

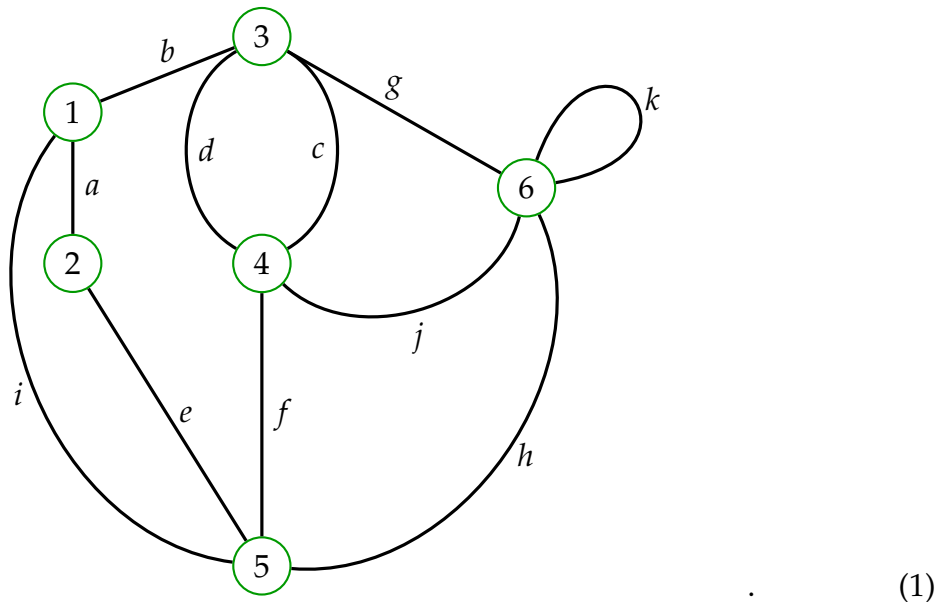
## 6. Math 235 Fall 2021, Worksheet 6: Graphs and some of their uses

On this worksheet, we will introduce some basic concepts from graph theory and see how they can be applied. This is not a replacement for an actual course on graph theory (such as the one I will give next Spring), nor for a textbook (such as most of the sources referenced in the bibliography of this worksheet).

As before,  $\mathbb{N}$  means the set  $\{0, 1, 2, \dots\}$ .

### 6.1. Graph basics

Informally, a *graph* is a collection of finitely many “vertices” (aka “nodes”) and finitely many “edges”. Each edge connects two nodes (which can be identical). Here is an example of a graph with 6 nodes and 11 edges, drawn in the obvious way (each node represented by a little circle, and each edge represented by a curve joining the two nodes that it connects):



In this example, the nodes are called 1, 2, 3, 4, 5, 6, and the edges are called  $a, b, c, \dots, j, k$ ; for example, the edge  $f$  connects nodes 4 and 5 (whereas the edge  $k$  connects node 6 with itself).

Such pictures can be convenient (at least when the graph is small; in more complex situations, some edges will have to cross). However, let us also give a formal definition of a graph:

**Definition 6.1.1. (a)** If  $V$  is any set, then  $\mathcal{P}_{1,2}(V)$  shall mean the set of all 1-element and all 2-element subsets of  $V$ . That is,

$$\mathcal{P}_{1,2}(V) := \{S \subseteq V \mid |S| \in \{1, 2\}\} = \{\{u, v\} \mid u, v \in V \text{ not necessarily distinct}\}.$$

(b) A graph is a triple  $(V, E, \varphi)$ , where  $V$  and  $E$  are two finite sets, and where  $\varphi : E \rightarrow \mathcal{P}_{1,2}(V)$  is any map.

(c) If  $G = (V, E, \varphi)$  is a graph, then we use the following terminology:

- The elements of  $V$  are called the *vertices* (or *nodes*) of  $G$ . The set  $V$  is called the *vertex set* of  $G$ .
- The elements of  $E$  are called the *edges* of  $G$ . The set  $E$  is called the *edge set* of  $G$ .
- If  $e \in E$  is an edge of  $G$ , then the elements of  $\varphi(e)$  are called the *endpoints* of  $e$ . Thus, each edge  $e$  has exactly 1 or 2 endpoints (since  $\varphi(e) \in \mathcal{P}_{1,2}(V)$ ). An edge  $e \in E$  that has only one endpoint will be called a *self-loop*.
- We say that two vertices  $u$  and  $v$  of  $V$  are *adjacent* if there exists an edge  $e \in E$  whose endpoints are  $u$  and  $v$ . (In other words,  $u$  and  $v$  are adjacent if  $\{u, v\} = \varphi(e)$  for some  $e \in E$ .) In this case, we say that the edge  $e$  *joins*  $u$  and  $v$ .
- A *neighbor* of a vertex  $u \in V$  shall mean a vertex  $v \in V$  that is adjacent to  $u$ .
- If  $v \in V$  is a vertex of  $G$ , and  $e \in E$  is an edge of  $G$ , then we say that  $e$  *contains*  $v$  if we have  $v \in \varphi(e)$  (that is, if  $v$  is an endpoint of  $e$ ).
- The *degree* of a vertex  $v \in V$  is defined to be the number of edges  $e \in E$  that contain  $v$ , where we agree that self-loops count twice (i.e., each self-loop that contains  $v$  counts as **two** edges containing  $v$ ). This number is denoted by  $\deg v$ .

Furthermore, we draw the graph  $G$  as above: Each node is represented by a little circle, and each edge is represented by a curve that joins the little circles that represent its endpoints.

**Example 6.1.2.** The graph visualized in (1) is the triple  $(V, E, \varphi)$ , where

$$\begin{aligned} V &= \{1, 2, 3, 4, 5, 6\}, & E &= \{a, b, c, \dots, j, k\}, \\ \varphi(a) &= \{1, 2\}, & \varphi(b) &= \{1, 3\}, & \varphi(c) &= \{3, 4\}, & \varphi(d) &= \{3, 4\}, \\ \varphi(e) &= \{2, 5\}, & \varphi(f) &= \{4, 5\}, & \varphi(g) &= \{3, 6\}, & \varphi(h) &= \{5, 6\}, \\ \varphi(i) &= \{1, 5\}, & \varphi(j) &= \{4, 6\}, & \varphi(k) &= \{6\}. \end{aligned}$$

In this graph,

- the vertices are  $1, 2, 3, 4, 5, 6$ ;
- the edges are  $a, b, c, \dots, j, k$ ;

- the endpoints of the edge  $g$  are 3 and 6, while the only endpoint of the edge  $k$  is 6;
- the edge  $k$  is a self-loop (and is the only edge of this graph that is a self-loop);
- the vertices 3 and 6 are adjacent (since they are the endpoints of the edge  $g$ ), while the vertices 1 and 4 are not;
- the neighbors of the vertex 3 are 1, 4 and 6;
- the edges that contain the vertex 3 are  $b, c, d, g$ ;
- the degrees of the vertices are

$$\begin{array}{llll} \deg 1 = 3, & \deg 2 = 2, & \deg 3 = 4, & \deg 4 = 4, \\ \deg 5 = 4, & \deg 6 = 5. & & \end{array}$$

Note that  $\deg 6 = 5$  is because the self-loop  $k$  counts twice; otherwise,  $\deg 6$  would be 4.

We note that our Definition 6.1.1 is not the only notion of “graph” used in mathematics. Other variants are simple graphs (which we will introduce below) and directed graphs (which we will not consider on this worksheet, but which are no less interesting<sup>1</sup>). To avoid ambiguity, we can refer to our notion of “graph” defined above as “*undirected multigraph*”. (The prefix “multi” here signals that a single pair of vertices can be joined by multiple edges, in contrast to the notion of a “simple graph” that we will define below.)

The claims of the following exercise are occasionally known as the *handshaking lemma*, and have a number of surprising applications ([Gijswi16]).

**Exercise 6.1.1.** Let  $G = (V, E, \varphi)$  be a graph.

- (a) Prove that the sum of the degrees of all vertices of  $G$  equals  $2 \cdot |E|$ .
- (b) Prove that the number of vertices of  $G$  that have odd degree is even.

*Solution idea.* (a) For each edge  $e \in E$ , let us arbitrarily choose one of the endpoints of  $e$  and denote it by  $\alpha(e)$ . The other endpoint of  $e$  will be denoted  $\beta(e)$ . (If  $e$  is a self-loop, then  $\alpha(e) = \beta(e)$ , since  $e$  has only one endpoint.)

For each  $v \in V$ , we have

$$\begin{aligned} \deg v &= (\text{the number of all } e \in E \text{ that contain } v, \\ &\quad \text{where self-loops are counted twice}) \\ &= (\text{the number of all } e \in E \text{ such that } v = \alpha(e)) \\ &\quad + (\text{the number of all } e \in E \text{ such that } v = \beta(e)). \end{aligned}$$

<sup>1</sup>See [Suksom20] for a nice selection of olympiad-level applications of directed graphs.

Summing this equality over all  $v \in V$ , we obtain

$$\begin{aligned}
 \sum_{v \in V} \deg v &= \sum_{v \in V} \underbrace{(\text{the number of all } e \in E \text{ such that } v = \alpha(e))}_{\substack{=(\text{the number of all } e \in E) \\ (\text{since each } e \in E \text{ is counted exactly once in this sum,} \\ \text{namely in the addend for } v = \alpha(e))}} \\
 &\quad + \sum_{v \in V} \underbrace{(\text{the number of all } e \in E \text{ such that } v = \beta(e))}_{\substack{=(\text{the number of all } e \in E) \\ (\text{since each } e \in E \text{ is counted exactly once in this sum,} \\ \text{namely in the addend for } v = \beta(e))}} \\
 &= (\text{the number of all } e \in E) + (\text{the number of all } e \in E) \\
 &= 2 \cdot \underbrace{(\text{the number of all } e \in E)}_{=|E|} = 2 \cdot |E|.
 \end{aligned}$$

In other words, the sum of the degrees of all vertices of  $G$  equals  $2 \cdot |E|$ . This solves Exercise 6.1.1 (a).

(b) Exercise 6.1.1 (a) shows that the sum of the degrees of all vertices of  $G$  equals  $2 \cdot |E|$ . Hence, this sum is even (since  $|E|$  is an integer). However, if a sum of integers is even, then the number of odd addends in this sum must be even<sup>2</sup>. Thus, the number of odd addends in the sum of the degrees of all vertices of  $G$  must be even. In other words, the number of vertices of  $G$  that have odd degree is even. This solves Exercise 6.1.1 (b).  $\square$

Exercise 6.1.1 (b) is often stated in a form like “in a meeting of  $n$  people, show that the number of people who have shaken an odd number of hands is always even”. (Generally, speaking of handshakes or friendships is a fairly transparent way to talk about graphs without saying “graph”.)

## 6.2. Simple graphs

A feature of our above definition of graphs (Definition 6.1.1) is that two vertices of a graph can be connected by several edges. (For example, the vertices 3 and 4 in (1) are connected by both  $c$  and  $d$ .) Sometimes this is undesirable, and so we want to have a notion of graphs that disallows it. The most convenient such notion is that of a *simple graph*, which also gets rid of the map  $\varphi$ :

**Definition 6.2.1.** A *simple graph* is a pair  $(V, E)$ , where  $V$  is a finite set, and where  $E$  is a subset of the set  $\mathcal{P}_2(V)$ . Here,  $\mathcal{P}_2(V)$  means the set of all 2-element subsets of  $V$ . (That is,  $\mathcal{P}_2(V) := \{S \subseteq V \mid |S| = 2\}$ .)

A simple graph  $(V, E)$  will be identified with the graph  $(V, E, \varphi)$ , where  $\varphi : E \rightarrow \mathcal{P}_{1,2}(V)$  is the canonical inclusion map (i.e., the map that sends each  $e \in E$  to itself).

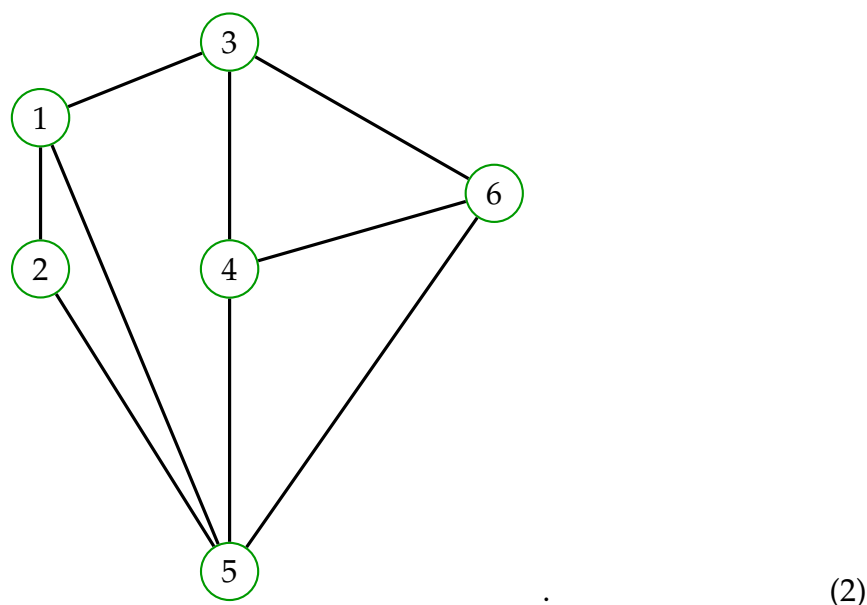
<sup>2</sup>This is because a sum of an odd number of odd integers is always odd.

Thus, there are two differences between a simple graph and a graph:

- An edge in a simple graph  $(V, E)$  is just a set of two vertices; its endpoints are its two elements. Meanwhile, an edge in a graph  $(V, E, \varphi)$  can be anything; its endpoints are the elements of its image under  $\varphi$ . (Thus, an edge in a simple graph “knows” its two endpoints, whereas an edge in a graph “outsources” this knowledge to the map  $\varphi$ .)
- Graphs can have self-loops, whereas simple graphs cannot (since an edge in a simple graph must be a 2-element set, not a 1-element set).

As a consequence of the first difference, a simple graph cannot have two distinct edges with the same two endpoints  $u$  and  $v$ ; in fact, these two edges would both have to be the set  $\{u, v\}$ , so they would not be distinct.

Here is an example of a simple graph with 6 nodes and 9 edges, drawn in the obvious way (each node represented by a little circle, and each edge represented by a curve joining the two nodes that it connects):



(This simple graph is what remains of the graph (1) if one removes the self-loop and replaces each edge  $e$  by  $\varphi(e)$ .)

**Convention 6.2.2.** We shall use the shorthand notation  $uv$  for a two-element set  $\{u, v\}$  when no ambiguities are possible. Thus, in particular, an edge of a simple graph will be written as  $uv$  if its endpoints are  $u$  and  $v$ .

For example, the edges of the simple graph shown in (2) are 12, 13, 15, 25, 34, 36, 45, 46, 56. It is clear that a simple graph with  $n$  vertices can have no more than  $\binom{n}{2}$  edges (whereas a graph can have any finite number of edges).

If  $G = (V, E)$  is a simple graph, then a *triangle* of  $G$  is defined to be a set  $\{u, v, w\}$  of three distinct vertices  $u, v, w$  of  $G$  such that  $uv$ ,  $vw$  and  $uw$  are edges of  $G$ .

**Exercise 6.2.1.** Let  $G = (V, E)$  be a simple graph that has no triangles. Let  $n = |V|$ . Prove that  $|E| \leq n^2/4$ .

*Solution idea.* This is known as *Mantel's theorem*. One of many proofs, here is one of the simplest:

Note that the assumption  $n = |V|$  says that our graph  $G$  has exactly  $n$  vertices.

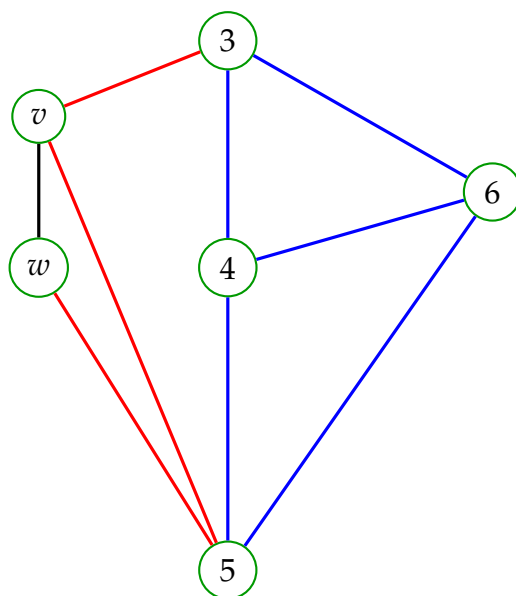
We shall solve the exercise by strong induction on  $n$ . Thus, we assume (as the induction hypothesis) that Exercise 6.2.1 is already solved for all graphs with fewer than  $n$  vertices. We must now prove it for our graph  $G = (V, E)$  with its  $n$  vertices.

We must prove that  $|E| \leq n^2/4$ . If  $E = \emptyset$ , then this is obvious (because in this case, we have  $|E| = 0 \leq n^2/4$ ). Hence, we WLOG assume that  $E \neq \emptyset$ . In other words, our graph  $G$  has at least one edge. Pick any such edge, and let  $v$  and  $w$  be its endpoints. Thus,  $vw$  is an edge of  $G$ , and therefore  $v \neq w$  (since  $G$  is a simple graph).

Let us now color each edge of  $G$  with one of three colors, by the following rule:

- The edge  $vw$  will be colored black.
- Each edge that contains exactly one of  $v$  and  $w$  will be colored red.
- Each edge that contains none of  $v$  and  $w$  will be colored blue.

For example, here is how this coloring will look like if  $G$  is the simple graph from (2) and if  $v = 1$  and  $w = 2$ :



(For those reading this in black and white: The edge  $vw$  is black; the edges  $v3, v5, w5$  are red; the remaining edges are blue.)

We now count the edges of each color (not just in the above example, but in general):

- There is exactly 1 black edge – namely, the edge  $vw$ .
- We claim that there are at most  $n - 2$  red edges. Why? Our graph  $G$  has  $n - 2$  vertices distinct from  $v$  and  $w$  (since  $v \neq w$ ), and each of these  $n - 2$  vertices is adjacent to at most one of  $v$  and  $w$  (because if some vertex  $u$  was adjacent to both  $v$  and  $w$ , then  $\{u, v, w\}$  would be a triangle, which would contradict our assumption that  $G$  has no triangles). Thus, each of these  $n - 2$  vertices is contained in at most one red edge. Hence,  $G$  has at most  $n - 2$  red edges.
- Finally, we claim that there are at most  $(n - 2)^2 / 4$  blue edges. Why? Let  $G'$  be the simple graph obtained from  $G$  by removing the two vertices  $v$  and  $w$  along with all black and all red edges (so that only the edges that contain neither  $v$  nor  $w$  remain). Formally speaking, this simple graph  $G'$  is defined to be the pair  $(V', E')$ , where

$$V' = V \setminus \{v, w\} \quad \text{and} \quad E' = \{\text{blue edges}\}.$$

This simple graph  $G'$  has  $n - 2$  vertices, thus fewer than  $n$  vertices. Moreover, it has no triangles (since any triangle of  $G'$  would be a triangle of  $G$ , but  $G$  has no triangles). Thus, by the induction hypothesis, we can apply Exercise 6.2.1 to  $G'$ ,  $V'$ ,  $E'$  and  $n - 2$  instead of  $G$ ,  $V$ ,  $E$  and  $n$ . As a result, we conclude that  $|E'| \leq (n - 2)^2 / 4$ . In other words, there are at most  $(n - 2)^2 / 4$  blue edges (since  $E' = \{\text{blue edges}\}$ ).

Since each edge of  $G$  is either black or red or blue, we thus conclude that the total number of edges of  $G$  is

$$\begin{aligned} & \underbrace{(\text{the number of black edges})}_{=1} + \underbrace{(\text{the number of red edges})}_{\leq n-2} \\ & + \underbrace{(\text{the number of blue edges})}_{\leq (n-2)^2/4} \\ & \leq 1 + (n - 2) + (n - 2)^2 / 4 = n^2 / 4 \quad (\text{by a straightforward computation}). \end{aligned}$$

In other words,  $|E| \leq n^2 / 4$ . Hence, Exercise 6.2.1 is solved for our graph  $G$ . This completes the induction step, and thus the exercise is solved.  $\square$

See [Jukna11, proofs of Theorem 4.7] for two other solutions to Exercise 6.2.1.

### 6.3. Walks, paths, circuits, cycles, connectivity, forests, trees

We can imagine the edges of a graph as roads that we can walk on. (Each edge is a two-way road, allowing us to walk from either endpoint to the other.) A longer walk can be obtained by successively following several edges, using the vertices as stops at which we can change from one edge to another. For example, in the graph shown in (1), we can walk from vertex 1 to vertex 3 using the edge  $b$ , then move on

to 4 using the edge  $d$ , then walk back to 3 using  $c$ . It is natural to wonder whether a given vertex can be reached from another by walking along a sequence of edges. This is a type of question that appears all over mathematics, so we introduce some terminology for it.

Recall that  $[m]$  means the set  $\{1, 2, \dots, m\}$  whenever  $m$  is an integer. (This set  $[m]$  is empty if  $m \leq 0$ .)

**Definition 6.3.1.** Let  $G = (V, E, \varphi)$  be a graph.

(a) A *walk* of  $G$  means a finite sequence  $(v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$  (we are labeling its entries alternatingly by  $v_i$  and  $e_i$ ) such that

- the entries  $v_0, v_1, \dots, v_k$  are vertices of  $G$ ,
- the entries  $e_1, e_2, \dots, e_k$  are edges of  $G$ , and
- each  $i \in [k]$  satisfies  $\varphi(e_i) = \{v_{i-1}, v_i\}$ .

In other words, a walk of  $G$  means a finite sequence whose entries are alternatingly vertices and edges of  $G$  (starting and ending with vertices), with the property that the endpoints of each edge in the sequence are the two vertices that appear immediately before and after it in the sequence.

(b) A walk  $(v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$  is said to be *closed* if  $v_k = v_0$ . A closed walk is also called a *circuit*.

(c) The *vertices* of a walk  $(v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$  are defined to be  $v_0, v_1, \dots, v_k$ . The *edges* of this walk are defined to be  $e_1, e_2, \dots, e_k$ . The *length* of this walk is defined to be the number  $k$ . (Thus, a walk of length  $k$  has  $k + 1$  vertices and  $k$  edges.)

(d) A *path* means a walk  $(v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$  whose vertices  $v_0, v_1, \dots, v_k$  are distinct.

(e) A *cycle* means a circuit (i.e., closed walk)  $(v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$  with the following properties:

- The edges  $e_1, e_2, \dots, e_k$  are distinct.
- The first  $k$  vertices  $v_0, v_1, \dots, v_{k-1}$  are distinct.
- We have  $k > 0$ .

(f) A walk  $(v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$  is said to *start* at  $v_0$  and to *end* at  $v_k$ .

(g) Let  $u$  and  $v$  be two vertices of  $G$ . A *walk from  $u$  to  $v$*  means a walk that starts at  $u$  and ends at  $v$ . If it is a path, then we also call it a *path from  $u$  to  $v$* .



**Example 6.3.2.** Let  $G$  be the graph shown in (1). Then:

- The sequence  $(1, b, 3, j, 6)$  is not a walk, since  $\varphi(j) \neq \{3, 6\}$ .
- The sequence  $(1, a, 2, e, 5)$  is a walk (since  $\varphi(a) = \{1, 2\}$  and  $\varphi(e) = \{2, 5\}$ ) and a path (since  $1, 2, 5$  are distinct). Its vertices are  $1, 2, 5$ , and its edges are  $a, e$ . Its length is 2. It is a path from 1 to 5.
- The sequence  $(1, a, 2, e, 5, i, 1)$  is a closed walk and a cycle. Its length is 3. It is not a path (since  $1, 2, 5, 1$  are not distinct).
- The sequence  $(1)$  is a closed walk of length 0, and is also a path. More generally, for any vertex  $v$  of  $G$ , the sequence  $(v)$  is a closed walk of length 0, and is also a path. These are the only closed walks that are paths.
- The sequence  $(3, c, 4, d, 3)$  is a cycle, whereas the sequence  $(3, c, 4, c, 3)$  is not (it is a closed walk, but its edges are not distinct).
- The sequence  $(1, b, 3, d, 4, j, 6, h, 5, e, 2, a, 1)$  is a cycle of length 6.
- The sequence  $(6, k, 6)$  is a cycle of length 1. Generally, any self-loop yields a cycle of length 1.
- The sequence  $(1, a, 2, e, 5, f, 4, j, 6, h, 5, i, 1)$  is a circuit, but not a cycle, since the vertices  $1, 2, 5, 4, 6, 5$  are not distinct.

Of course, all the concepts introduced in Definition 6.3.1 depend on the graph  $G$ . Thus, when  $G$  is not clear from the context, we speak of “walks of  $G$ ” and “paths of  $G$ ” and “circuits of  $G$ ” etc.

Here are some easy properties of paths:

**Proposition 6.3.3.** Let  $G$  be a graph that has  $n$  vertices. Then:

- (a) Any path of  $G$  has length  $\leq n - 1$ .
- (b) Any cycle of  $G$  has length  $\leq n$ .

*Proof of Proposition 6.3.3 (sketched).* (a) Let  $(v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$  be a path of  $G$ . Thus, the  $k + 1$  vertices  $v_0, v_1, \dots, v_k$  are distinct (by the definition of a path). Thus,  $G$  must have at least  $k + 1$  distinct vertices. Since  $G$  has  $n$  vertices, this entails  $n \geq k + 1$ . In other words,  $k \leq n - 1$ . Hence, our path has length  $\leq n - 1$ . This proves Proposition 6.3.3 (a).

(b) Analogous to part (a). □

**Proposition 6.3.4.** Let  $G$  be a graph. Let  $u$  and  $v$  be two vertices of  $G$ .

- (a) If  $G$  has a walk from  $u$  to  $v$ , then  $G$  has a path from  $u$  to  $v$ .
- (b) If  $G$  has (at least) two distinct paths from  $u$  to  $v$ , then  $G$  has a cycle.

*Proof of Proposition 6.3.4 (a) (sketched).* **(a)** Assume that  $G$  has a walk from  $u$  to  $v$ . Let  $\mathbf{w} = (v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$  be this walk. If the vertices  $v_0, v_1, \dots, v_k$  of this walk are distinct, then  $\mathbf{w}$  is a path from  $u$  to  $v$ , and thus we are done. If not, then there exist some  $i, j \in [k]$  such that  $i < j$  and  $v_i = v_j$ . In this case, we can choose such  $i$  and  $j$  and replace our walk  $\mathbf{w}$  by the shorter walk

$$\mathbf{w}' := (v_0, e_1, v_1, e_2, v_2, \dots, e_i, v_i, e_{j+1}, v_{j+1}, e_{j+2}, v_{j+2}, \dots, e_k, v_k)$$

(this is the walk  $\mathbf{w}$  with the entire part between  $v_i$  and  $v_j$  being cut out). We can continue doing this until we obtain a walk whose vertices are distinct (this will eventually happen, since the walk cannot get shorter and shorter indefinitely); this walk will then be a path from  $u$  to  $v$ . This proves Proposition 6.3.4 **(a)**.

See the Appendix (Section 6.6) for a proof of Proposition 6.3.4 **(b)**.  $\square$

**Proposition 6.3.5.** Let  $G$  be a graph. Let  $\mathbf{c} = (v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$  be a circuit of  $G$  such that the first  $k$  vertices  $v_0, v_1, \dots, v_{k-1}$  are distinct, and such that  $k > 2$ . Then,  $\mathbf{c}$  is a cycle.

*Proof of Proposition 6.3.5 (sketched).* This is intuitively clear but painful to prove rigorously. Skip this proof unless you really want to know.

We only need to prove that the  $k$  edges  $e_1, e_2, \dots, e_k$  of  $\mathbf{c}$  are distinct.

Assume the contrary. Thus, there exist two integers  $i$  and  $j$  with  $1 \leq i < j \leq k$  and  $e_i = e_j$ . Consider these  $i$  and  $j$ . Note that the three integers  $j-1, i-1$  and  $i$  all belong to the set  $\{0, 1, \dots, k-1\}$  (since  $1 < j \leq k$  and  $1 \leq i < k$ ). From  $i < j$ , we obtain  $i-1 < j-1$ , so that  $i-1 \neq j-1$ .

Write  $G$  in the form  $(V, E, \varphi)$ . Since  $\mathbf{c}$  is a walk, we have  $\varphi(e_i) = \{v_{i-1}, v_i\}$  and  $\varphi(e_j) = \{v_{j-1}, v_j\}$ . Hence,

$$v_{j-1} \in \{v_{j-1}, v_j\} = \varphi\left(\underbrace{e_j}_{=e_i}\right) = \varphi(e_i) = \{v_{i-1}, v_i\}.$$

In other words, we have  $v_{j-1} = v_{i-1}$  or  $v_{j-1} = v_i$ . Since the vertices  $v_0, v_1, \dots, v_{k-1}$  are distinct, this is only possible if  $j-1 = i-1$  or  $j-1 = i$  (since the integers  $j-1, i-1$  and  $i$  all belong to the set  $\{0, 1, \dots, k-1\}$ ). Hence, we must have  $j-1 = i-1$  or  $j-1 = i$ . Since  $j-1 = i-1$  is impossible (because  $i-1 \neq j-1$ ), we thus obtain  $j-1 = i$ . Thus,  $j = i+1$ .

Now, we shall show that  $j = k$ . Indeed, assume the contrary.<sup>3</sup> Thus,  $j \neq k$ . Hence,  $j < k$  (since  $j \leq k$  and  $j \neq k$ ). Therefore,  $j$  belongs to the set  $\{0, 1, \dots, k-1\}$ . Now, recall that we just showed that  $j = i+1$ . We can repeat the same argument with the roles of  $i$  and  $j$  interchanged (since the three integers  $i-1, j-1$  and  $j$  all belong to the set  $\{0, 1, \dots, k-1\}$ ), and thus obtain  $i = j+1$ . But this clearly

<sup>3</sup>Yes, we are making a proof by contradiction within a proof by contradiction. Awkward.

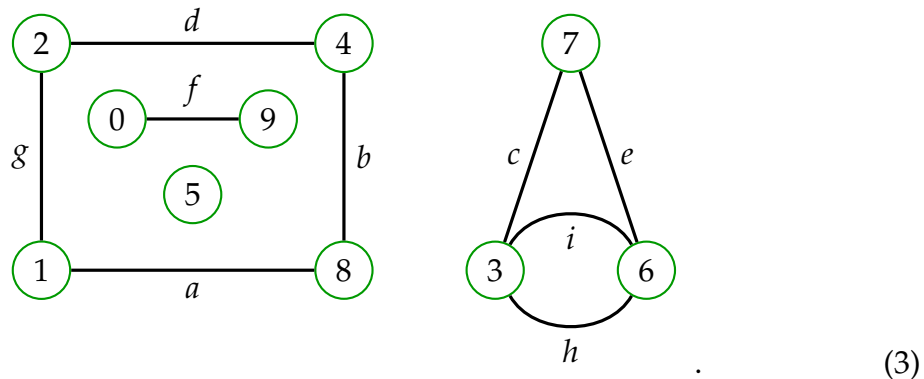
contradicts  $i < j$ . Thus, we have found a contradiction. Therefore, our assumption (that  $j \neq k$ ) was false.

Hence, we have proved that  $j = k$ . Thus,  $v_j = v_k = v_0$  (since  $\mathbf{c}$  is a circuit). Hence,  $v_0 = v_j \in \{v_{j-1}, v_j\} = \{v_{i-1}, v_i\}$ . In other words, we have  $v_0 = v_{i-1}$  or  $v_0 = v_i$ . Since the vertices  $v_0, v_1, \dots, v_{k-1}$  are distinct, this is only possible if  $0 = i - 1$  or  $0 = i$  (since the integers  $0, i - 1$  and  $i$  all belong to the set  $\{0, 1, \dots, k - 1\}$ ). Hence, we must have  $0 = i - 1$  or  $0 = i$ . Since  $0 = i$  is impossible (because  $1 \leq i$ ), we must thus have  $0 = i - 1$ . In other words,  $i = 1$ . However,  $j - 1 = i = 1$ , so that  $j = 2$ . Comparing this with  $j = k$ , we obtain  $k = 2$ . This contradicts  $k > 2$ . This contradiction shows that our assumption was false. Hence, the  $k$  edges  $e_1, e_2, \dots, e_k$  of  $\mathbf{c}$  are distinct. This completes the proof of Proposition 6.3.5.  $\square$

Using the notions of paths and walks, we can decompose a graph (more precisely, its vertex set) into *connected components*: sets of mutually accessible vertices. This formalizes the intuition of “islands” or “continents”. Here is the rigorous definition:

**Definition 6.3.6.** Let  $G$  be a graph. Two vertices  $u$  and  $v$  of  $G$  are said to be *path-connected* (in  $G$ ) if  $G$  has a path from  $u$  to  $v$ .

**Example 6.3.7.** Consider the following graph:



In this graph, the vertices 1 and 4 are path-connected (since  $(1, g, 2, d, 4)$  is a path from 1 to 4), but the vertices 1 and 3 are not (since there is no path from 1 to 3).

**Proposition 6.3.8.** Let  $G$  be a graph with vertex set  $V$ .

(a) Two vertices  $u$  and  $v$  of  $G$  are path-connected if and only if  $G$  has a walk from  $u$  to  $v$ .

(b) The relation “path-connected” (on the set  $V$ )<sup>4</sup> is an equivalence relation.

<sup>4</sup>By this, we mean the binary relation  $\sim$  on the set  $V$  that is defined as follows: Two vertices  $u, v \in V$  satisfy  $u \sim v$  if and only if  $u$  and  $v$  are path-connected.

*Proof of Proposition 6.3.8 (sketched).* **(a)** The “if” part follows from Proposition 6.3.4 **(a)**. The “only if” part is obvious (since any path is a walk).

**(b)** We check that the relation “path-connected” (on the set  $V$ ) is reflexive, symmetric and transitive:

- For each vertex  $u$ , the vertices  $u$  and  $u$  are path-connected (because the length-0 path  $(u)$  is a path from  $u$  to  $u$ ). Hence, the relation “path-connected” is reflexive.
- If  $u$  and  $v$  are two vertices such that the vertices  $u$  and  $v$  are path-connected, then the vertices  $v$  and  $u$  are path-connected. Indeed, if we pick any path from  $u$  and  $v$  and walk it in reverse, then we obtain a path from  $v$  to  $u$ <sup>5</sup>. Thus, the relation “path-connected” is symmetric.
- Finally, let us show that this relation is transitive. Indeed, let  $u, v$  and  $w$  be three vertices such that the vertices  $u$  and  $v$  are path-connected, and such that the vertices  $v$  and  $w$  are path-connected. Then, I claim that the vertices  $u$  and  $w$  are path-connected. Indeed, let  $\mathbf{p}$  be a path from  $u$  to  $v$ , and let  $\mathbf{q}$  be a path from  $v$  to  $w$ . Then, the last vertex of  $\mathbf{p}$  is the first vertex of  $\mathbf{q}$ . Hence, we can splice  $\mathbf{p}$  with  $\mathbf{q}$  at this vertex<sup>6</sup>; the result will be a walk from  $u$  to  $w$ . Hence,  $G$  has a walk from  $u$  to  $w$ . Therefore, Proposition 6.3.8 **(a)** (applied to  $w$  instead of  $v$ ) shows that the vertices  $u$  and  $w$  are path-connected. This shows that the relation “path-connected” is transitive.

Altogether, we have now shown that the relation “path-connected” (on the set  $V$ ) is reflexive, symmetric and transitive. Hence, this relation is an equivalence relation. This proves Proposition 6.3.8 **(b)**.  $\square$

**Definition 6.3.9.** Let  $G$  be a graph with vertex set  $V$ .

**(a)** The relation “path-connected” (on the set  $V$ ) is an equivalence relation (by Proposition 6.3.8 **(b)**). Its equivalence classes will be called the *connected components* of  $G$ . We will abbreviate the word “connected component” as “*component*”.

**(b)** The graph  $G$  is said to be *connected* if it has exactly 1 component.

**(c)** The graph  $G$  is said to be a *forest* if it has no cycles.

**(d)** The graph  $G$  is said to be a *tree* if it is a connected forest.

<sup>5</sup>Rigorously speaking, this is saying the following: If  $(v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$  is a path from  $u$  to  $v$ , then  $(v_k, e_k, v_{k-1}, e_{k-1}, v_{k-2}, \dots, e_1, v_0)$  is a path from  $v$  to  $u$ .

<sup>6</sup>Rigorously speaking, this means that we write the paths  $\mathbf{p}$  and  $\mathbf{q}$  in the forms  $\mathbf{p} = (v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$  and  $\mathbf{q} = (w_0, f_1, w_1, f_2, w_2, \dots, f_\ell, w_\ell)$ , and form the walk

$$(v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k, f_1, w_1, f_2, w_2, \dots, f_\ell, w_\ell)$$

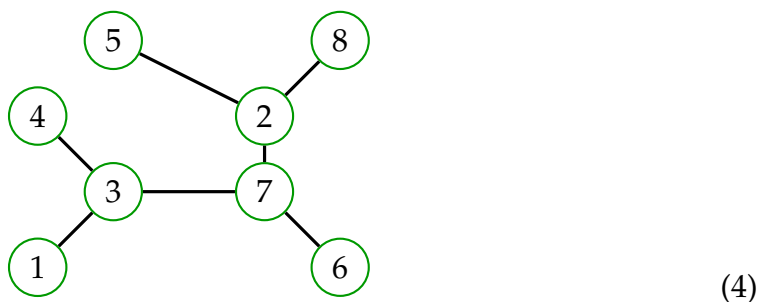
(noting that  $v_k = v = w_0$ ).

**Example 6.3.10.** The graph shown in (3) is not connected; instead, it has four components, namely

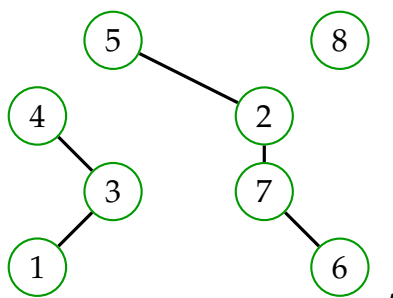
$$\{1, 2, 4, 8\}, \quad \{0, 9\}, \quad \{5\}, \quad \{3, 6, 7\}.$$

It is not a forest, however, since it has cycles (such as  $(3, h, 6, i, 3)$  and  $(1, g, 2, d, 4, b, 8, a, 1)$ ).

**Example 6.3.11.** The graph



(we are not labeling the edges since we will not refer to them) is a tree. Indeed, it is easy to see that it is connected and has no cycles. If we remove any edges from it, we obtain a forest. For instance, removing the edge that joins 3 with 7 and also removing the edge that joins 2 with 8, we obtain the forest



which has 3 components.

Note that each vertex of a graph  $G$  belongs to exactly one component of  $G$ . Indeed, this follows from a standard property of equivalence classes, since the components of  $G$  are the equivalence classes of the relation “path-connected”.

We furthermore note that the completely empty graph (which has 0 vertices and 0 edges) is not connected, since it has 0 (rather than 1) components. Thus, it is not a tree.

The components of a graph can be viewed as the smallest “inescapable” parts of its vertex set (“inescapable” in the sense that if one starts in a component and keeps moving along edges, one will never leave this component). The following easy exercise (a variant of Exercise 6.1.1) illustrates this:

**Exercise 6.3.1.** Let  $G = (V, E, \varphi)$  be a graph. Let  $C$  be a component of  $G$ .

(a) Prove that the sum of the degrees of all vertices in  $C$  equals  $2 \cdot |E_C|$ , where

$$E_C := \{e \in E \mid \text{both endpoints of } e \text{ belong to } C\}.$$

(b) Prove that the number of vertices in  $C$  that have odd degree is even.

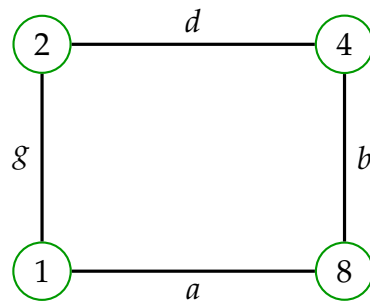
*Solution idea.* (a) The idea is to remove all vertices that don't belong to  $C$  (along with the edges that contain them) from the graph  $G$ , and apply Exercise 6.1.1 (a) to the graph that remains. Here are the details:

Define a subset  $E_C$  of  $E$  by

$$E_C := \{e \in E \mid \text{both endpoints of } e \text{ belong to } C\}.$$

Let  $G_C$  be the graph  $(C, E_C, \varphi|_{E_C})$ . (Why is this a well-defined graph? The definition of  $E_C$  shows that each edge  $e \in E_C$  has both its endpoints in  $C$ ; therefore, each edge  $e \in E_C$  satisfies  $\varphi(e) \in \mathcal{P}_{1,2}(C)$ . Thus,  $\varphi|_{E_C}$  is a map from  $E_C$  to  $\mathcal{P}_{1,2}(C)$ . Therefore,  $(C, E_C, \varphi|_{E_C})$  really is a well-defined graph.)

For example, if  $G$  is the graph shown in (3), and if  $C = \{1, 2, 4, 8\}$ , then  $G_C$  is the graph



Now, Exercise 6.1.1 (a) (applied to  $C, E_C, \varphi|_{E_C}$  and  $G_C$  instead of  $V, E, \varphi$  and  $G$ ) yields that the sum of the degrees of all vertices of  $G_C$  equals  $2 \cdot |E_C|$ . In other words, the sum of the degrees of all vertices in  $C$  equals  $2 \cdot |E_C|$  (since the vertices of  $G_C$  are precisely the vertices in  $C$ ). This solves Exercise 6.3.1 (a), right?

Not so fast! This cannot be a correct solution, since we have not used the assumption that  $C$  is a component of  $G$ , but Exercise 6.3.1 (a) clearly would not hold without this assumption.

Our mistake was subtle: We have forgotten that the degree of a vertex of a graph depends not just on the vertex, but also on the graph. Any  $v \in C$  is simultaneously a vertex of  $G$  and also a vertex of  $G_C$ , and thus has two degrees: one for  $G$  and one for  $G_C$ . The claim of Exercise 6.3.1 (a) clearly refers to the degree with respect to  $G$  (because the graph  $G_C$  does not appear in the exercise), but our application of Exercise 6.1.1 (a) gives a statement about the degree with respect to  $G_C$ .

Fortunately, the two degrees are equal; but we need to prove this. To make our path clearer, we introduce some less ambiguous terminology: If  $v$  is a vertex of a graph  $H$ , then the degree of  $v$  with respect to  $H$  (that is, the number of edges of

$H$  that contain  $v$ , where we agree that self-loops count twice) will be called the  $H$ -degree of  $v$ . With this language, we can no longer confuse degrees with respect to different graphs. We now claim the following:

*Claim 1:* Let  $v \in C$ . Then, the  $G$ -degree of  $v$  equals the  $G_C$ -degree of  $v$ .

[*Proof of Claim 1:* Any edge of  $G_C$  is an edge of  $G$  (since  $E_C \subseteq E$ ). Hence,

$$\{\text{edges of } G_C \text{ that contain } v\} \subseteq \{\text{edges of } G \text{ that contain } v\}.$$

More interestingly, we also have

$$\{\text{edges of } G \text{ that contain } v\} \subseteq \{\text{edges of } G_C \text{ that contain } v\}$$

<sup>7</sup>. Combining these two inclusions, we obtain

$$\{\text{edges of } G \text{ that contain } v\} = \{\text{edges of } G_C \text{ that contain } v\}.$$

In other words, the edges of  $G$  that contain  $v$  are precisely the edges of  $G_C$  that contain  $v$ .

However, the  $G$ -degree of  $v$  is defined to be the number of edges of  $G$  that contain  $v$ , whereas the  $G_C$ -degree of  $v$  is defined to be the number of edges of  $G_C$  that contain  $v$ . Therefore, these two degrees are counting the same thing (because we have shown that the edges of  $G$  that contain  $v$  are precisely the edges of  $G_C$  that contain  $v$ ); hence, they are equal. This proves Claim 1.]

Now, we can salvage our above argument as follows: Exercise 6.1.1 (a) (applied to  $C$ ,  $E_C$ ,  $\varphi|_{E_C}$  and  $G_C$  instead of  $V$ ,  $E$ ,  $\varphi$  and  $G$ ) yields that the sum of the  $G_C$ -degrees of all vertices of  $G_C$  equals  $2 \cdot |E_C|$ . In other words, the sum of the  $G_C$ -degrees of all vertices  $v \in C$  equals  $2 \cdot |E_C|$  (since the vertices of  $G_C$  are precisely the vertices  $v \in C$ ). However, Claim 1 shows that each of these  $G_C$ -degrees equals the corresponding  $G$ -degree; this allows us to rewrite the preceding sentence as follows: The sum of the  $G$ -degrees of all vertices  $v \in C$  equals  $2 \cdot |E_C|$ . This solves Exercise 6.3.1 (a).

---

<sup>7</sup>*Proof.* Let  $f \in \{\text{edges of } G \text{ that contain } v\}$ . We must show that  $f \in \{\text{edges of } G_C \text{ that contain } v\}$ .

We have  $f \in \{\text{edges of } G \text{ that contain } v\}$ ; in other words,  $f$  is an edge of  $G$  that contains  $v$ . Let  $u$  be the endpoint of  $f$  distinct from  $v$ . (If  $f$  is a self-loop, then we just set  $u = v$ .) Then,  $\varphi(f) = \{u, v\}$ . Hence,  $(u, f, v)$  is a walk of  $G$ . Hence,  $G$  has a walk from  $u$  to  $v$  (namely, this walk  $(u, f, v)$ ). According to Proposition 6.3.8 (a), this entails that the two vertices  $u$  and  $v$  of  $G$  are path-connected. In other words,  $u$  and  $v$  belong to the same component of  $G$  (since the components of  $G$  are just the equivalence classes of the relation “path-connected”). Since  $v$  belongs to  $C$ , we thus conclude that  $u$  must belong to  $C$  as well (because  $C$  is a component of  $G$ ). Now, both endpoints of the edge  $f$  belong to  $C$  (because the endpoints of the edge  $f$  are  $u$  and  $v$ , but we know that both  $u$  and  $v$  belong to  $C$ ). Hence,  $f \in \{e \in E \mid \text{both endpoints of } e \text{ belong to } C\} = E_C$ . Therefore,  $f$  is an edge of  $G_C$  (since  $G_C = (C, E_C, \varphi|_{E_C})$ ). Since  $f$  contains  $v$ , we thus conclude that  $f$  is an edge of  $G_C$  that contains  $v$ . In other words,  $f \in \{\text{edges of } G_C \text{ that contain } v\}$ .

Forget that we fixed  $f$ . We thus have shown that  $f \in \{\text{edges of } G_C \text{ that contain } v\}$  for each  $f \in \{\text{edges of } G \text{ that contain } v\}$ . In other words,  $\{\text{edges of } G \text{ that contain } v\} \subseteq \{\text{edges of } G_C \text{ that contain } v\}$ .

(b) Recall how we derived Exercise 6.1.1 (b) from Exercise 6.1.1 (a) a while ago. The same argument can be used to obtain Exercise 6.3.1 (b) from Exercise 6.3.1 (a).  $\square$

One useful property of cycles is that when we remove any edge of a cycle from a graph, then the remaining part of the cycle acts as a “diversion” for this missing edge, so that vertices that were path-connected before the removal remain path-connected after it. Conversely, if an edge has this property (that its removal does not disconnect any vertices), then it must belong to a cycle. This is an important result, so we state it properly. First, we define the removal of an edge from a graph:

**Definition 6.3.12.** Let  $G$  be a graph. Let  $e$  be any edge of  $G$ . Then,  $G \setminus e$  shall mean the graph obtained from  $G$  by removing the edge  $e$ . (That is, if  $G = (V, E, \varphi)$ , then  $G \setminus e = (V, E \setminus \{e\}, \varphi|_{E \setminus \{e\}})$ .)

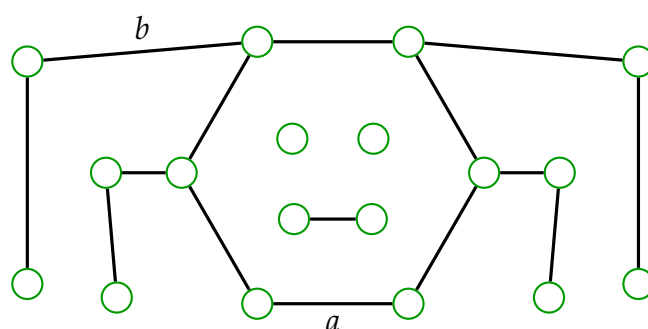
We can now state the above property of edges in cycles:

**Lemma 6.3.13.** Let  $G$  be a graph. Let  $e$  be any edge of  $G$ .

(a) If  $e$  is an edge of some cycle of  $G$ , then the components of  $G \setminus e$  are precisely the components of  $G$ .

(b) If  $e$  appears in no cycle of  $G$  (that is, there exists no cycle  $c$  of  $G$  such that  $e$  is an edge of  $c$ ), then the graph  $G \setminus e$  has one more component than  $G$ .

**Example 6.3.14.** Let  $G$  be the graph shown in the following picture:

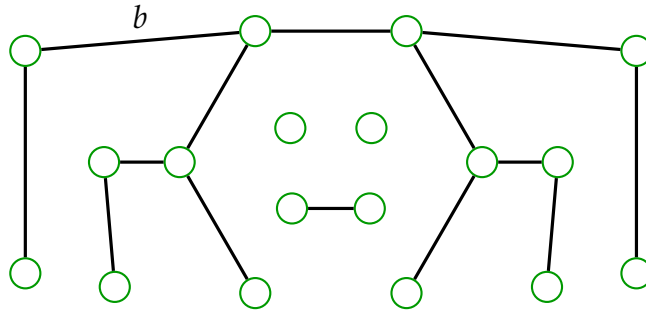


(5)

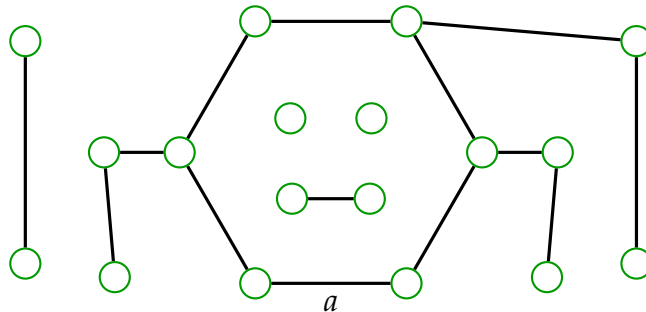
(where we have labeled only two edges). This graph has 4 components. The edge  $a$  is an edge of a cycle of  $G$ , whereas the edge  $b$  appears in no cycle of  $G$ . Thus, if we set  $e = a$ , then Lemma 6.3.13 (a) shows that the components of  $G \setminus e$



are precisely the components of  $G$ . This graph  $G \setminus e$  for  $e = a$  looks as follows:



and visibly has 4 components. On the other hand, if we set  $e = b$ , then Lemma 6.3.13 (b) shows that the graph  $G \setminus e$  has one more component than  $G$ . This graph  $G \setminus e$  for  $e = b$  looks as follows:



and visibly has 5 components.

Lemma 6.3.13 is a known result (see, e.g., [BonMur76, Theorem 2.3] for a proof). For the sake of completeness, we shall give a proof in an appendix (Section 6.7).

The following theorem can be compared to the pigeonhole principle:

**Theorem 6.3.15.** Let  $G = (V, E, \varphi)$  be a graph. Let  $n = |V|$ . Then:

- (a) If  $G$  is connected, then  $|E| \geq n - 1$ .
- (b) If  $G$  is connected and  $|E| = n - 1$ , then  $G$  is a tree.
- (c) If  $G$  is a forest and  $n > 0$ , then  $|E| \leq n - 1$ .
- (d) If  $G$  is a forest and  $|E| = n - 1$ , then  $G$  is a tree.
- (e) If  $G$  is a forest, then  $G$  has exactly  $n - |E|$  many components.

Theorem 6.3.15 is a classical result, and proofs of all five of its parts can be found in most good texts on graph theory.<sup>8</sup> For the sake of completeness, let us give a

<sup>8</sup>For example: Parts (b) and (d) appear in [Ruohon13, Theorem 2.1] and in [LeLeMe16, Theorem 12.11.6]. Part (a) is [BonMur76, Corollary 2.4.2]. Part (c) follows from [Wilson96, Corollary 9.2]. Part (e) appears in [Bona17, Proposition 10.6] and in [Bollob79, §I.2, Corollary 6].

proof here as well. First, we state an essentially obvious lemma:

**Lemma 6.3.16.** Let  $G = (V, E, \varphi)$  be a graph that has no edges. Let  $n = |V|$ . Then,  $G$  has exactly  $n$  components.

*Proof of Lemma 6.3.16.* The only paths in the graph  $G$  are the trivial paths  $(v)$  for  $v \in V$  (since any other path would have at least one edge, but  $G$  has no edges). Thus, in particular, a path from a vertex  $u$  to a vertex  $v$  cannot exist unless  $u = v$ . In other words, two vertices  $u$  and  $v$  of  $G$  cannot be path-connected in  $G$  unless  $u = v$ . Hence, the relation “path-connected” for the graph  $G$  is simply the equality relation. Therefore, its equivalence classes are the singleton sets  $\{v\}$  for all  $v \in V$ . The number of these equivalence classes is thus  $|V| = n$ . In other words, the number of components of  $G$  is  $n$  (since the components of  $G$  are defined to be the equivalence classes of the relation “path-connected”). This proves Lemma 6.3.16.  $\square$

*Proof of Theorem 6.3.15 (sketched).* Let  $e_1, e_2, \dots, e_k$  be all edges of  $G$  (listed without repetition). Thus,  $|E| = k$  and  $E = \{e_1, e_2, \dots, e_k\}$ . For each  $i \in \{0, 1, \dots, k\}$ , we let  $E_i$  be the subset  $\{e_1, e_2, \dots, e_i\}$  of  $E$ , and we let  $G_i$  be the graph  $(V, E_i, \varphi|_{E_i})$ . Thus,  $G_i$  is the graph that has the same vertices as  $G$ , but only the first  $i$  edges of  $G$  (that is, the edges  $e_1, e_2, \dots, e_i$ ). In other words,  $G_i$  is the graph obtained from  $G$  by removing its last  $k - i$  edges  $e_{i+1}, e_{i+2}, \dots, e_k$ . (Essentially, what we are doing is stepwise building up the graph  $G$  edge by edge, starting with a graph with no edges and then adding the edges one by one, in the order  $e_1, e_2, \dots, e_k$ . The graph  $G_i$  is what you obtain after  $i$  such steps.)

For each  $i \in \{0, 1, \dots, k\}$ , we let  $c_i$  be the number of components of the graph  $G_i$ . We break the essence of our proof up into seven claims:

*Claim 1:* We have  $c_0 = n$ .

*Claim 2:* We have  $G_k = G$ .

*Claim 3:* We have  $G_{i-1} = G_i \setminus e_i$  for each  $i \in [k]$ .

*Claim 4:* Let  $i \in [k]$  be such that  $e_i$  is an edge of some cycle of  $G_i$ . Then,  $c_i = c_{i-1}$ .

*Claim 5:* Let  $i \in [k]$  be such that  $e_i$  appears in no cycle of  $G_i$  (that is, there exists no cycle  $c$  of  $G_i$  such that  $e_i$  is an edge of  $c$ ). Then,  $c_i = c_{i-1} - 1$ .

*Claim 6:* For any  $i \in [k]$ , we have  $c_i \geq c_{i-1} - 1$ .

*Claim 7:* For any  $j \in \{0, 1, \dots, k\}$ , we have  $c_j \geq n - j$ .

The reader should have no trouble proving these seven claims (Lemma 6.3.13 is used for Claims 4 and 5); in any case, we give the proofs in an appendix (Section 6.8) below.

We note that the number of components of  $G_k$  is  $c_k$  (by the definition of  $c_k$ ). Since  $G_k = G$  (by Claim 2), we can rewrite this as follows: The number of components of  $G$  is  $c_k$ .

Now, we can easily finish off all four parts of Theorem 6.3.15:

**(a)** Assume that  $G$  is connected. In other words,  $G$  has exactly 1 component (by the definition of “connected”). In other words,  $c_k = 1$  (since the number of components of  $G$  is  $c_k$ ). However, Claim 7 (applied to  $j = k$ ) yields  $c_k \geq n - k$ . Hence,  $n - k \leq c_k = 1$  and thus  $k \geq n - 1$ . Thus,  $|E| = k \geq n - 1$ . This proves Theorem 6.3.15 **(a)**.

**(b)** Assume that  $G$  is connected and  $|E| = n - 1$ . We must prove that  $G$  is a tree.

Assume the contrary. Thus,  $G$  is not a tree. Since  $G$  is connected, we thus conclude that  $G$  cannot be a forest (because otherwise,  $G$  would be a connected forest, i.e., a tree). Hence,  $G$  must have a cycle (since otherwise,  $G$  would be a forest). Let  $\mathbf{c}$  be this cycle. (It does not matter which one we pick, if there are several.) Pick an arbitrary edge  $e$  of  $\mathbf{c}$ .

Recall that we have listed the edges of  $G$  as  $e_1, e_2, \dots, e_k$ . The order in which we have listed these edges was so far immaterial, but now let us agree to choose this order in such a way that  $e_k = e$  (in other words, we put the edge  $e$  at the very end of our list).

The edge  $e$  is an edge of some cycle of  $G$  (namely, of the cycle  $\mathbf{c}$ ). In other words, the edge  $e_k$  is an edge of some cycle of  $G_k$  (since  $e_k = e$  and  $G_k = G$ ). Hence, Claim 4 (applied to  $i = k$ ) yields  $c_k = c_{k-1}$ . That is,  $c_{k-1} = c_k$ . However,  $c_k = 1$  (this can be shown as in our above proof of Theorem 6.3.15 **(a)**).

However, Claim 7 (applied to  $j = k - 1$ ) yields  $c_{k-1} \geq n - (k - 1) = n - k + 1 > n - k$ . Hence,  $n - k < c_{k-1} = c_k = 1$ . In other words,  $k > n - 1$ . Thus,  $|E| = k > n - 1$ . But this contradicts  $|E| = n - 1$ . This contradiction shows that our assumption was false. Hence,  $G$  is a tree. This proves Theorem 6.3.15 **(b)**.

**(c)** Assume that  $G$  is a forest and  $n > 0$ . In particular, the graph  $G$  is a forest, i.e., has no cycles.

Let  $i \in [k]$  be arbitrary. Then, each edge of  $G_i$  is an edge of  $G$  (since the edge set  $E_i$  of  $G_i$  is a subset of  $E$ ). Hence, each cycle of  $G_i$  is a cycle of  $G$ . Since  $G$  has no cycles, we thus conclude that  $G_i$  has no cycles. Thus, in particular, the edge  $e_i$  appears in no cycle of  $G_i$ . Therefore, Claim 5 yields  $c_i = c_{i-1} - 1$ . In other words,  $c_{i-1} - c_i = 1$ .

Forget that we fixed  $i$ . We thus have proved the equality  $c_{i-1} - c_i = 1$  for each  $i \in [k]$ . If we sum these equalities over all  $i \in [k]$ , then we obtain

$$\sum_{i=1}^k (c_{i-1} - c_i) = \sum_{i=1}^k 1 = k \cdot 1 = k.$$

Hence,

$$\begin{aligned}
 k &= \sum_{i=1}^k (c_{i-1} - c_i) = (c_0 - c_1) + (c_1 - c_2) + (c_2 - c_3) + \cdots + (c_{k-1} - c_k) \\
 &= c_0 - c_k \quad (\text{by the telescope principle}) \\
 &= n - c_k \quad (\text{since Claim 1 yields } c_0 = n). \tag{6}
 \end{aligned}$$

However, the set  $V$  is nonempty (since  $|V| = n > 0$ ). This easily entails that  $G$  has at least one component<sup>9</sup>. In other words,  $c_k \geq 1$  (since the number of components of  $G$  is  $c_k$ ). Thus, (6) becomes  $k = n - \underbrace{c_k}_{\geq 1} \leq n - 1$ . Hence,  $|E| = k \leq n - 1$ . This

proves Theorem 6.3.15 (c).

(d) Assume that  $G$  is a forest and  $|E| = n - 1$ . We can then prove (6) (as we did in our proof of Theorem 6.3.15 (c)). However,  $k = |E| = n - 1$ . Comparing this with (6), we obtain  $n - c_k = n - 1$ . In other words,  $c_k = 1$ . In other words,  $G$  has exactly 1 component (since the number of components of  $G$  is  $c_k$ ). In other words,  $G$  is connected (by the definition of “connected”). Hence,  $G$  is a connected forest, i.e., a tree. Theorem 6.3.15 (d) is thus proved.

(e) Assume that  $G$  is a forest. We can then prove (6) (as we did in our proof of Theorem 6.3.15 (c)). Solving (6) for  $c_k$ , we obtain  $c_k = n - k$ . In other words,  $G$  has exactly  $n - k$  components (since the number of components of  $G$  is  $c_k$ ). This proves Theorem 6.3.15 (e).  $\square$

The following exercise has no immediately visible connection to graphs, yet can be solved by constructing an appropriate graph and applying Theorem 6.3.15 to it.

**Exercise 6.3.2.** Let  $n$  be a positive integer. Let  $A$  be an  $n \times n$ -matrix whose rows are distinct. (The entries of  $A$  can be arbitrary objects, not necessarily numbers.) Prove that we can choose one column of  $A$  such that if we remove this column, then the rows of the resulting  $n \times (n - 1)$ -matrix will still be distinct.

[**Example:** If  $n = 4$  and  $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ , and if we remove the last column

from  $A$ , then the rows of the resulting  $4 \times 3$ -matrix will still be distinct. For this specific matrix  $A$ , it is the only column with this property.]

*Solution idea.* We must prove that there exists some  $j \in [n]$  such that if we remove the  $j$ -th column from  $A$ , then the rows of the resulting  $n \times (n - 1)$ -matrix will still

<sup>9</sup>*Proof.* The components of  $G$  are equivalence classes of a certain equivalence relation on  $V$  (namely, of the relation “path-connected”). Since  $V$  is nonempty, we thus conclude that there exists at least one component of  $G$  (since an equivalence relation on a nonempty set must have at least one equivalence class). In other words,  $G$  has at least one component.

be distinct. In other words, we must prove that there exists some  $j \in [n]$  such that there are no two rows of  $A$  that differ only in their  $j$ -th entry.

Assume the contrary. Thus, for each  $j \in [n]$ , there are two rows of  $A$  that differ only in their  $j$ -th entry. In other words, for each  $j \in [n]$ , there exist two distinct elements  $p_j$  and  $q_j$  of  $[n]$  such that the  $p_j$ -th and  $q_j$ -th rows of  $A$  differ only in their  $j$ -th entry. Let us consider such  $p_j$  and  $q_j$ . (If there are many options for  $p_j$  and  $q_j$ , we just choose one.)

Now, let  $G$  be the graph with  $n$  vertices  $1, 2, \dots, n$  and  $n$  edges  $1, 2, \dots, n$  such that each edge  $j$  has endpoints  $p_j$  and  $q_j$ . (Yes, we are using the same numbers  $1, 2, \dots, n$  as the vertices and as the edges of  $G$ ; this is perfectly kosher, because Definition 6.1.1 never requires  $V$  and  $E$  to be disjoint! We could not do this with a simple graph, however.)

If this graph  $G$  was a forest, then it would have  $\leq n - 1$  edges (by Theorem 6.3.15 (c)), which would contradict the fact that it has  $n$  edges. Hence,  $G$  is not a forest. In other words,  $G$  has a cycle. Let  $(v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$  be this cycle; thus,  $k > 0$  and  $v_0 = v_k$ . Moreover, the edges  $e_1, e_2, \dots, e_k$  are distinct.

Note that the graph  $G$  has no self-loops (because the endpoints  $p_j$  and  $q_j$  of each edge  $j$  are distinct by their definition). Hence, the edge  $e_k$  is not a self-loop. Its two endpoints are therefore distinct. However, its two endpoints are  $v_{k-1}$  and  $v_k$  (since  $(v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$  is a cycle of  $G$ ). Thus,  $v_{k-1}$  and  $v_k$  are distinct. In other words,  $v_{k-1} \neq v_k$ .

Let  $i \in [k - 1]$ . Thus,  $i \in [k]$  and  $i \neq k$ . From  $i \neq k$ , we obtain  $e_i \neq e_k$  (since  $e_1, e_2, \dots, e_k$  are distinct). However,  $v_{i-1}$  and  $v_i$  are the two endpoints of the edge  $e_i$  (because  $(v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$  is a cycle of  $G$ ). Because of how we defined the graph  $G$ , this means that the  $v_{i-1}$ -th and  $v_i$ -th rows of  $A$  differ only in their  $e_i$ -th entry. Thus, the  $v_{i-1}$ -th and  $v_i$ -th rows of  $A$  do not differ in their  $e_k$ -th entry (since  $e_i \neq e_k$ ). Hence, these two rows have the same  $e_k$ -th entry. In other words, we have

$$\begin{aligned} & \text{(the } e_k\text{-th entry of the } v_{i-1}\text{-th row of } A) \\ &= \text{(the } e_k\text{-th entry of the } v_i\text{-th row of } A). \end{aligned} \tag{7}$$

Forget that we fixed  $i$ . We thus have proved the equality (7) for each  $i \in [k - 1]$ . Combining these equalities, we obtain

$$\begin{aligned} & \text{(the } e_k\text{-th entry of the } v_0\text{-th row of } A) \\ &= \text{(the } e_k\text{-th entry of the } v_1\text{-th row of } A) \\ &= \text{(the } e_k\text{-th entry of the } v_2\text{-th row of } A) \\ &= \dots \\ &= \text{(the } e_k\text{-th entry of the } v_{k-1}\text{-th row of } A). \end{aligned}$$

Thus,

$$\begin{aligned} & \text{(the } e_k\text{-th entry of the } v_{k-1}\text{-th row of } A) \\ &= \text{(the } e_k\text{-th entry of the } v_0\text{-th row of } A) \\ &= \text{(the } e_k\text{-th entry of the } v_k\text{-th row of } A) \quad (\text{since } v_0 = v_k). \end{aligned}$$

In other words, the  $v_{k-1}$ -th and  $v_k$ -th rows of  $A$  have the same  $e_k$ -th entries.

However,  $v_{k-1}$  and  $v_k$  are the two endpoints of the edge  $e_k$ . Because of how we defined the graph  $G$ , this means that the  $v_{k-1}$ -th and  $v_k$ -th rows of  $A$  differ only in their  $e_k$ -th entry. Since these two rows also have the same  $e_k$ -th entries (as we proved in the preceding paragraph), we thus conclude that these two rows do not differ at all. In other words, the  $v_{k-1}$ -th and  $v_k$ -th rows of  $A$  are completely identical. Hence, the matrix  $A$  has two identical rows (since  $v_{k-1} \neq v_k$ ). This contradicts the fact that the rows of  $A$  are distinct. This contradiction finishes our solution.  $\square$

For the sake of variety, let me also outline a different solution to Exercise 6.3.2 – this time, without any use of graphs. It illustrates a different tactic: the (arbitrary) choice of a total order.

*Second solution idea for Exercise 6.3.2.* We WLOG assume that the entries of  $A$  are integers<sup>10</sup>. Thus, we can compare them using the standard “smaller” and “greater” relations for integers.

Let  $R_1, R_2, \dots, R_n$  be the  $n$  rows of  $A$  (from top to bottom). Each of these  $n$  rows contains  $n$  integers, and thus can be viewed as an  $n$ -tuple in  $\mathbb{Z}^n$ .

We define a binary relation  $<$  on the set  $\mathbb{Z}^n$  as follows: For any two  $n$ -tuples  $\mathbf{a} = (a_1, a_2, \dots, a_n)$  and  $\mathbf{b} = (b_1, b_2, \dots, b_n)$  in  $\mathbb{Z}^n$ , we set  $\mathbf{a} < \mathbf{b}$  if and only if there exists some  $i \in [n]$  satisfying

$$a_i < b_i \quad \text{but } a_j = b_j \text{ for all } j < i.$$

(In other words, we set  $\mathbf{a} < \mathbf{b}$  if the first position at which the tuples  $\mathbf{a}$  and  $\mathbf{b}$  differ contains a smaller value in  $\mathbf{a}$  than it does in  $\mathbf{b}$ . This requires, in particular, that  $\mathbf{a}$  and  $\mathbf{b}$  differ somewhere – i.e., that  $\mathbf{a} \neq \mathbf{b}$ .)

For example,  $(1, 4, 2) < (1, 4, 5) < (1, 9, 0) < (2, 1, 3) < (2, 2, 4)$ .

It is not hard to see that the relation  $<$  on the set  $\mathbb{Z}^n$  is a strict total order – meaning that:

- No  $\mathbf{a} \in \mathbb{Z}^n$  satisfies  $\mathbf{a} < \mathbf{a}$ .
- If  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{Z}^n$  satisfy  $\mathbf{a} < \mathbf{b}$  and  $\mathbf{b} < \mathbf{c}$ , then  $\mathbf{a} < \mathbf{c}$ .
- Any two distinct  $n$ -tuples  $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$  satisfy either  $\mathbf{a} < \mathbf{b}$  or  $\mathbf{b} < \mathbf{a}$ .

This total order  $<$  is called the *lexicographic order*<sup>11</sup>.

<sup>10</sup>Indeed, we can achieve this by replacing the entries of  $A$  by integers (making sure that equal entries become equal integers, while distinct entries become distinct integers).

<sup>11</sup>The name comes from “lexicon”. And indeed, if we use the letters of the alphabet instead of the integers, then this order is precisely the order in which words appear in a dictionary – namely, in the order of the first letter, with ties being resolved using the second letter, with remaining ties being resolved using the third letter, and so on. However, the situation with words is somewhat more complicated, since words can have different lengths.

Now, recall that the rows  $R_1, R_2, \dots, R_n$  of our matrix  $A$  are distinct. Hence, by appropriately permuting them, we can order them in lexicographic order – i.e., we can permute them such that we get

$$R_1 < R_2 < \dots < R_n.$$

Clearly, the claim we are trying to prove does not change if we permute the rows; hence, we can WLOG assume that they have been permuted so that  $R_1 < R_2 < \dots < R_n$ . Assume this.<sup>12</sup>

For each  $i \in [n-1]$ , we have  $R_i < R_{i+1}$  (since  $R_1 < R_2 < \dots < R_n$ ). By the definition of the lexicographic order, this means the following: For each  $i \in [n-1]$ , there exists some  $k \in [n]$  such that the two  $n$ -tuples  $R_i$  and  $R_{i+1}$  agree in their first  $k-1$  entries, but the  $k$ -th entry of  $R_i$  is smaller than the  $k$ -th entry of  $R_{i+1}$ . Let us denote this  $k$  by  $k_i$ .<sup>13</sup>

Thus, we have defined  $n-1$  elements  $k_1, k_2, \dots, k_{n-1}$  of the set  $[n]$ . These  $n-1$  elements cannot cover the entire set  $[n]$  (that is, we cannot have  $[n] \subseteq \{k_1, k_2, \dots, k_{n-1}\}$ ), because the set  $[n]$  has more than  $n-1$  elements. Hence, there exists some  $j \in [n]$  such that  $j \notin \{k_1, k_2, \dots, k_{n-1}\}$ . Consider this  $j$ .<sup>14</sup>

Now, I claim that if we remove the  $j$ -th column from the matrix  $A$ , then the rows of the resulting  $n \times (n-1)$ -matrix will still be distinct. Clearly, this claim will solve Exercise 6.3.2.

Why is this claim true? Let  $R'_1, R'_2, \dots, R'_n$  be the rows of the matrix  $A$  after the  $j$ -th column has been removed.<sup>15</sup> Thus, we must show that these rows  $R'_1, R'_2, \dots, R'_n$  are distinct. Clearly, it suffices to show that  $R'_1 < R'_2 < \dots < R'_n$ . In other words, it suffices to show that  $R'_i < R'_{i+1}$  for each  $i \in [n-1]$ .

So let  $i \in [n-1]$  be arbitrary. We must prove that  $R'_i < R'_{i+1}$ . We have  $j \neq k_i$  (since  $j \notin \{k_1, k_2, \dots, k_{n-1}\}$ ). The definition of  $k_i$  shows that the rows  $R_i$  and  $R_{i+1}$  agree in their first  $k_i-1$  entries, but the  $k_i$ -th entry of  $R_i$  is smaller than the  $k_i$ -th entry of  $R_{i+1}$ . If we remove the  $j$ -th column from  $A$ , then both of these rows lose their  $j$ -th entries; however, their  $k_i$ -th entries survive (because  $j \neq k_i$ ), although they might “move one position to the left” (if  $j < k_i$ ). Thus, the relation  $R_i < R_{i+1}$  is preserved (because the entry of  $R'_i$  that used to be the  $k_i$ -th entry of  $R_i$  is still smaller than the corresponding entry of  $R'_{i+1}$ , and because all entries to its left are

<sup>12</sup>Here is an example of this situation (i.e., a matrix  $A$  satisfying  $R_1 < R_2 < \dots < R_n$ ):

$$A = \begin{pmatrix} 0 & 2 & 1 & 0 & 4 \\ 0 & 3 & 0 & 1 & 2 \\ 0 & 3 & 1 & 0 & 2 \\ 0 & 3 & 1 & 0 & 3 \\ 1 & 3 & 0 & 1 & 4 \end{pmatrix}. \quad (8)$$

Thus,  $R_1 = (0, 2, 1, 0, 4)$  and  $R_2 = (0, 3, 0, 1, 2)$ , etc.

<sup>13</sup>For example, if  $A$  is the matrix from (8), then  $k_1 = 2$  and  $k_2 = 3$  and  $k_3 = 5$  and  $k_4 = 1$ .

<sup>14</sup>For example, if  $A$  is the matrix from (8), then  $j = 4$ . (In this case, there is only one possibility for  $j$ ; in other cases, there can be many. But it suffices to pick one  $j$ .)

<sup>15</sup>For example, if  $A$  is the matrix from (8), then  $R'_1 = (0, 2, 1, 4)$  and  $R'_2 = (0, 3, 0, 2)$ , etc.

equal in  $R'_i$  and in  $R'_{i+1}$ ). In other words, we have  $R'_i < R'_{i+1}$ . But this is precisely what we wanted to prove. Thus, Exercise 6.3.2 is solved (again).  $\square$

**Exercise 6.3.3.** A country has finitely many towns and roads. Each road connects two distinct towns and can be ridden in two ways. Each town lies on an even number of roads. The road network connects all towns (i.e., for any two towns  $T$  and  $S$ , it is possible to get from  $T$  to  $S$  by traveling along a sequence of roads). Now, an arbitrary road gets closed for repairs. Prove that the road network still connects all towns.

*Solution idea.* First of all, the exercise is clearly a thinly veiled graph theory problem (with the towns standing for the vertices, and the roads standing for the edges). Let us translate it (back) into the language of graphs:

*Translated Exercise 6.3.3.* Let  $G$  be a connected graph such that each vertex of  $G$  has even degree. Let  $e$  be any edge of  $G$ . Prove that the graph  $G \setminus e$  is connected. (See Definition 6.3.12 for the definition of  $G \setminus e$ .)

Let us now solve this translated exercise:

Write the graph  $G$  as  $G = (V, E, \varphi)$ . We have assumed that the graph  $G$  is connected. In other words,  $G$  has exactly 1 component (by the definition of “connected”).

We must prove that the graph  $G \setminus e$  is connected. Assume the contrary. Thus, the graph  $G \setminus e$  is not connected. Hence, using Lemma 6.3.13 (a), we see that  $e$  appears in no cycle of  $G$  (that is, there exists no cycle  $c$  of  $G$  such that  $e$  is an edge of  $c$ )<sup>16</sup>.

Let  $u$  and  $v$  be the two endpoints of  $e$ . We claim that the vertices  $u$  and  $v$  are not path-connected in  $G \setminus e$ . Indeed, if they were, then the graph  $G \setminus e$  would have a path from  $u$  to  $v$  (by the definition of “path-connected”), and therefore  $e$  would be an edge of some cycle of  $G$  (namely, the cycle obtained by “closing” the path we just mentioned with the edge  $e$ )<sup>17</sup>; but this would contradict the fact that  $e$  appears in no cycle of  $G$ . Thus, the vertices  $u$  and  $v$  are not path-connected in  $G \setminus e$ . Hence, these two vertices  $u$  and  $v$  belong to different components of  $G \setminus e$  (since the components of  $G \setminus e$  are the equivalence classes of the relation “path-connected”).

Let  $C$  be the component of  $G \setminus e$  that contains  $u$ . Then,  $C$  contains  $u$ , but does not contain  $v$  (since we just have shown that  $u$  and  $v$  belong to different components of  $G \setminus e$ ). In other words,  $u \in C$  but  $v \notin C$ . This entails  $u \neq v$ ; thus, the edge  $e$  is not a self-loop (since  $u$  and  $v$  are the endpoints of  $e$ ).

Exercise 6.3.1 (b) (applied to  $G \setminus e$ ,  $E \setminus \{e\}$  and  $\varphi|_{E \setminus \{e\}}$  instead of  $G$ ,  $E$  and  $\varphi$ ) shows that the number of vertices in  $C$  that have odd degree with respect to the

<sup>16</sup>*Proof.* Assume the contrary. Thus,  $e$  is an edge of some cycle of  $G$ . Hence, Lemma 6.3.13 (a) shows that the components of  $G \setminus e$  are precisely the components of  $G$ . Thus,  $G \setminus e$  must have exactly 1 component (since  $G$  has exactly 1 component). In other words,  $G \setminus e$  is connected. But this contradicts the fact that  $G \setminus e$  is not connected. This contradiction shows that our assumption was false, qed.

<sup>17</sup>This argument is spelled out in more detail in the proof of Lemma 6.7.2 further below.



graph  $G \setminus e$  is even. We shall now show that this number is 1. This will clearly cause a contradiction, since 1 is not even.

To prove that this number is 1, we compare the degrees of the vertices of  $G$  with the degrees of the same vertices but regarded as vertices of  $G \setminus e$ . Recall that each vertex of  $G$  has even degree (by the assumptions of our exercise). As we pass from the graph  $G$  to  $G \setminus e$ , we lose the edge  $e$  (since  $G \setminus e$  was defined to be the graph  $G$  with its edge  $e$  removed), and thus the degrees of the vertices  $u$  and  $v$  get decremented by 1 (since these two vertices are contained in  $e$ ), while the degrees of all other vertices stay unchanged<sup>18</sup>. Thus, the degrees of the vertices  $u$  and  $v$  become odd (since they were even in  $G$ , but an even integer minus 1 yields an odd integer), while the degrees of all other vertices stay even (since they were even in  $G$ ). Hence, the only vertices of  $G \setminus e$  that have odd degree (regarded as vertices of  $G \setminus e$ ) are the two vertices  $u$  and  $v$ . Out of these two vertices, only  $u$  belongs to  $C$  (since  $u \in C$  and  $v \notin C$ ). Thus, the number of vertices in  $C$  that have odd degree with respect to the graph  $G \setminus e$  is 1. However, we have previously shown that this number is even. These two results contradict each other, since 1 is not even.

This contradiction shows that our assumption was false. Thus, the graph  $G \setminus e$  is connected. This solves Exercise 6.3.3.  $\square$

**Exercise 6.3.4.** Let  $G$  be a simple graph with at least one vertex. Let  $d > 1$  be an integer. Assume that each vertex of  $G$  has degree  $\geq d$ . Prove that  $G$  has a cycle of length  $\geq d + 1$ .

*Solution idea.* Let  $\mathbf{p} = (v_0, e_1, v_1, e_2, v_2, \dots, e_m, v_m)$  be a **longest** path of  $G$ . (Why does  $G$  have a longest path? Because  $G$  clearly has at least one path<sup>19</sup>, and because Proposition 6.3.3 (a) shows that the lengths of all paths of  $G$  are  $\leq n - 1$ , where  $n$  is the number of vertices of  $G$ .)

Since  $\mathbf{p}$  is a path, we see that the vertices  $v_0, v_1, \dots, v_m$  are distinct.

Recall the notion of a neighbor of a vertex (see Definition 6.1.1 (c)). Since  $G$  is a **simple** graph, the neighbors of  $v_0$  are in 1-to-1 correspondence with the edges that contain  $v_0$ : Namely, each neighbor  $w$  of  $v_0$  gives rise to an edge  $v_0w$  that contains  $v_0$ , and conversely, each edge that contains  $v_0$  contains precisely one neighbor of  $v_0$ . Thus, the number of all neighbors of  $v_0$  equals the number of all edges that contain  $v_0$ . But the latter number is the degree of  $v_0$  (by the definition of a degree), and thus is  $\geq d$  (since each vertex of  $G$  has degree  $\geq d$ ). Hence, the former number is  $\geq d$  as well. Thus, we have shown that the number of all neighbors of  $v_0$  is  $\geq d$ .

If all neighbors of  $v_0$  belonged to the set  $\{v_1, v_2, \dots, v_{d-1}\}$ <sup>20</sup>, then the number of all neighbors of  $v_0$  would be at most  $d - 1$ , which would contradict the preceding sentence. Thus, there exists at least one neighbor  $u$  of  $v_0$  that does not belong to

<sup>18</sup>Note that we are tacitly using the fact that  $u \neq v$  here. Indeed, if we had  $u = v$ , then the degree of the vertex  $u$  would get decremented by 2, not by 1.

<sup>19</sup>*Proof.* Recall that  $G$  has at least one vertex. Thus, we can pick a vertex  $v$  of  $G$ . The path  $(v)$  is then a path of  $G$ . Hence,  $G$  has at least one path.

<sup>20</sup>If  $d - 1 > m$ , then this set should be understood to mean  $\{v_1, v_2, \dots, v_m\}$ .

the set  $\{v_1, v_2, \dots, v_{d-1}\}$ . Consider this  $u$ . Let  $e_0$  be the edge  $uv_0$ . Thus,  $u \neq v_0$  (since an edge in a simple graph always has two distinct endpoints).

Attaching the vertex  $u$  and the edge  $e_0$  to the front of the path  $\mathbf{p}$ , we obtain a walk

$$\mathbf{p}' := (u, e_0, v_0, e_1, v_1, e_2, v_2, \dots, e_m, v_m).$$

If we had  $u \notin \{v_0, v_1, \dots, v_m\}$ , then this walk  $\mathbf{p}'$  would be a path; but this would contradict the fact that  $\mathbf{p}$  is a **longest** path of  $G$  (since  $\mathbf{p}'$  is clearly longer than  $\mathbf{p}$ ). Hence, we must have  $u \in \{v_0, v_1, \dots, v_m\}$ . In other words,  $u = v_i$  for some  $i \in \{0, 1, \dots, m\}$ . Consider this  $i$ . We have  $v_i = u \neq v_0$  and thus  $i \neq 0$ . Hence,  $i \in \{0, 1, \dots, m\} \setminus \{0\} = [m]$ . Thus,  $i > 0$ .

If we had  $i \leq d-1$ , then we would have  $u = v_i \in \{v_1, v_2, \dots, v_{d-1}\}$  (since  $i > 0$  and  $i \leq d-1$ ), which would contradict the fact that  $u$  does not belong to the set  $\{v_1, v_2, \dots, v_{d-1}\}$ . Hence, we cannot have  $i \leq d-1$ . Thus,  $i > d-1$ , so that  $i \geq d$ . Now, the walk

$$\mathbf{c} := (u, e_0, v_0, e_1, v_1, e_2, v_2, \dots, e_i, v_i)$$

is a circuit (since  $u = v_i$ ), and has length  $i+1 \geq d+1$  (since  $i \geq d$ ). Moreover, this circuit  $\mathbf{c}$  has length  $\underbrace{i}_{\geq d+1} + 1 > 1 + 1 = 2$ , and furthermore its first  $i+1$  vertices

$u, v_0, v_1, \dots, v_{i-1}$  are distinct<sup>21</sup>. However, Proposition 6.3.5 says that if a circuit of  $G$  has length  $k > 2$ , and if the first  $k$  vertices of this circuit are distinct, then this circuit is a cycle. Applying this to our circuit  $\mathbf{c}$  (and to  $k = i+1$ ), we thus conclude that  $\mathbf{c}$  is a cycle. Since  $\mathbf{c}$  has length  $\underbrace{i}_{\geq d} + 1 \geq d+1$ , we thus conclude that  $G$  has a cycle of length  $\geq d+1$  (namely,  $\mathbf{c}$ ). This solves Exercise 6.3.4.  $\square$

## 6.4. Class problems

The following problems are to be discussed during class.

The first is an alternative characterization of connected graphs:

**Exercise 6.4.1.** Let  $G = (V, E, \varphi)$  be a graph with at least one vertex. Show that the following two statements are equivalent:

- *Statement 1:* The graph  $G$  is connected.
- *Statement 2:* If  $A$  and  $B$  are any two disjoint nonempty subsets of  $V$  satisfying  $A \cup B = V$ , then there exists some edge of  $G$  that has one endpoint in  $A$  and its other endpoint in  $B$ .

<sup>21</sup>*Proof.* The vertices  $v_0, v_1, \dots, v_i$  are distinct (since the vertices  $v_0, v_1, \dots, v_m$  are distinct). In other words, the vertices  $v_i, v_0, v_1, \dots, v_{i-1}$  are distinct (since these are the same vertices as  $v_0, v_1, \dots, v_i$ , just in a different order). In other words, the vertices  $u, v_0, v_1, \dots, v_{i-1}$  are distinct (since  $u = v_i$ ).

**Exercise 6.4.2.** Let  $G = (V, E)$  be a simple graph. Set  $n = |V|$ . Assume that  $|E| > \binom{n-1}{2}$ . Prove that  $G$  is connected.

Note that Exercise 6.4.2 would not hold for a non-simple graph: For example, a graph could have three vertices 1, 2 and 3, and could have arbitrarily many edges connecting the same two vertices 1 and 2 while the vertex 3 would not be adjacent to anything.

The next exercise is a curious application of cycles in graphs to rounding numbers:

**Exercise 6.4.3.** A *row sum* of a matrix shall mean the sum of all entries in some row of this matrix. Likewise, a *column sum* of a matrix shall mean the sum of all entries in some column of this matrix. (Thus, an  $m \times n$ -matrix has  $m$  row sums and  $n$  column sums.)

Let  $A$  be an  $m \times n$ -matrix with real entries. Assume that all row sums of  $A$  and all column sums of  $A$  are integers. Prove that we can round each non-integer entry of  $A$  (that is, replace it either by the next-smaller integer or the next-larger integer) in such a way that the resulting matrix has the same row sums as  $A$  and the same column sums as  $A$ .

**Exercise 6.4.4.** An  $m \times n$  checkerboard is colored randomly: Each square is independently assigned red or black with probability  $\frac{1}{2}$ . We say that two squares  $p$  and  $q$  are in the same *room* if there is a sequence of squares, all of the same color, starting at  $p$  and ending at  $q$ , such that successive squares in the sequence share a common side. Show that the expected number of rooms is greater than  $\frac{mn}{8}$ . (This is Putnam 2004 problem A5.)

The next exercise is a finite version of a known fact about periodic sequences. To wit, if  $p$  and  $q$  are two periods of an infinite sequence, then  $\gcd(p, q)$  is a period of the sequence as well (see, e.g., [Grinbe20, Theorem 4.7.8 (d)]). The analogous property of finite sequences (i.e.,  $n$ -tuples) does not hold in general (e.g., the 6-tuple  $(0, 1, 0, 0, 1, 0)$  has periods 3 and 5, but does not have period  $\gcd(3, 5) = 1$ ). However, it can be salvaged if we require the periods  $p$  and  $q$  to be coprime and the  $n$ -tuple to have length  $n \geq p + q - 1$ .

**Exercise 6.4.5.** Let  $p$  and  $q$  be two coprime positive integers. Let  $n \in \mathbb{N}$  satisfy  $n \geq p + q - 1$ . Let  $(x_1, x_2, \dots, x_n)$  be an  $n$ -tuple of arbitrary objects. Assume that<sup>22</sup>

$$x_i = x_{i+p} \quad \text{for each } i \in [n - p].$$

Assume furthermore that

$$x_i = x_{i+q} \quad \text{for each } i \in [n - q].$$

Then, prove that  $x_1 = x_2 = \cdots = x_n$ .

[**Example:** If  $p = 3$  and  $q = 4$  and  $n = 6$ , then this exercise claims that any 6-tuple  $(x_1, x_2, \dots, x_6)$  satisfying  $x_1 = x_4$  and  $x_2 = x_5$  and  $x_3 = x_6$  and  $x_1 = x_5$  and  $x_2 = x_6$  must satisfy  $x_1 = x_2 = \cdots = x_6$ .]

## 6.5. Homework exercises

Solve 3 of the 6 exercises below and upload your solutions on gradescope by November 7.

**Exercise 6.5.1.** Consider a conference with an even (and nonzero) number of participants. Prove that there exist two distinct participants that have an even number of common friends. (We are assuming here that friendship is a mutual relation – i.e., if  $a$  is a friend of  $b$ , then  $b$  is a friend of  $a$ . Furthermore, no person  $a$  counts as his own friend.)

The next exercise generalizes Mantel's theorem (Exercise 6.2.1) to simple graphs that may have triangles:

**Exercise 6.5.2.** Let  $G = (V, E)$  be a simple graph. Set  $n = |V|$ . Prove that we can find some edges  $e_1, e_2, \dots, e_k$  of  $G$  and some triangles  $t_1, t_2, \dots, t_\ell$  of  $G$  such that  $k + \ell \leq n^2/4$  and such that each edge  $e \in E \setminus \{e_1, e_2, \dots, e_k\}$  is a subset of (at least) one of the triangles  $t_1, t_2, \dots, t_\ell$ .

(Actually, “(at least) one” can be replaced by “exactly one” in Exercise 6.5.2, but this would make the exercise significantly harder.)

Our next exercise generalizes Exercise 6.4.5:

**Exercise 6.5.3.** Let  $p$  and  $q$  be two positive integers. Let  $g = \gcd(p, q)$ . Let  $n \in \mathbb{N}$  satisfy  $n \geq p + q - g$ . Let  $(x_1, x_2, \dots, x_n)$  be an  $n$ -tuple of arbitrary objects. Assume that

$$x_i = x_{i+p} \quad \text{for each } i \in [n - p].$$

Assume furthermore that

$$x_i = x_{i+q} \quad \text{for each } i \in [n - q].$$

Then, prove that  $x_i = x_{i+g}$  for each  $i \in [n - g]$ .

<sup>22</sup>We recall that  $[m]$  denotes the set  $\{1, 2, \dots, m\}$  whenever  $m \in \mathbb{N}$ .

**Exercise 6.5.4.** Let  $V$  be a nonempty finite set. Let  $G$  and  $H$  be two graphs with vertex set  $V$ . Assume that for each  $u \in V$  and  $v \in V$ , there exists a path from  $u$  to  $v$  in  $G$  or a path from  $u$  to  $v$  in  $H$ . Prove that at least one of the graphs  $G$  and  $H$  is connected.

**Exercise 6.5.5.** Let  $G$  be a graph. Let  $d > 2$  be an integer. Assume that each vertex of  $G$  has degree  $\geq d$ . Prove that  $G$  has a cycle whose length is not divisible by  $d$ .

**Exercise 6.5.6.** Among  $n$  senators, some are enemies. It is assumed that the “enemy” relation is mutual – i.e., if  $a$  is an enemy of  $b$ , then  $b$  is an enemy of  $a$ . A set  $S$  of senators is said to be *odious* if each senator not in  $S$  has at least one enemy in  $S$ . Prove that the number of odious sets of senators is odd.

(Note that the set of all  $n$  senators is always odious, for vacuous reasons.)

[**Example:** If  $n = 3$  and the three senators are labelled 1, 2, 3, and the only pairs of mutual enemies are  $\{1, 2\}$  and  $\{1, 3\}$ , then the odious sets are  $\{1\}$ ,  $\{1, 2\}$ ,  $\{1, 3\}$ ,  $\{2, 3\}$  and  $\{1, 2, 3\}$ .]

## 6.6. Appendix: Backtrack-free walks and a proof of Proposition 6.3.4 (b)

The claim of Proposition 6.3.4 (b) is fairly intuitive: Having two distinct paths between the same two vertices surely suggests that the two paths must “diverge” at some point, only to “converge” back at some later point; there should thus be a cycle “in there somewhere” between these two points. However, some more work is needed to obtain a rigorous proof (in particular, the two paths may converge and diverge several times, and then one has to be careful about which points one chooses). We will give a proof based on the notion of a *backtrack-free walk*:<sup>23</sup>

**Definition 6.6.1.** Let  $G$  be a graph. Let  $\mathbf{w} = (v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$  be a walk of  $G$ . We say that this walk  $\mathbf{w}$  is *backtrack-free* if it satisfies

$$e_i \neq e_{i+1} \quad \text{for each } i \in [k-1].$$

(In other words, we say that a walk  $\mathbf{w}$  is backtrack-free if no two consecutive edges of  $\mathbf{w}$  are identical.)

Note that each cycle is backtrack-free (since Definition 6.3.1 (e) shows that all its edges are distinct). Each path is backtrack-free as well; indeed, this follows from the following stronger fact:

<sup>23</sup>We are using the notation  $[m]$  for the set  $\{1, 2, \dots, m\}$  whenever  $m \in \mathbb{Z}$ . If  $m \leq 0$ , then this set is empty.

**Proposition 6.6.2.** Let  $G$  be a graph. Let  $\mathbf{p}$  be a path of  $G$ . Then, the edges of  $\mathbf{p}$  are distinct.

*Proof of Proposition 6.6.2 (sketched).* This is similar to the proof of Proposition 6.3.5; we give the proof only for the sake of completeness.

Write the path  $\mathbf{p}$  in the form  $\mathbf{p} = (v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$ . Thus, we must prove that the  $k$  edges  $e_1, e_2, \dots, e_k$  of  $\mathbf{p}$  are distinct.

Assume the contrary. Thus, there exist two distinct integers  $i, j \in [k]$  satisfying  $e_i = e_j$ . Consider these  $i$  and  $j$ . Note that the four integers  $i-1, j-1, i$  and  $j$  all belong to the set  $\{0, 1, \dots, k\}$  (since  $i, j \in [k]$ ). We have  $i \neq j$  (since  $i$  and  $j$  are distinct), so that  $i-1 \neq j-1$ .

Write  $G$  in the form  $(V, E, \varphi)$ . Since  $\mathbf{c}$  is a walk, we have  $\varphi(e_i) = \{v_{i-1}, v_i\}$  and  $\varphi(e_j) = \{v_{j-1}, v_j\}$ . Hence,

$$v_{j-1} \in \{v_{j-1}, v_j\} = \varphi\left(\underbrace{e_j}_{=e_i}\right) = \varphi(e_i) = \{v_{i-1}, v_i\}.$$

In other words, we have  $v_{j-1} = v_{i-1}$  or  $v_{j-1} = v_i$ . Since the vertices  $v_0, v_1, \dots, v_k$  are distinct, this is only possible if  $j-1 = i-1$  or  $j-1 = i$  (since the integers  $j-1, i-1$  and  $i$  all belong to the set  $\{0, 1, \dots, k\}$ ). Hence, we must have  $j-1 = i-1$  or  $j-1 = i$ . Since  $j-1 = i-1$  is impossible (because  $i-1 \neq j-1$ ), we thus obtain  $j-1 = i$ . Hence,  $i = j-1 < j$ . The same argument (with the roles of  $i$  and  $j$  interchanged) yields  $j < i$ . But these two inequalities clearly contradict one another. Hence, our assumption was wrong. This completes the proof of Proposition 6.6.2.  $\square$

An example of a backtrack-free walk that is neither a path nor a cycle is the walk  $(1, b, 3, d, 4, j, 6, g, 3, d, 4)$  in the graph shown in (1). The two equal edges  $d$  in this walk do not violate its backtrack-freeness, since they are not consecutive but rather spread apart by two other edges.

We now claim the following fact (which will be used in the proof of Proposition 6.3.4 (b)):

**Proposition 6.6.3.** Let  $G$  be a graph. Let  $\mathbf{c}$  be a backtrack-free circuit of  $G$  that has length  $> 0$ . Then,  $G$  has a cycle whose edges are edges of  $\mathbf{c}$ .

*Proof of Proposition 6.6.3 (sketched).* Write the graph  $G$  as  $G = (V, E, \varphi)$ . Write the circuit  $\mathbf{c}$  as

$$\mathbf{c} = (v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k).$$

Thus,  $v_k = v_0$  (since  $\mathbf{c}$  is a circuit). Moreover,  $k$  is the length of  $\mathbf{c}$ , and therefore we have  $k > 0$  (since  $\mathbf{c}$  was assumed to have length  $> 0$ ).

Define a *loop* to be a pair  $(i, j)$  of integers with  $k \geq i > j \geq 0$  and  $v_i = v_j$ . Thus,  $(k, 0)$  is a loop (since  $k > 0$  and  $v_k = v_0$ ), so that the set of all loops is nonempty.

This set is, of course, finite. Thus, there exists a loop  $(i, j)$  with smallest difference  $i - j$ . (There might exist several such loops with equal difference; in this case, we just pick one of them.) Pick such a loop  $(i, j)$  with smallest difference  $i - j$ . Thus,  $i$  and  $j$  are two integers with  $k \geq i > j \geq 0$  and  $v_i = v_j$ . From  $i > j$ , we obtain  $i - j > 0$ . The vertices  $v_j, v_{j+1}, \dots, v_{i-1}$  are distinct<sup>24</sup>. Moreover, the edges  $e_{j+1}, e_{j+2}, \dots, e_i$  are distinct<sup>25</sup>.

Now, let  $\mathbf{d}$  be the sequence  $(v_j, e_{j+1}, v_{j+1}, e_{j+2}, v_{j+2}, \dots, e_i, v_i)$ . This sequence  $\mathbf{d}$  is just the piece of our circuit  $\mathbf{c}$  starting at  $v_j$  and ending at  $v_i$ . Thus,  $\mathbf{d}$  is a walk. Furthermore,  $\mathbf{d}$  is a circuit, since  $v_j = v_i$ . This circuit  $\mathbf{d}$  is a cycle (since its edges  $e_{j+1}, e_{j+2}, \dots, e_i$  are distinct, and since the vertices  $v_j, v_{j+1}, \dots, v_{i-1}$  are distinct, and since  $i - j > 0$ ), and its edges are edges of  $\mathbf{c}$  (by its construction). Hence, the graph  $G$  has a cycle whose edges are edges of  $\mathbf{c}$  (namely, the cycle  $\mathbf{d}$ ). This proves Proposition 6.6.3.  $\square$

We can now prove a generalization of Proposition 6.3.4 (b), in which “paths” are replaced by “backtrack-free walks”:

**Proposition 6.6.4.** Let  $G$  be a graph. Let  $u$  and  $v$  be two vertices of  $G$ . If  $G$  has (at least) two distinct backtrack-free walks from  $u$  to  $v$ , then  $G$  has a cycle.

*Proof of Proposition 6.6.4 (sketched).* Assume that  $G$  has (at least) two distinct backtrack-free walks from  $u$  to  $v$ .

<sup>24</sup>*Proof.* Assume the contrary. Then, two of these vertices  $v_j, v_{j+1}, \dots, v_{i-1}$  are equal. In other words, we have  $v_p = v_q$  for some integers  $p$  and  $q$  satisfying  $j \leq p < q \leq i - 1$ . Consider these  $p$  and  $q$ . Then,  $k \geq i \geq i - 1 \geq q$  and  $p \geq j \geq 0$ , so that  $k \geq q > p \geq 0$ . Moreover,  $v_q = v_p$  (since  $v_p = v_q$ ). Hence,  $(q, p)$  is a loop. The difference  $q - p$  of this loop is smaller than the difference  $i - j$  of the loop  $(i, j)$  (because  $\underbrace{q}_{\leq i-1 < i} - \underbrace{p}_{\geq j} > i - j$ ). But this contradicts the fact that  $(i, j)$  is a loop with

smallest difference  $i - j$ . This contradiction shows that our assumption was false. Qed.

<sup>25</sup>*Proof.* Assume the contrary. Thus, two of these edges are equal. In other words, we have  $e_p = e_q$  for some integers  $p$  and  $q$  satisfying  $j + 1 \leq p < q \leq i$ . Consider these  $p$  and  $q$ . Note that  $p \leq i - 1$  (since  $p < i$ ) and  $p - 1 \leq p \leq i - 1$  and  $q - 1 \leq i - 1$  (since  $q \leq i$ ). Hence, all three integers  $q - 1, p - 1$  and  $p$  belong to  $\{j, j + 1, \dots, i - 1\}$  (because  $j + 1 \leq p < q$  shows also that all these three integers are  $\geq j$ ).

However, we have  $\varphi(e_p) = \{v_{p-1}, v_p\}$  (since  $\mathbf{c}$  is a walk) and  $\varphi(e_q) = \{v_{q-1}, v_q\}$  (similarly). Thus,

$$v_{q-1} \in \{v_{q-1}, v_q\} = \varphi\left(\underbrace{e_q}_{=e_p}\right) = \varphi(e_p) = \{v_{p-1}, v_p\}.$$

In other words,  $v_{q-1} = v_{p-1}$  or  $v_{q-1} = v_p$ . However, since the vertices  $v_j, v_{j+1}, \dots, v_{i-1}$  are distinct, this is only possible if  $q - 1 = p - 1$  or  $q - 1 = p$  (since the three integers  $q - 1, p - 1$  and  $p$  all belong to the set  $\{j, j + 1, \dots, i - 1\}$ ). Thus, we must have  $q - 1 = p - 1$  or  $q - 1 = p$ . Since  $q - 1 = p - 1$  is impossible (because  $p < q$  entails  $p - 1 < q - 1$ ), we thus obtain  $q - 1 = p$ . Hence,  $e_{q-1} = e_p = e_q$ . However,  $e_{q-1} \neq e_q$  (since the circuit  $\mathbf{c}$  is backtrack-free). The last two sentences are in contradiction to each other. Hence, our assumption was wrong, qed.

We define a *lune* to be a pair  $(\mathbf{p}, \mathbf{q})$ , where

$$\begin{aligned}\mathbf{p} &= (p_0, a_1, p_1, a_2, p_2, \dots, a_k, p_k) & \text{and} \\ \mathbf{q} &= (q_0, b_1, q_1, b_2, q_2, \dots, b_\ell, q_\ell)\end{aligned}$$

are two distinct backtrack-free walks satisfying  $p_0 = q_0$  and  $p_k = q_\ell$ . In other words, a lune means a pair of two distinct backtrack-free walks that start at the same vertex and end at the same vertex. We have assumed that  $G$  has two distinct backtrack-free walks from  $u$  to  $v$ ; thus,  $G$  has a lune.

The *size* of a lune  $(\mathbf{p}, \mathbf{q})$  will be defined as the sum of the lengths of  $\mathbf{p}$  and  $\mathbf{q}$ . This is always a nonnegative integer.

Let us pick a lune  $(\mathbf{p}, \mathbf{q})$  of smallest possible size. (This clearly exists, since  $G$  has a lune.) Write the two backtrack-free walks  $\mathbf{p}$  and  $\mathbf{q}$  in the form

$$\begin{aligned}\mathbf{p} &= (p_0, a_1, p_1, a_2, p_2, \dots, a_k, p_k) & \text{and} \\ \mathbf{q} &= (q_0, b_1, q_1, b_2, q_2, \dots, b_\ell, q_\ell).\end{aligned}$$

Since  $(\mathbf{p}, \mathbf{q})$  is a lune, we thus have  $p_0 = q_0$  and  $p_k = q_\ell$  and  $\mathbf{p} \neq \mathbf{q}$  (because a lune must be a pair of two **distinct** walks).

In particular,  $p_k = q_\ell$ . In other words, the last vertex of  $\mathbf{p}$  is the last vertex of  $\mathbf{q}$ . Hence, we obtain a walk  $\mathbf{c}$  by first following  $\mathbf{p}$  from  $p_0$  to  $p_k$  and then following  $\mathbf{q}$  in reverse direction from  $q_\ell$  to  $q_0$ . Rigorously speaking, this walk  $\mathbf{c}$  is defined to be

$$\left( \overbrace{p_0, a_1, p_1, a_2, p_2, \dots, a_k, p_k}^{\text{this is the walk } \mathbf{p}}, \underbrace{b_\ell, q_{\ell-1}, b_{\ell-1}, q_{\ell-2}, b_{\ell-2}, \dots, q_1, b_1, q_0}_{\text{this is the walk } \mathbf{q}, \text{ followed in reverse (since } p_k = q_\ell)} \right).$$

This walk  $\mathbf{c}$  is a circuit, since  $p_0 = q_0$ . Furthermore, it has length  $k + \ell$ , and thus has length  $> 0$ <sup>26</sup>. Hence, if we can show that this circuit  $\mathbf{c}$  is backtrack-free, then Proposition 6.6.3 will yield that  $G$  has a cycle whose edges are edges of  $\mathbf{c}$ ; this will yield the claim of Proposition 6.6.4.

Thus, it remains to prove that  $\mathbf{c}$  is backtrack-free. To prove this, we assume the contrary. Thus,  $\mathbf{c}$  is not backtrack-free. In other words, two consecutive edges of  $\mathbf{c}$  are identical. Since the edges of  $\mathbf{c}$  are  $a_1, a_2, \dots, a_k, b_\ell, b_{\ell-1}, \dots, b_1$  in this order, this means that we must be in one of the following three cases:

*Case 1:* We have  $a_i = a_{i+1}$  for some  $i \in [k-1]$ .

*Case 2:* We have  $a_k = b_\ell$  (and  $k > 0$  and  $\ell > 0$ ).

*Case 3:* We have  $b_i = b_{i-1}$  for some  $i \in \{2, 3, \dots, \ell\}$ .

However, Case 1 is impossible, since the walk  $\mathbf{p} = (p_0, a_1, p_1, a_2, p_2, \dots, a_k, p_k)$  is backtrack-free. Similarly, Case 3 is impossible, since the walk  $\mathbf{q}$  is backtrack-free.

<sup>26</sup>*Proof.* We must show that  $k + \ell > 0$ .

Assume the contrary. Thus,  $k + \ell \leq 0$ . Since  $k$  and  $\ell$  are nonnegative integers, this entails that  $k = 0$  and  $\ell = 0$ . Now,  $\mathbf{p} = (p_0, a_1, p_1, a_2, p_2, \dots, a_k, p_k) = (p_0)$  (because  $k = 0$ ) and  $\mathbf{q} = (q_0)$  (similarly). However,  $p_0 = q_0$  and thus  $(p_0) = (q_0)$ . Hence,  $\mathbf{p} = (p_0) = (q_0) = \mathbf{q}$ . This contradicts  $\mathbf{p} \neq \mathbf{q}$ . This contradiction shows that our assumption was false, qed.



Hence, we must be in Case 2. In other words, we have  $a_k = b_\ell$  (and  $k > 0$  and  $\ell > 0$ ). Now,  $\varphi(a_k) = \{p_{k-1}, p_k\}$  (since  $\mathbf{p}$  is a walk). In other words, the endpoints of  $a_k$  are  $p_{k-1}$  and  $p_k$ . Hence,

$$p_{k-1} = (\text{the endpoint of } a_k \text{ distinct from } p_k) \quad (9)$$

(with the understanding that if  $a_k$  is a self-loop, then  $p_{k-1} = p_k$ ). Similarly,

$$q_{\ell-1} = (\text{the endpoint of } b_\ell \text{ distinct from } q_\ell) \quad (10)$$

(with the understanding that if  $b_\ell$  is a self-loop, then  $q_{\ell-1} = q_\ell$ ). However, the right hand sides of the two equalities (9) and (10) are equal (since  $a_k = b_\ell$  and  $p_k = q_\ell$ ). Thus, their left hand sides are equal as well. In other words, we have  $p_{k-1} = q_{\ell-1}$ .

Now, if we remove the last vertex and the last edge from the backtrack-free walk  $\mathbf{p}$ , then we obtain a new backtrack-free walk

$$\mathbf{p}' := (p_0, a_1, p_1, a_2, p_2, \dots, a_{k-1}, p_{k-1}).$$

Similarly, if we remove the last vertex and the last edge from the backtrack-free walk  $\mathbf{q}$ , then we obtain a new backtrack-free walk

$$\mathbf{q}' := (q_0, b_1, q_1, b_2, q_2, \dots, b_{\ell-1}, q_{\ell-1}).$$

These two backtrack-free walks  $\mathbf{p}'$  and  $\mathbf{q}'$  are distinct (because if they were equal, then the two original walks  $\mathbf{p} = (p_0, a_1, p_1, a_2, p_2, \dots, a_k, p_k)$  and  $\mathbf{q} = (q_0, b_1, q_1, b_2, q_2, \dots, b_\ell, q_\ell)$  would also be equal<sup>27</sup>; but this would contradict  $\mathbf{p} \neq \mathbf{q}$ ) and satisfy  $p_0 = q_0$  and  $p_{k-1} = q_{\ell-1}$ . Hence,  $(\mathbf{p}', \mathbf{q}')$  is a lune. This lune has smaller size than the original lune  $(\mathbf{p}, \mathbf{q})$  (indeed, the lune  $(\mathbf{p}, \mathbf{q})$  has size  $k + \ell$ , whereas the lune  $(\mathbf{p}', \mathbf{q}')$  has size  $(k - 1) + (\ell - 1) < k + \ell$ ). This, however, contradicts the fact that  $(\mathbf{p}, \mathbf{q})$  is a lune of smallest possible size (by its definition). This contradiction shows that our assumption was false.

Hence, we have shown that  $\mathbf{c}$  is backtrack-free. Thus, Proposition 6.6.3 yields that  $G$  has a cycle whose edges are edges of  $\mathbf{c}$ . Hence, in particular,  $G$  has a cycle. Proposition 6.6.4 is thus proved.  $\square$

Proposition 6.3.4 (b) follows from Proposition 6.6.4, since any path is a backtrack-free walk.

## 6.7. Appendix: Proof of Lemma 6.3.13

After Example 6.3.14, Lemma 6.3.13 should be really intuitive. Indeed, let us consider what happens when we remove the edge  $e$  from the graph  $G$ :

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<sup>27</sup>because  $a_k = b_\ell$  and  $p_k = q_\ell$

- If  $e$  is an edge of some cycle of  $G$ , then any path that uses  $e$  can get “diverted” through the rest of this cycle, and thus any two vertices that are path-connected in  $G$  stay path-connected in  $G \setminus e$ . (Technical point: The “diversion” might turn the path into a walk, because some vertices might be traversed twice. However, by Proposition 6.3.8 (a), a walk is sufficient.)
- If  $e$  appears in no cycle of  $G$ , then the two endpoints of  $e$  (which are clearly path-connected in  $G$ ) should not be path-connected in  $G \setminus e$  any more (because otherwise, we could pick a path in  $G \setminus e$  that connects these two endpoints, and append the edge  $e$  to it, thus obtaining a cycle of  $G$  that contains the edge  $e$ ). Thus, the component of  $G$  that contains these two endpoints should split into two components of  $G \setminus e$ . (Why only two? This takes a bit of thought.) Meanwhile, all remaining components of  $G$  should not be affected by the removal of  $e$ , and thus remain components of  $G \setminus e$ .

What follows is a long and cumbersome but fairly straightforward formalization of these intuitive arguments. I cannot recommend reading it, but I hope its presence helps to lend this worksheet an aura of completeness and autarky. (Much of the difficulty in formalizing this argument stems from the arduousness of comparing the equivalence classes of different equivalence relations.)

Before we get to the proof of Lemma 6.3.13, let us first show a simple fact about walks:

**Lemma 6.7.1.** Let  $G$  be a graph. Let  $\mathbf{w}$  be a walk of  $G$ . Let  $w$  be a vertex of  $\mathbf{w}$ . Let  $e$  be an edge of  $\mathbf{w}$ . Let  $u$  and  $v$  be the endpoints of  $e$ . Then, there exists a walk of  $G \setminus e$  that starts at  $w$  and ends at  $u$  or  $v$ .

*Proof of Lemma 6.7.1.* Write the walk  $\mathbf{w}$  in the form

$$\mathbf{w} = (w_0, f_1, w_1, f_2, w_2, \dots, f_n, w_n).$$

We introduce some terminology:

- A *forward subwalk* of  $\mathbf{w}$  shall mean a walk of the form

$$(w_i, f_{i+1}, w_{i+1}, f_{i+2}, w_{i+2}, \dots, f_j, w_j)$$

for two integers  $i$  and  $j$  satisfying  $0 \leq i \leq j \leq n$ . (Thus, in particular, the walk  $\mathbf{w}$  itself is a forward subwalk of  $\mathbf{w}$ , obtained by setting  $i = 0$  and  $j = n$ . Also, the length-0 walks  $(w_i)$  for all  $i \in \{0, 1, \dots, n\}$  are forward subwalks of  $\mathbf{w}$ . A less trivial example of a forward subwalk of  $\mathbf{w}$  is  $(w_2, f_3, w_3, f_4, w_4)$  (assuming that  $n \geq 4$ ).)

- A *backward subwalk* of  $\mathbf{w}$  shall mean a walk of the form

$$(w_j, f_j, w_{j-1}, f_{j-1}, w_{j-2}, \dots, f_{i+1}, w_i)$$

for two integers  $i$  and  $j$  satisfying  $0 \leq i \leq j \leq n$ . (In other words, a backward subwalk of  $\mathbf{w}$  is a forward subwalk of  $\mathbf{w}$  walked backwards. A less trivial example of a backward subwalk of  $\mathbf{w}$  is  $(w_5, f_5, w_4, f_4, w_3)$  (assuming that  $n \geq 5$ ).)

- A *subwalk* of  $\mathbf{w}$  shall mean a forward subwalk or a backward subwalk of  $\mathbf{w}$ . (Thus, roughly speaking, a subwalk of  $\mathbf{w}$  is a contiguous part of the walk  $\mathbf{w}$ , walked in either forward or backward direction.)

We note that the walk  $\mathbf{w}$  contains the edge  $e$ , whose endpoints are  $u$  and  $v$ . Thus,  $u$  and  $v$  are vertices of  $\mathbf{w}$ . Also,  $w$  is a vertex of  $\mathbf{w}$ .

We shall say that a walk is *nice* if it starts at  $w$  and ends at  $u$  or  $v$ . Clearly, the walk  $\mathbf{w}$  has a subwalk that starts at  $w$  and ends at  $u$  (in fact, we know that  $u$  and  $w$  are vertices of  $\mathbf{w}$ , so all we need is to locate  $u$  and  $w$  on  $\mathbf{w}$  and walk the part of  $\mathbf{w}$  from  $w$  to  $u$  <sup>28</sup>). Thus,  $\mathbf{w}$  has a nice subwalk (since a subwalk that starts at  $w$  and ends at  $u$  is automatically nice).

Now, let us pick a **shortest** nice subwalk  $\mathbf{w}'$  of  $\mathbf{w}$ . (This is well-defined, because we have just explained that  $\mathbf{w}$  has a nice subwalk.) This subwalk  $\mathbf{w}'$  cannot contain the edge  $e$  (because otherwise, we could remove the edge  $e$  and all successive edges from  $\mathbf{w}'$ , and we would then get a shorter subwalk of  $\mathbf{w}$  that is still nice<sup>29</sup>; but this would contradict the fact that  $\mathbf{w}'$  is a **shortest** nice subwalk of  $\mathbf{w}$ ). Hence, all edges of  $\mathbf{w}'$  are edges of the graph  $G \setminus e$ ; therefore,  $\mathbf{w}'$  is a walk of  $G \setminus e$ . Since  $\mathbf{w}'$  is nice, we thus conclude that there exists a nice walk of  $G \setminus e$  (namely,  $\mathbf{w}'$ ). In other words, there exists a walk of  $G \setminus e$  that starts at  $w$  and ends at  $u$  or  $v$  (because this is what it means to be “nice”). This proves Lemma 6.7.1.  $\square$

We shall need two more simple lemmas:

**Lemma 6.7.2.** Let  $G$  be a graph. Let  $e$  be an edge of  $G$ . Let  $u$  and  $v$  be the two endpoints of  $e$ . If the graph  $G \setminus e$  has a path from  $u$  to  $v$ , then  $e$  is an edge of some cycle of  $G$ .

*Proof of Lemma 6.7.2.* Write the graph  $G$  as  $G = (V, E, \varphi)$ .

Assume that the graph  $G \setminus e$  has a path from  $u$  to  $v$ . Let  $\mathbf{p} = (w_0, f_1, w_1, f_2, w_2, \dots, f_k, w_k)$  be this path (so that  $w_0 = u$  and  $w_k = v$ ). Since  $u$  and  $v$  are the two endpoints of  $e$ ,

<sup>28</sup>Here is this argument in detail: We know that  $u$  and  $w$  are vertices of  $\mathbf{w}$ . In other words, there exist integers  $p \in \{0, 1, \dots, n\}$  and  $q \in \{0, 1, \dots, n\}$  such that  $u = w_p$  and  $w = w_q$ . Consider these  $p$  and  $q$ . Now, we need to find a subwalk of  $\mathbf{w}$  that starts at  $w$  and ends at  $u$ . If  $p \geq q$ , then the forward subwalk

$$(w_q, f_{q+1}, w_{q+1}, f_{q+2}, w_{q+2}, \dots, f_p, w_p)$$

of  $\mathbf{w}$  is such a subwalk. If  $p < q$ , then we instead have to use the backward subwalk

$$(w_q, f_q, w_{q-1}, f_{q-1}, w_{q-2}, \dots, f_{p+1}, w_p).$$

In either case, we have found a subwalk of  $\mathbf{w}$  that starts at  $w$  and ends at  $u$ .

<sup>29</sup>It is still nice because it starts at  $w$  (since  $\mathbf{w}'$  starts at  $w$ ) and ends at  $u$  or  $v$  (because it ends at an endpoint of  $e$ , according to its construction).

we have  $\varphi(e) = \{u, v\} = \{v, u\} = \{w_k, w_0\}$  (since  $v = w_k$  and  $u = w_0$ ). Hence, the tuple  $(w_0, f_1, w_1, f_2, w_2, \dots, f_k, w_k, e, w_0)$  (which is obtained by appending  $e$  and  $w_0$  to the end of  $\mathbf{p}$ ) is a circuit of  $G$ . Moreover, the edges  $f_1, f_2, \dots, f_k, e$  of this circuit are distinct<sup>30</sup>, and its first  $k + 1$  vertices  $w_0, w_1, \dots, w_k$  are distinct as well<sup>31</sup>. Thus, this circuit is a cycle (since its length is  $k + 1 > 0$ ). Clearly,  $e$  is an edge of this cycle. Hence,  $e$  is an edge of some cycle of  $G$ . This proves Lemma 6.7.2.  $\square$

**Lemma 6.7.3.** Let  $G$  be a graph. Let  $e$  be an edge of  $G$ . Let  $u$  and  $v$  be the two endpoints of  $e$ . Clearly,  $u$  and  $v$  are path-connected in  $G$  (by Proposition 6.3.8 (a), since  $(u, e, v)$  is a walk from  $u$  to  $v$ ). Thus,  $u$  and  $v$  belong to one and the same component of  $G$ . Let  $C$  be this component. Let  $A$  be the component of  $G \setminus e$  that contains  $u$ . Let  $B$  be the component of  $G \setminus e$  that contains  $v$ .

Let  $\mathbf{w}$  be a walk of  $G$  such that  $e$  is an edge of  $\mathbf{w}$ . Let  $w$  be any vertex of  $\mathbf{w}$ . Then:

- (a) We have  $w \in C$ .
- (b) We have  $w \in A \cup B$ .

*Proof of Lemma 6.7.3.* Lemma 6.7.1 yields that there exists a walk of  $G \setminus e$  that starts at  $w$  and ends at  $u$  or  $v$ . Let  $\mathbf{w}'$  be this walk. Thus,  $\mathbf{w}'$  starts at  $w$  and ends at  $u$  or  $v$ . Note that  $u$  and  $v$  play symmetric roles in our setup. More precisely, we can swap  $u$  with  $v$  at will, as long as we simultaneously swap  $A$  with  $B$ ; this leaves the claim of Lemma 6.7.3 unchanged. Hence, we can WLOG assume that  $\mathbf{w}'$  ends at  $u$  (since we know that  $\mathbf{w}'$  ends at  $u$  or  $v$ ). Assume this.

Now, the walk  $\mathbf{w}'$  is a walk from  $w$  to  $u$  (since it starts at  $w$  and ends at  $u$ ). Hence, the graph  $G \setminus e$  has a walk from  $w$  to  $u$  (namely,  $\mathbf{w}'$ ). Consequently, Proposition 6.3.8 (a) (applied to  $G \setminus e$ ,  $w$  and  $u$  instead of  $G$ ,  $u$  and  $v$ ) shows that the two vertices  $w$  and  $u$  of  $G \setminus e$  are path-connected. Therefore, the two vertices  $w$  and  $u$  of  $G$  are path-connected, too (since any path of  $G \setminus e$  is a path of  $G$ ). Hence,  $w$  and  $u$  belong to the same component of  $G$  (since the components of  $G$  are the equivalence classes of the relation “path-connected”). In other words, the component of  $G$  that contains  $u$  also contains  $w$ . In other words,  $C$  also contains  $w$  (since we defined  $C$  to be the component of  $G$  that contains  $u$ ). In other words,  $w \in C$ . This proves Lemma 6.7.3 (a).

(b) We have already shown that the two vertices  $w$  and  $u$  of  $G \setminus e$  are path-connected. Thus,  $w$  and  $u$  belong to the same component of  $G \setminus e$  (since the components of  $G \setminus e$  are the equivalence classes of the relation “path-connected”). In other words, the component of  $G \setminus e$  that contains  $u$  also contains  $w$ . In other words,  $A$  also contains  $w$  (since we defined  $A$  to be the component of  $G \setminus e$  that contains  $u$ ). Hence,  $w \in A \subseteq A \cup B$ . This proves Lemma 6.7.3 (b).  $\square$

<sup>30</sup>*Proof.* Recall that  $(w_0, f_1, w_1, f_2, w_2, \dots, f_k, w_k)$  is a path in  $G \setminus e$ . Hence,  $f_1, f_2, \dots, f_k$  are distinct edges of  $G \setminus e$ . In other words,  $f_1, f_2, \dots, f_k$  are distinct edges of  $G$  that are distinct from  $e$ . Hence, the edges  $f_1, f_2, \dots, f_k, e$  of  $G$  are distinct.

<sup>31</sup>since  $(w_0, f_1, w_1, f_2, w_2, \dots, f_k, w_k)$  is a path

*Proof of Lemma 6.3.13 (sketched).* If  $H$  is any graph, then we shall denote the relation “path-connected” (defined with respect to  $H$ ) by  $\overset{H}{\sim}$ . That is,  $\overset{H}{\sim}$  is the binary relation on the vertex set of  $H$  that is defined as follows: Two vertices  $u$  and  $v$  of  $H$  satisfy  $u \overset{H}{\sim} v$  if and only if  $u$  and  $v$  are path-connected (in the graph  $H$ ).

Therefore,  $\overset{H}{\sim}$  is an equivalence relation, and the components of a graph  $H$  are the equivalence classes of this relation  $\overset{H}{\sim}$  (since they are defined to be the equivalence classes of the relation “path-connected”).

Write the graph  $G$  as  $G = (V, E, \varphi)$ . Thus,  $V$  is the vertex set of  $G$ . Clearly,  $V$  is also the vertex set of  $G \setminus e$ .

(a) Assume that  $e$  is an edge of some cycle of  $G$ . Let  $\mathbf{c}$  be this cycle. Write the cycle  $\mathbf{c}$  in the form

$$\mathbf{c} = (v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k).$$

Then,  $e = e_i$  for some  $i \in [k]$  (since  $e$  is an edge of  $\mathbf{c}$ ). Consider this  $i$ . Moreover,  $v_0 = v_k$  (since  $\mathbf{c}$  is a cycle). Clearly, the two tuples

$$(v_0, e_1, v_1, e_2, v_2, \dots, e_{i-1}, v_{i-1}) \quad \text{and} \quad (v_i, e_{i+1}, v_{i+1}, e_{i+2}, v_{i+2}, \dots, e_k, v_k)$$

are walks of the graph  $G$  (since they are pieces of the cycle  $\mathbf{c}$ ). Moreover,  $\mathbf{c}$  is a cycle, thus a walk; therefore,  $\varphi(e_i) = \{v_{i-1}, v_i\}$ .

The edges of  $\mathbf{c}$  are distinct (since  $\mathbf{c}$  is a cycle); in other words, the  $k$  edges  $e_1, e_2, \dots, e_k$  are distinct. Hence, the edges  $e_1, e_2, \dots, e_{i-1}$  are distinct from  $e_i$ . In other words, the edges  $e_1, e_2, \dots, e_{i-1}$  are distinct from  $e$  (since  $e = e_i$ ). Hence, these edges  $e_1, e_2, \dots, e_{i-1}$  are edges of the graph  $G \setminus e$ . Thus, the tuple  $(v_0, e_1, v_1, e_2, v_2, \dots, e_{i-1}, v_{i-1})$  is a walk of the graph  $G \setminus e$  (since we know that this tuple is a walk of the graph  $G$ ). This walk is clearly a walk from  $v_0$  to  $v_{i-1}$ . Hence, the graph  $G \setminus e$  has a walk from  $v_0$  to  $v_{i-1}$  (namely, the walk we just mentioned). Thus, the vertices  $v_0$  and  $v_{i-1}$  of  $G \setminus e$  are path-connected (by Proposition 6.3.8 (a), applied to  $G \setminus e$ ,  $v_0$  and  $v_{i-1}$  instead of  $G$ ,  $u$  and  $v$ ). In other words,  $v_0 \overset{G \setminus e}{\sim} v_{i-1}$  (by the definition of our notation  $\overset{G \setminus e}{\sim}$ ).

A similar argument (using the walk  $(v_i, e_{i+1}, v_{i+1}, e_{i+2}, v_{i+2}, \dots, e_k, v_k)$  instead of the walk  $(v_0, e_1, v_1, e_2, v_2, \dots, e_{i-1}, v_{i-1})$ ) shows that  $v_i \overset{G \setminus e}{\sim} v_k$ . In view of  $v_0 = v_k$ , this rewrites as  $v_i \overset{G \setminus e}{\sim} v_0$ . Combining this with  $v_0 \overset{G \setminus e}{\sim} v_{i-1}$ , we obtain  $v_i \overset{G \setminus e}{\sim} v_{i-1}$  (since  $\overset{G \setminus e}{\sim}$  is an equivalence relation). Therefore,  $v_{i-1} \overset{G \setminus e}{\sim} v_i$  (since  $\overset{G \setminus e}{\sim}$  is an equivalence relation).

We need to show that the components of  $G \setminus e$  are precisely the components of  $G$ . To do so, it clearly suffices to show that the “path-connectedness” relation of the graph  $G \setminus e$  is precisely the “path-connectedness” relation of the graph  $G$  (because the components of a graph are the equivalence classes of its “path-connectedness” relation). In other words, it suffices to show that the relation  $\overset{G \setminus e}{\sim}$  is precisely the relation  $\overset{G}{\sim}$  (since these two relations are the “path-connectedness” relations of the

graphs  $G \setminus e$  and  $G$ , respectively). In other words, it suffices to show that if  $u$  and  $v$  are two vertices of  $G$ , then  $u \stackrel{G \setminus e}{\sim} v$  holds if and only if  $u \stackrel{G}{\sim} v$  holds.

So let us show this. Let  $u$  and  $v$  be two vertices of  $G$ . We must show that  $u \stackrel{G \setminus e}{\sim} v$  holds if and only if  $u \stackrel{G}{\sim} v$  holds. The “only if” part of this is clear (since any path of  $G \setminus e$  is a path of  $G$ ); thus, we only need to prove the “if” part. So we assume that  $u \stackrel{G}{\sim} v$ . Our goal is to show that  $u \stackrel{G \setminus e}{\sim} v$ .

We know that  $u \stackrel{G}{\sim} v$ . In other words,  $u$  and  $v$  are path-connected in  $G$ . In other words, the graph  $G$  has a path from  $u$  to  $v$ . Let  $\mathbf{p}$  be such a path. This path  $\mathbf{p}$  is a path of  $G$ . We are in one of the following two cases:

Case 1: The edge  $e$  is not an edge of  $\mathbf{p}$ .

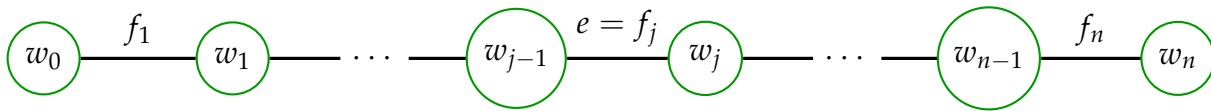
Case 2: The edge  $e$  is an edge of  $\mathbf{p}$ .

Let us first consider Case 1. In this case,  $e$  is not an edge of  $\mathbf{p}$ . Hence, all edges of  $\mathbf{p}$  are edges of  $G$  distinct from  $e$ . In other words, all edges of  $\mathbf{p}$  are edges of  $G \setminus e$ . Thus,  $\mathbf{p}$  is a path of  $G \setminus e$  as well (since  $\mathbf{p}$  is a path of  $G$ ). Therefore,  $G \setminus e$  has a path from  $u$  to  $v$  (namely, the path  $\mathbf{p}$ ). In other words,  $u$  and  $v$  are path-connected in  $G \setminus e$ . In other words,  $u \stackrel{G \setminus e}{\sim} v$ . Hence,  $u \stackrel{G \setminus e}{\sim} v$  is proved in Case 1.

Let us now consider Case 2. In this case,  $e$  is an edge of  $\mathbf{p}$ . Write the path  $\mathbf{p}$  in the form

$$\mathbf{p} = (w_0, f_1, w_1, f_2, w_2, \dots, f_n, w_n).$$

Since  $\mathbf{p}$  is a path from  $u$  to  $v$ , we thus must have  $w_0 = u$  and  $w_n = v$ . Furthermore,  $e = f_j$  for some  $j \in [n]$  (since  $e$  is an edge of  $\mathbf{p}$ ). Consider this  $j$ . Hence,  $f_j = e = e_i$ . Here is a picture showing the path  $\mathbf{p}$ :



(note that all vertices  $w_0, w_1, \dots, w_n$  are distinct since  $\mathbf{p}$  is a path).

Note that  $\mathbf{p}$  is a path, thus a walk; therefore,  $\varphi(f_j) = \{w_{j-1}, w_j\}$ . However, from  $f_j = e_i$ , we obtain  $\varphi(f_j) = \varphi(e_i)$ . In other words,  $\{w_{j-1}, w_j\} = \{v_{i-1}, v_i\}$  (since  $\varphi(f_j) = \{w_{j-1}, w_j\}$  and  $\varphi(e_i) = \{v_{i-1}, v_i\}$ ). Thus, we have

$$\text{either } (w_{j-1} = v_{i-1} \text{ and } w_j = v_i) \text{ or } (w_{j-1} = v_i \text{ and } w_j = v_{i-1}).$$

In either of these cases, we conclude that  $w_{j-1} \stackrel{G \setminus e}{\sim} w_j$  (since  $v_i \stackrel{G \setminus e}{\sim} v_{i-1}$  and  $v_{i-1} \stackrel{G \setminus e}{\sim} v_i$ ).

Clearly, the two tuples

$$(w_0, f_1, w_1, f_2, w_2, \dots, f_{j-1}, w_{j-1}) \quad \text{and} \quad (w_j, f_{j+1}, w_{j+1}, f_{j+2}, \dots, f_n, w_n)$$

are walks of the graph  $G$  (since they are pieces of the walk  $\mathbf{p}$ ).

However, the edges of  $\mathbf{p}$  are distinct (by Proposition 6.6.2); in other words, the  $n$  edges  $f_1, f_2, \dots, f_n$  are distinct. Hence, the edges  $f_1, f_2, \dots, f_{j-1}$  are distinct from  $f_j$ . In other words, the edges  $f_1, f_2, \dots, f_{j-1}$  are distinct from  $e$  (since  $e = f_j$ ). Hence, these edges  $f_1, f_2, \dots, f_{j-1}$  are edges of the graph  $G \setminus e$ . Thus, the tuple  $(w_0, f_1, w_1, f_2, w_2, \dots, f_{j-1}, w_{j-1})$  is a walk of the graph  $G \setminus e$  (since we know that this tuple is a walk of the graph  $G$ ). This walk is clearly a walk from  $w_0$  to  $w_{j-1}$ . Hence, the graph  $G \setminus e$  has a walk from  $w_0$  to  $w_{j-1}$  (namely, the walk we just mentioned). Thus, the vertices  $w_0$  and  $w_{j-1}$  of  $G \setminus e$  are path-connected (by Proposition 6.3.8 (a), applied to  $G \setminus e$ ,  $w_0$  and  $w_{j-1}$  instead of  $G$ ,  $u$  and  $v$ ). In other words,  $w_0 \stackrel{G \setminus e}{\sim} w_{j-1}$  (by the definition of our notation  $\stackrel{G \setminus e}{\sim}$ ).

A similar argument (using the walk  $(w_j, f_{j+1}, w_{j+1}, f_{j+2}, \dots, f_n, w_n)$  instead of  $(w_0, f_1, w_1, f_2, w_2, \dots, f_{j-1}, w_{j-1})$ ) shows that  $w_j \stackrel{G \setminus e}{\sim} w_n$ .

Combining  $w_0 \stackrel{G \setminus e}{\sim} w_{j-1}$  with  $w_{j-1} \stackrel{G \setminus e}{\sim} w_j$  and  $w_j \stackrel{G \setminus e}{\sim} w_n$ , we obtain  $w_0 \stackrel{G \setminus e}{\sim} w_n$  (since  $\stackrel{G \setminus e}{\sim}$  is an equivalence relation). In other words,  $u \stackrel{G \setminus e}{\sim} v$  (since  $w_0 = u$  and  $w_n = v$ ). Thus,  $u \stackrel{G \setminus e}{\sim} v$  is proved in Case 2.

Now we have proved  $u \stackrel{G \setminus e}{\sim} v$  in both Cases 1 and 2. Hence,  $u \stackrel{G \setminus e}{\sim} v$  always holds. As we explained above, this completes the proof of Lemma 6.3.13 (a).

**(b)** Assume that  $e$  appears in no cycle of  $G$  (that is, there exists no cycle  $\mathbf{c}$  of  $G$  such that  $e$  is an edge of  $\mathbf{c}$ ). We must prove that the graph  $G \setminus e$  has one more component than  $G$ . We shall achieve this as follows:

We let  $u$  and  $v$  be the two endpoints of  $e$ . Clearly,  $u$  and  $v$  are path-connected in  $G$  (by Proposition 6.3.8 (a), since  $(u, e, v)$  is a walk from  $u$  to  $v$ ). Thus,  $u$  and  $v$  belong to one and the same component of  $G$ . Let  $C$  be this component. Let  $A$  be the component of  $G \setminus e$  that contains  $u$ . Let  $B$  be the component of  $G \setminus e$  that contains  $v$ . We shall now show the following:

*Claim 1:* We have  $A \neq B$ .

*Claim 2:* We have  $A \cup B = C$ .

*Claim 3:* The components of  $G$  distinct from  $C$  are precisely the components of  $G \setminus e$  distinct from  $A$  and  $B$ .

Once these three claims are proved, it will be clear how the components of  $G \setminus e$  differ from those of  $G$ : Namely, the component  $C$  of  $G$  breaks apart into two distinct components  $A$  and  $B$  of  $G \setminus e$  (by Claim 1 and Claim 2), whereas all other components of  $G$  remain components of  $G \setminus e$  (by Claim 3). Thus, it will follow immediately that  $G \setminus e$  has one more component than  $G$ . This will complete the proof of Lemma 6.3.13 (b).

Thus, it remains to prove the three Claims 1, 2 and 3. We begin with Claim 1:

[*Proof of Claim 1:* Assume the contrary. Thus,  $A = B$ . The definition of  $A$  yields  $u \in A$ . The definition of  $B$  yields  $v \in B$ . In view of  $A = B$ , this rewrites as  $v \in A$ . Now, the vertices  $u$  and  $v$  both lie in  $A$  (since  $u \in A$  and  $v \in A$ ). Hence,  $u$  and  $v$  lie in the same component of  $G \setminus e$  (since  $A$  is a component of  $G \setminus e$ ). In other words,  $u \stackrel{G \setminus e}{\sim} v$  (since the components of  $G \setminus e$  are the equivalence classes of the relation  $\stackrel{G \setminus e}{\sim}$ ). Due to the definition of  $\stackrel{G \setminus e}{\sim}$ , this is saying that the graph  $G \setminus e$  has a path from  $u$  to  $v$ . Hence, Lemma 6.7.2 shows that  $e$  is an edge of some cycle of  $G$ . However, this contradicts our assumption that  $e$  appears in no cycle of  $G$ . This contradiction shows that our assumption was false; thus, Claim 1 has been proved.]

In order to prove Claim 2, we shall split it into two smaller claims:

*Claim 2A:* We have  $A \cup B \subseteq C$ .

*Claim 2B:* We have  $A \cup B \supseteq C$ .

[*Proof of Claim 2A:* Let  $w \in A \cup B$ . We shall show that  $w \in C$ .

We have  $w \in A$  or  $w \in B$  (since  $w \in A \cup B$ ). Thus, we WLOG assume that  $w \in A$  (since the case  $w \in B$  is analogous<sup>32</sup>). In other words,  $w$  belongs to the component of  $G \setminus e$  that contains  $u$  (since  $A$  was defined to be the component of  $G \setminus e$  that contains  $u$ ). Equivalently, we have  $w \stackrel{G \setminus e}{\sim} u$  (since the components of  $G \setminus e$  are the equivalence classes of the relation  $\stackrel{G \setminus e}{\sim}$ ). Hence,  $w \stackrel{G}{\sim} u$  (since any path of  $G \setminus e$  is a path of  $G$ ). In other words,  $w$  belongs to the component of  $G$  that contains  $u$  (since the components of  $G$  are the equivalence classes of the relation  $\stackrel{G}{\sim}$ ). Since the latter component is  $C$  (by the definition of  $C$ ), we have thus shown that  $w \in C$ .

Forget that we fixed  $w$ . We thus have proved that  $w \in C$  for each  $w \in A \cup B$ . In other words,  $A \cup B \subseteq C$ . This proves Claim 2A.]

[*Proof of Claim 2B:* Let  $w \in C$ . We shall show that  $w \in A \cup B$ .

We have  $w \in C$ . In other words,  $w$  belongs to the component of  $G$  that contains  $u$  (since the components of  $G$  are the equivalence classes of the relation  $\stackrel{G}{\sim}$ ). Equivalently, we have  $w \stackrel{G}{\sim} u$  (since the components of  $G$  are the equivalence classes of the relation  $\stackrel{G}{\sim}$ ). In other words,  $G$  has a path from  $w$  to  $u$  (by the definition of the relation  $\stackrel{G}{\sim}$ ). Let  $\mathbf{p} = (w_0, f_1, w_1, f_2, w_2, \dots, f_n, w_n)$  be this path (so that  $w_0 = w$  and  $w_n = u$ ). Thus,  $(w_0, f_1, w_1, f_2, w_2, \dots, f_n, w_n, e, v)$  is a walk of  $G$  (since  $\varphi(e) = \{u, v\} = \{w_n, v\}$  (because  $u = w_n$ )). Let  $\mathbf{w}$  be this walk. Then,  $e$  is an edge of  $\mathbf{w}$  (namely, the last edge), and  $w$  is a vertex of  $\mathbf{w}$  (since  $w = w_0$ ). Hence, Lemma 6.7.3 (b) yields  $w \in A \cup B$ .

Forget that we fixed  $w$ . We thus have proved that  $w \in A \cup B$  for each  $w \in C$ . In other words,  $A \cup B \supseteq C$ . This proves Claim 2B.]

<sup>32</sup>In fact, our situation does not change if we swap  $u$  with  $v$  while simultaneously swapping  $A$  with  $B$ .



[Proof of Claim 2: Combining Claim 2A with Claim 2B, we obtain  $C = A \cup B$ . Thus, Claim 2 follows.]

In order to prove Claim 3, we shall need the following three auxiliary claims:

*Claim 3A:* Let  $x \in V$  and  $w \in V$  be such that  $w \notin C$ . Then,  $x \stackrel{G}{\sim} w$  holds if and only if  $x \stackrel{G \setminus e}{\sim} w$ .

*Claim 3B:* Each component of  $G$  distinct from  $C$  is a component of  $G \setminus e$  distinct from  $A$  and  $B$ .

*Claim 3C:* Each component of  $G \setminus e$  distinct from  $A$  and  $B$  is a component of  $G$  distinct from  $C$ .

[Proof of Claim 3A: We need to prove the two implications  $(x \stackrel{G}{\sim} w) \implies (x \stackrel{G \setminus e}{\sim} w)$  and  $(x \stackrel{G \setminus e}{\sim} w) \implies (x \stackrel{G}{\sim} w)$ . Since the implication  $(x \stackrel{G \setminus e}{\sim} w) \implies (x \stackrel{G}{\sim} w)$  is obvious (because each path of  $G \setminus e$  is a path of  $G$ ), it thus only remains to prove the implication  $(x \stackrel{G}{\sim} w) \implies (x \stackrel{G \setminus e}{\sim} w)$ . So let us assume that  $x \stackrel{G}{\sim} w$ . We must show that  $x \stackrel{G \setminus e}{\sim} w$ .

We have  $x \stackrel{G}{\sim} w$ . In other words,  $G$  has a path from  $x$  to  $w$  (by the definition of the relation  $\stackrel{G}{\sim}$ ). Let  $\mathbf{p}$  be this path. Clearly,  $w$  is a vertex of this path  $\mathbf{p}$ . If  $e$  was an edge of this path  $\mathbf{p}$ , then Lemma 6.7.3 (a) (applied to  $\mathbf{w} = \mathbf{p}$ ) would yield that  $w \in C$  (since  $\mathbf{p}$  is a path, thus a walk); but this would contradict  $w \notin C$ . Thus,  $e$  is not an edge of the path  $\mathbf{p}$ . Therefore, all edges of  $\mathbf{p}$  are distinct from  $e$  and thus are edges of  $G \setminus e$ . In other words,  $\mathbf{p}$  is a path of  $G \setminus e$ . Hence,  $G \setminus e$  has a path from  $x$  to  $w$  (namely,  $\mathbf{p}$ ). In other words,  $x \stackrel{G \setminus e}{\sim} w$  (by the definition of the relation  $\stackrel{G \setminus e}{\sim}$ ). Thus, we have proved the implication  $(x \stackrel{G}{\sim} w) \implies (x \stackrel{G \setminus e}{\sim} w)$ . The proof of Claim 3A is thus complete.]

[Proof of Claim 3B: Let  $D$  be a component of  $G$  distinct from  $C$ . We must prove that  $D$  is a component of  $G \setminus e$  distinct from  $A$  and  $B$ .

The set  $D$  is a component of  $G$ , thus an equivalence class of the relation  $\stackrel{G}{\sim}$  (by the definition of a component). In other words,  $D$  is the equivalence class of some vertex  $w \in V$  with respect to the relation  $\stackrel{G}{\sim}$ . Consider this  $w$ . Thus,

$$\begin{aligned} D &= \left( \text{the equivalence class of } \stackrel{G}{\sim} \text{ containing } w \right) \\ &= \left\{ x \in V \mid x \stackrel{G}{\sim} w \right\}. \end{aligned} \tag{11}$$

Hence,  $w \in D$ .

However, the components of  $G$  are the equivalence classes of the equivalence relation  $\sim^G$ ; thus, they are mutually disjoint (since the equivalence classes of an equivalence relation are always disjoint). In other words, any two distinct components of  $G$  are disjoint. Hence,  $C$  and  $D$  are disjoint (since  $C$  and  $D$  are two distinct components of  $G$ ). In other words, an element of  $D$  cannot be an element of  $C$ . Hence,  $w \notin C$  (since  $w \in D$ ).

Therefore, Claim 3A shows that a vertex  $x \in V$  satisfies  $x \sim^G w$  if and only if it satisfies  $x \sim^{G \setminus e} w$ . Thus,  $\{x \in V \mid x \sim^G w\} = \{x \in V \mid x \sim^{G \setminus e} w\}$ . Hence, (11) becomes

$$\begin{aligned} D &= \{x \in V \mid x \sim^G w\} = \{x \in V \mid x \sim^{G \setminus e} w\} \\ &= \left( \text{the equivalence class of } \sim^{G \setminus e} \text{ containing } w \right). \end{aligned}$$

This shows that  $D$  is an equivalence class of  $\sim^{G \setminus e}$ . In other words,  $D$  is a component of  $G \setminus e$  (by the definition of a component). It remains to show that  $D$  is distinct from  $A$  and  $B$ .

We have  $A \subseteq A \cup B = C$  (by Claim 2). Thus, from  $w \notin C$ , we obtain  $w \notin A$ . Comparing this with  $w \in D$ , we obtain  $A \neq D$  (since  $D$  contains  $w$ , but  $A$  does not). In other words,  $D$  is distinct from  $A$ . Similarly,  $D$  is distinct from  $B$ . This completes the proof of Claim 3B.]

[*Proof of Claim 3C:* Let  $D$  be a component of  $G \setminus e$  distinct from  $A$  and  $B$ . We must prove that  $D$  is a component of  $G$  distinct from  $C$ .

The set  $D$  is a component of  $G \setminus e$ , thus an equivalence class of the relation  $\sim^{G \setminus e}$  (by the definition of a component). In other words,  $D$  is the equivalence class of some vertex  $w \in V$  with respect to the relation  $\sim^{G \setminus e}$ . Consider this  $w$ . Thus,

$$\begin{aligned} D &= \left( \text{the equivalence class of } \sim^{G \setminus e} \text{ containing } w \right) \\ &= \{x \in V \mid x \sim^{G \setminus e} w\}. \end{aligned} \tag{12}$$

Hence,  $w \in D$ .

However, the components of  $G \setminus e$  are the equivalence classes of the equivalence relation  $\sim^{G \setminus e}$ ; thus, they are mutually disjoint (since the equivalence classes of an equivalence relation are always disjoint). In other words, any two distinct components of  $G \setminus e$  are disjoint. Hence,  $A$  and  $D$  are disjoint (since  $A$  and  $D$  are two distinct components of  $G \setminus e$ ). In other words, an element of  $D$  cannot be an element of  $A$ . Hence,  $w \notin A$  (since  $w \in D$ ). Similarly,  $w \notin B$ . Combining  $w \notin A$  with  $w \notin B$ , we obtain  $w \notin A \cup B$ . In other words,  $w \notin C$  (since Claim 2 yields  $C = A \cup B$ ).

Hence, Claim 3A shows that a vertex  $x \in V$  satisfies  $x \stackrel{G}{\sim} w$  if and only if it satisfies  $x \stackrel{G \setminus e}{\sim} w$ . Thus,  $\{x \in V \mid x \stackrel{G}{\sim} w\} = \{x \in V \mid x \stackrel{G \setminus e}{\sim} w\}$ . Comparing this with (12), we obtain

$$D = \{x \in V \mid x \stackrel{G}{\sim} w\} = \left( \text{the equivalence class of } \stackrel{G}{\sim} \text{ containing } w \right).$$

This shows that  $D$  is an equivalence class of  $\stackrel{G}{\sim}$ . In other words,  $D$  is a component of  $G$  (by the definition of a component). It remains to show that  $D$  is distinct from  $C$ . However, this follows by observing that  $D$  contains  $w$  (since  $w \in D$ ) but  $C$  does not (since  $w \notin C$ ). This completes the proof of Claim 3C.]

[*Proof of Claim 3:* Combining Claim 3B with Claim 3C, we see that the components of  $G$  distinct from  $C$  are precisely the components of  $G \setminus e$  distinct from  $A$  and  $B$ . This proves Claim 3.]

Now, as we said, we can finish our proof of Lemma 6.3.13 **(b)**. In fact, we know that  $A$  and  $B$  are two distinct components of  $G \setminus e$  (indeed, Claim 1 says that they are distinct). Hence,

$$\begin{aligned} & (\text{the number of components of } G \setminus e) \\ &= (\text{the number of components of } G \setminus e \text{ distinct from } A \text{ and } B) + 2. \end{aligned}$$

On the other hand, since  $C$  is a component of  $G$ , we have

$$\begin{aligned} & (\text{the number of components of } G) \\ &= \underbrace{(\text{the number of components of } G \text{ distinct from } C)}_{\substack{= (\text{the number of components of } G \setminus e \text{ distinct from } A \text{ and } B) \\ \text{(by Claim 3)}}} + 1 \\ &= (\text{the number of components of } G \setminus e \text{ distinct from } A \text{ and } B) + 1. \end{aligned}$$

The right hand sides of these two equalities clearly differ by 1. Thus, the left hand sides differ by 1 as well. In other words, we have

$$(\text{the number of components of } G \setminus e) = (\text{the number of components of } G) + 1.$$

In other words, the graph  $G \setminus e$  has one more component than  $G$ . This proves Lemma 6.3.13 **(b)** at last.  $\square$

## 6.8. Appendix: Details omitted from the proof of Theorem 6.3.15

*Details for the proof of Theorem 6.3.15.* Back in our proof of Theorem 6.3.15, we have left seven claims (Claims 1–7) unproved. Here are their proofs:

[Proof of Claim 1: The definition of  $E_0$  yields  $E_0 = \{e_1, e_2, \dots, e_0\} = \emptyset$ . However, the definition of  $G_0$  yields  $G_0 = (V, E_0, \varphi|_{E_0})$ . Thus, the graph  $G_0$  has edge set  $E_0$ . Since  $E_0 = \emptyset$ , this means that the graph  $G_0$  has edge set  $\emptyset$ . In other words, the graph  $G_0$  has no edges. Hence, Lemma 6.3.16 (applied to  $G_0$ ,  $E_0$  and  $\varphi|_{E_0}$  instead of  $G$ ,  $E$  and  $\varphi$ ) yields that  $G_0$  has exactly  $n$  components. In other words,  $c_0 = n$  (since  $c_0$  was defined as the number of components of  $G_0$ ). This proves Claim 1.]

[Proof of Claim 2: The definition of  $E_k$  yields  $E_k = \{e_1, e_2, \dots, e_k\} = E$ . Thus,  $\varphi|_{E_k} = \varphi|_E = \varphi$ . The definition of  $G_k$  yields

$$\begin{aligned} G_k &= (V, E_k, \varphi|_{E_k}) = (V, E, \varphi) && (\text{since } E_k = E \text{ and } \varphi|_{E_k} = \varphi) \\ &= G. \end{aligned}$$

This proves Claim 2.]

[Proof of Claim 3: Let  $i \in [k]$ . The definitions of  $E_i$  and  $E_{i-1}$  yield  $E_i = \{e_1, e_2, \dots, e_i\}$  and  $E_{i-1} = \{e_1, e_2, \dots, e_{i-1}\}$ . However, the edges  $e_1, e_2, \dots, e_k$  are distinct (by their definition). Thus,  $\{e_1, e_2, \dots, e_i\} \setminus \{e_i\} = \{e_1, e_2, \dots, e_{i-1}\}$ . In view of  $E_i = \{e_1, e_2, \dots, e_i\}$  and  $E_{i-1} = \{e_1, e_2, \dots, e_{i-1}\}$ , this rewrites as  $E_i \setminus \{e_i\} = E_{i-1}$ . The definition of  $G_i$  yields  $G_i = (V, E_i, \varphi|_{E_i})$ ; therefore,  $e_i$  is an edge of  $G_i$  (since  $e_i \in \{e_1, e_2, \dots, e_i\} = E_i$ ). Hence, the definition of  $G_i \setminus e_i$  yields that

$$G_i \setminus e_i = (V, E_i \setminus \{e_i\}, \varphi|_{E_i \setminus \{e_i\}}) = (V, E_{i-1}, \varphi|_{E_{i-1}})$$

(since  $E_i \setminus \{e_i\} = E_{i-1}$ ). On the other hand,  $G_{i-1} = (V, E_{i-1}, \varphi|_{E_{i-1}})$  (by the definition of  $G_{i-1}$ ). Comparing these two equalities, we obtain  $G_{i-1} = G_i \setminus e_i$ . This proves Claim 3.]

[Proof of Claim 4: Lemma 6.3.13 (a) (applied to  $G_i$  and  $e_i$  instead of  $G$  and  $e$ ) yields that the components of  $G_i \setminus e_i$  are precisely the components of  $G_i$  (since  $e_i$  is an edge of some cycle of  $G_i$ ). Since  $G_{i-1} = G_i \setminus e_i$ , we can restate this as follows: The components of  $G_{i-1}$  are precisely the components of  $G_i$ . Hence, the number of components of  $G_{i-1}$  equals the number of components of  $G_i$ . Since the former number is  $c_{i-1}$ , and since the latter number is  $c_i$ , we can rewrite this as follows:  $c_{i-1} = c_i$ . In other words,  $c_i = c_{i-1}$ . This proves Claim 4.]

[Proof of Claim 5: The definition of  $G_i$  yields  $G_i = (V, E_i, \varphi|_{E_i})$ ; therefore,  $e_i$  is an edge of  $G_i$  (since  $e_i \in \{e_1, e_2, \dots, e_i\} = E_i$ ). Hence, Lemma 6.3.13 (b) (applied to  $G_i$  and  $e_i$  instead of  $G$  and  $e$ ) yields that the graph  $G_i \setminus e_i$  has one more component than  $G_i$  (since  $e_i$  appears in no cycle of  $G_i$ ). Since  $G_{i-1} = G_i \setminus e_i$ , we can restate this as follows: The graph  $G_{i-1}$  has one more component than  $G_i$ . In other words, the number of components of  $G_{i-1}$  equals the number of components of  $G_i$  plus 1. Since the former number is  $c_{i-1}$ , and since the latter number is  $c_i$ , we can rewrite this as follows:  $c_{i-1} = c_i + 1$ . In other words,  $c_i = c_{i-1} - 1$ . This proves Claim 5.]

[Proof of Claim 6: Let  $i \in [k]$ . We must prove that  $c_i \geq c_{i-1} - 1$ . If  $e_i$  is an edge of some cycle of  $G_i$ , then this follows from Claim 4 (since Claim 4 yields  $c_i = c_{i-1} \geq c_{i-1} - 1$  in this case). Thus, we WLOG assume that  $e_i$  is not an edge of any cycle of  $G_i$ . In other words,  $e_i$  appears in no cycle of  $G_i$ . Hence, Claim 5 yields  $c_i = c_{i-1} - 1 \geq c_{i-1} - 1$ . This proves Claim 6.]

[Proof of Claim 7: We shall prove Claim 7 by induction on  $j$ :

*Induction base:* By Claim 1, we have  $c_0 = n = n - 0 \geq n - 0$ . In other words, Claim 7 holds for  $j = 0$ .

*Induction step:* Let  $i \in [k]$ . Assume that Claim 7 holds for  $j = i - 1$ . We must prove that Claim 7 holds for  $j = i$ .

We have assumed that Claim 7 holds for  $j = i - 1$ . In other words, we have  $c_{i-1} \geq n - (i - 1)$ . However, Claim 6 yields  $c_i \geq \underbrace{c_{i-1}}_{\geq n-(i-1)} - 1 \geq n - (i - 1) - 1 = n - i$ . In other words, Claim 7 holds for  $j = i$ . This completes the induction step. Thus, Claim 7 is proved.]  $\square$

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