Let $P$ be a regular $n$-gon. We label the consecutive vertices of this $n$-gon $P$ by $A_0$, $A_1$, ..., $A_{n-1}$, and we let $A_n = A_0$.

Let $M$ be a point in the plane, and let $B_k$ be the orthogonal projection of this point $M$ on the line $A_kA_{k+1}$ for each $k \in \{0, 1, ..., n-1\}$. Assume that this projection $B_k$ lies inside the segment $A_kA_{k+1}$ for each $k \in \{0, 1, ..., n-1\}$. Prove that

$$\sum_{k=0}^{n-1} \text{Area}(\triangle (MA_kB_k)) = \frac{1}{2} \text{Area}(P).$$

Solution by Darij Grinberg.

We denote the area of any triangle $XYZ$ by $|XYZ|$ (instead of the lengthy notation $\text{Area}(\triangle (XYZ))$).

We set $A_{n+1} = A_1$ just as the problem author set $A_n = A_0$.

We WLOG assume that the $n$-gon $P$ is directed counter-clockwise.

For every $k \in \{0, 1, ..., n-1\}$, let $C_k$ denote the midpoint of the side $A_kA_{k+1}$ of $P$. Also, let $O$ be the center of $P$. Due to the symmetry of $P$, we have $OC_k \perp A_kA_{k+1}$ for every $k \in \{0, 1, ..., n-1\}$. Let $2a$ be the sidelength of $P$, and let $d$ be the distance from $O$ to every side of $P$.

Let $D_k$ be the foot of the perpendicular from $M$ to $OC_k$ for every $k \in \{0, 1, ..., n-1\}$. Then, $MB_kC_kD_k$ is a rectangle for every $k \in \{0, 1, ..., n-1\}$ (due to right angles at $B_k$, $C_k$ and $D_k$).

We will use directed segments, denoting the directed length of any segment $XY$ by $\overrightarrow{XY}$. Of course, this directed length is well-defined only if the points $X$ and $Y$ lie on some directed line. For every $k \in \{0, 1, ..., n-1\}$,

- we direct the line $A_kA_{k+1}$ in such a way that $\overrightarrow{A_kA_{k+1}} > 0$ (so that $\overrightarrow{A_kA_{k+1}} = 2a$),
- we direct the line $MB_k$ in such a way that $\overrightarrow{MB_k} > 0$,
- we direct the line $OC_k$ in such a way that $\overrightarrow{OC_k} > 0$ (so that $\overrightarrow{OC_k} = d$),
- we direct the line $MD_k$ in the same way as the line $A_kA_{k+1}$ (to which it is parallel, because $MB_kC_kD_k$ is a rectangle)$^1$,
- we direct the line $OM$ in such a way that $\overrightarrow{OM} > 0$.

For any two directed lines $g$ and $h$, we can not only endow segments along these lines with signs (what leads to directed segments), but also define a directed angle $\angle (g, h)$ between the directed lines $g$ and $h$; this is the angle about which $g$ must be rotated in order to end up parallel and equidirected to $h$. This angle $\angle (g, h)$ is an element of the group $\mathbb{R} \setminus \{2\pi \mathbb{Z}\}$ (in other words, it is an angle defined up to integral multiples of $2\pi$).

\footnote{If $M$ coincides with $D_k$, the line $MD_k$ has to be understood as the perpendicular from $M$ to $OC_k$ (remember the definition of $D_k$).}
Define an angle $\rho \in \mathbb{R} \setminus (2\pi \mathbb{Z})$ by $\rho = \angle (A_kA_{k+1}, A_{k+1}A_{k+2})$ for every $k \in \{0, 1, ..., n-1\}$ (this is possible since all angles $\angle (A_kA_{k+1}, A_{k+1}A_{k+2})$ are equal, because $P$ is a regular $n$-gon). Then, $n\rho = 0$ (since $n\rho = \sum_{k=0}^{n-1} \rho = \sum_{k=0}^{n-1} \angle (A_kA_{k+1}, A_{k+1}A_{k+2}) = \angle (A_0A_1, A_nA_{n+1}) = \angle (A_0A_1, A_0A_1) = 0$) and thus $n \cdot 2\rho = 0$, but $\rho \neq 0$ and $2\rho \neq 0$ (since the lines $A_kA_{k+1}$ and $A_{k+1}A_{k+2}$ are not parallel). Let $\phi = \angle (OM, A_0A_1)$. Then,

$$\angle (OM, A_kA_{k+1}) = \angle (OM, A_0A_1) + \sum_{i=0}^{k-1} \angle (A_iA_{i+1}, A_{i+1}A_{i+2}) = \phi + \sum_{i=0}^{k-1} \rho = \phi + k \rho$$

for every $k \in \{0, 1, ..., n-1\}$. Besides, $\angle (A_kA_{k+1}, OC_k) = \frac{\pi}{2}$ (in fact, $OC_k \perp A_kA_{k+1}$ yields $\angle (A_kA_{k+1}, OC_k) = \pm \frac{\pi}{2}$, and the $\pm$ becomes a $+$ since the $n$-gon $P$ is directed counter-clockwise), so that

$$\angle (OM, OC_k) = \angle (OM, A_kA_{k+1}) + \angle (A_kA_{k+1}, OC_k) = \phi + k \rho + \frac{\pi}{2}$$

for every $k \in \{0, 1, ..., n-1\}$.

Since triangle $MA_kB_k$ is right-angled at $B_k$, we have $|MA_kB_k| = \frac{1}{2} \cdot MB_k \cdot A_kB_k$.

Since triangle $OA_kC_k$ is right-angled at $C_k$, we have $|OA_kC_k| = \frac{1}{2} \cdot OC_k \cdot A_kC_k$. Notice that $A_kC_k = a$ (since $C_k$ is the midpoint of $A_kA_{k+1}$, and $A_kA_{k+1} = 2a$) and $OC_k = d$, so this becomes $|OA_kC_k| = \frac{1}{2} \cdot d \cdot a$.

The rectangle $MB_kC_kD_k$ yields $DC_kC_k = MB_k$. On the other hand, $C_kB_k$ is the orthogonal projection of the segment $OM$ onto the line $A_kA_{k+1}$, so that $\frac{C_kB_k}{OM} = \cos \angle (OM, A_kA_{k+1})$. Besides, $OD_k$ is the orthogonal projection of the segment
$OM$ onto the line $OC_k$, so that $\overrightarrow{OD_k} = \overrightarrow{OM} \cdot \cos \angle (OM, OC_k)$. Thus,

$$\sum_{k=0}^{n-1} |MA_k B_k| = \sum_{k=0}^{n-1} \frac{1}{2} \cdot MB_k \cdot \overrightarrow{A_k B_k} = \frac{1}{2} \sum_{k=0}^{n-1} \frac{MB_k}{\overrightarrow{OC_k - OD_k}} - \frac{A_k B_k}{\overrightarrow{a + C_k B_k}} = \frac{1}{2} \sum_{k=0}^{n-1} (d - \overrightarrow{OM} \cdot \cos \angle (OM, OC_k)) \cdot (a + \overrightarrow{OM} \cdot \cos \angle (OM, A_k A_{k+1}))$$

$$= \frac{1}{2} \sum_{k=0}^{n-1} \left( d - \overrightarrow{OM} \cdot \cos \left( \phi + k\rho + \frac{\pi}{2} \right) \right) \cdot \left( a + \overrightarrow{OM} \cdot \cos (\phi + k\rho) \right)$$

$$= \frac{1}{2} \sum_{k=0}^{n-1} \left( da + d \cdot \overrightarrow{OM} \cdot \cos (\phi + k\rho) + a \cdot \overrightarrow{OM} \cdot \sin (\phi + k\rho) + \overrightarrow{OM}^2 \cdot \sin (\phi + k\rho) \cos (\phi + k\rho) \right)$$

$$= \frac{1}{2} \sum_{k=0}^{n-1} da + \frac{1}{2} d \cdot \overrightarrow{OM} \cdot \sum_{k=0}^{n-1} \cos (\phi + k\rho) + \frac{1}{2} a \cdot \overrightarrow{OM} \cdot \sum_{k=0}^{n-1} \sin (\phi + k\rho) + \frac{1}{4} \overrightarrow{OM}^2 \cdot \sum_{k=0}^{n-1} \sin (2\phi + k \cdot 2\rho).$$

(11392.1)

Now, we will show that any two angles $\phi$ and $\rho$ such that $n\rho = 0$ and $\rho \neq 0$ satisfy

$$\sum_{k=0}^{n-1} \cos (\phi + k\rho) = 0;$$

(11392.2)

$$\sum_{k=0}^{n-1} \sin (\phi + k\rho) = 0,$$

(11392.3)

and that any two angles $\phi$ and $\rho$ such that $n \cdot 2\rho = 0$ and $2\rho \neq 0$ satisfy

$$\sum_{k=0}^{n-1} \sin (2\phi + k \cdot 2\rho) = 0.$$

(11392.4)

In fact, $n\rho = 0$ yields $e^{i\cdot n\rho} = 1$, but $\rho \neq 0$ yields $e^{i\rho} \neq 1$. Thus,

$$0 = e^{i\phi} \frac{1 - 1}{e^{i\rho} - 1} = e^{i\phi} \frac{e^{i\cdot n\rho} - 1}{e^{i\rho} - 1} = e^{i\phi} \sum_{k=0}^{n-1} e^{i\cdot k\rho} = \sum_{k=0}^{n-1} e^{i\cdot (\phi + k\rho)} = \sum_{k=0}^{n-1} (\cos (\phi + k\rho) + i \sin (\phi + k\rho)).$$

Taking the real part of this equation, we obtain (11392.2); the imaginary part yields (11392.3). The identity (11392.4) is nothing but (11392.3) applied to the angles $2\phi$ and $2\rho$ instead of $\phi$ and $\rho$.

Using (11392.2)-(11392.4), our equation (11392.1) simplifies to

$$\sum_{k=0}^{n-1} |MA_k B_k| = \frac{1}{2} \sum_{k=0}^{n-1} da = \frac{1}{2} \sum_{k=0}^{n-1} d \cdot a = \sum_{k=0}^{n-1} |O A_k C_k|.$$

(11392.5)
By the symmetry of the regular $n$-gon $P$, we have $\sum_{k=0}^{n-1} |OA_kC_k| = \sum_{k=0}^{n-1} |OC_kB_k|$, while obviously $\sum_{k=0}^{n-1} |OA_kC_k| + \sum_{k=0}^{n-1} |OC_kB_k| = \text{Area } P$. Thus, $\sum_{k=0}^{n-1} |OA_kC_k| = \frac{1}{2} \text{Area } P$, so that (11392.5) becomes $\sum_{k=0}^{n-1} |MA_kB_k| = \frac{1}{2} \text{Area } P$, qed.

**Remark.** A user of the MathLinks webforum called Myth (Mikhail Leptchinski in real life) found this problem in 2005: