

Math 331, Fall 2026: homework set 1

due date: Wednesday 7 October 2026 at 3:30 PM on Canvas.

Please solve 5 of the 10 exercises!

Exercise 1. For the following five binary operations $*$ on the set $\mathbb{Z}$, determine whether $(\mathbb{Z}, *)$ is a semigroup, a monoid, a group, or none of these:

- (a) the operation $*$ given by $a * b = \max\{a, b\}$;
- (b) the operation $*$ given by $a * b = a + b + 1$;
- (c) the operation $*$ given by $a * b = |ab|$;
- (d) the operation $*$ given by $a * b = ab + a + b + 1$;
- (e) the operation $*$ given by $a * b = ab - a - b + 2$.

Proofs are not required in this exercise.

If s is an element of a monoid $(S, *)$, and if n is a nonnegative integer, then s^n denotes the n -fold product $\underbrace{s * s * \cdots * s}_{n \text{ times } s}$. For $n = 0$, this is simply the neutral element of $(S, *)$. If s is furthermore invertible, then s^n is defined for negative integers n as well, namely by setting $s^n = (s^{-1})^{-n}$ when n is negative.

Exercise 2. Let $(S, *)$ be a monoid with neutral element e . Let $s \in S$.

- (a) Prove that if $s^3 = e$ and $s^5 = e$, then $s = e$.
- (b) Prove that if $s^6 = e$ and $s^{10} = e$, then $s^2 = e$.
- (c) Is it true that if $s^6 = e$ and $s^{10} = e$, then $s = e$?

Exercise 3. Let $(S, *)$ be a semigroup. Let $a, b \in S$ be such that ab is central in S . Prove that

$$(ab)^n = a^n b^n \quad \text{for all positive integers } n.$$

Exercise 4. Let $(S, *)$ be a monoid, and let $a \in S$ be an invertible element. Let $b \in S$.

- (a) Prove that $(aba^{-1})^k = ab^k a^{-1}$ for each nonnegative integer k .
- (b) Now assume that b , too, is invertible. Prove that $(aba^{-1})^k = ab^k a^{-1}$ holds for all integers k , including the negative ones.

Exercise 5. Consider a group $G = \{e, a, b, c, d\}$ with five elements, of which you only know that $ee = e$ and $aa = b$ and $ab = c$ and $ac = d$. Reconstruct the entire multiplication table of this group, i.e., compute all products xy with $x, y \in G$.

xy	$y = e$	$y = a$	$y = b$	$y = c$	$y = d$
$x = e$	e				
$x = a$		b	c	d	
$x = b$					
$x = c$					
$x = d$					

Just fill in this table; don't show your proof.

[You should not assume that e is the neutral element just because I called it e . But it follows from $ee = e$ – do you see how?]

Exercise 6. The following is the multiplication table of a group $\{e, p, q, r, u, v\}$ with 6 elements:

xy	$y = e$	$y = p$	$y = q$	$y = r$	$y = u$	$y = v$
$x = e$	e					
$x = p$		e			q	
$x = q$			e			
$x = r$						
$x = u$		r			v	e
$x = v$						

Fill in the missing values. (Again, don't write down your argument.)

Exercise 7. Let $(S, *)$ be a nonempty semigroup. Assume that for each $a \in S$, the maps

$$L_a : S \rightarrow S,$$

$$x \mapsto a * x$$

and

$$R_a : S \rightarrow S,$$

$$x \mapsto x * a$$

are bijective. Prove that $(S, *)$ is a group.

[Hint: How would you construct a neutral element?]

A **submonoid** of a monoid $(S, *)$ is defined to be a subset T of S that contains the neutral element of S and is closed under the operation $*$ (that is, satisfies $a * b \in T$ for all $a, b \in T$). For example, the set $\mathbb{N} = \{0, 1, 2, \dots\}$ is a submonoid of $(\mathbb{Z}, +)$. Note that a submonoid T of a monoid $(S, *)$ always becomes a monoid itself if we restrict the operation $*$ to it (more precisely: to $T \times T$).

Exercise 8. Let $\mathbb{Z}_{\text{ev}}$ be the set of all even integers, and let $\mathbb{Z}_{\text{od}}$ be the set of all odd integers. Consider the two monoids $(\mathbb{Z}, +)$ and $(\mathbb{Z}, \cdot)$.

- (a) Is $\mathbb{Z}_{\text{ev}}$ a submonoid of $(\mathbb{Z}, +)$?
- (b) Is $\mathbb{Z}_{\text{od}}$ a submonoid of $(\mathbb{Z}, +)$?
- (c) Is $\mathbb{Z}_{\text{ev}}$ a submonoid of $(\mathbb{Z}, \cdot)$?
- (d) Is $\mathbb{Z}_{\text{od}}$ a submonoid of $(\mathbb{Z}, \cdot)$?
- (e) Is $\{1\} \cup \mathbb{Z}_{\text{ev}}$ a submonoid of $(\mathbb{Z}, \cdot)$?

Exercise 9. Let $(S, *)$ be a monoid with neutral element e . Let $a \in S$ be any element. The **centralizer** of a is defined to be the set C_a of all $s \in S$ that commute with a (that is, that satisfy $as = sa$).

- (a) Prove that C_a is a submonoid of S (that is: $e \in C_a$ and $st \in C_a$ for all $s, t \in C_a$).
- (b) Assume that S is a group. Prove that C_a is a subgroup of S (that is: in addition to what was shown in part (a), we also have $s^{-1} \in C_a$ for each $s \in C_a$).
- (c) What is C_e ?

Recall that a semigroup $(S, *)$ is said to be **abelian** if $ab = ba$ for all $a, b \in S$.

Exercise 10. Let $(S, *)$ be a monoid with neutral element e . Let n be a positive integer. Define the set

$$T_n := \{s \in S \mid s^n = e\}.$$

- (a) Prove that if $(S, *)$ is abelian, then T_n is a submonoid of S .
- (b) Prove that this does not necessarily hold when $(S, *)$ is not abelian.

[Hint: Try the group from Exercise 6 and $n = 2$.]