coeff}\left(f_{2(k-1)+1}, k\right)}_{\substack{=0,\text{ since} \\ \deg f_{2(k-1)+1} = k-1 < k}} + \operatorname{coeff}\left(f_{2(k-1)}, k-1\right) \right) \\
&= \underbrace{(2k-1)}_{>0} \underbrace{\operatorname{coeff}\left(f_{2(k-1)}, k-1\right)}_{>0} > 0.
\end{aligned}$$

These, combined, yield $\deg f_{2k} = k$.

Furthermore, Corollary 2 (applied to $n = 2k+1$ and $s = k$) yields

$$\begin{aligned}\deg f_{2k+1} &\leq \max\left\{\deg f_{(2k+1)-1}, 1 + \deg f_{(2k+1)-2}\right\} = \max\left\{\deg f_{2k}, 1 + \deg f_{2(k-1)+1}\right\} \\ &= \max\{k, 1 + (k-1)\} = \max\{k, k\} = k\end{aligned}$$

and

$$\begin{aligned}\operatorname{coeff}(f_{2k+1}, k) &= ((2k+1)-1)\left(\operatorname{coeff}\left(f_{(2k+1)-1}, k\right) + \operatorname{coeff}\left(f_{(2k+1)-2}, k-1\right)\right) \\ &= \underbrace{2k}_{>0}\left(\underbrace{\operatorname{coeff}(f_{2k}, k)}_{>0} + \underbrace{\operatorname{coeff}\left(f_{2(k-1)+1}, k-1\right)}_{>0}\right) > 0.\end{aligned}$$

These, combined, yield $\deg f_{2k+1} = k$.

Altogether, we have thus shown $\deg f_{2k} = \deg f_{2k+1} = k$, $\operatorname{coeff}(f_{2k}, k) > 0$ and $\operatorname{coeff}(f_{2k+1}, k) > 0$. In other words, we have shown that Theorem 3 holds for $u = k$. This completes the induction step. Thus, the proof of Theorem 3 is complete.

To conclude, here is a formula for $\deg f_n$:

Corollary 4. For every integer $n \geq 0$, we have $\deg f_n = \begin{cases} \left\lfloor \dfrac{n}{2} \right\rfloor, & \text{if } n \neq 1; \\ -\infty, & \text{if } n = 1 \end{cases}$ (where we consider $\deg 0$ to be $-\infty$).

Proof of Corollary 4. If $n = 0$, then $f_n(x) = f_0(x) = 1$, so that $\deg f_n = 0 = \left\lfloor \dfrac{0}{2} \right\rfloor = \left\lfloor \dfrac{n}{2} \right\rfloor$.

If $n = 1$, then $f_n(x) = f_1(x) = 0$, so that $\deg f_n = -\infty$.

If n is even and $n > 1$, then there exists a positive integer u such that $n = 2u$, so that

$$\begin{aligned}\deg f_n &= \deg f_{2u} = u \qquad \text{(by Theorem 3)} \\ &= \lfloor u \rfloor = \left\lfloor \frac{2u}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor.\end{aligned}$$

If n is odd and $n > 1$, then there exists a positive integer u such that $n = 2u+1$, so that

$$\begin{aligned}\deg f_n &= \deg f_{2u+1} = u \qquad \text{(by Theorem 3)} \\ &= \left\lfloor u + \frac{1}{2} \right\rfloor = \left\lfloor \frac{2u+1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor.\end{aligned}$$

Thus, for every integer $n \geq 0$, we have

$$\deg f_n = \begin{cases} \left\lfloor \dfrac{n}{2} \right\rfloor, & \text{if } n = 0; \\ -\infty, & \text{if } n = 1; \\ \left\lfloor \dfrac{n}{2} \right\rfloor, & \text{if } n \text{ is even and } n > 1; \\ \left\lfloor \dfrac{n}{2} \right\rfloor, & \text{if } n \text{ is odd and } n > 1 \end{cases} = \begin{cases} \left\lfloor \dfrac{n}{2} \right\rfloor, & \text{if } n = 0; \\ -\infty, & \text{if } n = 1; \\ \left\lfloor \dfrac{n}{2} \right\rfloor, & \text{if } n > 1 \end{cases} = \begin{cases} \left\lfloor \dfrac{n}{2} \right\rfloor, & \text{if } n \neq 1; \\ -\infty, & \text{if } n = 1 \end{cases}.$$

Corollary 4 is proven.