American Mathematical Monthly Problem 11409 by Paolo Perfetti.

Let α and β be positive reals such that $\beta > \alpha$. Let

$$S(\alpha,\beta,N) = \sum_{n=2}^{N} n(\log n)(-1)^n \prod_{k=2}^{n} \frac{\alpha + k\log k}{\beta + (k+1)\log(k+1)}.$$

Prove that the limit $\lim\limits_{N\to\infty} S(\alpha,\beta,N)$ exists.

Solution (by Darij Grinberg).

EDIT: This solution is slightly flawed. Can you find the flaw? – See the remark at the end of the solution for the answer and a workaround.

Since $n\log n \prod\limits_{k=2}^{n} \dfrac{\alpha + k\log k}{\beta + (k+1)\log(k+1)}$ is positive for every $n \geq 2$, the series

$$\sum_{n=2}^{\infty} n\log n(-1)^n \prod_{k=2}^{n} \frac{\alpha + k\log k}{\beta + (k+1)\log(k+1)}$$

is alternating. Thus, by Leibniz's criterion, in order to prove its convergence, it is enough to show that

$$\lim_{n\to\infty}\left(n\log n \prod_{k=2}^{n} \frac{\alpha + k\log k}{\beta + (k+1)\log(k+1)}\right) = 0.$$

Since

$$n\log n \prod_{k=2}^{n} \frac{\alpha + k\log k}{\beta + (k+1)\log(k+1)} = n\log n \frac{\prod\limits_{k=2}^{n}(\alpha + k\log k)}{\prod\limits_{k=2}^{n}(\beta + (k+1)\log(k+1))}$$

$$= n\log n \frac{\prod\limits_{k=2}^{n}(\alpha + k\log k)}{\prod\limits_{k=3}^{n+1}(\beta + k\log k)} = (\beta + 2\log 2)\frac{n\log n}{\beta + (n+1)\log(n+1)} \frac{\prod\limits_{k=2}^{n}(\alpha + k\log k)}{\prod\limits_{k=2}^{n}(\beta + k\log k)}$$

$$= (\beta + 2\log 2)\frac{n\log n}{\beta + (n+1)\log(n+1)} \prod_{k=2}^{n} \frac{\alpha + k\log k}{\beta + k\log k},$$

this is equivalent to proving that

$$\lim_{n\to\infty}\left((\beta + 2\log 2)\frac{n\log n}{\beta + (n+1)\log(n+1)} \prod_{k=2}^{n} \frac{\alpha + k\log k}{\beta + k\log k}\right) = 0.$$

Since $\beta + 2\log 2$ is just a constant, and

$$\lim_{n\to\infty} \frac{n\log n}{\beta + (n+1)\log(n+1)} = \lim_{n\to\infty} \frac{1}{\dfrac{\beta}{n\log n} + \dfrac{n+1}{n}\dfrac{\log(n+1)}{\log n}}$$

$$= \frac{1}{\underbrace{\lim_{n\to\infty} \dfrac{\beta}{n\log n}}_{=0} + \underbrace{\lim_{n\to\infty} \dfrac{n+1}{n}}_{=1} \underbrace{\lim_{n\to\infty} \dfrac{\log(n+1)}{\log n}}_{=1}} = \frac{1}{0 + 1\cdot 1} = 1$$

(where we used that $\lim_{n\to\infty} \dfrac{\log(n+1)}{\log n} = 1$, which is easy to see[1]), this reduces to showing that

$$\lim_{n\to\infty} \left(\prod_{k=2}^{n} \frac{\alpha + k\log k}{\beta + k\log k} \right) = 0.$$

Now, set $\gamma = \beta - \alpha$; then, $\beta > \alpha$ yields $\gamma > 0$, so that

$$\prod_{k=2}^{n} \frac{\alpha + k\log k}{\beta + k\log k} = \prod_{k=2}^{n} \frac{1}{\dfrac{\beta - \alpha}{\alpha + k\log k} + 1} = \prod_{k=2}^{n} \frac{1}{\dfrac{\gamma}{\alpha + k\log k} + 1}.$$

Thus, it remains to prove that

$$\lim_{n\to\infty} \left(\prod_{k=2}^{n} \frac{1}{\dfrac{\gamma}{\alpha + k\log k} + 1} \right) = 0,$$

what is obviously equivalent to

$$\lim_{n\to\infty} \left(\prod_{k=2}^{n} \left(\frac{\gamma}{\alpha + k\log k} + 1 \right) \right) = \infty.$$

Now,

$$\prod_{k=2}^{n} \left(\frac{\gamma}{\alpha + k\log k} + 1 \right) \geq \sum_{k=2}^{n} \frac{\gamma}{\alpha + k\log k}$$

[2], so it will be enough to prove that

$$\lim_{n\to\infty} \left(\sum_{k=2}^{n} \frac{\gamma}{\alpha + k\log k} \right) = \infty.$$

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[1]In fact, $\dfrac{\log(n+1)}{\log n} > 1$ (since log is increasing) and $\dfrac{\log(n+1)}{\log n} < \dfrac{\log(2n)}{\log n} = \dfrac{\log 2 + \log n}{\log n} = \underbrace{\dfrac{\log 2}{\log n}}_{\substack{\to 0 \text{ for} \\ n\to\infty}} + 1$ for $n \geq 2$, so that the sequence $\left(\dfrac{\log(n+1)}{\log n}\right)_{n\geq 2}$ is enclosed between two sequences that both converge to 1 for $n \to \infty$, and thus converges to 1.

[2]This crude estimate becomes clear by multiplying out the product on the left hand side - you obtain numerous addends, among them all those appearing in the sum on the right hand side, and some more (which are all positive, so they can be omitted).

But this can be done in the same way as one usually shows that the harmonic series diverges:

$$\begin{aligned}
\sum_{k=2}^{n} \frac{\gamma}{\alpha + k\log k} &\geq \sum_{k=2}^{2^{\lfloor \log_2 n \rfloor}} \frac{\gamma}{\alpha + k\log k} \qquad \left(\text{since } n \geq 2^{\lfloor \log_2 n \rfloor}\right) \\
&= \sum_{i=1}^{\lfloor \log_2 n \rfloor - 1} \sum_{k=2^i}^{2^{i+1}-1} \frac{\gamma}{\alpha + k\log k} > \sum_{i=1}^{\lfloor \log_2 n \rfloor - 1} \sum_{k=2^i}^{2^{i+1}-1} \frac{\gamma}{\alpha + 2^{i+1}\log 2^{i+1}} \qquad \left(\text{since } k < 2^{i+1}\right) \\
&= \sum_{i=1}^{\lfloor \log_2 n \rfloor - 1} 2^i \cdot \frac{\gamma}{\alpha + 2^{i+1}\log 2^{i+1}} = \sum_{i=1}^{\lfloor \log_2 n \rfloor - 1} \frac{\gamma}{\dfrac{\alpha}{2^i} + 2\underbrace{\log 2^{i+1}}_{=(i+1)\log 2}} \\
&= \sum_{i=1}^{\lfloor \log_2 n \rfloor - 1} \frac{\gamma \big/ (2\log 2)}{\dfrac{\alpha}{2^{i+1}\log 2} + (i+1)} > \sum_{i=1}^{\lfloor \log_2 n \rfloor - 1} \frac{\gamma \big/ (2\log 2)}{\lceil \alpha \rceil + (i+1)} \qquad \left(\text{since } \frac{\alpha}{2^{i+1}\log 2} < \alpha < \lceil \alpha \rceil\right) \\
&= \frac{\gamma}{2\log 2} \sum_{i=1}^{\lfloor \log_2 n \rfloor - 1} \frac{1}{\lceil \alpha \rceil + (i+1)} = \frac{\gamma}{2\log 2} \sum_{i=1}^{\lfloor \log_2 n \rfloor - 1} \frac{1}{i + (\lceil \alpha \rceil + 1)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_{n\to\infty} \left(\sum_{k=2}^{n} \frac{\gamma}{\alpha + k\log k} \right) &\geq \frac{\gamma}{2\log 2} \lim_{n\to\infty} \left(\sum_{i=1}^{\lfloor \log_2 n \rfloor - 1} \frac{1}{i + (\lceil \alpha \rceil + 1)} \right) = \frac{\gamma}{2\log 2} \lim_{n\to\infty} \left(\underbrace{\sum_{i=\lceil \alpha \rceil + 2}^{\lceil \alpha \rceil + \lfloor \log_2 n \rfloor} \frac{1}{i}}_{\substack{\text{this is an evergrowing patch} \\ \text{of the harmonic series, and thus} \\ \text{tends to } \infty \text{ for } n\to\infty}} \right) \\
&= \frac{\gamma}{2\log 2} \infty = \infty,
\end{aligned}$$

qed.

Remark. Have you found the mistake in the above solution? It is in the application of Leibniz's criterion. In order to use it to prove the convergence of the alternating series

$$\sum_{n=2}^{\infty} n\log n\,(-1)^n \prod_{k=2}^{n} \frac{\alpha + k\log k}{\beta + (k+1)\log(k+1)},$$

one not only has to show that

$$\lim_{n\to\infty} \left(n\log n \prod_{k=2}^{n} \frac{\alpha + k\log k}{\beta + (k+1)\log(k+1)} \right) = 0,$$

but also must prove that the sequence

$$\left(n\log n \prod_{k=2}^{n} \frac{\alpha + k\log k}{\beta + (k+1)\log(k+1)} \right)_{n\in\mathbb{N}}$$

is monotonically decreasing from some n onwards. How to show this? Here is one possible way: We have to prove the existence of some $N \in \mathbb{N}$ such that the inequality

$$n \log n \prod_{k=2}^{n} \frac{\alpha + k \log k}{\beta + (k+1) \log (k+1)} \geq (n+1) \log (n+1) \prod_{k=2}^{n+1} \frac{\alpha + k \log k}{\beta + (k+1) \log (k+1)}$$

holds for every $n > N$. This inequality is easily rewritten as

$$\frac{n \log n}{(n+1) \log (n+1)} \geq \frac{\alpha + (n+1) \log (n+1)}{\beta + (n+2) \log (n+2)}. \tag{1}$$

Now, in order to prove (1), we introduce some notation.

A function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$ is called *neat* if is either constantly 0 or has only finitely many zeroes.

A *1-logarithmic term* on an interval $I \subseteq \mathbb{R}$ will mean a term of the form $\sum\limits_{k=1}^{u} p_k \log q_k$, where $p_1, p_2, ..., p_u, q_1, q_2, ..., q_u$ are finitely many rational functions in one variable x over $\mathbb{R}$ such that $q_1, q_2, ..., q_u$ are all positive on I.

A *2-logarithmic term* on an interval $I \subseteq \mathbb{R}$ will mean a term of the form $\sum\limits_{k=1}^{u} p_k \log q_k \log r_k$, where $p_1, p_2, ..., p_u, q_1, q_2, ..., q_u, r_1, r_2, ..., r_u$ are finitely many rational functions in one variable x over $\mathbb{R}$ such that $q_1, q_2, ..., q_u, r_1, r_2, ..., r_u$ are all positive on I.

Obviously, any 1-logarithmic term defines a function $I \to \mathbb{R}$, and any 2-logarithmic term defines a function $I \to \mathbb{R}$. A function $I \to \mathbb{R}$ will be called *1-logarithmic* if it can be represented by a 1-logarithmic term on I, and similarly it will be called *2-logarithmic* if it can be represented by a 2-logarithmic term on I.

First, we notice an easy property:

Lemma 1. a) If a function $f : I \to \mathbb{R}$ is 1-logarithmic, then so is $f^{(\ell)}$ for every $\ell \in \mathbb{N}$.

b) If a function $f : I \to \mathbb{R}$ is 2-logarithmic, then so is $f^{(\ell)}$ for every $\ell \in \mathbb{N}$.

Proof of Lemma 1. **a)** This will follow by induction once we show that if a function $f : I \to \mathbb{R}$ is 1-logarithmic, then so is f'. But this is clear because

$$\left(\sum_{k=1}^{u} p_k \log q_k \right)' = \sum_{k=1}^{u} \left(p_k' \log q_k + \frac{p_k q_k'}{q_k} \right) = \sum_{k=1}^{u} \left(p_k' \log q_k + \frac{p_k q_k'}{q_k} \log e \right).$$

b) This can be proven similarly to **a)**, but we won't use it here, so we restrain from giving the proof.

Thus, Lemma 1 is proven.

Now we claim:

Lemma 2. a) Let $I \subseteq \mathbb{R}$ be an interval. Then, any 1-logarithmic function on I is neat.

b) Let $I \subseteq \mathbb{R}$ be an interval. Then, any 2-logarithmic function on I is neat.

Proof of Lemma 2. **a)** Let $\sum\limits_{k=1}^{u} p_k \log q_k$ be a 1-logarithmic term on I; this means that $p_1, p_2, ..., p_u, q_1, q_2, ..., q_u$ are finitely many rational functions in one variable x over $\mathbb{R}$ such that $q_1, q_2, ..., q_u$ are all positive on I. We must prove that the function $\sum\limits_{k=1}^{u} p_k \log q_k : I \to \mathbb{R}$ is neat.

We can WLOG assume that p_1, p_2, ..., p_u are polynomials (else, just multiply the rational functions p_1, p_2, ..., p_u by their common denominator).

Notice that if $f : I \to \mathbb{R}$ is a differentiable function such that f' is neat, then f is neat as well (since Rolle's theorem asserts the existence of a zero of f' between any two zeroes of f). By induction, this yields that if $f^{(\ell)}$ is neat for some $\ell \in \mathbb{N}$, then f is neat. Hence, in order to prove that the function $\sum\limits_{k=1}^{u} p_k \log q_k$ is neat, it is enough to show that there exists an $\ell \in \mathbb{N}$ such that $\left(\sum\limits_{k=1}^{u} p_k \log q_k\right)^{(\ell)}$ is a rational function (since any rational function is neat). This can be proven by induction over $\max\{\deg p_k \mid k \in \{1, 2, ..., u\}\}$: If $\max\{\deg p_k \mid k \in \{1, 2, ..., u\}\} < 0$, then $p_k = 0$ for every k, so that $\sum\limits_{k=1}^{u} p_k \log q_k = 0$ and everything is obvious. If not, then

$$\left(\sum_{k=1}^{u} p_k \log q_k\right)' = \sum_{k=1}^{u} \left(p_k' \log q_k + \frac{p_k q_k'}{q_k}\right) = \sum_{k=1}^{u} p_k' \log q_k + \sum_{k=1}^{u} \frac{p_k q_k'}{q_k}.$$

Here, $\sum\limits_{k=1}^{u} p_k' \log q_k$ is a 1-logarithmic term which satisfies $\max\{\deg p_k' \mid k \in \{1, 2, ..., u\}\} < \max\{\deg p_k \mid k \in \{1, 2, ..., u\}\}$. Hence, by induction, there exists some $\ell \in \mathbb{N}$ such that $\left(\sum\limits_{k=1}^{u} p_k' \log q_k\right)^{(\ell)}$ is a rational function. Thus

$$\begin{aligned}\left(\sum_{k=1}^{u} p_k \log q_k\right)^{(\ell+1)} &= \left(\left(\sum_{k=1}^{u} p_k \log q_k\right)'\right)^{(\ell)} = \left(\sum_{k=1}^{u} p_k' \log q_k + \sum_{k=1}^{u} \frac{p_k q_k'}{q_k}\right)^{(\ell)} \\ &= \underbrace{\left(\sum_{k=1}^{u} p_k' \log q_k\right)^{(\ell)}}_{\text{a rational function}} + \underbrace{\left(\sum_{k=1}^{u} \frac{p_k q_k'}{q_k}\right)^{(\ell)}}_{\text{a rational function}}\end{aligned}$$

is a rational function, and the induction step is done.

b) Let $\sum\limits_{k=1}^{u} p_k \log q_k \log r_k$ be a 2-logarithmic term on I; this means that p_1, p_2, ..., p_u, q_1, q_2, ..., q_u, r_1, r_2, ..., r_u are finitely many rational functions in one variable x over $\mathbb{R}$ such that q_1, q_2, ..., q_u, r_1, r_2, ..., r_u are all positive on I. We must prove that the function $\sum\limits_{k=1}^{u} p_k \log q_k \log r_k : I \to \mathbb{R}$ is neat.

As in the proof of Lemma 2 **a)**, we can WLOG assume that p_1, p_2, ..., p_u are polynomials. Again, we remember that if a function $f : I \to \mathbb{R}$ is such that $f^{(\ell)}$ is neat for some $\ell \in \mathbb{N}$, then f is neat. Hence, in order to prove that the function $\sum\limits_{k=1}^{u} p_k \log q_k \log r_k$ is neat, it is enough to show that there exists an $\ell \in \mathbb{N}$ such that $\left(\sum\limits_{k=1}^{u} p_k \log q_k \log r_k\right)^{(\ell)}$ is a 1-logarithmic function (since Lemma 2 **a)** states that any 1-logarithmic function is neat). This can be proven by induction over $\max\{\deg p_k \mid k \in \{1, 2, ..., u\}\}$: If $\max\{\deg p_k \mid k \in \{1, 2, ..., u\}\} < 0$, then $p_k = 0$ for every k, so that $\sum\limits_{k=1}^{u} p_k \log q_k \log r_k =$

0 and everything is obvious. If not, then

$$\begin{aligned}\left(\sum_{k=1}^{u} p_k \log q_k \log r_k\right)' &= \sum_{k=1}^{u} \left(p_k' \log q_k \log r_k + \frac{p_k q_k'}{q_k} \log r_k + \frac{p_k r_k'}{r_k} \log q_k\right) \\ &= \sum_{k=1}^{u} p_k' \log q_k \log r_k + \sum_{k=1}^{u} \left(\frac{p_k q_k'}{q_k} \log r_k + \frac{p_k r_k'}{r_k} \log q_k\right).\end{aligned}$$

Here, $\sum\limits_{k=1}^{u} p_k' \log q_k \log r_k$ is a 2-logarithmic term which satisfies $\max\{\deg p_k' \mid k \in \{1, 2, ..., u\}\} < \max\{\deg p_k \mid k \in \{1, 2, ..., u\}\}$. Hence, by induction, there exists some $\ell \in \mathbb{N}$ such that $\left(\sum\limits_{k=1}^{u} p_k' \log q_k \log r_k\right)^{(\ell)}$ is a 1-logarithmic function. Besides, $\sum\limits_{k=1}^{u} \left(\frac{p_k q_k'}{q_k} \log r_k + \frac{p_k r_k'}{r_k} \log q_k\right)$ is a 1-logarithmic function, so that $\left(\sum\limits_{k=1}^{u} \left(\frac{p_k q_k'}{q_k} \log r_k + \frac{p_k r_k'}{r_k} \log q_k\right)\right)^{(\ell)}$ is a 1-logarithmic function as well (by Lemma 1 **a)**). Thus,

$$\begin{aligned}\left(\sum_{k=1}^{u} p_k \log q_k \log r_k\right)^{(\ell+1)} &= \left(\left(\sum_{k=1}^{u} p_k \log q_k \log r_k\right)'\right)^{(\ell)} \\ &= \left(\sum_{k=1}^{u} p_k' \log q_k \log r_k + \sum_{k=1}^{u} \left(\frac{p_k q_k'}{q_k} \log r_k + \frac{p_k r_k'}{r_k} \log q_k\right)\right)^{(\ell)} \\ &= \underbrace{\left(\sum_{k=1}^{u} p_k' \log q_k \log r_k\right)^{(\ell)}}_{\text{a 1-logarithmic function}} + \underbrace{\left(\sum_{k=1}^{u} \left(\frac{p_k q_k'}{q_k} \log r_k + \frac{p_k r_k'}{r_k} \log q_k\right)\right)^{(\ell)}}_{\text{a 1-logarithmic function}}\end{aligned}$$

is a 1-logarithmic function, and the induction step is done.

The proof of Lemma 2 is thus complete.

Back to our problem. Define a function $g : \mathbb{R}^+ \to \mathbb{R}$ by

$$g(x) = x \log x \cdot (\beta + (x+2) \log(x+2)) - (\alpha + (x+1) \log(x+1)) \cdot (x+1) \log(x+1).$$

This function g is 2-logarithmic (in order to see it, just replace β and α by $\beta \log e$ and $\alpha \log e$, respectively, and multiply out), and therefore neat (by Lemma 2 **b)**). In other words, g is constantly 0 or has only finitely many zeroes on $\mathbb{R}^+$. In both of these cases, we conclude that there exists some $N \in \mathbb{N}$ such that the number $g(x)$ has the same sign for all real $x > N$ (because g is continuous, and thus cannot change signs without having a zero). Thus, either

$$\frac{x \log x}{(x+1) \log(x+1)} \geq \frac{\alpha + (x+1) \log(x+1)}{\beta + (x+2) \log(x+2)} \qquad \text{for all } x > N,$$

or

$$\frac{x \log x}{(x+1) \log(x+1)} \leq \frac{\alpha + (x+1) \log(x+1)}{\beta + (x+2) \log(x+2)} \qquad \text{for all } x > N.$$

The first of these two cases yields that (1) holds for every $n > N$, and thus the sequence $\left(n \log n \prod\limits_{k=2}^{n} \frac{\alpha + k \log k}{\beta + (k+1) \log(k+1)}\right)_n$ is decreasing from some n onwards. The second of these two cases is impossible, since it would (similarly) yield that the sequence $\left(n\log n\prod_{k=2}^{n}\frac{\alpha+k\log k}{\beta+(k+1)\log(k+1)}\right)_n$ is *increasing* from some n onwards, what is absurd because all its values are positive while its limit is 0 (as we have shown in the solution of the problem). Thus, the first case must hold, and we conclude that the sequence $\left(n\log n\prod_{k=2}^{n}\frac{\alpha+k\log k}{\beta+(k+1)\log(k+1)}\right)_n$ is decreasing from some n onwards, so the gap in our solution of the problem is (finally) filled.

Thanks to mathmanman for noticing the mistake!