

# Math 701 Fall 2026, Lecture 7: Skew Schur polynomials continued

**website:** <https://www.cip.ifi.lmu.de/~grinberg/t/26fs/>  
(GPT-6 was used to create some of the examples below.)

## 1.6.1. The Bender–Knuth involutions (continued)

We continue where we left off in Lecture 6:

*Finishing the proof of Theorem 1.6.14.* We continue with the notation from Lecture 6. In particular, fix  $k \in [N - 1]$ . We now define a map

$$\beta_k : \text{SSYT}(\lambda/\mu, N) \rightarrow \text{SSYT}(\lambda/\mu, N)$$

as follows: Given  $T \in \text{SSYT}(\lambda/\mu, N)$ , we perform the following procedure:

- For each row of  $T$ , if there are  $a$  free  $k$ 's and  $b$  free  $(k + 1)$ 's in this row, then we replace them by  $b$  entries  $k$  followed by  $a$  entries  $k + 1$ , from left to right in the same cells. All other entries remain unchanged.

We define  $\beta_k(T)$  to be the result of this procedure. We still need to prove the following:

*Observation 2:* This map  $\beta_k$  is well-defined, i.e., the tableau  $\beta_k(T)$  belongs to  $\text{SSYT}(\lambda/\mu, N)$ .

Before we prove this, let us give an example of  $\beta_k$ .

**Example 1.6.17.** Let  $k = 2$  and  $N = 4$ . We use the tableau  $T$  from Example 1.6.16 in Lecture 6, of shape  $(9, 8, 6) / (3, 1)$ . The map  $\beta_2$  acts as follows:

$$T = \begin{array}{cccccc} & & 1 & 1 & 1 & \mathbf{2} & \mathbf{3} & 4 \\ & 1 & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{3} & \mathbf{3} & 4 \\ 1 & \mathbf{2} & \mathbf{3} & 4 & 4 & 4 & & \end{array} \mapsto \beta_2(T) = \begin{array}{cccccc} & & 1 & 1 & 1 & \mathbf{2} & \mathbf{2} & 4 \\ & 1 & \mathbf{2} & \mathbf{2} & \mathbf{3} & \mathbf{3} & \mathbf{3} & 4 \\ 1 & \mathbf{3} & \mathbf{3} & 4 & 4 & 4 & & \end{array}.$$

As before, the 2's and 3's are boldfaced, and the matched entries are shaded. The shaded entries stay unchanged. The free blocks change as follows:

$$\begin{array}{ccc} \text{row 1} & \text{row 2} & \text{row 3} \\ (3) \mapsto (2) & (2, 2, 3) \mapsto (2, 3, 3) & (2) \mapsto (3) \end{array}$$

All entries other than 2 and 3 stay unchanged as well. Thus, the five 2's and four 3's in  $T$  become four 2's and five 3's in  $\beta_2(T)$ . Notice that in the second row, we replace 223 by 233, not by 332: the new free block must still be weakly increasing.

*Proof of Observation 2.* The entries of  $\beta_k(T)$  are clearly in  $[N]$ , since  $k, k+1 \in [N]$ . It remains to show that it is semistandard.

- Why are its rows weakly increasing? Consider a row of  $T$ . In  $\beta_k(T)$ , some of its free entries may have changed from  $k$  to  $k+1$  or vice versa; no other entries changed. But Observation 1 from Lecture 6 shows that the matched  $k$ 's are to the left of the free  $k$ 's and the matched  $(k+1)$ 's are to the right of the free  $(k+1)$ 's; thus, all the free entries form a single contiguous block in our row. By definition, this block stays weakly increasing in  $\beta_k(T)$ . All the entries in this block are either  $k$ 's or  $(k+1)$ 's in both  $T$  and  $\beta_k(T)$ ; all the entries to the left of this block are  $\leq k$ ; all the entries to the right of this block are  $\geq k+1$ . Thus, the row is still weakly increasing in  $\beta_k(T)$ .
- Why are its columns strictly increasing? Because we only changed free  $k$ 's and free  $(k+1)$ 's, and there are no other  $k$ 's or  $(k+1)$ 's in their columns to mess up the inequalities. (For instance, changing a free  $k$  to a  $k+1$  still keeps it smaller than the neighboring entry to its south (if there is one), since that entry cannot be a  $k+1$  because the  $k$  was free.)

So  $\beta_k(T)$  is semistandard, and Observation 2 is proved.  $\square$

*Observation 3:* This map  $\beta_k$  is an involution, i.e., we have  $\beta_k \circ \beta_k = \text{id}$ .

*Proof of Observation 3.* We must show that

$$\beta_k(\beta_k(T)) = T \quad \text{for each } T \in \text{SSYT}(\lambda/\mu, N).$$

Fix  $T \in \text{SSYT}(\lambda/\mu, N)$ . To compute  $\beta_k(T)$  from  $T$ , we identify the free and matched  $k$ 's and  $(k+1)$ 's, and flip the numbers of free  $k$ 's and free  $(k+1)$ 's in each row. Note that the matched entries of  $T$  are unchanged in  $\beta_k(T)$  and thus, in particular, remain matched in  $\beta_k(T)$  (since their "partners" also don't change). On the other hand, the free entries of  $T$  are replaced by new entries, which are again free (since all the other entries in their respective columns are  $< k$  or  $> k+1$ , and thus don't change). Hence, if a row of  $T$  had  $a$  free  $k$ 's and  $b$  free  $(k+1)$ 's, then the same row of  $\beta_k(T)$  has  $b$  free  $k$ 's and  $a$  free  $(k+1)$ 's.

Now, when we compute  $\beta_k(\beta_k(T))$  from  $\beta_k(T)$  (by the same procedure that gave us  $\beta_k(T)$  from  $T$ ), we flip the numbers of free  $k$ 's and free  $(k+1)$ 's in each row again. Hence, if a row of  $T$  had  $a$  free  $k$ 's and  $b$  free  $(k+1)$ 's, then the same row of  $\beta_k(T)$  has  $b$  free  $k$ 's and  $a$  free  $(k+1)$ 's (as we saw above), and therefore the same row of  $\beta_k(\beta_k(T))$  will have  $a$  free  $k$ 's and  $b$  free  $(k+1)$ 's again, which is the same distribution that  $T$  used to have. Since the other entries in that row don't change at all, this shows that the whole row is the same in  $\beta_k(\beta_k(T))$  as it was in  $T$ . We have proved this for every row; thus,  $\beta_k(\beta_k(T)) = T$ .  $\square$

*Observation 4:* For each  $T \in \text{SSYT}(\lambda/\mu, N)$ , we have

$$\begin{aligned} (\# \text{ of } k\text{'s in } \beta_k(T)) &= (\# \text{ of } (k+1)\text{'s in } T) & \text{and} \\ (\# \text{ of } (k+1)\text{'s in } \beta_k(T)) &= (\# \text{ of } k\text{'s in } T) \end{aligned}$$

and furthermore

$$(\# \text{ of } i\text{'s in } \beta_k(T)) = (\# \text{ of } i\text{'s in } T) \quad \text{for each } i \notin \{k, k+1\}.$$

*Proof of Observation 4.* In the proof of Observation 3, we observed that, as we transform  $T$  into  $\beta_k(T)$ , all matched entries stay matched, while all free entries remain free (though they can change their values). The former fact yields

$$\begin{aligned} (\# \text{ of matched } k\text{'s in } \beta_k(T)) &= (\# \text{ of matched } k\text{'s in } T) \\ &= (\# \text{ of matched } (k+1)\text{'s in } T) \end{aligned}$$

(as we saw above). The latter fact shows that for each  $i \geq 1$ , we have

$$\begin{aligned} (\# \text{ of free } k\text{'s in the } i\text{-th row of } \beta_k(T)) \\ = (\# \text{ of free } (k+1)\text{'s in the } i\text{-th row of } T) \end{aligned}$$

(because if the  $i$ -th row of  $T$  has  $a$  free  $k$ 's and  $b$  free  $(k+1)$ 's, then, in  $\beta_k(T)$ , these entries become  $b$  free  $k$ 's and  $a$  free  $(k+1)$ 's and remain free; therefore the  $i$ -th row of  $\beta_k(T)$  has  $b$  free  $k$ 's). Summing this equality over all  $i \geq 1$ , we obtain

$$(\# \text{ of free } k\text{'s in } \beta_k(T)) = (\# \text{ of free } (k+1)\text{'s in } T).$$

However, each  $k$  in  $\beta_k(T)$  is either free or matched. Thus,

$$\begin{aligned} (\# \text{ of } k\text{'s in } \beta_k(T)) &= \underbrace{(\# \text{ of matched } k\text{'s in } \beta_k(T))}_{=(\# \text{ of matched } (k+1)\text{'s in } T)} + \underbrace{(\# \text{ of free } k\text{'s in } \beta_k(T))}_{=(\# \text{ of free } (k+1)\text{'s in } T)} \\ &= (\# \text{ of matched } (k+1)\text{'s in } T) + (\# \text{ of free } (k+1)\text{'s in } T) \\ &= (\# \text{ of } (k+1)\text{'s in } T). \end{aligned}$$

This proves the first equality of Observation 4. The second equality is similar; the third is trivial.  $\square$

*Observation 5:* For each  $T \in \text{SSYT}(\lambda/\mu, N)$ , we have

$$x_{\beta_k(T)} = s_k \cdot x_T.$$

*Proof of Observation 5.* We have

$$x_T = \prod_{i=1}^N x_i^{(\# \text{ of } i\text{'s in } T)} \quad \text{and} \quad x_{\beta_k(T)} = \prod_{i=1}^N x_i^{(\# \text{ of } i\text{'s in } \beta_k(T))}.$$

These two monomials have largely the same exponents, by Observation 4; only the exponents on  $x_k$  and on  $x_{k+1}$  get swapped. This is precisely saying that  $s_k$  takes one monomial into the other.  $\square$

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Now,

$$\begin{aligned}
 s_{\lambda/\mu} &= \sum_{T \in \text{SSYT}(\lambda/\mu, N)} x_T = \sum_{T \in \text{SSYT}(\lambda/\mu, N)} x_{\beta_k(T)} \\
 &\quad \left( \begin{array}{c} \text{here, we substituted } \beta_k(T) \text{ for } T, \text{ since } \beta_k \text{ is a bijection} \\ \text{(by Observation 3)} \end{array} \right) \\
 &= \sum_{T \in \text{SSYT}(\lambda/\mu, N)} s_k \cdot x_T \quad (\text{by Observation 5}) \\
 &= s_k \cdot \underbrace{\sum_{T \in \text{SSYT}(\lambda/\mu, N)} x_T}_{=s_{\lambda/\mu}} = s_k \cdot s_{\lambda/\mu}.
 \end{aligned}$$

That is,  $s_k \cdot s_{\lambda/\mu} = s_{\lambda/\mu}$ . Since we have proved this for each  $k \in [N-1]$ , we conclude (by Lemma 1.2.9 from Lecture 3) that  $s_{\lambda/\mu}$  is symmetric.  $\square$

As we mentioned, for  $\mu = \emptyset = ()$ , we have  $s_{\lambda/\mu} = s_\lambda$ , so we conclude that  $s_\lambda$  is symmetric.

## 1.7. The Littlewood–Richardson rule

What happens when we multiply two Schur polynomials? For instance,

$$\begin{aligned}
 s_{(2,1)}s_{(1,1)} &= s_{(2,1,1,1)} + s_{(2,2,1)} + s_{(3,1,1)} + s_{(3,2)}; \\
 s_{(2,1)}s_{(2,1)} &= s_{(2,2,2)} + 2s_{(3,2,1)} + s_{(3,3)} + s_{(4,1,1)} \\
 &\quad + s_{(4,2)} + s_{(2,2,1,1)} + s_{(3,1,1,1)}.
 \end{aligned}$$

The fact that such an expression always exists is not too surprising; indeed, the  $s_\lambda$  for  $\lambda$  being an  $N$ -partition form a basis of  $\mathcal{S}$ , as we will later show. That the same expansion works for every  $N$  is also unsurprising once we get to the  $N \rightarrow \infty$  case (next chapter). But the fact that there are no minus signs in this expansion requires explanation! Originally the explanation came from Schur's and Weyl's work on representations of  $\text{GL}(N)$ . But the Littlewood–Richardson rule claims a combinatorial formula for the expansion. Since Stembridge's proof in 2002, this combinatorial formula has become very accessible.

To state it, we will need a bit more notation.

**Definition 1.7.1.** Let  $\lambda$  and  $\mu$  be two partitions. Let  $T \in \text{SSYT}(\lambda/\mu, N)$ . We define the **content** of  $T$  to be the  $N$ -tuple  $\text{cont } T = (a_1, a_2, \dots, a_N)$ , where

$$a_i = (\# \text{ of } i\text{'s in } T) = |\{c \in Y(\lambda/\mu) \mid T(c) = i\}|.$$

For example, if  $N = 5$ , then

$$\text{cont} \left( \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 3 & 3 & & \\ \hline \end{array} \right) = (2, 1, 2, 1, 0).$$

Note that

$$x_T = x^{\text{cont } T} \quad \text{for each } T \in \text{SSYT}(\lambda/\mu, N),$$

where, for any  $N$ -tuple  $(a_1, a_2, \dots, a_N) \in \mathbb{N}^N$ , we set  $x^{(a_1, a_2, \dots, a_N)} = \prod_{i=1}^N x_i^{a_i}$ .

**Definition 1.7.2.** Let  $\lambda/\mu$  be a skew partition. Let  $T$  be a tableau of shape  $\lambda/\mu$ . Let  $j$  be a positive integer. Then,  $\text{col}_{\geq j} T$  means the restriction of  $T$  to columns  $j, j+1, j+2, \dots$  (that is, the result of removing the first  $j-1$  columns from  $T$ ). The remaining cells retain their original coordinates.

For example,  $\text{col}_{\geq 1} T = T$ , whereas  $\text{col}_{\geq j} T$  is empty for any sufficiently large  $j$ .

For example,

$$\text{col}_{\geq 3} \left( \begin{array}{|c|c|c|c|} \hline & & 1 & 1 & 2 \\ \hline & 1 & 2 & 3 & \\ \hline 3 & 4 & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}.$$

Note that the shape of  $\text{col}_{\geq j} T$  is still a skew Young diagram.

**Definition 1.7.3.** Let  $\lambda/\mu$  be a skew partition, and let  $\nu$  be an  $N$ -partition. A tableau  $T \in \text{SSYT}(\lambda/\mu, N)$  is said to be  $\nu$ -**Yamanouchi** if it has the property that for each positive integer  $j$ , the  $N$ -tuple  $\nu + \text{cont}(\text{col}_{\geq j} T)$  is an  $N$ -partition (i.e., weakly decreasing).

**Remark 1.7.4.** To conceptualize this definition, think of  $T$  as a voting process. You have  $N$  candidates labelled  $1, 2, \dots, N$ ; each entry  $k$  of  $T$  is a vote for candidate  $k$ . You start counting votes, where  $\nu$  is a tally of votes already counted (so you start with  $\nu_1$  votes for 1, with  $\nu_2$  votes for 2, and so on). Then you record the votes in the last column of  $T$  (note that each candidate gets at most one vote from the given column). Then you move on to the next column to its left and record its votes. Then you continue, moving left, until all  $T$ -votes are counted. The tableau  $T$  is  $\nu$ -Yamanouchi if and only if the candidates retain their relative order (i.e., candidate 1 always has at least as many votes as 2, who in turn has at least as many votes as 3, and so on) throughout the process (meaning, after each single column).

**Example 1.7.5.**

- (a) Let  $N = 3$  and  $\nu = \emptyset = (0,0,0)$ . Which of the following seven tableaux are  $\nu$ -Yamanouchi?

$$\begin{array}{ccc}
 T_1 = \begin{array}{|c|c|c|} \hline & 1 & 1 \\ \hline 2 & 2 & \\ \hline \end{array} & T_2 = \begin{array}{|c|c|c|} \hline & 1 & 1 \\ \hline 2 & 3 & \\ \hline \end{array} & T_3 = \begin{array}{|c|c|c|} \hline & 1 & 1 \\ \hline 1 & 2 & \\ \hline \end{array} \\
 \\
 T_4 = \begin{array}{|c|c|c|} \hline & & 1 \\ \hline 1 & & \\ \hline 2 & & \\ \hline \end{array} & T_5 = \begin{array}{|c|c|c|} \hline & & 1 \\ \hline 1 & & \\ \hline 3 & & \\ \hline \end{array} & T_6 = \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline & 1 & 2 & 2 \\ \hline 2 & 3 & & \\ \hline \end{array} \\
 \\
 T_7 = \begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline 2 & 2 & \\ \hline \end{array}
 \end{array}$$

Answer:  $T_1, T_3, T_4$  and  $T_6$  are;  $T_2, T_5$  and  $T_7$  are not. Here is a column-by-column check. Write

$$C_j(T) := \text{cont}(\text{col}_{\geq j} T).$$

Starting on the right and adding one column at a time, we obtain the following contents. The boldfaced tuples are the ones that are not weakly decreasing:

|       | $C_4(T)$ | $C_3(T)$       | $C_2(T)$       | $C_1(T)$       |
|-------|----------|----------------|----------------|----------------|
| $T_1$ | (0,0,0)  | (1,0,0)        | (2,1,0)        | (2,2,0)        |
| $T_2$ | (0,0,0)  | (1,0,0)        | <b>(2,0,1)</b> | (2,1,1)        |
| $T_3$ | (0,0,0)  | (1,0,0)        | (2,1,0)        | (3,1,0)        |
| $T_4$ | (0,0,0)  | (0,0,0)        | (1,0,0)        | (2,1,0)        |
| $T_5$ | (0,0,0)  | (0,0,0)        | (1,0,0)        | <b>(2,0,1)</b> |
| $T_6$ | (1,1,0)  | (2,2,0)        | (3,2,1)        | (3,3,1)        |
| $T_7$ | (0,0,0)  | <b>(0,1,0)</b> | <b>(1,2,0)</b> | <b>(1,3,0)</b> |

For  $j \geq 5$ , all these contents are (0,0,0). Thus, the table checks every condition in the definition. In particular,  $T_2$  fails at  $j = 2$  because there is a 3 but no 2 in its last two columns;  $T_5$  fails at  $j = 1$  for the same reason. The tableau  $T_7$  already fails at  $j = 3$ , since its last column contains a 2 but no 1. Notice that the full content of  $T_2$  is weakly decreasing; checking just the full content is therefore not enough.

- (b) What  $\nu$  should you take to make  $T_2$  and  $T_5$  become  $\nu$ -Yamanouchi? Try to find a  $\nu$  that is as small in size as possible.

We can take  $\nu = (1, 1, 0)$ . After adding each nonempty column, from right to left, the tuples  $\nu + C_j(T)$  are

$$\begin{array}{l|l} T_2 & (2, 1, 0) \quad (3, 1, 1) \quad (3, 2, 1) \\ T_5 & (2, 1, 0) \quad (3, 1, 1) \end{array}$$

All these tuples are weakly decreasing, as is the initial tuple  $\nu$ . This proves that both tableaux are  $\nu$ -Yamanouchi.

Why is size 2 minimal? For  $T_2$  at  $j = 2$ , and for  $T_5$  at  $j = 1$ , the condition requires  $\nu + (2, 0, 1)$  to be weakly decreasing. Thus,  $\nu_2 \geq \nu_3 + 1 \geq 1$ . Since  $\nu$  itself is weakly decreasing, we also have  $\nu_1 \geq \nu_2 \geq 1$ . Therefore,  $|\nu| \geq 2$ , and our choice attains this bound for each of the two tableaux.

To make  $T_7$   $\nu$ -Yamanouchi, we can take  $\nu = (2, 0, 0)$ . The successive tuples, again after adding each column from right to left, are

$$(2, 1, 0), \quad (3, 2, 0), \quad (3, 3, 0),$$

so this choice works. It also has the smallest possible size: the full content of  $T_7$  is  $(1, 3, 0)$ , so any suitable  $\nu$  must satisfy  $\nu_1 + 1 \geq \nu_2 + 3$ , or equivalently  $\nu_1 \geq \nu_2 + 2$ . Hence  $|\nu| \geq 2$ .

We can finally state the Littlewood–Richardson rule:

**Theorem 1.7.6** (Zelevinsky’s generalized Littlewood–Richardson rule, in Yamanouchi form for finite  $N$ ).

Let  $\lambda/\mu$  be a skew partition and let  $\nu$  be an  $N$ -partition. Then,

$$s_\nu \cdot s_{\lambda/\mu} = \sum_{\substack{T \in \text{SSYT}(\lambda/\mu, N); \\ T \text{ is } \nu\text{-Yamanouchi}}} s_{\nu + \text{cont } T}.$$

Some comments:

- The term  $s_{\nu + \text{cont } T}$  in the above sum is always well-defined: if  $T$  is  $\nu$ -Yamanouchi, then

$$\nu + \text{cont } T = \nu + \text{cont } (\text{col}_{\geq 1} T)$$

is an  $N$ -partition.

- By picking  $\mu = \emptyset$ , we get a formula for  $s_\nu \cdot s_\lambda$ , that is, for any product of two Schur polynomials as a sum of Schur polynomials.

We will actually prove something slightly different:

$$a_{\nu+\rho} \cdot s_{\lambda/\mu} \stackrel{?}{=} \sum_{\substack{T \in \text{SSYT}(\lambda/\mu, N); \\ T \text{ is } \nu\text{-Yamanouchi}}} a_{\nu + \text{cont } T + \rho}.$$

Once this formula is proved, we will apply it twice – once to get  $a_{\rho}s_{\lambda} = a_{\lambda+\rho}$  for  $\ell(\lambda) \leq N$ , and again to get the Littlewood–Richardson rule.

Before we get to the proof, let us get a basic fact out of the way:

**Proposition 1.7.7.** If  $\lambda$  is a partition of length  $> N$ , then  $s_{\lambda} = 0$ .

*Proof.* Let  $\lambda$  be a partition of length  $> N$ . Then,  $\text{SSYT}(\lambda, N)$  is empty, since the first column of an SSYT of shape  $\lambda$  must contain at least  $\ell(\lambda) > N$  distinct entries (strictly increasing!), but we have only  $N$  numbers to use. Thus, the sum defining  $s_{\lambda}$  is empty and equals 0.  $\square$

In contrast, a skew Schur polynomial  $s_{\lambda/\mu}$  can be nonzero even if  $\lambda$  and  $\mu$  are arbitrarily long. For instance, for any positive integer  $k$ , setting  $\lambda = (k, k-1, \dots, 2, 1)$  and  $\mu = (k-1, k-2, \dots, 1)$ , we can fill the  $k$  cells of  $\lambda/\mu$  with 1's, and this always produces a tableau  $T \in \text{SSYT}(\lambda/\mu, N)$  for any  $N \geq 1$ , so that  $s_{\lambda/\mu}$  contains at least an  $x_1^k$  term.