

Math 701 Fall 2026, Lecture 6: Skew Schur polynomials

website: <https://www.cip.ifi.lmu.de/~grinberg/t/26fs/>
(GPT-6 was used to create some of the examples below.)

1.6. Skew Schur polynomials

1.6.1. Skew partitions, tableaux and Schur functions

We recall that the length of a partition λ (that is, the # of its nonzero entries¹) is denoted by $\ell(\lambda)$.

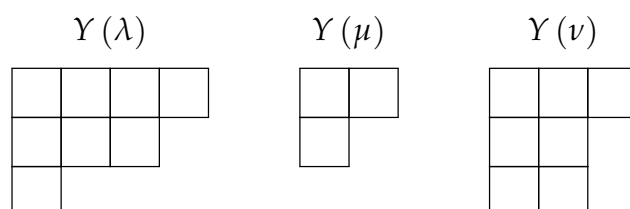
Definition 1.6.1. Let λ and μ be two partitions.

We say that μ is **contained** in λ if $Y(\mu) \subseteq Y(\lambda)$. We write “ $\mu \subseteq \lambda$ ” for this statement. Equivalently, $\mu \subseteq \lambda$ holds if and only if

$$\mu_i \leq \lambda_i \quad \text{for each } i \geq 1$$

(where μ_i means the i -th entry of μ , and similarly λ_i is understood). Here we are again identifying partitions with infinite sequences (by padding them with infinitely many zeroes), so we have $\lambda_i = 0$ for all $i > \ell(\lambda)$.

Example 1.6.2. Let $\lambda = (4, 3, 1)$, $\mu = (2, 1)$ and $\nu = (3, 2, 2)$. Their Young diagrams are



We have $\mu \subseteq \lambda$, since $2 \leq 4$ and $1 \leq 3$ and $0 \leq 1$ (and $0 \leq 0$ for all the remaining entries). Likewise, we have $\mu \subseteq \nu$. However, we don't have $\nu \subseteq \lambda$, since $\nu_3 = 2 > 1 = \lambda_3$. Thus, the cell $(3, 2)$ belongs to $Y(\nu)$ but not to $Y(\lambda)$. Notice that $|\nu| = 7 < 8 = |\lambda|$; having fewer boxes does not guarantee containment.

Definition 1.6.3. Let λ and μ be two partitions such that $\mu \subseteq \lambda$. Then, the pair (λ, μ) is called a **skew partition**, and is denoted by λ/μ .

¹The symbol “#” means “number”.

The **skew Young diagram** $Y(\lambda/\mu)$ of λ/μ is defined to be the set

$$\begin{aligned} Y(\lambda) \setminus Y(\mu) &= \{(i, j) \in \{1, 2, 3, \dots\}^2 \mid j \in [\lambda_i] \setminus [\mu_i]\} \\ &= \{(i, j) \in \{1, 2, 3, \dots\}^2 \mid \mu_i < j \leq \lambda_i\}. \end{aligned}$$

The original Young diagrams $Y(\lambda)$ will occasionally be called **straight Young diagrams** in order to distinguish them from skew ones.

As usual, we will often omit the word “Young”.

Example 1.6.4. For $\lambda = (4, 3, 1)$ and $\mu = (2, 1)$ as above, we have

$$Y(\lambda/\mu) = \begin{array}{cccc} & & \square & \square \\ & \square & \square & \\ \square & & & \end{array}.$$

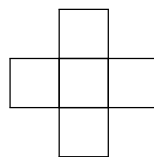
The missing boxes at the start of the first two rows are the boxes of $Y(\mu)$; the remaining boxes of $Y(\lambda)$ stay in their original positions. More explicitly,

$$Y(\lambda/\mu) = \{(1, 3), (1, 4), (2, 2), (2, 3), (3, 1)\}.$$

This diagram has $5 = |\lambda| - |\mu|$ boxes.

Of course, each straight Young diagram $Y(\lambda)$ is a skew Young diagram, namely $Y(\lambda/())$. We often denote the empty partition $()$ by $\emptyset$, so we have $Y(\lambda) = Y(\lambda/\emptyset)$.

Skew Young diagrams (unlike straight ones, i.e., ones of the form $Y(\lambda)$) need not be left-aligned. Nevertheless, their rows and their columns still don't have “gaps” (i.e., any space between two cells in the same row or column is also occupied by cells). This is a necessary, but not a sufficient condition for a set of cells to be a skew Young diagram; for example, the set



has this property as well but is not a skew Young diagram. However, we can say more about skew Young diagrams. The most comfortable way to do so is by introducing a partial order on $\mathbb{Z}^2$:

Definition 1.6.5. We define a partial order $\leq$ on the set of all cells $(i, j) \in \mathbb{Z}^2$ by setting

$$(a, b) \leq (c, d) \quad \text{if and only if} \quad (a \leq c \text{ and } b \leq d).$$

This is the **entrywise partial order** on $\mathbb{Z}^2$, and is also called the **northwest order** (since cells become smaller as you move northwest).

For example, $(1,1) \leq (2,4) \leq (3,4) \leq (3,5)$, but the two cells $(1,2)$ and $(2,1)$ are incomparable.

We can now characterize straight and skew Young diagrams in terms of this order:

Lemma 1.6.6 (northwest-closure of straight Young diagrams). Let λ be a partition. Let c and d be two cells in $\{1,2,3,\dots\}^2$ such that $c \leq d$ (in the northwest order) and $d \in Y(\lambda)$. Then, $c \in Y(\lambda)$.

Proof. Write the cells c and d as $c = (i', j')$ and $d = (i, j)$. Then, $(i, j) = d \in Y(\lambda)$, so that $j \in [\lambda_i]$. That is, $j \leq \lambda_i$. But $c \leq d$ means that $i' \leq i$ and $j' \leq j$ (since $c = (i', j')$ and $d = (i, j)$). Since λ is a partition, we have $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$ and thus $\lambda_{i'} \geq \lambda_i$ (because $i' \leq i$). Now, $j' \leq j \leq \lambda_i \leq \lambda_{i'}$ (since $\lambda_{i'} \geq \lambda_i$). In other words, $j' \in [\lambda_{i'}]$, so that $(i', j') \in Y(\lambda)$. That is, $c \in Y(\lambda)$, and Lemma 1.6.6 is proved. $\square$

Lemma 1.6.7 (convexity of skew Young diagrams). Let λ/μ be a skew partition. Let c, d, e be three cells in $\mathbb{Z}^2$ such that $c \leq d \leq e$ (in the northwest order) and $c \in Y(\lambda/\mu)$ and $e \in Y(\lambda/\mu)$. Then, $d \in Y(\lambda/\mu)$.

Proof. First of all, $c \in Y(\lambda/\mu) \subseteq \{1,2,3,\dots\}^2$. That is, both coordinates of the cell c are ≥ 1 (where the “coordinates” of a cell (i, j) are i and j , as you would expect from the picture). But $c \leq d$ shows that the coordinates of the cell c are $\leq$ to the respective coordinates of the cell d . Since the former coordinates are ≥ 1 , this entails that the latter are ≥ 1 as well. In other words, $d \in \{1,2,3,\dots\}^2$.

From $d \in \{1,2,3,\dots\}^2$ and $d \leq e$ and $e \in Y(\lambda/\mu) = Y(\lambda) \setminus Y(\mu) \subseteq Y(\lambda)$, we obtain $d \in Y(\lambda)$ (by Lemma 1.6.6, applied to d and e instead of c and d).

Now, we shall prove that $d \notin Y(\mu)$. Indeed, assume the contrary. Then, $d \in Y(\mu)$. From $c \in \{1,2,3,\dots\}^2$ and $c \leq d$ and $d \in Y(\mu)$, we then obtain $c \in Y(\mu)$ (by Lemma 1.6.6, applied to μ instead of λ). But this contradicts $c \in Y(\lambda/\mu) = Y(\lambda) \setminus Y(\mu)$. This contradiction shows that our assumption was false. Hence, $d \notin Y(\mu)$ is proved.

Combining $d \in Y(\lambda)$ with $d \notin Y(\mu)$, we obtain $d \in Y(\lambda) \setminus Y(\mu) = Y(\lambda/\mu)$, qed. $\square$

It turns out that, among finite subsets of $\{1,2,3,\dots\}^2$, Lemma 1.6.6 and Lemma 1.6.7 actually characterize straight Young diagrams and skew Young diagrams, respectively. This is easy to prove for straight Young diagrams (see [24s, Proposition 5.1.7 (a)] for a proof); the version for skew ones will not be needed.

Note that a Young diagram $Y(\lambda)$ uniquely determines the partition λ (see [24s, Proposition 5.1.7 (c)] for a proof), but a skew Young diagram $Y(\lambda/\mu)$

does not uniquely determine λ and μ . For example, $(3,2) / (3) \neq (2,2) / (2)$, but

$$Y((3,2) / (3)) = Y((2,2) / (2)) = \{(2,1), (2,2)\}.$$

Let us now define Young tableaux of skew shapes. This is done in the exact same way as for straight shapes:

Definition 1.6.8. Let λ/μ be a skew partition.

A **Young tableau** of shape $Y(\lambda/\mu)$ (or just of shape λ/μ) means a map $T : Y(\lambda/\mu) \rightarrow \{1, 2, 3, \dots\}$, that is, a way to fill the boxes of $Y(\lambda/\mu)$ with positive integers (one per box). Such Young tableaux are often called **skew Young tableaux**. Again, we omit “Young”.

If we don't have $\mu \subseteq \lambda$, then, by convention, there are no Young tableaux of shape λ/μ .

The notions of “**semistandard**” and “**standard**” are defined for skew Young tableaux in the same way as for straight ones: “semistandard” means “weakly increasing along rows, strictly increasing down columns”, whereas “standard” additionally requires that the entries are $1, 2, \dots, |Y(\lambda/\mu)|$ with no repetition. Again, we abbreviate “semistandard Young tableau” as “SSYT”.

We again use the notation $\text{SSYT}(\lambda/\mu)$ for the set of all SSYTs of shape $Y(\lambda/\mu)$, and the notation $\text{SSYT}(\lambda/\mu, k)$ for the set of those SSYTs of shape $Y(\lambda/\mu)$ whose entries belong to $[k]$.

Lemma 1.6.9. Let λ and μ be two partitions. Let T be an SSYT of shape λ/μ . Then:

- (a) If two cells $c, d \in Y(\lambda/\mu)$ satisfy $c \leq d$ (in the northwest order), then $T(c) \leq T(d)$.
- (b) If two cells $c, d \in Y(\lambda/\mu)$ satisfy $c \leq d$ but do not lie in the same row, then $T(c) < T(d)$.

Proof sketch. We outline the proofs; details can be found in [21s, §B.10, detailed proof of Lemma 7.3.17] (for N -partitions instead of partitions, but this does not affect the proof).

(a) Let $c, d \in Y(\lambda/\mu)$ be two cells that satisfy $c \leq d$. Thus, we can walk from c to d by a sequence of steps, each of which is either an east-step (going from a cell (i, j) to the cell $(i, j+1)$) or a south-step (going from a cell (i, j) to the cell $(i+1, j)$). Each of the intermediate cells p we traverse during this walk satisfies $c \leq p \leq d$ and thus belongs to $Y(\lambda/\mu)$ (by Lemma 1.6.7, applied to p and d instead of d and e); thus, it has an entry in T . The definition of an SSYT ensures that the entries in these intermediate cells on our path increase weakly as we traverse the path (since the entries of SSYT weakly increase along rows

and strictly increase down columns); thus, in particular, the entry $T(c)$ (in our starting cell) must be $\leq$ to the entry $T(d)$ (in our ending cell). This proves part (a).

(b) Proceed as in the proof of part (a) above, but now additionally observe that our walk must contain at least one south-step (since c and d do not lie in the same row), and the entries of T increase strictly as we make a south-step (since they strictly increase down columns). Thus, the final inequality $T(c) \leq T(d)$ must be strict, and this proves part (b). $\square$

The following two definitions are direct calques of Definition 1.5.17 and Definition 1.5.19:

Definition 1.6.10. Let λ/μ be a skew partition. Let T be any skew tableau of shape λ/μ whose entries belong to $[N]$. Then, we define the corresponding monomial

$$x_T := \prod_{c \text{ is a cell of } Y(\lambda/\mu)} x_{T(c)} = \prod_{(i,j) \in Y(\lambda/\mu)} x_{T(i,j)} = \prod_{k=1}^N x_k^{(\# \text{ of } k\text{'s in } T)}$$

in the indeterminates $x_1, x_2, \dots, x_N$. (Of course, “# of k ’s in T ” means the number of boxes of T with a k in them.)

Definition 1.6.11. Let λ/μ be a skew partition. Then, the (skew) Schur polynomial $s_{\lambda/\mu} \in \mathcal{P}$ is defined by

$$s_{\lambda/\mu} := \sum_{T \in \text{SSYT}(\lambda/\mu, N)} x_T.$$

Example 1.6.12. Assume that $N \geq 3$. The skew diagram $Y((2,2)/(1))$ and a tableau of this shape T look as follows:

$$Y((2,2)/(1)) = \begin{array}{|c|c|} \hline & \\ \hline \square & \square \\ \hline \end{array}, \quad T = \begin{array}{|c|c|} \hline & i \\ \hline j & k \\ \hline \end{array}.$$

Such a tableau T belongs to $\text{SSYT}((2,2)/(1), N)$ if and only if $i, j, k \in [N]$ and $i < k$ and $j \leq k$. Its monomial is $x_T = x_i x_j x_k$. Thus,

$$\begin{aligned} s_{(2,2)/(1)} &= \sum_{i < k; j \leq k} x_i x_j x_k \quad \left(\begin{array}{l} \text{where all summation indices are} \\ \text{understood to belong to } [N] \end{array} \right) \\ &= 2 \sum_{u < v < w} x_u x_v x_w + \sum_{u < v} x_u^2 x_v + \sum_{u < v} x_u x_v^2 \\ &\quad \left(\begin{array}{l} \text{by a computation similar to that done in} \\ \text{Example 1.5.20 (c) (exercise!)} \end{array} \right) \\ &= 2m_{(1,1,1)} + m_{(2,1)}. \end{aligned}$$

This is the same as $s_{(2,1)}$ from Example 1.5.20 (c). We will later see why this is not a coincidence.

Example 1.6.13. The skew diagram $Y((3,2,1)/(2,1))$ has three boxes, lying in three different rows and three different columns:

$$Y((3,2,1)/(2,1)) = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}, \quad T = \begin{array}{|c|} \hline i \\ \hline j \\ \hline k \\ \hline \end{array}.$$

Thus, every tableau of shape $Y((3,2,1)/(2,1))$ is automatically semistandard (there are no row or column inequalities to check); each of its entries i, j, k can be chosen independently from $[N]$. We obtain

$$s_{(3,2,1)/(2,1)} = \sum_{i,j,k \in [N]} x_i x_j x_k = (x_1 + x_2 + \cdots + x_N)^3 = e_1^3 = s_{(1)}^3.$$

Now we generalize the symmetry of s_λ (Theorem 1.5.22 (a)) as follows:

Theorem 1.6.14. Let λ/μ be any skew partition. Then, the polynomial $s_{\lambda/\mu}$ is symmetric.

Setting $\mu = \emptyset = ()$, this recovers the symmetry of s_λ .

We will next prove Theorem 1.6.14 bijectively, using the so-called **Bender–Knuth involutions**.

Remark 1.6.15. The polynomial $s_{\lambda/\mu}$ is clearly homogeneous of degree $|Y(\lambda/\mu)| = |\lambda| - |\mu|$, since each tableau T of shape $Y(\lambda/\mu)$ has $|Y(\lambda/\mu)|$ entries.

1.6.2. The Bender–Knuth involutions

We begin the proof of Theorem 1.6.14 now; we will conclude it in Lecture 7.

Beginning of the proof of Theorem 1.6.14. For each $k \in [N-1]$, let $s_k \in S_N$ be the transposition that swaps k with $k+1$. As we know (by Lemma 1.2.9 from Lecture 3), it suffices to prove that $s_k \cdot s_{\lambda/\mu} = s_{\lambda/\mu}$ for each $k \in [N-1]$.

So we fix some $k \in [N-1]$. We will construct a bijection

$$\beta_k : \text{SSYT}(\lambda/\mu, N) \rightarrow \text{SSYT}(\lambda/\mu, N)$$

(called the k -th **Bender–Knuth involution**) that

- interchanges the # of k 's with the # of $(k+1)$'s in a tableau (i.e., if a given tableau $T \in \text{SSYT}(\lambda/\mu, N)$ has i many k 's and j many $(k+1)$'s, then its image $\beta_k(T)$ will have j many k 's and i many $(k+1)$'s), and

- leaves all other entries of the tableau unchanged.

Once such a bijection β_k is constructed, we will easily conclude that

$$x_{\beta_k(T)} = s_k \cdot x_T \quad \text{for all } T \in \text{SSYT}(\lambda/\mu, N),$$

and thus applying s_k to $s_{\lambda/\mu} = \sum_{T \in \text{SSYT}(\lambda/\mu, N)} x_T$ will merely permute the addends of this sum, showing that $s_k \cdot s_{\lambda/\mu} = s_{\lambda/\mu}$.

In order to construct the map β_k , we analyze the k 's and the $(k+1)$'s in a semistandard tableau more closely. Let $T \in \text{SSYT}(\lambda/\mu, N)$. We focus on the k 's and the $(k+1)$'s in T . An entry k in T will be called **matched** if there is a $k+1$ in the same column (thus necessarily in the neighboring cell to its south²). An entry $k+1$ in T will be called **matched** if there is a k in the same column (thus necessarily in the neighboring cell to its north³). All other k 's and $(k+1)$'s in T will be called **free**.

Example 1.6.16. Let $k = 2$ and $N = 4$. Consider the following SSYT of shape $(9, 8, 6) / (3, 1)$ (we have set all the k 's and the $(k+1)$'s in boldface for convenience):

$$T = \begin{array}{cccccccc} & & & & 1 & 1 & 1 & \mathbf{2} & \mathbf{3} & 4 \\ & & & & & & & & & \\ & & 1 & \mathbf{2} & \mathbf{2} & \mathbf{2} & 3 & \mathbf{3} & 4 & \\ & 1 & \mathbf{2} & \mathbf{3} & 4 & 4 & 4 & & & \end{array}.$$

The shaded boxes contain the matched entries. There are two matched pairs: one in column 3 and one in column 7. The free 2's are in cells $(2, 4)$, $(2, 5)$ and $(3, 2)$; the free 3's are in cells $(1, 8)$ and $(2, 6)$. In particular, there are three free 2's and two free 3's. The entries 1 and 4 are neither matched nor free, since these terms refer only to k and $k+1$.

Clearly,

$$(\# \text{ of matched } k\text{'s}) = (\# \text{ of matched } (k+1)\text{'s}),$$

since matched entries come in pairs of adjacent entries (each matched k has a matched $k+1$ in its neighboring cell to its south, and each matched $k+1$ has a

²Why? Because

- if this $k+1$ were in a cell further north than the k , then we would immediately get a contradiction to the "increase strictly down each column" condition for a semistandard tableau;
- if this $k+1$ were more than one row further south than the k , then there would be at least one further entry between the k and the $k+1$; then (again by the "increase strictly down each column" condition) this entry would have to be simultaneously larger than k and smaller than $k+1$, which is impossible.

So the only place the $k+1$ can be is in the neighboring cell to the south of the k .

³for a similar reason as in the previous sentence

matched k in its neighboring cell to its north; this gives a 1-to-1 correspondence between the matched k 's and the matched $(k+1)$'s. Each column of T that contains a matched entry will contain a matched k , a matched $k+1$ and no other k 's or $(k+1)$'s (since entries in a column strictly increase).

Now we focus on the rows. We claim the following:

Observation 1: Each row of T can be subdivided into the following six blocks (which may be empty):

entries $< k$ matched k 's free k 's free $(k+1)$'s matched $(k+1)$'s entries $> k+1$.

appearing in this order from left to right. (That is, the row starts with entries $< k$, if it has any; then, as we move right, come the matched k 's; then come the free k 's; then the free $(k+1)$'s; then the matched $(k+1)$'s; and then come the entries $> k+1$ at last.)

Proof of Observation 1. The entries in a row weakly increase from left to right; thus, the row has the form

entries $< k$ k 's $(k+1)$'s entries $> k+1$.

It remains to check that all matched k 's come before all free k 's, and that all free $(k+1)$'s come before all matched $(k+1)$'s. In other words, we must check that there cannot be free k 's to the left of matched k 's (in the same row), nor can there be matched $(k+1)$'s to the left of free $(k+1)$'s.

We start by proving the former. If there was a free k to the left of a matched k , then these two k 's along with their neighboring cells to the south would form a configuration of the form

free k	matched k
?	$k+1$

(where the two displayed columns need not be adjacent). The “?” is an actual cell of $Y(\lambda/\mu)$ (by Lemma 1.6.7) and thus must be filled with $k+1$ (since whatever entry lies in this cell must be $> k$ but simultaneously $\leq k+1$), thus rendering the k above it matched; this contradicts the freeness of the left k . So we have shown that there cannot be free k 's to the left of matched k 's (in the same row).

For a similar reason, there cannot be matched $(k+1)$'s to the left of free $(k+1)$'s. Indeed, if there were, then we would get a configuration

k	?
matched $k+1$	free $k+1$

 ,

where the “?” would again be an actual cell (by Lemma 1.6.7) and would be filled with k (since its entry would be $\geq k$ but $< k + 1$), which would render the $k + 1$ underneath it matched and cause a contradiction again.

So Observation 1 is proved. $\square$

This completes the first step of the construction. It remains to define β_k using the free entries and verify its required properties.

New starting time: 10:58.

References

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