American Mathematical Monthly Problem 11398 by Stanley Huang

Let ABC be an acute-angled triangle such that its angle at A is the middle-sized among its three angles. Further assume that the incenter I of triangle ABC is equidistant from the circumcenter O and the orthocenter H. Prove that $\measuredangle CAB = 60^\circ$, and that the circumradius of triangle IBC equals the circumradius of triangle ABC.

Solution (by Darij Grinberg).

Before we come to the solution of the problem, we recapitulate two known facts from triangle geometry:

> **Lemma 1.** Let ABC be a triangle, and let X, Y, Z be the midpoints of the arcs BC, CA, AB (not containing A, B, C, respectively) of the circumcircle of triangle ABC. Then, the incenter I of triangle ABC is the orthocenter of triangle XYZ.

Proof of Lemma 1. I really don't wish to repeat this proof here, since it is more than well-known. I gave it in

`http://www.mathlinks.ro/Forum/viewtopic.php?t=6095`

post #2; if my memory doesn't betray me, the proof can also be found in the solution of an AMM problem from a few years ago.

> **Lemma 2, the Sylvester theorem.** If O is the circumcenter and H is the orthocenter of a triangle ABC, then $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$.

Proof of Lemma 2. It is a known fact (the Euler line theorem) that the points O, G and H are collinear, and $\overrightarrow{HG} = 2 \cdot \overrightarrow{GO}$, where G is the centroid of triangle ABC. This yields $\overrightarrow{HO} = \overrightarrow{HG} + \overrightarrow{GO} = 2 \cdot \overrightarrow{GO} + \overrightarrow{GO} = 3 \cdot \overrightarrow{GO}$; in other words, $\overrightarrow{OH} = 3 \cdot \overrightarrow{OG}$. Since $\overrightarrow{OG} = \dfrac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}$ (what follows from G being the centroid of triangle ABC), this becomes $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$, and Lemma 2 is proven.

Now let us solve the problem. We are going to use complex numbers. For every point named by a capital letter - say, P - we denote its affix (this means the complex number corresponding to this point) by the corresponding lower letter - in this case, p. In particular, we denote the affix of the incenter I by i; we will *not* use the letter i for $\sqrt{-1}$ (we will simply write $\sqrt{-1}$ for $\sqrt{-1}$).

We can WLOG assume that the vertices a, b, c of triangle ABC lie on the unit circle $\left\{t \in \mathbb{C} \mid t\overline{t} = 1\right\}$. This yields $a\overline{a} = b\overline{b} = c\overline{c} = 1$ and $o = 0$. Let $\measuredangle A$, $\measuredangle B$, $\measuredangle C$ denote the three angles $\measuredangle CAB$, $\measuredangle ABC$, $\measuredangle BCA$ of triangle ABC.

Let X, Y, Z be the midpoints of the arcs BC, CA, AB (not containing A, B, C, respectively) of the unit circle (which, of course, is the circumcircle of triangle ABC). Then, Lemma 1 yields that I is the orthocenter of triangle XYZ, whereas it is clear that O is the circumcenter of this triangle. Thus, applying Lemma 2 to triangle XYZ

in lieu of triangle ABC yields $\overrightarrow{OI}=\overrightarrow{OX}+\overrightarrow{OY}+\overrightarrow{OZ}$. Translating into complex numbers, this becomes $i-o=(x-o)+(y-o)+(z-o)$. Since $o=0$, this becomes $i=x+y+z$.

On the other hand, directly applying Lemma 2 to triangle ABC leads to $\overrightarrow{OH}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}$, what becomes $h-o=(a-o)+(b-o)+(c-o)$ when translated into complex numbers, and thus $h=a+b+c$ since $o=0$.

Next we claim that:

Lemma 3. We have $a=-\dfrac{yz}{x}$, $b=-\dfrac{zx}{y}$ and $c=-\dfrac{xy}{z}$.

Proof of Lemma 3. For any complex number $t\neq 0$, we denote by $\arg t$ the *principal value* of the argument of t (that is, the value lying in the interval $[0,2\pi)$), and we denote by $\sqrt{t}$ the square root of t that satisfies $\arg\sqrt{t}<\pi$. Obviously, if t lies on the unit circle, then so does $\sqrt{t}$, and we have $\arg\sqrt{t}=\dfrac{1}{2}\arg t$. Now, WLOG assume that the triangle ABC is directed clockwise. Then, the arc BC on the unit circle is the arc that goes in the counter-clockwise direction from C to B; hence, the midpoint X of this arc has the affix $c\sqrt{\dfrac{b}{c}}$ [1]. In other words, $x=c\sqrt{\dfrac{b}{c}}$. Similarly, $y=a\sqrt{\dfrac{c}{a}}$ and $z=b\sqrt{\dfrac{a}{b}}$. It is easy to see that $\sqrt{\dfrac{b}{c}}\cdot\sqrt{\dfrac{c}{a}}\cdot\sqrt{\dfrac{a}{b}}=-1$ (since

$$\left|\sqrt{\frac{b}{c}}\cdot\sqrt{\frac{c}{a}}\cdot\sqrt{\frac{a}{b}}\right|=\sqrt{\frac{|b|}{|c|}}\cdot\sqrt{\frac{|c|}{|a|}}\cdot\sqrt{\frac{|a|}{|b|}}=1$$

and

$$\begin{aligned}\arg\left(\sqrt{\frac{b}{c}}\cdot\sqrt{\frac{c}{a}}\cdot\sqrt{\frac{a}{b}}\right)&\equiv\arg\sqrt{\frac{b}{c}}+\arg\sqrt{\frac{c}{a}}+\arg\sqrt{\frac{a}{b}}=\frac{1}{2}\arg\frac{b}{c}+\frac{1}{2}\arg\left(\frac{c}{a}\right)+\frac{1}{2}\arg\left(\frac{a}{b}\right)\\&=\frac{1}{2}\left(\arg\frac{b}{c}+\arg\frac{c}{a}+\arg\frac{a}{b}\right)=\frac{1}{2}\left(\underbrace{\measuredangle COB+\measuredangle BOA+\measuredangle AOC}_{\text{these angles mean directed angles}}\right)\\&=\frac{1}{2}\cdot 360^\circ=180^\circ \bmod 360^\circ\end{aligned}$$

). Besides, $\left(\sqrt{\dfrac{b}{c}}\right)^2=\dfrac{b}{c}$. Thus,

$$\frac{yz}{x}=\frac{a\sqrt{\frac{c}{a}}\cdot b\sqrt{\frac{a}{b}}}{c\sqrt{\frac{b}{c}}}=\frac{a\sqrt{\frac{c}{a}}\cdot b\sqrt{\frac{a}{b}}\cdot\sqrt{\frac{b}{c}}}{c\sqrt{\frac{b}{c}}\cdot\sqrt{\frac{b}{c}}}=\frac{ab\cdot\sqrt{\frac{b}{c}}\cdot\sqrt{\frac{c}{a}}\cdot\sqrt{\frac{a}{b}}}{c\left(\sqrt{\frac{b}{c}}\right)^2}=\frac{ab\cdot(-1)}{c\cdot\frac{b}{c}}=-a,$$

[1] One may be tempted to "simplify" this to $\sqrt{bc}$, but this can turn out false - we may have $c\sqrt{\dfrac{b}{c}}=\sqrt{bc}$ but we also may have $c\sqrt{\dfrac{b}{c}}=-\sqrt{bc}$.

so that $a=-\frac{yz}{x}$. Similarly, $b=-\frac{zx}{y}$ and $c=-\frac{xy}{z}$.

Thus, $h=a+b+c$ becomes $h=\left(-\frac{yz}{x}\right)+\left(-\frac{zx}{y}\right)+\left(-\frac{xy}{z}\right)=-xyz\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)$, so that

$$i-h=(x+y+z)-\left(-xyz\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)\right)=(x+y+z)+xyz\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right). \tag{11398.1}$$

Now, the condition of the problem states that $|IO|=|IH|$. Thus, $|i-o|=|i-h|$; that is, $|i|=|i-h|$. Hence, $|i|^2=|i-h|^2$, what rewrites as $i\overline{i}=(i-h)\left(\overline{i-h}\right)$. Using $i=x+y+z$ and (11398.1), this rewrites as

$$(x+y+z)\left(\overline{x}+\overline{y}+\overline{z}\right)=\left((x+y+z)+xyz\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)\right)\left(\left(\overline{x}+\overline{y}+\overline{z}\right)+\overline{xyz}\left(\frac{1}{\overline{x}^2}+\frac{1}{\overline{y}^2}+\frac{1}{\overline{z}^2}\right)\right).$$

Since the points X, Y, Z lie on the unit circle, their affixes x, y, z satisfy $x\overline{x}=1$, $y\overline{y}=1$, $z\overline{z}=1$, so we can replace $\overline{x}$, $\overline{y}$, $\overline{z}$ by $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ in this equation, and obtain

$$\begin{aligned}&(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\\&=\left((x+y+z)+xyz\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)\right)\\&\cdot\left(\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+\frac{1}{x}\cdot\frac{1}{y}\cdot\frac{1}{z}\left(\frac{1}{\left(1/x\right)^2}+\frac{1}{\left(1/y\right)^2}+\frac{1}{\left(1/z\right)^2}\right)\right).\end{aligned}$$

After some work, this equation simplifies to

$$\left(y^2+yz+z^2\right)\left(z^2+zx+x^2\right)\left(x^2+xy+y^2\right)=0.$$

Hence, one of the numbers y^2+yz+z^2, z^2+zx+x^2 and x^2+xy+y^2 must be 0.

First we consider the case $y^2+yz+z^2=0$. Since $y^2+yz+z^2=\left(y+e^{2\pi\sqrt{-1}/3}z\right)\left(y-e^{2\pi\sqrt{-1}/3}z\right)$, we must have $y+e^{2\pi\sqrt{-1}/3}z=0$ or $y-e^{2\pi\sqrt{-1}/3}z$ in this case, so that $\frac{y}{z}=\pm e^{2\pi\sqrt{-1}/3}$. Using Lemma 3, this yields $\frac{c}{b}=\frac{-\frac{xy}{z}}{-\frac{zx}{y}}=\left(\frac{y}{z}\right)^2=\left(\pm e^{2\pi\sqrt{-1}/3}\right)^2=e^{4\pi\sqrt{-1}/3}$. Consequently, $\measuredangle BOC=\arg\frac{b}{c}=\arg e^{4\pi\sqrt{-1}/3}=\frac{4\pi}{3}$, so that $\measuredangle COB=2\pi-\measuredangle BOC=2\pi-\frac{4\pi}{3}=\frac{2\pi}{3}$. But by the central angle theorem for triangle ABC, we have $\measuredangle COB=2\cdot\measuredangle CAB$, so this becomes $\measuredangle CAB=\frac{\pi}{3}=60^\circ$. In other words, $\measuredangle A=60^\circ$.

Hence, in the case $y^2+yz+z^2=0$, we have obtained $\measuredangle A=60^\circ$. Similarly, in the case $z^2+zx+x^2=0$ we obtain $\measuredangle B=60^\circ$, and in the case $x^2+xy+y^2=0$ we conclude that $\measuredangle C=60^\circ$. Thus, one of the three angles $\measuredangle A$, $\measuredangle B$, $\measuredangle C$ of triangle ABC must be equal to 60°. But 60° is also the average of these angles $\measuredangle A$, $\measuredangle B$, $\measuredangle C$ (since $\measuredangle A+\measuredangle B+\measuredangle C=180^\circ$ and thus $\frac{\measuredangle A+\measuredangle B+\measuredangle C}{3}=60^\circ$), and if one of the three angles $\measuredangle A$, $\measuredangle B$, $\measuredangle C$ equals to the average of these angles, then it must be the middle-sized angle. Hence, the middle-sized angle of triangle ABC equals $60°$; in other words, $\measuredangle A = 60°$ (since the problem requires that $\measuredangle A$ is the middle-sized angle of triangle ABC).

It remains to prove that the circumradius of triangle IBC is the same as that of ABC. This is easy now: By the extended law of sines, the circumradius of triangle IBC is $\dfrac{BC}{2\sin\measuredangle BIC}$, while the circumradius of triangle ABC is $\dfrac{BC}{2\sin\measuredangle A}$. Hence, it remains to show that $\sin\measuredangle BIC = \sin\measuredangle A$. But

$$\begin{aligned}\measuredangle BIC &= 180° - \measuredangle IBC - \measuredangle ICB = 180° - \frac{\measuredangle B}{2} - \frac{\measuredangle C}{2} \\ &\qquad \left(\begin{array}{c}\text{since } I \text{ is the incenter of triangle } ABC, \text{ and thus lies on the} \\ \text{angle bisectors of its angles } \measuredangle B \text{ and } \measuredangle C\end{array}\right) \\ &= 90° + \frac{180° - \measuredangle B - \measuredangle C}{2} = 90° + \frac{\measuredangle A}{2} = 90° + \frac{60°}{2} = 120°\end{aligned}$$

so that $\sin\measuredangle BIC = \sin 120° = \sin 60° = \sin\measuredangle A$, qed.

Remark. Nowhere in this solution did we need the assumption that triangle ABC be acute.