American Mathematical Monthly Problem 11458 by Cezar Lupu, Bucharest, Romania and Vicenţiu Rădulescu, Bucharest, Romania.

Let a_1, a_2, ..., a_n be nonnegative reals. Let r be a positive integer. Prove that

$$\left(\sum_{1\leq i,j\leq n} \frac{i^r j^r a_i a_j}{i+j-1}\right)^2 \leq \sum_{m=1}^{n} m^{r-1} a_m \cdot \sum_{1\leq i,j,k\leq n} \frac{i^r j^r k^r a_i a_j a_k}{i+j+k-2}.$$

Solution by Darij Grinberg.

First, we set $b_u = u^r a_u$ for every $u \in \{1, 2, ..., n\}$. Then, we are left with proving that

$$\left(\sum_{1\leq i,j\leq n} \frac{b_i b_j}{i+j-1}\right)^2 \leq \sum_{m=1}^{n} \frac{b_m}{m} \cdot \sum_{1\leq i,j,k\leq n} \frac{b_i b_j b_k}{i+j+k-2}. \qquad (1)$$

This is, however, an easy application of the Cauchy-Schwarz inequality for integrals: Define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sum\limits_{m=1}^{n} b_m x^{m-1}$ for every $x \in \mathbb{R}$. Then,

$$\int_0^1 f(x)\,dx = \sum_{m=1}^{n} b_m \underbrace{\int_0^1 x^{m-1}dx}_{=\frac{1}{m}} = \sum_{m=1}^{n} \frac{b_m}{m}. \qquad (2)$$

Besides, $f(x) = \sum\limits_{m=1}^{n} b_m x^{m-1}$ yields $(f(x))^2 = \left(\sum\limits_{m=1}^{n} b_m x^{m-1}\right)^2 = \sum\limits_{1\leq i,j\leq n} b_i b_j x^{i+j-2}$ and therefore

$$\int_0^1 (f(x))^2\,dx = \sum_{1\leq i,j\leq n} b_i b_j \underbrace{\int_0^1 x^{i+j-2}dx}_{=\frac{1}{i+j-1}} = \sum_{1\leq i,j\leq n} \frac{b_i b_j}{i+j-1}. \qquad (3)$$

Furthermore, $f(x) = \sum\limits_{m=1}^{n} b_m x^{m-1}$ yields $(f(x))^3 = \left(\sum\limits_{m=1}^{n} b_m x^{m-1}\right)^3 = \sum\limits_{1\leq i,j,k\leq n} b_i b_j b_k x^{i+j+k-3}$ and therefore

$$\int_0^1 (f(x))^3\,dx = \sum_{1\leq i,j,k\leq n} b_i b_j b_k \underbrace{\int_0^1 x^{i+j+k-3}dx}_{=\frac{1}{i+j+k-2}} = \sum_{1\leq i,j,k\leq n} \frac{b_i b_j b_k}{i+j+k-2}. \qquad (4)$$

The Cauchy-Schwarz inequality for integrals (applied to the functions $x \mapsto \sqrt{f(x)}$ and $x \mapsto \sqrt{(f(x))^3}$) yields

$$\int_0^1 f(x)\,dx \cdot \int_0^1 (f(x))^3\,dx \geq \left(\int_0^1 \underbrace{\sqrt{f(x) \cdot (f(x))^3}}_{=\sqrt{(f(x))^4}=(f(x))^2} dx \right)^2 = \left(\int_0^1 (f(x))^2\,dx \right)^2.$$

Using (2), (3) and (4), this inequality transforms into (1), and our solution is complete.

Remark. The nonnegativity of a_1, a_2, ..., a_n was only used when we applied Cauchy-Schwarz to the functions $x \mapsto \sqrt{f(x)}$ and $x \mapsto \sqrt{(f(x))^3}$. Of course, it can be replaced by the weaker condition that the polynomial $f(x)$ be nonnegative on the interval $[0, 1]$.