For every \( k \in \mathbb{N} \), for every \( k \)-tuple \( v \in \{0, 1\}^k \) and every \( i \in \{1, 2, ..., k\} \), let us denote by \( v_i \) the \( i \)-th component of the \( k \)-tuple \( v \) (remember that \( v \) is an element of \( \{0, 1\}^k \), that is, a \( k \)-tuple of elements of \( \{0, 1\} \)). Then, every \( v \in \{0, 1\}^k \) satisfies \( v = (v_1, v_2, ..., v_k) \).

Let \( n > 1 \) be an integer. For any \( n \)-tuple \( a \in \{0, 1\}^n \), we define two integers \( F(a) \) and \( G(a) \) by

\[
F(a) = |\{ i \in \{1, 2, ..., n-1\} \mid a_i = a_{i+1} = 0 \}|;
\]

\[
G(a) = |\{ i \in \{1, 2, ..., n-1\} \mid a_i = a_{i+1} = 1 \}|.
\]

Set

\[
T = \{ a \in \{0, 1\}^n \mid F(a) = G(a) \}.
\]

Prove that

\[
|T| = \begin{cases} 
2 \binom{n-2}{(n-1)/2}, & \text{if } n \text{ is even;} \\
2 \binom{n-2}{(n-1)/2}, & \text{if } n \text{ is odd}
\end{cases} = 2 \binom{n-2}{\lfloor (n-2)/2 \rfloor}.
\]

**Example.** For instance, if \( n = 4 \), then

\[
T = \{ (0,0,1,1), (0,1,0,1), (1,0,1,0), (1,1,0,0) \},
\]

so that \( |T| = 4 \).

**Solution by Darij Grinberg.**

We are going to introduce some more notations:

- For any assertion \( \mathcal{U} \), we denote by \( [\mathcal{U}] \) the Boolean value of the assertion \( \mathcal{U} \) (that is, \( [\mathcal{U}] = \begin{cases} 1, & \text{if } \mathcal{U} \text{ is true}; \\ 0, & \text{if } \mathcal{U} \text{ is false}. \end{cases} \)).

  It is then clear that if \( B \) is a set, and \( \mathcal{U}(a) \) is an assertion for every element \( a \) of \( B \), then

  \[
  \sum_{a \in B} [\mathcal{U}(a)] = |\{ a \in B \mid \mathcal{U}(a) \text{ is true} \}|.
  \]

  Also, if \( \mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_m \) are \( m \) assertions, then \( [\mathcal{U}_1 \text{ and } \mathcal{U}_2 \text{ and } ... \text{ and } \mathcal{U}_m] = \prod_{j=1}^{m} [\mathcal{U}_j] \).

- If \( n \) is an integer and \( k \) is a real, then we define the binomial coefficient \( \binom{n}{k} \) by

  \[
  \binom{n}{k} = \begin{cases} 
  \frac{n(n-1)...(n-k+1)}{k!}, & \text{if } k \in \mathbb{N} \;
  \\
  0, & \text{otherwise}
  \end{cases}
  \]
where \( \mathbb{N} \) denotes the set \( \{0, 1, 2, \ldots\} \). This definition agrees with the standard definition of \( \binom{n}{k} \) in the case when \( k \in \mathbb{Z} \).

A consequence of this definition is that if \( S \) is a finite set and \( k \) is a real, then
\[
\binom{|S|}{k} = |\{ A \in \mathcal{P}(S) \mid |A| = k \}|.
\]

It is immediate that
\[
[a = 1] = \alpha \quad \text{for every } \alpha \in \{0, 1\}, \quad \text{and} \quad (1)
\]
\[
[a = 0] = 1 - \alpha \quad \text{for every } \alpha \in \{0, 1\}. \quad (2)
\]

Now, let us solve the problem.

Define a map \( f : \{0, 1\}^n \rightarrow \mathcal{P}(\{1, 2, \ldots, n\}) \) by
\[
f(a) = \{ i \in \{1, 2, \ldots, n\} \mid a_i = 1 \}
\]
for any \( n \)-tuple \( a \in \{0, 1\}^n \).

It is known that this map \( f : \{0, 1\}^n \rightarrow \mathcal{P}(\{1, 2, \ldots, n\}) \) is a bijection.

For any \( n \)-tuple \( a \in \{0, 1\}^n \), we have
\[
|f(a)| = |\{ i \in \{1, 2, \ldots, n\} \mid a_i = 1 \}| = \sum_{i \in \{1, 2, \ldots, n\}} [a_i = 1] = \sum_{i \in \{1, 2, \ldots, n\}} a_i.
\]

Thus, for any real \( k \), we have
\[
\left| \left\{ a \in \{0, 1\}^n \mid \sum_{i \in \{1, 2, \ldots, n\}} a_i = k \right\} \right| = \left| \left\{ a \in \{0, 1\}^n \mid |f(a)| = k \right\} \right|
\]
\[
= |f^{-1}(\{ A \in \mathcal{P}(\{1, 2, \ldots, n\}) \mid |A| = k \})| = \left| \left\{ A \in \mathcal{P}(\{1, 2, \ldots, n\}) \mid |A| = k \right\} \right|
\]
(since \( f \) is a bijection)
\[
= \left| \{1, 2, \ldots, n| \right| = \binom{n}{k}.
\]

Applying this to \( n-2 \) instead of \( n \), we obtain
\[
\left| \left\{ a \in \{0, 1\}^{n-2} \mid \sum_{i \in \{1, 2, \ldots, n-2\}} a_i = k \right\} \right| = \binom{n-2}{k}. \quad (3)
\]
Now, for any \( n \)-tuple \( a \in \{0, 1\}^n \), we have

\[
F(a) = |\{ i \in \{1, 2, \ldots, n-1\} \mid a_i = a_{i+1} = 0 \}| = \sum_{i \in \{1, 2, \ldots, n-1\}} [a_i = a_{i+1} = 0]
\]

\[
= \sum_{i \in \{1, 2, \ldots, n-1\}} [a_i = 0 \text{ and } a_{i+1} = 0] = \sum_{i \in \{1, 2, \ldots, n-1\}} [a_i = 0] [a_{i+1} = 0] = \sum_{i \in \{1, 2, \ldots, n-1\}} (1 - a_i) (1 - a_{i+1})
\]

(since \( a_i = 0 \) and \( a_{i+1} = 0 \) as \( a_i, a_{i+1} \in \{0, 1\} \)

\[
G(a) = |\{ i \in \{1, 2, \ldots, n-1\} \mid a_i = a_{i+1} = 1 \}| = \sum_{i \in \{1, 2, \ldots, n-1\}} [a_i = a_{i+1} = 1]
\]

\[
= \sum_{i \in \{1, 2, \ldots, n-1\}} [a_i = 1 \text{ and } a_{i+1} = 1] = \sum_{i \in \{1, 2, \ldots, n-1\}} [a_i = 1] [a_{i+1} = 1] = \sum_{i \in \{1, 2, \ldots, n-1\}} a_i a_{i+1}
\]

(since \( a_i = 1 \) and \( a_{i+1} = 1 \) as \( a_i, a_{i+1} \in \{0, 1\} \)

so that

\[
F(a) - G(a) = \sum_{i \in \{1, 2, \ldots, n-1\}} (1 - a_i) (1 - a_{i+1}) - \sum_{i \in \{1, 2, \ldots, n-1\}} a_i a_{i+1}
\]

\[
= \sum_{i \in \{1, 2, \ldots, n-1\}} (1 - a_i) + \sum_{i \in \{1, 2, \ldots, n-1\}} a_i - \sum_{i \in \{1, 2, \ldots, n-1\}} a_{i+1}
\]

\[
= (n - 1) - \left( a_1 + \sum_{i \in \{2, 3, \ldots, n-1\}} a_i \right) - \left( \sum_{i \in \{2, 3, \ldots, n-1\}} a_i + a_n \right)
\]

\[
= (n - 1) - a_1 - a_n - 2 \sum_{i \in \{2, 3, \ldots, n-1\}} a_i.
\]

(4)

Obviously, the map \( v : \{0, 1\}^n \rightarrow \{0, 1\} \times \{0, 1\}^{n-2} \times \{0, 1\} \) defined by

\[
v(a) = (a_1, (a_2, a_3, \ldots, a_{n-1}), a_n)
\]

for any \( n \)-tuple \( a \in \{0, 1\}^n \)

is a bijection.
Now, for any \( x \in \{0, 1\} \) and \( y \in \{0, 1\} \), we have

\[
\{ a \in T \mid (a_1, a_n) = (x, y) \} = \{ a \in \{0, 1\}^n \mid F(a) = G(a) \text{ and } (a_1, a_n) = (x, y) \} = \{ a \in \{0, 1\}^n \mid F(a) - G(a) = 0 \text{ and } a_1 = x \text{ and } a_n = y \}
\]

\[
= \left\{ a \in \{0, 1\}^n \mid (n - 1) - a_1 - a_n - 2 \sum_{i \in \{2, 3, \ldots, n-1\}} a_i = 0 \text{ and } a_1 = x \text{ and } a_n = y \right\}
\]

(by (4))

\[
= \left\{ a \in \{0, 1\}^n \mid \sum_{i \in \{2, 3, \ldots, n-1\}} a_i = \frac{(n - 1) - a_1 - a_n}{2} \text{ and } a_1 = x \text{ and } a_n = y \right\}
\]

\[
= \left\{ a \in \{0, 1\}^n \mid \sum_{i \in \{2, 3, \ldots, n-1\}} a_i = \frac{(n - 1) - x - y}{2} \text{ and } a_1 = x \text{ and } a_n = y \right\}
\]

\[
v^{-1} \left( \{ x \} \times \left\{ b \in \{0, 1\}^{n-2} \mid \sum_{i \in \{1, 2, \ldots, n-2\}} b_i = \frac{(n - 1) - x - y}{2} \right\} \times \{ y \} \right).
\]

Since \( v \) is a bijection, this yields

\[
|\{ a \in T \mid (a_1, a_n) = (x, y) \}| = \left| \{ x \} \times \left\{ b \in \{0, 1\}^{n-2} \mid \sum_{i \in \{1, 2, \ldots, n-2\}} b_i = \frac{(n - 1) - x - y}{2} \right\} \times \{ y \} \right|
\]

\[
= \left| \left\{ b \in \{0, 1\}^{n-2} \mid \sum_{i \in \{1, 2, \ldots, n-2\}} b_i = \frac{(n - 1) - x - y}{2} \right\} \right|
\]

\[
= \left( \frac{n - 2}{2} \right) \left( \frac{n - 1}{2} - x - y \right) \quad \text{(by (3), applied to } k = \frac{(n - 1) - x - y}{2}).
\]

(5)

Now,

\[
|T| = |\{ a \in T \mid (a_1, a_n) = (0, 0) \}| + |\{ a \in T \mid (a_1, a_n) = (0, 1) \}| + |\{ a \in T \mid (a_1, a_n) = (1, 0) \}| + |\{ a \in T \mid (a_1, a_n) = (1, 1) \}|
\]

\[
= \left( \frac{n - 2}{2} \right) \left( \frac{n - 1}{2} - 0 - 0 \right) + \left( \frac{n - 2}{2} \right) \left( \frac{n - 1}{2} - 0 - 1 \right) + \left( \frac{n - 2}{2} \right) \left( \frac{n - 1}{2} - 1 - 0 \right) + \left( \frac{n - 2}{2} \right) \left( \frac{n - 1}{2} - 1 - 1 \right)
\]

(by (5))

\[
= \frac{n - 2}{2} \left( \frac{n - 1}{2} \right) + 2 \left( \frac{n - 2}{2} \right) \left( \frac{n - 2}{2} \right) + \left( \frac{n - 2}{2} \right) \left( \frac{n - 3}{2} \right).
\]

(6)
Now, if $n$ is odd, then $\frac{n-2}{2} \notin \mathbb{N}$, and thus $\left(\frac{n-2}{2}\right) = 0$, so this becomes

$$|T| = \left(\frac{n-2}{n-1}\right) + \left(\frac{n-2}{n-3}\right) = \left(\frac{n-2}{n-1}\right) + \left(\frac{n-2}{n-3}\right) = \left(\frac{n-1}{n-2}\right)$$

(by the recurrence of the binomial coefficients). On the other hand, if $n$ is even, then $\frac{n-1}{2} \notin \mathbb{N}$ and $\frac{n-3}{2} \notin \mathbb{N}$, so that $\left(\frac{n-2}{n-1}\right) = 0$ and $\left(\frac{n-2}{n-3}\right) = 0$, and thus (6) becomes

$$|T| = 2\left(\frac{n-2}{n-2}\right).$$

Thus, altogether we see that

$$|T| = \begin{cases} 
2\left(\frac{n-2}{n-2}\right), & \text{if } n \text{ is even;} \\
\frac{2}{n-1}, & \text{if } n \text{ is odd}
\end{cases}$$

for any $n > 1$. This simplifies to $|T| = 2\left(\frac{n-2}{n-2}\right)$ (because in the case of $n$ even, we have $\frac{n-2}{2} = \left\lfloor \frac{n-2}{2} \right\rfloor$, while in the case of $n$ odd, we notice that

$$\left(\frac{n-1}{n-1}\right) = 2\left(\frac{n-1}{n-1}\right) - 1 \quad \left(\text{since } \left(\frac{2\eta}{\eta}\right) = 2\left(\frac{2\eta-1}{\eta-1}\right) \text{ for any integer } \eta > 0\right)$$

$$= 2\left(\left\lfloor \frac{n-2}{2} \right\rfloor\right)$$

). The problem is thus solved.