

Math 701 Fall 2026, Lecture 5: Schur polynomials

website: <https://www.cip.ifi.lmu.de/~grinberg/t/26fs/>
(GPT-6 was used to create some of the examples below.)

We will soon see more examples. First, two notations:

Convention 1.5.5. The length of a partition λ is denoted by $\ell(\lambda)$. As we recall, this is the number of nonzero entries of λ .

For instance, $\ell((2, 2, 1)) = 3$.

Convention 1.5.6. We add and subtract N -tuples of integers entrywise. That is, if $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in \mathbb{Z}^N$ and $\beta = (\beta_1, \beta_2, \dots, \beta_N) \in \mathbb{Z}^N$, then we set

$$\begin{aligned}\alpha + \beta &:= (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_N + \beta_N) & \text{and} \\ \alpha - \beta &:= (\alpha_1 - \beta_1, \alpha_2 - \beta_2, \dots, \alpha_N - \beta_N).\end{aligned}$$

Of course, if $\alpha \in \mathbb{N}^N$ and $\beta \in \mathbb{N}^N$, then $\alpha + \beta$ belongs to $\mathbb{N}^N$ as well (but $\alpha - \beta$ does not always).

Recall that an N -partition is a weakly decreasing N -tuple of nonnegative integers. Recall furthermore that ρ is the specific N -partition $(N-1, N-2, \dots, 1, 0)$.

Lemma 1.5.7. If $\lambda \in \mathbb{N}^N$ is an N -partition, then $\lambda + \rho$ is a **strictly** decreasing N -tuple of nonnegative integers.

Conversely, if $\alpha \in \mathbb{N}^N$ is a strictly decreasing N -tuple of nonnegative integers, then $\alpha - \rho$ is an N -partition.

Thus, we obtain a bijection

$$\begin{aligned}\{N\text{-partitions}\} &\rightarrow \{\text{strictly decreasing } N\text{-tuples } \alpha \in \mathbb{N}^N\}, \\ \lambda &\mapsto \lambda + \rho\end{aligned}$$

(with inverse map given by $\alpha \mapsto \alpha - \rho$).

Proof. We shall only prove the “Conversely” claim, since the rest is (even more) trivial.

So let $\alpha \in \mathbb{N}^N$ be a strictly decreasing N -tuple of nonnegative integers. Write α as $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$; thus, $\alpha_1 > \alpha_2 > \dots > \alpha_N \geq 0$. Likewise, write ρ as $\rho = (\rho_1, \rho_2, \dots, \rho_N)$, so that $\rho_i = N - i$ for each $i \in [N]$.

We WLOG assume that $N > 0$ (else, the claim is trivial).

Now, let $i \in [N-1]$. Then, $(\alpha - \rho)_i = \alpha_i - \underbrace{\rho_i}_{=N-i} = \alpha_i - (N - i) = \alpha_i - N + i$
and likewise $(\alpha - \rho)_{i+1} = \alpha_{i+1} - N + (i + 1)$. But $\alpha_1 > \alpha_2 > \dots > \alpha_N$, so that

$\alpha_i > \alpha_{i+1}$, and therefore $\alpha_i \geq \alpha_{i+1} + 1$ (since α_i and α_{i+1} are integers). Hence,

$$\begin{aligned} (\alpha - \rho)_i &= \underbrace{\alpha_i}_{\geq \alpha_{i+1} + 1} - N + i \\ &\geq \alpha_{i+1} + 1 - N + i = \alpha_{i+1} - N + (i + 1) = (\alpha - \rho)_{i+1}. \end{aligned}$$

We have proved this for each $i \in [N - 1]$. In other words,

$$(\alpha - \rho)_1 \geq (\alpha - \rho)_2 \geq \cdots \geq (\alpha - \rho)_N.$$

Since we furthermore have $(\alpha - \rho)_N = \alpha_N - \underbrace{\rho_N}_{=0} = \alpha_N \geq 0$, we thus obtain

$$(\alpha - \rho)_1 \geq (\alpha - \rho)_2 \geq \cdots \geq (\alpha - \rho)_N \geq 0.$$

Hence, $\alpha - \rho$ is a weakly decreasing N -tuple of nonnegative integers, i.e., an N -partition. This proves the “Conversely” claim. $\square$

Example 1.5.8. Assume that $N > 0$. Example 1.5.4 from Lecture 4 showed that

$$a_{(1)+\rho} = a_\rho \cdot (x_1 + x_2 + \cdots + x_N).$$

(Keep in mind that we identify partitions of length $\leq N$ with N -partitions, so that $(1) = (1, 0, 0, \dots, 0)$ with $N - 1$ many zeroes. The N -tuple α from Example 1.5.4 is what we now call $(1) + \rho$.) Thus,

$$a_{(1)+\rho}/a_\rho = x_1 + x_2 + \cdots + x_N.$$

And of course,

$$a_{() + \rho}/a_\rho = 1.$$

More generally, it can be shown that

$$a_{(n)+\rho}/a_\rho = h_n \quad \text{for each } n \in \mathbb{N}$$

(this can be derived, e.g., by Laplace expansion from [Nica22, Theorem 1]). Furthermore, using the shorthand (1^n) for the n -tuple $(1, 1, \dots, 1)$, we have

$$a_{(1^n)+\rho}/a_\rho = e_n \quad \text{for each } n \in \{0, 1, \dots, N\}$$

(see [Prasol94, 1.15] for a particularly nice proof).

For another example, assume that $N \geq 3$. Then, it can be shown that

$$\begin{aligned} a_{(2,1)+\rho}/a_\rho &= \underbrace{\sum_{i < j} (x_i^2 x_j + x_i x_j^2)}_{=m_{(2,1)}} + 2 \underbrace{\sum_{i < j < k} x_i x_j x_k}_{=m_{(1,1,1)}} \\ &= m_{(2,1)} + 2m_{(1,1,1)}. \end{aligned}$$

Do we see any patterns here? How can we generalize this?

Today we shall give a general answer to “what is $a_{\lambda+\rho}/a_\rho$, combinatorially?”. In the next few lectures, we will prove this and more.

1.5.2. Young diagrams, Young tableaux and Schur polynomials

We first define some combinatorial notions. In [21s, Chapter 7], they are defined for N -partitions; here we define them for partitions (of arbitrary length) instead, thus avoiding an unnecessary dependence on N . Recall that $[k]$ denotes the set $\{1, 2, \dots, k\}$ whenever k is an integer.

Definition 1.5.9. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ be a partition (written as an infinite sequence by inserting infinitely many zeroes at its end).

The **Young diagram** of λ is defined to be the finite set

$$\{(i, j) \mid i \in \{1, 2, 3, \dots\} \text{ and } j \in [\lambda_i]\} \subseteq \{1, 2, 3, \dots\}^2,$$

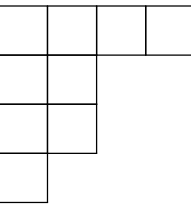
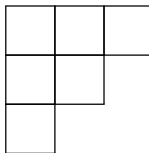
that is, the set of all pairs (i, j) of positive integers i and j satisfying $j \leq \lambda_i$.

We visualize each pair $(i, j) \in \mathbb{Z}^2$ as a square of size 1 in the plane, centered at the point with matrix coordinates (i, j) . **Matrix coordinates** are Cartesian coordinates where the x-axis runs top-to-bottom and the y-axis runs left-to-right. (This is the coordinate system used to index the entries of a matrix: the usual way to write down a matrix puts its (i, j) -th entry at matrix coordinates (i, j) .)

So a Young diagram is a set of squares, which are called its **cells** or **boxes**. It looks like a table of left-aligned rows, with the i -th row (counted from the top) having λ_i boxes. (By definition, it has infinitely many rows, but only finitely many of them are nonempty.) Formally speaking, a **cell** (aka **box**) is just a pair (i, j) of integers.

We denote the Young diagram of λ by $Y(\lambda)$. It contains $|\lambda|$ boxes.

Example 1.5.10. Here are some partitions λ and (underneath them) their Young diagrams $Y(\lambda)$ (visualized):

λ	(3)	(1, 1, 1)	(2, 1)	(4, 2, 2, 1)	(3, 2, 1)
$Y(\lambda)$					

Formally speaking,

$$\begin{aligned} Y((3)) &= \{(1, 1), (1, 2), (1, 3)\}, \\ Y((1, 1, 1)) &= \{(1, 1), (2, 1), (3, 1)\}, \\ Y((2, 1)) &= \{(1, 1), (1, 2), (2, 1)\}, \end{aligned}$$

and so on. Note that $Y((\))$ is the empty set $\emptyset$.

Definition 1.5.11. Let λ be a partition.

A **Young tableau** of **shape** $Y(\lambda)$ means a map $T : Y(\lambda) \rightarrow \{1, 2, 3, \dots\}$. We visualize it as a way of filling the boxes of the Young diagram $Y(\lambda)$ with positive integers (one integer per box), simply by writing into each box $c \in Y(\lambda)$ the corresponding value $T(c)$.

We often omit the word “Young” in “Young tableau” (so we just speak of “a tableau”). We furthermore write “of shape λ ” instead of “of shape $Y(\lambda)$ ”.

The plural of the word “tableau” is “tableaux”.

Warning 1.5.12. In [21s, Chapter 7], I define Young tableaux only for N -partitions λ (not for all partitions λ), and I require their entries to belong to $[N]$ (see [21s, Definition 7.3.5]). Here I do not impose this requirement, since it would limit us too much.

Example 1.5.13. Here is a Young tableau of shape $(4, 3, 3)$:

5	11	8	2
3	1	3	
4	5	5	

Formally speaking, this is a map $Y((4, 3, 3)) \rightarrow \{1, 2, 3, \dots\}$ sending $(1, 1)$ to 5, sending $(1, 2)$ to 11 and so on.

The visual interpretation of a tableau as a Young diagram filled with numbers suggests some rather natural concepts to define:

- The **entry** of a tableau T in a cell (i, j) means the value $T(i, j)$.
- The u -**th row** of a tableau T means the sequence of all entries of T in the cells (i, j) with $i = u$, read from left to right.
- The v -**th column** of a tableau T means the sequence of all entries of T in the cells (i, j) with $j = v$, read from top to bottom.
- The **boxes** (or **cells**) of a tableau T are just the boxes of its shape $Y(\lambda)$.
- Two boxes are said to be **adjacent** if they have an edge in common in the picture, i.e., if one of them has the form (i, j) and the other has the form $(i + 1, j)$ or $(i, j + 1)$.
- The words “north”, “west”, etc. mean what they would mean in the picture: e.g., the box $(2, 4)$ lies one step north and three steps west of $(3, 7)$.

Some Young tableaux are nicer than others:

Definition 1.5.14. Let λ be a partition.

A Young tableau T of shape λ is said to be **semistandard** if its entries

- increase weakly along each row from left to right (i.e., we have $T(i, j) \leq T(i, j+1)$ whenever (i, j) and $(i, j+1)$ belong to $Y(\lambda)$), and
- increase strictly down each column from top to bottom (i.e., we have $T(i, j) < T(i+1, j)$ whenever (i, j) and $(i+1, j)$ belong to $Y(\lambda)$).

We let $\text{SSYT}(\lambda)$ denote the set of all semistandard Young tableaux of shape λ . Furthermore, for any $k \in \mathbb{N}$, we let $\text{SSYT}(\lambda, k)$ denote the set of all semistandard Young tableaux of shape λ whose entries belong to $[k]$. We also abbreviate the word “semistandard Young tableau” as “**SSYT**”.

Definition 1.5.15. Furthermore, a Young tableau T of shape λ is said to be **standard** if it is semistandard and its entries are $1, 2, \dots, |Y(\lambda)|$ with no repetition. (Keep in mind: $|Y(\lambda)|$ is the number of boxes in $Y(\lambda)$.)

Example 1.5.16. Here are five tableaux of shape $(3, 2)$:

1	3	2
2	4	

T_1

1	2	3
2	2	

T_2

1	1	2
2	3	

T_3

1	2	4
3	6	

T_4

1	2	4
3	5	

T_5

- (a) The tableaux T_1 and T_2 are **not semistandard**. In T_1 , the first row decreases from 3 to 2 (although the columns increase strictly). In T_2 , the rows increase weakly, but the second column has equal entries.
- (b) The tableaux T_3 and T_4 are **semistandard but not standard**. Their rows increase weakly and their columns increase strictly. However, T_3 has repeated entries, while T_4 has entries 1, 2, 3, 4, 6 instead of 1, 2, 3, 4, 5. Thus, distinct entries alone do not make a semistandard tableau standard.
- (c) The tableau T_5 is **standard**: it is semistandard and contains each of 1, 2, 3, 4, 5 exactly once.

Definition 1.5.17. Let T be a Young tableau of shape λ whose entries belong to $[N]$. Then, we define the corresponding monomial

$$x_T := \prod_{c \text{ is a cell of } Y(\lambda)} x_{T(c)} = \prod_{(i,j) \in Y(\lambda)} x_{T(i,j)} = \prod_{k=1}^N x_k^{(\# \text{ of } k\text{'s in } T)}$$

in the indeterminates $x_1, x_2, \dots, x_N$. (Of course, “# of k 's in T ” means the number of boxes of T with a k in them.)

Example 1.5.18. Let $N = 3$. Consider the tableau

$$T = \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 3 & \\ \hline \end{array}, \quad \text{with corresponding monomial } x_T = x_1 x_1 x_3 x_2 x_3 = x_1^2 x_2 x_3^2.$$

The rows increase weakly and the columns increase strictly, so this tableau of shape $(3, 2)$ is semistandard. It is not standard, since some entries repeat.

Definition 1.5.19. Let λ be a partition. Then, the **Schur polynomial** $s_\lambda \in \mathcal{P}$ is defined by

$$s_\lambda := \sum_{T \in \text{SSYT}(\lambda, N)} x_T.$$

Example 1.5.20.

(a) Assume that $N > 0$. Let $n \in \mathbb{N}$. Then,

$$s_{(n)} = h_n,$$

because an SSYT of shape (n) with entries in $[N]$ is just a weakly increasing n -tuple of elements of $[N]$ (the entries of its single row).

(b) Let $n \in \{0, 1, \dots, N\}$. Then,

$$s_{(1^n)} = e_n,$$

because an SSYT of shape (1^n) with entries in $[N]$ is just a strictly increasing n -tuple of elements of $[N]$ (the entries of its single column).

(c) Assume that $N \geq 3$. An SSYT of shape $(2, 1)$ with entries in $[N]$ is a tableau of the form

$$\begin{array}{|c|c|} \hline i & j \\ \hline k & \\ \hline \end{array} \quad \text{with } i, j, k \in [N], \quad i \leq j \text{ and } i < k.$$

Thus, we have

$$\begin{aligned} s_{(2,1)} &= \sum_{i \leq j; i < k} x_i x_j x_k && \left(\begin{array}{l} \text{where all summation indices are} \\ \text{understood to belong to } [N] \end{array} \right) \\ &= \sum_{\substack{i \leq j; i < k; \\ j < k}} x_i x_j x_k + \sum_{\substack{i \leq j; i < k; \\ j \geq k}} x_i x_j x_k \end{aligned}$$

$$\begin{aligned}
& \left(\begin{array}{c} \text{here, we have split the sum} \\ \text{according to whether } j < k \text{ or } j \geq k \end{array} \right) \\
&= \sum_{i \leq j < k} x_i x_j x_k + \sum_{i < k \leq j} x_i x_j x_k \\
&= \sum_{i < j < k} x_i x_j x_k + \sum_{i=j < k} x_i x_j x_k + \sum_{i < k < j} x_i x_j x_k + \sum_{i < k=j} x_i x_j x_k \\
& \left(\begin{array}{c} \text{here, we have split the second sum} \\ \text{according to whether } k < j \text{ or } k=j, \\ \text{and the first sum according to} \\ \text{whether } i < j \text{ or } i=j \end{array} \right) \\
&= \sum_{u < v < w} x_u x_v x_w + \sum_{u < v} x_u^2 x_v + \sum_{u < v < w} \underbrace{x_u x_w x_v}_{=x_u x_v x_w} + \sum_{u < v} x_u x_v^2 \\
&= 2 \underbrace{\sum_{u < v < w} x_u x_v x_w}_{=m_{(1,1,1)}} + \underbrace{\sum_{u < v} x_u^2 x_v + \sum_{u < v} x_u x_v^2}_{=m_{(2,1)}} \\
&= 2m_{(1,1,1)} + m_{(2,1)}.
\end{aligned}$$

Example 1.5.21. Let $N = 3$, and write x, y, z for x_1, x_2, x_3 . The eight tableaux $T \in \text{SSYT}((2,1), 3)$ are shown below, with their respective monomials x_T underneath:

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y^2z	yz^2	xyz	xyz																

Hence,

$$s_{(2,1)} = x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2 + 2xyz.$$

In particular, the xyz -coefficient 2 comes from the two different semistandard tableaux that use 1, 2, 3 exactly once. These are precisely the standard tableaux of shape $(2,1)$.

These examples seem to suggest that s_λ is always symmetric. Not only that, but it also resembles the $a_{\lambda+\rho}/a_\rho$ that we have seen above. Both of these are true, and we will prove them next week:

Theorem 1.5.22. Let λ be a partition. Then:

- (a) The polynomial s_λ is symmetric.
- (b) If $\ell(\lambda) \leq N$ (that is, if λ is an N -partition), then

$$a_{\lambda+\rho} = a_\rho \cdot s_\lambda$$

(where $\lambda + \rho$ is the entrywise sum of the two N -partitions $\lambda, \rho \in \mathbb{N}^N \subseteq \mathbb{Z}^N$).

In particular, Conjecture 1.5.3 from Lecture 4 will follow from this, since s_λ obviously has nonnegative coefficients.

We will prove Theorem 1.5.22 (a) soon; but first we will generalize our Schur polynomials s_λ to the **skew Schur polynomials** $s_{\lambda/\mu}$.

References

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