American Mathematical Monthly Problem 11403 by Yaming Yu, Irvine, CA.

For every integer $n \geq 0$, define a polynomial $f_n \in \mathbb{Q}[x]$ by

$$f_n(x) = \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j).$$

Find $\deg f_n$ for every $n > 1$.

**Solution by Darij Grinberg.**

We claim that $\deg f_n = \left\lfloor \frac{n}{2} \right\rfloor$ for every integer $n > 1$. The key to the proof is the following recurrence relation for our polynomials:

$$f_n(x) = (n-1)(f_{n-1}(x) + xf_{n-2}(x)) \quad (11403.1)$$

for every integer $n > 1$.

**Proof of (11403.1).** Let $n > 1$. Then,

$$f_n(x) = \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j) = \sum_{i=0}^{n} \left( \binom{n-1}{i-1} + \binom{n-1}{i} \right) (-x)^{n-i} \prod_{j=0}^{i-1} (x+j)$$

$$= \sum_{i=0}^{n} \binom{n-1}{i-1} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j) + \sum_{i=0}^{n} \binom{n-1}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j)$$

(here we substituted $i+1$ for $i$ in the first sum)

$$= \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i} \prod_{j=0}^{i} (x+j) + \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j)$$

(here we removed a zero addend from each of the two sums)

$$= \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i} (x+i) \prod_{j=0}^{i-1} (x+j) + \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i-1} (-x) \prod_{j=0}^{i-1} (x+j)$$

$$= \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i} (x+i) + (-x)^{n-i} \prod_{j=0}^{i-1} (x+j) = \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j)$$

$$= \sum_{i=0}^{n-1} (n-1) \cdot \binom{n-2}{i-1} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j) \quad \text{(since } i \cdot \binom{n-1}{i} = (n-1) \cdot \binom{n-2}{i-1})$$

$$= (n-1) \cdot \sum_{i=0}^{n-1} \binom{n-2}{i-1} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j). \quad (11403.2)$$
But adding up the equalities

\[ f_{n-1}(x) = \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{(n-1)-i} \prod_{j=0}^{i-1} (x+j) = \sum_{i=0}^{n-1} \binom{n-2}{i} \prod_{j=0}^{i-1} (x+j) \]

and

\[ xf_{n-2}(x) = x \sum_{i=0}^{n-2} \binom{n-2}{i} (-x)^{(n-2)-i} \prod_{j=0}^{i-1} (x+j) = -\sum_{i=0}^{n-2} \binom{n-2}{i} (-x)^{(n-2)-i} \prod_{j=0}^{i-1} (x+j) \]

yields

\[ f_{n-1}(x) + xf_{n-2}(x) = \sum_{i=0}^{n-1} \binom{n-2}{i} \prod_{j=0}^{i-1} (x+j) \]

(since two of the three sums cancel out), and thus \((f_{n-1}(x) + xf_{n-2}(x))\) becomes \(f_n(x) = (n-1) \cdot \)\((f_{n-1}(x) + xf_{n-2}(x))\). This proves \((11403.1)\).

Next, we introduce a notation: For any polynomial \(p \in \mathbb{Q}[x]\), and for any integer \(k \geq 0\), we denote by \(\text{coeff}(p,k)\) the coefficient of \(p\) before \(x^k\). Then, every polynomial \(p \in \mathbb{Q}[x]\) satisfies \(p(x) = \sum_{k \geq 0} \text{coeff}(p,k) \cdot x^k\).

The recurrence \((11403.2)\) immediately yields the relations

\[ \deg f_n \leq \max \{\deg f_{n-1}, 1 + \deg f_{n-2}\} \]  \hspace{1cm} (11403.3)

and

\[ \text{coeff}(f_n, s) = (n-1) \cdot \text{coeff}(f_{n-1}, s) + \text{coeff}(f_{n-2}, s-1) \]  \hspace{1cm} (11403.4)

for every positive integers \(n > 3\) and \(s\). Now, a straightforward induction (using \(f_2(x) = x\) and \(f_3(x) = 2x\) as the induction base, and \((11403.2)\) and \((11403.4)\) for the induction step) shows the three relations

\[ \deg f_{2u} = \deg f_{2u+1} = u, \]  \hspace{1cm} (11403.5)

\[ \text{coeff}(f_{2u}, u) > 0, \]  \hspace{1cm} (11403.6)

\[ \text{coeff}(f_{2u+1}, u) > 0 \]  \hspace{1cm} (11403.7)

for every integer \(u \geq 1\) (we won’t use the relations \((11403.6)\) and \((11403.7)\) anymore, but we need them to survive in the induction step, since they ensure that the leading terms of the polynomials \(f_{n-1}(x)\) and \(xf_{n-2}(x)\) don’t cancel out each other in \((11403.1)\)). In particular, \((11403.5)\) yields our assertion that \(\deg f_n = \left\lfloor \frac{n}{2} \right\rfloor\) for every integer \(n > 1\).