American Mathematical Monthly Problem 11435 by Panagiote Ligouras, Noci, Italy.

Given a triangle $ABC$ with sidelengths $a$, $b$, $c$, inradius $r$ and circumradius $R$. Prove the inequality

$$\frac{a^2bc}{(a+b)(a+c)} + \frac{b^2ca}{(b+c)(b+a)} + \frac{c^2ab}{(c+a)(c+b)} \leq \frac{9}{2} r R. $$

Solution by Darij Grinberg.

Let $\Delta$ be the area of triangle $ABC$, and $s = \frac{1}{2} (a+b+c)$ its semiperimeter. It is known that $\Delta = sr$, so that $r = \frac{\Delta}{s} = \frac{\Delta}{\frac{1}{2} (a+b+c)} = \frac{2\Delta}{a+b+c}$, and it is also known that $R = \frac{abc}{4\Delta}$. Thus, $\frac{9}{2} r R = \frac{9}{2} \cdot \frac{2\Delta}{a+b+c} \cdot \frac{abc}{4\Delta} = \frac{9abc}{4(a+b+c)}$. Thus, the inequality that we have to prove,

$$\frac{a^2bc}{(a+b)(a+c)} + \frac{b^2ca}{(b+c)(b+a)} + \frac{c^2ab}{(c+a)(c+b)} \leq \frac{9}{2} r R,$$

rewrites as

$$\frac{a^2bc}{(a+b)(a+c)} + \frac{b^2ca}{(b+c)(b+a)} + \frac{c^2ab}{(c+a)(c+b)} \leq \frac{9abc}{4(a+b+c)}.$$

Upon multiplication by $\frac{a+b+c}{abc}$, this becomes

$$\frac{a(a+b+c)}{(a+b)(a+c)} + \frac{b(a+b+c)}{(b+c)(b+a)} + \frac{c(a+b+c)}{(c+a)(c+b)} \leq \frac{9}{4}. \quad (1)$$

Now, consider a new triangle with sidelengths $a' = b+c$, $b' = c+a$, $c' = a+b$ (such a triangle actually exists because $b+c$, $c+a$, $a+b$ satisfy the triangle inequalities: for instance, $(c+a)+(a+b) > (b+c)$). Let $A'$, $B'$, $C'$ be the three angles of this triangle, and let $s' = \frac{1}{2} (a'+b'+c')$ be the semiperimeter of this triangle. Then,

$$s' = \frac{1}{2} (a'+b'+c') = \frac{1}{2} ((b+c) + (c+a) + (a+b)) = a+b+c,$$

and a standard formula yields

$$\cos^2 \frac{A'}{2} = \frac{s' (s'-a')}{b'c'} = \frac{(a+b+c)((a+b+c)-(b+c))}{(c+a)(a+b)} = \frac{a(a+b+c)}{(a+b)(a+c)},$$

and similarly $\cos^2 \frac{B'}{2} = \frac{b(a+b+c)}{(b+c)(b+a)}$ and $\cos^2 \frac{C'}{2} = \frac{c(a+b+c)}{(c+a)(c+b)}$. Hence, the inequality in question, (1), is equivalent to

$$\cos^2 \frac{A'}{2} + \cos^2 \frac{B'}{2} + \cos^2 \frac{C'}{2} \leq \frac{9}{4}.$$
But this is easy using Jensen’s inequality: The function 

\[ [0, \pi] \to \mathbb{R}, \quad \varphi \mapsto \cos^2 \frac{\varphi}{2}, \]

is concave (since \( \cos^2 \frac{\varphi}{2} = \frac{\cos \varphi + 1}{2} \), and the function 

\[ [0, \pi] \to \mathbb{R}, \quad \varphi \mapsto \cos \varphi \]

is concave), and thus the Jensen inequality yields

\[
\cos^2 \frac{A'}{2} + \cos^2 \frac{B'}{2} + \cos^2 \frac{C'}{2} \leq 3 \cos^2 \frac{\left( \frac{A' + B' + C'}{3} \right)}{2} = 3 \cos^2 \frac{\left( \frac{A' + B' + C'}{3} \right)}{6}
\]

\[
= 3 \cos^2 \frac{\pi}{6} \quad \text{(since } A', B', C' \text{ are angles of a triangle)}
\]

\[
= 3 \cdot \frac{3}{4} = \frac{9}{4},
\]

qed.