Note on the Young–Jucys–Murphy elements

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Abstract. We give a new proof that the k-th Young–Jucys–Murphy element in the group algebra of S_n is annihilated by the polynomial $\prod\limits_{i=-k+1}^{k-1} (t-i)$. This proof is inspired by Igor Makhlin's proof on MathOverflow #83150, but uses no linear algebra whatsoever. At its core lies an algebraic computation found with the help of LLMs.

This note is meant to give a new answer to MathOverflow question #83150 and an answer to MathOverflow question #420318.

Acknowledgments. The idea of this is note is owed to Igor Makhlin's MathOverflow question and answer [Makhli11]. The proof of Lemma 3.3 is based upon somewhat rough and inchoate suggestions of two LLMs: GPT-5.4 and Claude Opus 4.6 (see below).

1. The Young–Jucys–Murphy elements

We let $\mathbb{N}$ denote the set $\{0, 1, 2, \ldots\}$. We fix an $n \in \mathbb{N}$, and consider the symmetric group S_n; this group consists of the permutations of the set $[n] := \{1, 2, \ldots, n\}$. Furthermore, let $\mathbf{k}$ be any commutative ring, and let $\mathbf{k}[S_n]$ be the group algebra of S_n over $\mathbf{k}$.

For any two distinct elements i, j of $[n]$, we let $t_{i,j} \in S_n$ be the transposition that swaps i with j. The *Young–Jucys–Murphy elements* $J_1, J_2, \ldots, J_n$ of $\mathbf{k}[S_n]$ are defined by

$$J_k := \sum_{i=1}^{k-1} t_{i,k} \in \mathbf{k}[S_n] \qquad \text{for each } k \in [n]$$

(so that $J_1 = (\text{empty sum}) = 0$). It is known that these elements $J_1, J_2, \ldots, J_n$ commute (see, e.g., [Grinbe25, Theorem 3.4.2], where they are denoted by $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n$). This fact has a very easy proof. The following fact is also well-known ([Grinbe25, Theorem 3.4.5]), but less easy:

Theorem 1.1. For each $k \in [n]$, we have $\prod\limits_{i=-k+1}^{k-1} (J_k - i) = 0$.

When $\mathbf{k}$ is a field of characteristic 0, this can be reinterpreted as saying that the linear operator on $\mathbf{k}[S_n]$ given by left multiplication by J_k (that is, the operator sending each $a \in \mathbf{k}[S_n]$ to $J_k a$) is diagonalizable, and that all its eigenvalues are integers between $-k+1$ and $k-1$ (inclusive). In this form, this theorem originates in Murphy's 1981 paper [Murphy81][1]; indeed, he shows a stronger result [Murphy81, (3.8)] which describes an eigenbasis of this operator and matches the eigenvectors to the eigenvalues (which can be used to compute their multiplicities). Unsurprisingly, his eigenbasis relies on the irreducible representations of S_n (the Specht modules). In a later paper [Murphy92, (5.2)], he states Theorem 1.1 explicitly (even in the more general setting of Hecke algebras). A more conservative variant of Murphy's proof is given by Garsia in [GarEge20, Theorem 2.13]. A different proof, also using representations, is given by Lascoux in [Lascou02, Lemma 43][2]. A different tradition of proofs of Theorem 1.1, more self-contained but using some linear algebra, goes back to the work of Okounkov and Vershik [VerOko05, Theorem 5.1] (from which the theorem easily follows, even if it is not stated there); a particularly beautiful example is Igor Makhlin's proof in [Makhli11]. Yet, even this proof trades the representation theory for a use of the spectral theorem (every symmetric matrix over $\mathbb{R}$ is diagonalizable); while fairly basic, it is a tool rather alien to the combinatorial essence of the theorem.

In this note, we will give a completely elementary proof of Theorem 1.1, relying only on a tricky inductive computation.

2. The antipode and the positivity trick

Before we come to the proof, we state a simple lemma about the group algebra of S_n that will come surprisingly handy.

We say that a commutative ring $\mathbf{k}$ is *formally real* if it has the following property: If m elements $a_1, a_2, \ldots, a_m$ of $\mathbf{k}$ satisfy $a_1^2 + a_2^2 + \cdots + a_m^2 = 0$, then $a_1 = a_2 = \cdots = a_m = 0$. In other words, a commutative ring $\mathbf{k}$ is formally real if and only if a (finite) sum of squares of elements of $\mathbf{k}$ is never 0 unless all the elements being squared are 0. For instance, any (totally) ordered integral domain is formally real (because a square a^2 in such a domain is always positive unless $a = 0$, and a nonempty sum of positive elements cannot be 0). In particular, $\mathbb{Z}$ is formally real. This is all we need in the following.

Next, we define a well-known map on the group algebra $\mathbf{k}[S_n]$.

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[1] He denotes the J_k as L_k.

[2] He denotes the J_k as ξ_k. Note also that he misspells $(x-n+1)\cdots x\cdots(x+n-1)$ as $(x-n)\cdots x\cdots(x+n)$.

We let $S : \mathbf{k}[S_n] \to \mathbf{k}[S_n]$ be the $\mathbf{k}$-linear map that sends each permutation $w \in S_n$ to w^{-1}. This is well-defined, since $(w)_{w\in S_n}$ is a basis of the $\mathbf{k}$-module $\mathbf{k}[S_n]$. This map S is called the *antipode* of $\mathbf{k}[S_n]$.

The following is well-known and easy to check ([Grinbe25, Theorem 3.11.14]):

Theorem 2.1. **(a)** The map $S : \mathbf{k}[S_n] \to \mathbf{k}[S_n]$ is a $\mathbf{k}$-algebra anti-automorphism.

(b) It is furthermore an involution (i.e., it satisfies $S \circ S = \operatorname{id}$).

Furthermore, any transposition $t_{i,k} \in S_n$ is fixed under the antipode S (since the definition of S yields $S(t_{i,k}) = t_{i,k}^{-1} = t_{i,k}$). Hence, any sum of transpositions in $\mathbf{k}[S_n]$ is also fixed under S (since S is $\mathbf{k}$-linear). But each Young–Jucys–Murphy element J_k (for any $k \in [n]$) is a sum of transpositions $t_{i,k}$, and thus is fixed under S (according to the previous sentence). In other words, we have proved the following:

Lemma 2.2. For any $k \in [n]$, we have $S(J_k) = J_k$.

Over a formally real commutative ring, we furthermore have the following:

Lemma 2.3. Let $\mathbf{k}$ be a formally real commutative ring. Let $a \in \mathbf{k}[S_n]$ be such that $S(a) \cdot a = 0$. Then, $a = 0$.

Proof. Expand a in the basis $(w)_{w\in S_n}$ of $\mathbf{k}[S_n]$. That is, write a as

$$a = \sum_{w\in S_n} a_w w \qquad \text{for some scalars } a_w \in \mathbf{k}.$$

Then,

$$\begin{aligned} S(a) &= S\left(\sum_{w\in S_n} a_w w\right) = \sum_{w\in S_n} a_w \underbrace{S(w)}_{\substack{=w^{-1} \\ \text{(by the definition of } S)}} \qquad \left(\begin{array}{c}\text{since the map } S \\ \text{is } \mathbf{k}\text{-linear}\end{array}\right) \\ &= \sum_{w\in S_n} a_w w^{-1} = \sum_{v\in S_n} a_v v^{-1}. \end{aligned}$$

Multiplying this equality with the equality $a = \sum\limits_{w \in S_n} a_w w$, we obtain

$$S(a) \cdot a = \left(\sum_{v \in S_n} a_v v^{-1} \right) \left(\sum_{w \in S_n} a_w w \right) = \sum_{v \in S_n} \sum_{w \in S_n} \underbrace{a_v v^{-1} a_w w}_{= a_v a_w v^{-1} w}$$

$$= \sum_{v \in S_n} \sum_{w \in S_n} a_v a_w v^{-1} w = \sum_{v \in S_n} \sum_{u \in S_n} a_v a_{vu} \underbrace{v^{-1} v u}_{= u}$$

$$\left(\begin{array}{c} \text{here, we have substituted } vu \text{ for } w \text{ in the inner sum,} \\ \text{since the map } S_n \to S_n,\ u \mapsto vu \text{ (for given } v \in S_n\text{)} \\ \text{is a bijection (because } S_n \text{ is a group)} \end{array} \right)$$

$$= \sum_{v \in S_n} \sum_{u \in S_n} a_v a_{vu} u = \sum_{u \in S_n} \left(\sum_{v \in S_n} a_v a_{vu} \right) u.$$

Thus,

$$\sum_{u \in S_n} \left(\sum_{v \in S_n} a_v a_{vu} \right) u = S(a) \cdot a = 0$$

(by assumption). Note that the left hand side of this equality is a $\mathbf{k}$-linear combination of the family $(u)_{u \in S_n}$ with coefficients $\sum\limits_{v \in S_n} a_v a_{vu}$. Since the family $(u)_{u \in S_n}$ is $\mathbf{k}$-linearly independent (being a basis of the group algebra $\mathbf{k}[S_n]$), this equality thus entails that all the coefficients $\sum\limits_{v \in S_n} a_v a_{vu}$ on the left hand side must be 0. In other words,

$$\sum_{v \in S_n} a_v a_{vu} = 0 \qquad \text{for each } u \in S_n.$$

Applying this to $u = \operatorname{id}$, we obtain $\sum\limits_{v \in S_n} a_v a_{v \operatorname{id}} = 0$. In other words, $\sum\limits_{v \in S_n} a_v^2 = 0$ (since $a_v a_{v \operatorname{id}} = a_v a_v = a_v^2$ for each $v \in S_n$). Since the ring $\mathbf{k}$ is formally real, this entails that

$$a_v = 0 \qquad \text{for all } v \in S_n \tag{1}$$

(because a (finite) sum of squares of elements of $\mathbf{k}$ is never 0 unless all the elements being squared are 0). Therefore,

$$a = \sum_{w \in S_n} \underbrace{a_w}_{\substack{=0 \\ \text{(by (1))}}} w = \sum_{w \in S_n} 0w = 0.$$

This proves Lemma 2.3. □

3. The recursion lemma

Now comes the main tool of the proof of Theorem 1.1: a lemma that I conjectured in MathOverflow question #420318. First a piece of notation: For each positive integer k, we define the polynomial

$$p_k(t) := \prod_{i=-k+1}^{k-1} (t-i) \in \mathbb{Z}[t].$$

For instance,

$$\begin{aligned}
p_1(t) &= t; \\
p_2(t) &= (t+1)\,t\,(t-1); \\
p_3(t) &= (t+2)(t+1)\,t\,(t-1)(t-2); \\
p_4(t) &= (t+3)(t+2)(t+1)\,t\,(t-1)(t-2)(t-3).
\end{aligned}$$

Before we state our main lemma, we shall show two easy properties of these polynomials p_k:

Lemma 3.1. Let k be a positive integer. Then, the polynomial $p_k \in \mathbb{Z}[t]$ is odd, i.e., is a $\mathbb{Z}$-linear combination of the powers t^m for odd m.

Proof. By definition of p_k, we have

$$\begin{aligned}
p_k(t) &= \prod_{i=-k+1}^{k-1}(t-i) = \left(\prod_{i=-k+1}^{-1}(t-i)\right)(t-0)\left(\prod_{i=1}^{k-1}(t-i)\right) \\
&\qquad \left(\begin{array}{c}\text{here, we have broken up the product into}\\ \text{three parts: the product of the factors for } i<0\text{, the}\\ \text{factor for } i=0\text{, and the product of the factors for } i>0\end{array}\right) \\
&= \left(\prod_{i=1}^{k-1}\underbrace{(t-(-i))}_{=t+i}\right)\underbrace{(t-0)}_{=t}\left(\prod_{i=1}^{k-1}(t-i)\right) \qquad \left(\begin{array}{c}\text{here, we have substituted } -i\\ \text{for } i \text{ in the first product}\end{array}\right) \\
&= \left(\prod_{i=1}^{k-1}(t+i)\right)t\left(\prod_{i=1}^{k-1}(t-i)\right) = t\underbrace{\left(\prod_{i=1}^{k-1}(t+i)\right)\left(\prod_{i=1}^{k-1}(t-i)\right)}_{=\prod_{i=1}^{k-1}((t+i)(t-i))} \\
&= t\prod_{i=1}^{k-1}\underbrace{((t+i)(t-i))}_{=t^2-i^2} = t\prod_{i=1}^{k-1}\left(t^2-i^2\right). \qquad (2)
\end{aligned}$$

Clearly, the product $\prod_{i=1}^{k-1}(t^2-i^2)$ is a polynomial in t^2. In other words, it can be written as

$$\prod_{i=1}^{k-1}\left(t^2-i^2\right) = \sum_{s=0}^{r} a_s \left(t^2\right)^s \qquad (3)$$

for some $r \in \mathbb{N}$ and some coefficients $a_0, a_1, \ldots, a_r \in \mathbb{Z}$. Consider these r and $a_0, a_1, \ldots, a_r$. Now, substituting (3) into (2), we find

$$p_k(t) = t \sum_{s=0}^{r} a_s \left(t^2\right)^s = \sum_{s=0}^{r} a_s \underbrace{t\left(t^2\right)^s}_{=t^{2s+1}} = \sum_{s=0}^{r} a_s t^{2s+1}.$$

This shows that $p_k(t)$ is a $\mathbb{Z}$-linear combination of odd powers of t (since $2s+1$ is always odd when $s \in \mathbb{N}$). In other words, $p_k(t)$ is an odd polynomial of t. This proves Lemma 3.1. □

Lemma 3.2. Let k be a positive integer. Then,

$$p_{k+1}(t) = (t+k)(t-k)\, p_k(t) \tag{4}$$

and

$$p_{k+1}(t) = (t+k)(t+k-1)\, p_k(t-1). \tag{5}$$

Proof. The definition of p_k yields $p_k(t) = \prod\limits_{i=-k+1}^{k-1} (t-i)$. Substituting $t-1$ for t in this equality, we obtain

$$\begin{aligned} p_k(t-1) &= \prod_{i=-k+1}^{k-1} \underbrace{((t-1)-i)}_{=t-(i+1)} = \prod_{i=-k+1}^{k-1} (t-(i+1)) \\ &= \prod_{i=-k+2}^{k} (t-i) \end{aligned} \tag{6}$$

(here, we have substituted i for $i+1$ in the product).

The definition of p_{k+1} yields

$$\begin{aligned} p_{k+1}(t) &= \prod_{i=-(k+1)+1}^{(k+1)-1} (t-i) = \prod_{i=-k}^{k} (t-i) \qquad \left(\begin{array}{c} \text{since } -(k+1)+1 = -k \\ \text{and } (k+1)-1 = k \end{array}\right) \\ &= (t-(-k)) \left(\prod_{i=-k+1}^{k-1} (t-i)\right)(t-k) \end{aligned}$$

(here, we have split off the factors for $i = -k$ and for $i = k$ from the product). Thus,

$$\begin{aligned} p_{k+1}(t) &= \underbrace{(t-(-k))}_{=t+k} \underbrace{\left(\prod_{i=-k+1}^{k-1} (t-i)\right)}_{\substack{=p_k(t) \\ \text{(by the definition of } p_k)}} (t-k) \\ &= (t+k) \cdot p_k(t) \cdot (t-k) = (t+k)(t-k)\, p_k(t), \end{aligned}$$

which proves (4).

Furthermore, as we have already seen,

$$p_{k+1}(t) = \prod_{i=-k}^{k} (t-i) = (t-(-k))(t-(-k+1)) \prod_{i=-k+2}^{k} (t-i)$$

(here, we have split off the factors for $i = -k$ and for $i = -k+1$ from the product). Thus,

$$p_{k+1}(t) = \underbrace{(t-(-k))}_{=t+k} \underbrace{(t-(-k+1))}_{=t+k-1} \underbrace{\prod_{i=-k+2}^{k} (t-i)}_{\substack{=p_k(t-1) \\ \text{(by (6))}}} = (t+k)(t+k-1)\, p_k(t-1).$$

This proves (5). Thus the proof of Lemma 3.2 is complete. □

We shall now state the crucial lemma:

Lemma 3.3. Let k be a positive integer. Let x, y, s be three elements of a ring R (not necessarily commutative) that satisfy the conditions

$$xy = yx, \tag{7}$$

$$s^2 = 1, \tag{8}$$

$$sy = xs + 1, \tag{9}$$

$$p_k(x) = 0. \tag{10}$$

Then,

$$p_k(y) \cdot p_{k+1}(y) = 0 \tag{11}$$

and

$$p_{k+1}^2(y) = 0. \tag{12}$$

Proof. The condition (7) shows that the elements x and y of R commute. Thus, the $\mathbb{Z}$-subalgebra of R generated by x and y is commutative. Let us denote this commutative $\mathbb{Z}$-subalgebra by C. Of course, $x, y \in C$.

From $s^2 = 1$, we obtain $s^2ys = 1ys = ys$, so that

$$\begin{aligned} ys = s^2ys = s \underbrace{sy}_{\substack{=xs+1 \\ \text{(by (9))}}} s = s(xs+1)s = sx \underbrace{ss}_{=s^2=1} + \underbrace{ss}_{=s^2=1} \\ = sx + 1. \end{aligned} \tag{13}$$

There is a certain symmetry in our situation: If we read all the products in R backwards (i.e., replace each product $r_1 r_2 \cdots r_k$ by $r_k r_{k-1} \cdots r_1$), then the conditions (7), (8), (9) and (10) remain true. Indeed, the equalities (8) and (10) remain unchanged because they only contain polynomials of a single element (s or x, respectively); the equality (7) turns into $yx = xy$, which is equivalent to it; and finally, the equality (9) turns into (13), which we know to be true. This symmetry will be useful to us later; we call it the *reversal symmetry*.

Now we claim the following:

Claim 1: For any polynomial $f \in \mathbb{Z}[t]$, we have

$$f(y) - f(x) = (sf(y) - f(x)s)(y - x).$$

Proof of Claim 1. Both sides of this equality are $\mathbb{Z}$-linear in f. Thus, by linearity, we can WLOG assume that f is a monomial t^m for some $m \in \mathbb{N}$ (since any polynomial $f \in \mathbb{Z}[t]$ is a $\mathbb{Z}$-linear combination of monomials). In other words, it suffices to prove that

$$y^m - x^m = (sy^m - x^m s)(y - x) \tag{14}$$

for each $m \in \mathbb{N}$.

So let us prove (14). We induct on m:

Base case: For $m = 0$, the equality (14) holds because both of its sides are 0.

Induction step: Let $m \in \mathbb{N}$. Assume (as the induction hypothesis) that (14) holds for m. We must prove that (14) also holds for $m + 1$ instead of m. That is, we must prove that $y^{m+1} - x^{m+1} = \left(sy^{m+1} - x^{m+1}s\right)(y - x)$. But this follows from

$$\begin{aligned}
&\underbrace{\left(sy^{m+1} - x^{m+1}s\right)}_{\substack{=sy^m y - x^m xs \\ =(sy^m - x^m s)y + x^m(sy - xs)}} (y - x) \\
&= ((sy^m - x^m s)y + x^m(sy - xs))(y - x) \\
&= (sy^m - x^m s) \underbrace{y(y - x)}_{\substack{=(y-x)y \\ \text{(since } x \text{ and } y \text{ belong to} \\ \text{the commutative ring } C\text{)}}} + x^m \underbrace{(sy - xs)}_{\substack{=1 \\ \text{(by (9))}}} (y - x) \\
&= \underbrace{(sy^m - x^m s)(y - x)}_{\substack{=y^m - x^m \\ \text{(by (14),} \\ \text{since we assumed that (14)} \\ \text{holds for } m\text{)}}} y + x^m(y - x) \\
&= (y^m - x^m)y + x^m(y - x) = y^m y - x^m x = y^{m+1} - x^{m+1}.
\end{aligned}$$

Thus, the induction step is complete, and (14) is proved. Hence, Claim 1 follows. □

Claim 2: For any polynomial $f \in \mathbb{Z}[t]$, we have

$$f(y) - f(x) = (y - x)(f(y)s - sf(x)).$$

Proof of Claim 2. Reversal symmetry shows that if we read all the products in Claim 1 backwards, then we still obtain a true fact. But this fact is precisely Claim 2.

(To put it differently: A proof of Claim 2 can be obtained from the above proof of Claim 1 by reading all products backwards, and by using (13) instead of (9).) □

Now, define two elements

$$u := p_k(y) \in C \qquad \text{(since } y \in C\text{)}$$

and

$$d := y - x \in C \qquad \text{(since } x, y \in C\text{)}.$$

Thus, $du = ud$ (since d and u both lie in the commutative ring C).

Now we claim:

Claim 3: We have

$$su = ud = du = us.$$

Proof of Claim 3. Applying Claim 1 to $f = p_k$, we obtain

$$p_k(y) - p_k(x) = (sp_k(y) - p_k(x)s)(y - x).$$

Since $p_k(x) = 0$ (by (10)) and $p_k(y) = u$ (by the definition of u) and $y - x = d$ (by the definition of d), we can rewrite this as

$$u - 0 = (su - 0s)d.$$

In other words, $u = sud$. Multiplying this equality by s from the left, we find

$$su = \underbrace{ss}_{=s^2=1} ud = ud. \tag{15}$$

By reversal symmetry, the same must hold if we read all the products backwards; that is, we have

$$us = du. \tag{16}$$

(Alternatively, this can be obtained from Claim 2 in the same way as (15) was derived from Claim 1.) Recall that $du = ud$, so that $ud = du$. Altogether,

$$su = ud = du = us \qquad \text{(by (16))}.$$

This proves Claim 3. □

Claim 4: We have $d^2u = u$.

Proof of Claim 4. Claim 3 yields $du = us$. Thus,

$$d^2 u = d \underbrace{du}_{=us} = \underbrace{du}_{=us} s = u \underbrace{ss}_{=s^2=1} = u.$$

This proves Claim 4. ☐

Claim 5: We have $(dy-1)\,u = dxu$.

Proof of Claim 5. We have

$$\begin{aligned}(dy-1)\,u - dxu &= dyu - u - dxu = \underbrace{dyu - dxu}_{=d(y-x)u} - u \\ &= d \underbrace{(y-x)}_{\substack{=d \\ \text{(by the definition of } d\text{)}}} u - u = \underbrace{ddu}_{\substack{=d^2u=u \\ \text{(by Claim 4)}}} - u = u - u = 0.\end{aligned}$$

That is, $(dy-1)\,u = dxu$. This proves Claim 5. ☐

Claim 6: For any polynomial $f \in \mathbb{Z}[t]$, we have

$$f(dy-1) \cdot u = f(dx) \cdot u.$$

Proof of Claim 6. Both sides of this equality are $\mathbb{Z}$-linear in f. Thus, it suffices to prove this equality in the case when f is a monomial t^m for some $m \in \mathbb{N}$. In other words, it suffices to prove that

$$(dy-1)^m \cdot u = (dx)^m \cdot u \tag{17}$$

for each $m \in \mathbb{N}$. We shall prove (17) by induction on m:

Base case: For $m = 0$, the equality (17) is simply saying that $u = u$, which is clearly true.

Induction step: Let $m \in \mathbb{N}$. Assume (as the induction hypothesis) that $(dy-1)^m \cdot u = (dx)^m \cdot u$. We must then prove that $(dy-1)^{m+1} \cdot u = (dx)^{m+1} \cdot u$.

The elements d, x, u all lie in the commutative ring C, and thus all commute. Hence, $dxu = udx$. Now,

$$\begin{aligned}\underbrace{(dy-1)^{m+1}}_{=(dy-1)^m \cdot (dy-1)} \cdot u &= (dy-1)^m \cdot \underbrace{(dy-1)\,u}_{\substack{=dxu \\ \text{(by Claim 5)}}} = (dy-1)^m \cdot \underbrace{dxu}_{=udx} \\ &= \underbrace{(dy-1)^m \cdot u}_{\substack{=(dx)^m \cdot u \\ \text{(by the induction hypothesis)}}} dx = (dx)^m \cdot \underbrace{udx}_{\substack{=dxu \\ \text{(since } dxu=udx\text{)}}} \\ &= \underbrace{(dx)^m \cdot dx}_{=(dx)^{m+1}} u = (dx)^{m+1} \cdot u.\end{aligned}$$

This completes the induction step. Thus, (17) is proved by induction. Thus, the proof of Claim 6 is complete. ☐

Claim 7: For any even $m \in \mathbb{N}$, we have $d^m u = u$.

Proof of Claim 7. In other words, we must show that $d^{2i} u = u$ for each $i \in \mathbb{N}$ (since any even $m \in \mathbb{N}$ can be written as $2i$ for some $i \in \mathbb{N}$). But this follows by a simple induction on i. (The *base case* $i = 0$ is obvious. The *induction step* from i to $i + 1$ proceeds by assuming that $d^{2i} u = u$, and arguing that $\underbrace{d^{2(i+1)}}_{=d^{2i+2}=d^{2i}d^2} u = d^{2i} \underbrace{d^2 u}_{\substack{=u \\ \text{(by Claim 4)}}} = d^{2i} u = u$.) Thus, Claim 7 follows. □

Claim 8: For any odd polynomial $f \in \mathbb{Z}[t]$, we have

$$f(y)\, u = f(dy)\, du.$$

Proof of Claim 8. Both sides of this equality are $\mathbb{Z}$-linear in f. Thus, it suffices to prove this equality in the case when f is a monomial t^m for some odd $m \in \mathbb{N}$ (since any odd polynomial $f \in \mathbb{Z}[t]$ is a $\mathbb{Z}$-linear combination of such monomials). In other words, it suffices to prove that

$$y^m u = (dy)^m\, du \tag{18}$$

for each odd $m \in \mathbb{N}$.

So let us do this. Let $m \in \mathbb{N}$ be odd. Then, $m + 1$ is even, so that Claim 7 (applied to $m + 1$ instead of m) yields $d^{m+1} u = u$. But the elements y, u, d all lie in the commutative ring C, so all their products and powers commute. Hence, $(dy)^m = d^m y^m = y^m d^m$ and therefore

$$(dy)^m\, du = y^m \underbrace{d^m d}_{=d^{m+1}} u = y^m \underbrace{d^{m+1} u}_{=u} = y^m u.$$

This proves (18). Thus, the proof of Claim 8 is complete. □

Claim 9: For any odd polynomial $f \in \mathbb{Z}[t]$, we have

$$f(x)\, u = f(dx)\, du.$$

Proof of Claim 9. Analogous to Claim 8, just using x instead of y. □

Now, Lemma 3.1 shows that the polynomial p_k is odd. Hence, Claim 9 (applied to $f = p_k$) yields $p_k(x)\, u = p_k(dx)\, du$. Thus,

$$p_k(dx)\, du = \underbrace{p_k(x)}_{\substack{=0 \\ \text{(by (10))}}} u = 0. \tag{19}$$

But $ud = du$ (since both d and u lie in the commutative ring C) and thus $dud = ddu = d^2 u = u$ (by Claim 4), so that $u = dud$. Hence,

$$p_k(dx) \cdot \underbrace{u}_{=dud} = \underbrace{p_k(dx)\, du}_{\substack{=0 \\ \text{(by (19))}}} d = 0. \tag{20}$$

On the other hand, Lemma 3.1 (applied to $k+1$ instead of k) shows that the polynomial p_{k+1} is odd. Hence, Claim 8 (applied to $f = p_{k+1}$) yields

$$p_{k+1}(y)\, u = p_{k+1}(dy)\, du = p_{k+1}(dy)\, ud \tag{21}$$

(since $du = ud$).

Substituting dy for t in the equality (5), we find

$$p_{k+1}(dy) = (dy+k)(dy+k-1)\, p_k(dy-1).$$

Multiplying both sides of this equality by u, we obtain

$$\begin{aligned} p_{k+1}(dy) \cdot u &= (dy+k)(dy+k-1) \underbrace{p_k(dy-1) \cdot u}_{\substack{=p_k(dx)\cdot u \\ \text{(by Claim 6, applied to } f=p_k)}} \\ &= (dy+k)(dy+k-1) \underbrace{p_k(dx) \cdot u}_{\substack{=0 \\ \text{(by (20))}}} = 0. \end{aligned}$$

Hence, (21) becomes

$$p_{k+1}(y)\, u = p_{k+1}(dy)\, ud = \underbrace{p_{k+1}(dy) \cdot u}_{=0} \cdot d = 0.$$

In view of $u = p_k(y)$, we can rewrite this as

$$p_{k+1}(y) \cdot p_k(y) = 0.$$

Since $p_k(y)$ and $p_{k+1}(y)$ both lie in the commutative ring C (since $y \in C$), we have

$$p_k(y) \cdot p_{k+1}(y) = p_{k+1}(y) \cdot p_k(y) = 0.$$

This proves (11).

Furthermore, substituting y for t in the equality (4), we obtain

$$p_{k+1}(y) = (y+k)(y-k)\, p_k(y).$$

Now,

$$p_{k+1}^2(y) = \underbrace{p_{k+1}(y)}_{=(y+k)(y-k)p_k(y)} \cdot p_{k+1}(y) = (y+k)(y-k) \cdot \underbrace{p_k(y) \cdot p_{k+1}(y)}_{\substack{=0 \\ \text{(by (11))}}} = 0.$$

This proves (12). Thus, Lemma 3.3 is proved. □

The above proof has been obtained in collaboration with GPT-5.4 and Claude Opus 4.6. Neither model came up with a correct proof on its own; both made repeated mistakes of the "division by zero" kind (e.g., dividing by $y-x$ or by $y-x-1$, neither of which is guaranteed to be invertible), along with occasionally more basic computational mistakes. Yet, the ideas of Claim 1 and of studying the element $u = p_k(y)$ were suggested by the models.

We note that $p_{k+1}(y)$ is not always 0 in the general setup of Lemma 3.3; counterexamples can be computed using Gröbner bases (see `https://mathoverflow.net/questions/420318`).

4. Proving Theorem 1.1

We are now close to proving Theorem 1.1 by induction on k. In the induction step from k to $k+1$, we shall apply Lemma 3.3 to $R = \mathbb{Z}\left[S_n\right]$ and $s = t_{k,k+1}$ and $x = J_k$ and $y = J_{k+1}$. In order to justify this, we need the following:

Lemma 4.1. Let $k \in [n-1]$. Set $s_k := t_{k,k+1} \in S_n$. Then, in $\mathbf{k}\left[S_n\right]$, we have

$$s_k J_{k+1} = J_k s_k + 1.$$

Proof. It is easy to see that each $i \in [k-1]$ satisfies

$$s_k t_{i,k+1} = t_{i,k} s_k \qquad (22)$$

(in fact, both sides of this equality equal the 3-cycle $\operatorname{cyc}_{i,k,k+1} \in S_n$ that sends the elements $i, k, k+1$ to $k, k+1, i$ and leaves all other elements of $[n]$ unchanged).

Furthermore, $s_k = t_{k,k+1}$, so that $s_k t_{k,k+1} = t_{k,k+1} t_{k,k+1} = t_{k,k+1}^2 = \mathrm{id}$ (since $t_{k,k+1}$ is a transposition, and thus squares to the identity).

By the definition of J_k, we have $J_k = \sum\limits_{i=1}^{k-1} t_{i,k}$. Hence,

$$J_k s_k = \left(\sum_{i=1}^{k-1} t_{i,k}\right) s_k = \sum_{i=1}^{k-1} t_{i,k} s_k. \qquad (23)$$

By the definition of J_{k+1}, we have

$$J_{k+1} = \sum_{i=1}^{(k+1)-1} t_{i,k+1} = \sum_{i=1}^{k} t_{i,k+1} = \sum_{i=1}^{k-1} t_{i,k+1} + t_{k,k+1}$$

(here, we have split off the addend for $i = k$ from the sum). Hence,

$$\begin{aligned} s_k J_{k+1} &= s_k \left(\sum_{i=1}^{k-1} t_{i,k+1} + t_{k,k+1}\right) = \sum_{i=1}^{k-1} \underbrace{s_k t_{i,k+1}}_{\substack{= t_{i,k} s_k \\ \text{(by (22))}}} + \underbrace{s_k t_{k,k+1}}_{= \mathrm{id} = 1} = \underbrace{\sum_{i=1}^{k-1} t_{i,k} s_k}_{\substack{= J_k s_k \\ \text{(by (23))}}} + 1 \\ &= J_k s_k + 1. \end{aligned}$$

This proves Lemma 4.1. □

Proof of Theorem 1.1. We must prove that $\prod\limits_{i=-k+1}^{k-1} \left(J_k - i\right) = 0$ for each $k \in [n]$. Let us first prove this in the case when $\mathbf{k} = \mathbb{Z}$. Thus, we assume that $\mathbf{k} = \mathbb{Z}$.

We must prove that $\prod\limits_{i=-k+1}^{k-1} \left(J_k - i\right) = 0$ for each $k \in [n]$. In other words, we must prove that

$$p_k\left(J_k\right) = 0 \qquad \text{for each } k \in [n] \qquad (24)$$

(since the definition of p_k yields $p_k\left(J_k\right) = \prod\limits_{i=-k+1}^{k-1} \left(J_k - i\right)$).

We shall prove (24) by induction on k:

Base case: For $k = 1$, we have $p_k\left(t\right) = p_1\left(t\right) = t$ and thus $p_k\left(J_k\right) = J_k = J_1 = 0$. Hence, (24) is proved for $k = 1$.

Induction step: Fix $k \in [n-1]$. Assume (as the induction hypothesis) that (24) holds for k. We must prove that (24) also holds for $k+1$ instead of k. In other words, we must prove that $p_{k+1}\left(J_{k+1}\right) = 0$.

Our induction hypothesis says that $p_k\left(J_k\right) = 0$.

Set $s_k := t_{k,k+1} \in S_n$. Then, $s_k^2 = 1$ (since $s_k = t_{k,k+1}$ is a transposition) and $J_k J_{k+1} = J_{k+1} J_k$ (since the elements $J_1, J_2, \ldots, J_n$ commute) and $s_k J_{k+1} = J_k s_k + 1$ (by Lemma 4.1). Therefore, Lemma 3.3 (applied to $R = \mathbf{k}\left[S_n\right]$ and $x = J_k$ and $y = J_{k+1}$ and $s = s_k$) yields

$$p_k\left(J_{k+1}\right) \cdot p_{k+1}\left(J_{k+1}\right) = 0$$

and

$$p_{k+1}^2\left(J_{k+1}\right) = 0. \tag{25}$$

On the other hand, Lemma 2.2 (applied to $k+1$ instead of k) yields $S\left(J_{k+1}\right) = J_{k+1}$. But S is a $\mathbf{k}$-algebra anti-morphism (by Theorem 2.1 **(a)**); thus, each $a \in \mathbf{k}\left[S_n\right]$ and each polynomial $f \in \mathbb{Z}\left[t\right]$ satisfy $S\left(f\left(a\right)\right) = f\left(S\left(a\right)\right)$ (since $\mathbf{k}$-algebra anti-morphisms respect polynomials, just like $\mathbf{k}$-algebra morphisms do). Applying this to $a = J_{k+1}$ and $f = p_{k+1}$, we obtain

$$S\left(p_{k+1}\left(J_{k+1}\right)\right) = p_{k+1}\left(\underbrace{S\left(J_{k+1}\right)}_{=J_{k+1}}\right) = p_{k+1}\left(J_{k+1}\right).$$

Therefore,

$$\begin{aligned} \underbrace{S\left(p_{k+1}\left(J_{k+1}\right)\right)}_{=p_{k+1}\left(J_{k+1}\right)} \cdot p_{k+1}\left(J_{k+1}\right) &= p_{k+1}\left(J_{k+1}\right) \cdot p_{k+1}\left(J_{k+1}\right) = \left(p_{k+1}\left(J_{k+1}\right)\right)^2 \\ &= p_{k+1}^2\left(J_{k+1}\right) = 0 \qquad \left(\text{by (25)}\right). \end{aligned}$$

Therefore, Lemma 2.3 (applied to $a = p_{k+1}\left(J_{k+1}\right)$) yields

$$p_{k+1}\left(J_{k+1}\right) = 0$$

(since the ring $\mathbf{k} = \mathbb{Z}$ is formally real). This completes the induction step. Thus, (24) is proved by induction.

As we said above, this completes the proof of Theorem 1.1 for $\mathbf{k} = \mathbb{Z}$. In other words, we have now proved that for each $k \in [n]$, we have

$$\prod_{i=-k+1}^{k-1} \left(J_k - i\right) = 0 \qquad \text{in } \mathbb{Z}\left[S_n\right]. \tag{26}$$

Now let us return to the general case, in which $\mathbf{k}$ is an arbitrary commutative ring. There is a canonical ring morphism $g : \mathbb{Z} \to \mathbf{k}$, and this morphism g induces a base change morphism

$$g_* : \mathbb{Z}[S_n] \to \mathbf{k}[S_n]$$

between the group rings of S_n over $\mathbb{Z}$ and $\mathbf{k}$ (see [Grinbe25, Definition 3.12.22]). The latter morphism g_* clearly sends each Young–Jucys–Murphy element J_k of $\mathbb{Z}[S_n]$ to the corresponding element J_k of $\mathbf{k}[S_n]$ (since it preserves each transposition $t_{i,k}$). Thus, applying this morphism g_* to both sides of the equality (26), we obtain

$$\prod_{i=-k+1}^{k-1} (J_k - i) = 0 \qquad \text{in } \mathbf{k}[S_n].$$

This proves Theorem 1.1, now in the general case. □

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