

Math 331, Fall 2026: homework set 2

due date: Wednesday 14 October 2026 at 3:30 PM on Canvas.

Please solve 5 of the 10 exercises!

As discussed in §1.14, we write all semigroups (including monoids and groups) multiplicatively unless we say otherwise.

Exercise 1. Let S be a semigroup, written multiplicatively. Let a and b be two elements of S that commute.

- (a) Prove that $a^i b = b a^i$ for each positive integer i .
- (b) Prove that $a^i b^j = b^j a^i$ for all positive integers i and j .
- (c) Prove that $(ab)^n = a^n b^n$ for all positive integers n .

Exercise 2. Let a be an invertible element of a monoid M . Prove that $o(a^{-1}) = o(a)$. (In particular, this means that $o(a)$ and $o(a^{-1})$ are either both finite or both infinite.)

Exercise 3. Find a nonabelian monoid of size 3.

[Hint: Start with a nonabelian semigroup of size 2.]

Exercise 4. Let S be a semigroup. Prove that we can obtain a monoid $S \cup \{1\}$ from S by adjoining a new element – let us call it 1 – and setting $s * 1 := s$ and $1 * s := s$ and $1 * 1 := 1$ for all $s \in S$. (This is called **adjoining a neutral element** to S . Note that if S already had a neutral element 1_{old} , then the new 1 will dethrone 1_{old} , since $1 * 1_{\text{old}}$ is no longer 1.)

Exercise 5. Let S be a monoid. Let $a \in S$ have an inverse. Prove that

$$(a^n)^m = a^{nm} \quad \text{for all integers } n \text{ and } m.$$

[Hint: Make sure your argument allows any or both of n and m to be negative.]

Exercise 6. Let G be a group. Prove that G is abelian if and only if all $a, b \in G$ satisfy $(ab)^{-1} = a^{-1}b^{-1}$.

Exercise 7. Let S be a monoid, written multiplicatively. Let a and b be two elements of S that commute and have inverses.

- (a) Prove that $a^i b = b a^i$ for each integer i .
- (b) Prove that $a^i b^j = b^j a^i$ for all integers i and j .
- (c) Prove that $(ab)^n = a^n b^n$ for all integers n .

[Hint: You are allowed to use all preceding exercises even if you didn't solve them.]

Exercise 8. Let G be a group. Let $u, v \in G$. Prove that $o(uv) = o(vu)$.

Exercise 9. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be the map that sends each $n \in \mathbb{N}$ to $n + 1$. (Recall that $\mathbb{N} = \{0, 1, 2, \dots\}$.)

Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be the map that sends each $n \in \mathbb{N}$ to $\max\{n - 1, 0\}$ (that is, it sends $0, 1, 2, 3, 4, \dots$ to $0, 0, 1, 2, 3, \dots$, respectively).

Prove that $g \circ f = \text{id}$ but $f \circ g \neq \text{id}$. Conclude that the preceding exercise does not generalize to monoids. (What is the monoid here?)

Exercise 10. Consider the two permutations $u = t_{1,2}t_{3,4}t_{5,6}$ and $v = t_{2,3}t_{4,5}$ in the symmetric group $S_6 = S_{\{1,2,3,4,5,6\}}$. Here, $t_{i,j}$ denotes the transposition that swaps two given elements i and j .

(a) Compute uv .

(b) Compute $o(u)$ and $o(v)$ and $o(uv)$.