Math 530 Spring 2022, Lecture 7: multigraphs

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

1. Multigraphs

1.1. Definitions

So far, we have been working with simple graphs. We shall now introduce several other kinds of graphs, starting with the **multigraphs**.

Definition 1.1.1. Let V be a set. Then, $\mathcal{P}_{1,2}(V)$ shall mean the set of all 1-element or 2-element subsets of V. In other words,

$$\begin{aligned}\mathcal{P}_{1,2}(V) &:= \{S \subseteq V \mid |S| \in \{1,2\}\} \\ &= \{\{u,v\} \mid u,v \in V \text{ not necessarily distinct}\}.\end{aligned}$$

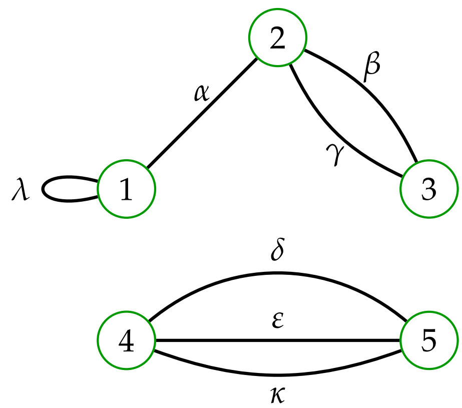
For instance,

$$\mathcal{P}_{1,2}(\{1,2,3\}) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}.$$

We can now define multigraphs:

Definition 1.1.2. A **multigraph** is a triple (V,E,φ), where V and E are two finite sets, and $\varphi : E \to \mathcal{P}_{1,2}(V)$ is a map.

Example 1.1.3. Here is a multigraph:



Formally speaking, this multigraph is the triple (V,E,φ), where

$$V = \{1,2,3,4,5\}, \qquad E = \{\alpha,\beta,\gamma,\delta,\varepsilon,\kappa,\lambda\},$$

and where $\varphi : E \to \mathcal{P}_{1,2}(V)$ is the map that sends $\alpha,\beta,\gamma,\delta,\varepsilon,\kappa,\lambda$ to $\{1,2\},\{2,3\},\{2,3\},\{4,5\},\{4,5\},\{4,5\},\{1\}$, respectively. (Of course, you can write $\{1\}$ as $\{1,1\}$.)

This suggests the following terminology (most of which is a calque of our previously defined terminology for simple graphs):

Definition 1.1.4. Let $G = (V, E, \varphi)$ be a multigraph. Then:

(a) The elements of V are called the **vertices** of G.

The set V is called the **vertex set** of G, and is denoted $\operatorname{V}(G)$.

(b) The elements of E are called the **edges** of G.

The set E is called the **edge set** of G, and is denoted $\operatorname{E}(G)$.

(c) If e is an edge of G, then the elements of $\varphi(e)$ are called the **endpoints** of e.

(d) We say that an edge e **contains** a vertex v if $v \in \varphi(e)$ (in other words, if v is an endpoint of e).

(e) Two vertices u and v are said to be **adjacent** if there exists an edge $e \in E$ whose endpoints are u and v.

(f) Two edges e and f are said to be **parallel** if $\varphi(e) = \varphi(f)$. (In the above example, any two of the edges $\delta, \varepsilon, \kappa$ are parallel.)

(g) We say that G has **no parallel edges** if G has no two distinct edges that are parallel.

(h) An edge e is called a **loop** (or **self-loop**) if $\varphi(e)$ is a 1-element set (i.e., if e has only one endpoint). (In Example 1.1.3, the edge λ is a loop.)

(i) We say that G is **loopless** if G has no loops (among its edges).

(j) The **degree** $\deg v$ (also written $\deg_G v$) of a vertex v of G is defined to be the number of edges that contain v, where loops are counted twice. In other words,

$$\begin{aligned}\deg v &= \deg_G v \\ &:= \underbrace{\left|\{e \in E \mid v \in \varphi(e)\}\right|}_{\substack{\text{this counts all edges} \\ \text{that contain } v}} + \underbrace{\left|\{e \in E \mid \varphi(e) = \{v\}\}\right|}_{\substack{\text{this counts all loops} \\ \text{that contain } v \text{ once again}}}.\end{aligned}$$

(Note that, unlike in the case of a simple graph, $\deg v$ is **not** the number of neighbors of v, unless it happens that v is not contained in any loops or parallel edges.)

(For example, in Example 1.1.3, we have $\deg 1 = 3$ and $\deg 2 = 3$ and $\deg 3 = 2$ and $\deg 4 = 3$ and $\deg 5 = 3$.)

(k) A **walk** in G means a list of the form

$$(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k) \qquad (\text{with } k \geq 0),$$

where $v_0, v_1, \ldots, v_k$ are vertices of G, where $e_1, e_2, \ldots, e_k$ are edges of G, and where each $i \in \{1, 2, \ldots, k\}$ satisfies

$$\varphi(e_i) = \{v_{i-1}, v_i\}$$

(that is, the endpoints of each edge e_i are v_{i-1} and v_i). Note that we have to record both the vertices **and** the edges in our walk, since we want the walk to "know" which edges it traverses. (For instance, in Example 1.1.3, the two walks $(1, \alpha, 2, \beta, 3)$ and $(1, \alpha, 2, \gamma, 3)$ are distinct.)

The **vertices** of a walk $(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ are $v_0, v_1, \ldots, v_k$; the **edges** of this walk are $e_1, e_2, \ldots, e_k$. This walk is said to **start** at v_0 and **end** at v_k; it is also said to be a **walk from** v_0 **to** v_k. Its **starting point** is v_0, and its **ending point** is v_k. Its **length** is k.

(l) A **path** means a walk whose vertices are distinct.

(m) The notions of "**path-connected**" and "**connected**" and "**component**" are defined exactly as for simple graphs. The symbol $\simeq_G$ still means "path-connected".

(n) A **closed walk** (or **circuit**) means a walk $(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ with $v_k = v_0$.

(o) A **cycle** means a closed walk $(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ such that

- the vertices $v_0, v_1, \ldots, v_{k-1}$ are distinct;
- the edges $e_1, e_2, \ldots, e_k$ are distinct;
- we have $k \geq 1$.

(Note that we are not requiring $k \geq 3$ any more, as we did for simple graphs. Thus, in Example 1.1.3, both $(2, \beta, 3, \gamma, 2)$ and $(1, \lambda, 1)$ are cycles, but $(2, \beta, 3, \beta, 2)$ is not. The purpose of the "$k \geq 3$" requirement for cycles in simple graphs was to disallow closed walks such as $(2, \beta, 3, \beta, 2)$ from being cycles; but they are now excluded by the "the edges $e_1, e_2, \ldots, e_k$ are distinct" condition.)

(p) Hamiltonian paths and cycles are defined as for simple graphs.

(q) We draw a multigraph by drawing each vertex as a point, each edge as a curve, and labeling both the vertices and the edges (or not, if we don't care about what they are). An example of such a drawing appeared in Example 1.1.3.

So there are two differences between simple graphs and multigraphs:

1. A multigraph can have loops, whereas a simple graph cannot.

2. In a simple graph, an edge e **is** a set of two vertices, whereas in a multigraph, an edge e **has** a set of two vertices (possibly two equal ones, if e is a loop) assigned to it by the map φ. This not only allows for parallel edges, but also lets us store some information in the "identities" of the edges.

Nevertheless, the two notions have much in common; thus, they are both called "graphs":

Convention 1.1.5. The word "**graph**" means either "simple graph" or "multigraph". The precise meaning should usually be understood from the context. (I will try not to use it when it could cause confusion.)

Fortunately, simple graphs and multigraphs have many properties in common, and often it is not hard to derive a result about multigraphs from the analogous result about simple graphs or vice versa. We will soon explore how some of the properties we have seen in the previous lectures can be adapted to multigraphs. First, however, let us explain how to convert multigraphs into simple graphs and vice versa.

1.2. Conversions

We can turn each multigraph into a simple graph, but at a cost of losing some information:

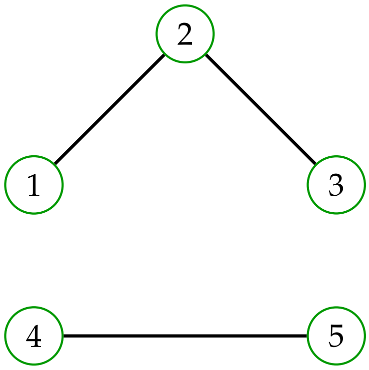
Definition 1.2.1. Let $G = (V, E, \varphi)$ be a multigraph. Then, the **underlying simple graph** G^{simp} of G means the simple graph

$$(V, \ \{\varphi(e) \ \mid \ e \in E \text{ is not a loop}\}).$$

In other words, it is the simple graph with vertex set V in which two distinct vertices u and v are adjacent if and only if u and v are adjacent in G. Thus, G^{simp} is obtained from G by removing loops and "collapsing" parallel edges to a single edge.

For example, the underlying simple graph of the multigraph G in Example

1.1.3 would be



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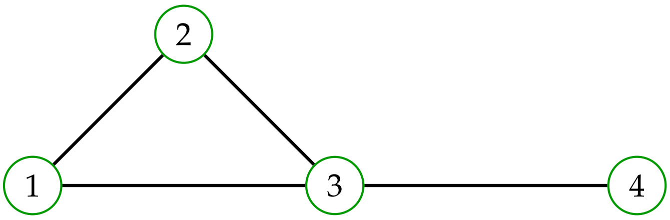
Conversely, each simple graph can be viewed as a multigraph:

Definition 1.2.2. Let $G = (V, E)$ be a simple graph. Then, the **corresponding multigraph** G^{mult} is defined to be the multigraph

$$(V, E, \iota),$$

where $\iota : E \to \mathcal{P}_{1,2}(V)$ is the map sending each $e \in E$ to e itself.

Example 1.2.3. If

$G =$ 

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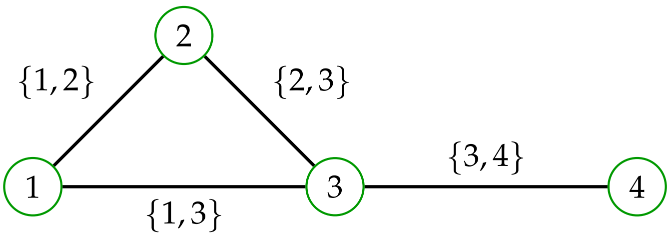
then

$G^{\text{mult}} =$ 

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As we said, the "underlying simple graph" construction $G \mapsto G^{\text{simp}}$ destroys information, so it is irreversible. This being said, the two constructions $G \mapsto G^{\text{simp}}$ and $G \mapsto G^{\text{mult}}$ come fairly close to undoing one another:[1]

[1]In the following proposition, we will use the notion of an "isomorphism of multigraphs". A rigorous definition of this notion is given in Definition 1.3.8 further below (but it is more or less what you would expect: it is a way to relabel the vertices and the edges of one multigraph to obtain those of another).

Proposition 1.2.4.

(a) If G is a simple graph, then $\left(G^{\text{mult}}\right)^{\text{simp}} = G$.

(b) If G is a loopless multigraph that has no parallel edges, then $\left(G^{\text{simp}}\right)^{\text{mult}} \cong G$. (This is just an isomorphism, not an equality, since the "identities" of the edges of G have been forgotten in G^{simp} and cannot be recovered.)

(c) If G is a multigraph that has loops or (distinct) parallel edges, then the multigraph $\left(G^{\text{simp}}\right)^{\text{mult}}$ has fewer edges than G and thus is not isomorphic to G.

Proof. A matter of understanding the definitions. □

We will often identify a simple graph G with the corresponding multigraph G^{mult}. This may be dangerous, because we have defined notions such as adjacency, walks, paths, cycles, etc. both for simple graphs and for multigraphs; thus, when we identify a simple graph G with the multigraph G^{mult}, we are potentially inviting ambiguity (for example, does "cycle of G" mean a cycle of the simple graph G or of the multigraph G^{mult} ?). Fortunately, this ambiguity is harmless, because whenever G is a simple graph, any of the notions we defined for G is equivalent to the corresponding notion for the multigraph G^{mult}. For example, for the notions of a cycle, we have the following:

Proposition 1.2.5. Let G be a simple graph. Then:

(a) If $(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ is a cycle of the multigraph G^{mult}, then $(v_0, v_1, \ldots, v_k)$ is a cycle of the simple graph G.

(b) Conversely, if $(v_0, v_1, \ldots, v_k)$ is a cycle of the simple graph G, then $(v_0, \{v_0, v_1\}, v_1, \{v_1, v_2\}, v_2, \ldots, v_{k-1}, \{v_{k-1}, v_k\}, v_k)$ is a cycle of the multigraph G^{mult}.

Proof. This is not completely obvious, since our definitions of a cycle of a simple graph and of a cycle of a multigraph were somewhat different. The proof boils down to checking the following two statements:

1. If $(v_0, v_1, \ldots, v_k)$ is a cycle of the simple graph G, then its edges $\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}$ are distinct.
2. If $(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ is a cycle of the multigraph G^{mult}, then $k \geq 3$.

Checking statement 2 is easy (we cannot have $k = 1$ since G^{mult} has no loops, and we cannot have $k = 2$ since this would lead to $e_1 = e_2$). Statement 1 is also clear, since the distinctness of the k vertices $v_0, v_1, \ldots, v_{k-1}$ forces the 2-element sets formed from these k vertices to also be distinct (and since the edges $\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\} = \{v_{k-1}, v_0\}$ are such 2-element sets). □

For all other notions discussed above, it is even more obvious that there is no ambiguity.

1.3. Generalizing from simple graphs to multigraphs

Now, as promised, we shall revisit the results of the first 6 lectures, and see which of them also hold for multigraphs instead of simple graphs.

1.3.1. Lecture 1

In Lecture 1, we proved the following:

Proposition 1.3.1. Let G be a simple graph with $|\text{V}(G)| \geq 6$ (that is, G has at least 6 vertices). Then, at least one of the following two statements holds:

- *Statement 1:* There exist three distinct vertices a, b and c of G such that ab, bc and ca are edges of G.
- *Statement 2:* There exist three distinct vertices a, b and c of G such that none of ab, bc and ca is an edge of G.

This is still true for multigraphs[2], because replacing a multigraph G by the underlying simple graph G^{simp} does not change the meaning of the statement.

1.3.2. Lecture 2

In Lecture 2, we defined the degree of a vertex v in a simple graph $G = (V, E)$ by

$$\begin{aligned}
\deg v &:= (\text{the number of edges } e \in E \text{ that contain } v) \\
&= (\text{the number of neighbors of } v) \\
&= |\{u \in V \mid uv \in E\}| \\
&= |\{e \in E \mid v \in e\}| .
\end{aligned}$$

These equalities **no longer hold** when G is a multigraph. Parallel edges correspond to the same neighbor, so the number of neighbors of v is only a lower bound on $\deg v$.

[2]Of course, we should understand it appropriately: i.e., we should read "ab is an edge" as "there is an edge with endpoints a and b".

Here is another proposition we proved in Lecture 2:

Proposition 1.3.2. Let G be a simple graph with n vertices. Let v be a vertex of G. Then,

$$\deg v \in \{0, 1, \ldots, n-1\}.$$

This proposition, too, **no longer holds** for multigraphs, because you can have arbitrarily many edges in a multigraph with just 1 or 2 vertices. (You can even have parallel loops!)

We also proved the following:

Proposition 1.3.3 (Euler 1736)**.** Let G be a simple graph. Then, the sum of the degrees of all vertices of G equals twice the number of edges of G. In other words,

$$\sum_{v \in \operatorname{V}(G)} \deg v = 2 \cdot |\operatorname{E}(G)|.$$

Is this true for multigraphs? Yes, because we have said that loops should count twice in the definition of the degree. The proof needs some tweaking, though. Let me give a slightly different proof; but first, let me state the claim for multigraphs as a proposition of its own:

Proposition 1.3.4 (Euler 1736 for multigraphs)**.** Let G be a multigraph. Then, the sum of the degrees of all vertices of G equals twice the number of edges of G. In other words,

$$\sum_{v \in \operatorname{V}(G)} \deg v = 2 \cdot |\operatorname{E}(G)|.$$

Proof. Write G as $G = (V, E, \varphi)$; thus, $\operatorname{V}(G) = V$ and $\operatorname{E}(G) = E$.

For each edge e, let us (arbitrarily) choose one endpoint of e and denote it by $\alpha(e)$. The other endpoint will be called $\beta(e)$. If e is a loop, then we set $\beta(e) = \alpha(e)$. Then, for each vertex v, we have

$$\begin{aligned} \deg v = & \left(\text{the number of } e \in E \text{ such that } v = \alpha(e)\right) \\ & + \left(\text{the number of } e \in E \text{ such that } v = \beta(e)\right) \end{aligned}$$

(note how loops get counted twice on the right hand side, because if $e \in E$ is a loop, then v is both $\alpha(e)$ and $\beta(e)$ at the same time). Summing up this equality over all $v \in V$, we obtain

$$\begin{aligned} \sum_{v \in V} \deg v = & \sum_{v \in V} \left(\text{the number of } e \in E \text{ such that } v = \alpha(e)\right) \\ & + \sum_{v \in V} \left(\text{the number of } e \in E \text{ such that } v = \beta(e)\right). \end{aligned}$$

However,

$$\sum_{v\in V}(\text{the number of } e\in E \text{ such that } v=\alpha(e)) = |E|,$$

since each edge $e \in E$ is counted in exactly one addend of this sum. Similarly,

$$\sum_{v\in V}(\text{the number of } e\in E \text{ such that } v=\beta(e)) = |E|.$$

Thus, the above equality becomes

$$\begin{aligned}\sum_{v\in V}\deg v &= \underbrace{\sum_{v\in V}(\text{the number of } e\in E \text{ such that } v=\alpha(e))}_{=|E|} \\ &\qquad + \underbrace{\sum_{v\in V}(\text{the number of } e\in E \text{ such that } v=\beta(e))}_{=|E|} \\ &= |E|+|E| = 2\cdot|E|.\end{aligned}$$

This proves Proposition 1.3.4. □

This is a good motivation for counting loops twice in the definition of a degree.

Here is another fact we saw in Lecture 2:

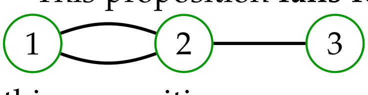
Corollary 1.3.5 (handshake lemma)**.** Let G be a simple graph. Then, the number of vertices v of G whose degree $\deg v$ is odd is even.

This **still holds for multigraphs**, and it follows from Proposition 1.3.4 in the same way as for simple graphs.

Here is another result from Lecture 2:

Proposition 1.3.6. Let G be a simple graph with at least two vertices. Then, there exist two distinct vertices v and w of G that have the same degree.

This proposition **fails for multigraphs**. For example, the multigraph



has three vertices with degrees $1, 2, 3$. Fortunately, this proposition was more of a curiosity than a useful fact.

What about Mantel's theorem?

Theorem 1.3.7 (Mantel's theorem)**.** Let G be a simple graph with n vertices and e edges. Assume that $e > n^2/4$. Then, G has a triangle (i.e., three distinct vertices that are pairwise adjacent).

This theorem **fails for multigraphs**, because we can join two vertices with a lot of parallel edges and thus satisfy $e > n^2/4$ for stupid reasons without ever creating a triangle. Thus, Turan's theorem also fails for multigraphs.

Graph isomorphy (and isomorphisms) can still be defined for multigraphs, but the definition is not the same as for simple graphs. Graph isomorphisms can no longer be defined merely as bijections between the vertex sets, since we also need to specify what they do to the edges. Instead, we define them as follows:

Definition 1.3.8. Let $G = (V, E, \varphi)$ and $H = (W, F, \psi)$ be two multigraphs.

(a) A **graph isomorphism** (or **isomorphism**) from G to H means a **pair** (α, β) of bijections

$$\alpha : V \to W \qquad \text{and} \qquad \beta : E \to F$$

with the property that if $e \in E$, then the endpoints of $\beta(e)$ are the images under α of the endpoints of e. (This property can also be restated as a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\beta} & F \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathcal{P}_{1,2}(V) & \xrightarrow[\mathcal{P}(\alpha)]{} & \mathcal{P}_{1,2}(W) \end{array} ,$$

where $\mathcal{P}(\alpha)$ is the map from $\mathcal{P}_{1,2}(V)$ to $\mathcal{P}_{1,2}(W)$ that sends each subset $\{u, v\} \in \mathcal{P}_{1,2}(V)$ to $\{\alpha(u), \alpha(v)\} \in \mathcal{P}_{1,2}(W)$. If you are used to category theory, this restatement may look more natural to you.)

(b) We say that G and H are **isomorphic** (this is written $G \cong H$) if there exists a graph isomorphism from G to H.

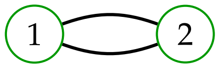
Again, isomorphy of multigraphs is an equivalence relation.

1.3.3. Lecture 3

In Lecture 3, we defined the complete graphs K_n, the path graphs P_n and the cycle graphs C_n as simple graphs. Thus, all of them can be viewed as multigraphs if one so desires (since each simple graph G gives rise to a multigraph G^{mult}).

However, using multigraphs, we can extend our definition of n-th cycle graphs C_n to the case $n = 1$ and also tweak it in the case $n = 2$ to make it more natural. We do this as follows:

Definition 1.3.9. We modify the definition of cycle graphs as follows:

- **(a)** We **redefine** the 2-nd cycle graph C_2 to be the multigraph with two vertices 1 and 2 and two parallel edges with endpoints 1 and 2. (We don't care what the edges are, only that there are two of them and each has endpoints 1 and 2.) Thus, it looks as follows: 

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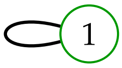
- **(b)** We define the 1-st cycle graph C_1 to be the multigraph with one vertex 1 and one edge (which is necessarily a loop). Thus, it looks as follows: 

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This has the effect that the n-th cycle graph C_n has exactly n edges for each $n \geq 1$ (rather than having 1 edge for $n = 2$, as it did back when it was a simple graph).

Next, let us define submultigraphs of a multigraph:

Definition 1.3.10. A **submultigraph** of a multigraph $G = (V, E, \varphi)$ is a multigraph of the form (W, F, ψ), where $W \subseteq V$ and $F \subseteq E$ and $\psi = \varphi\mid_F$.

With these definitions, we can now identify cycles in a multigraph with subgraphs isomorphic to a cycle graph: A cycle of length n in a multigraph G is "the same as" a submultigraph of G isomorphic to C_n. (We leave the details to the reader.)