University of Minnesota, School of Mathematics

Math 4281: Introduction to Modern Algebra, Spring 2019: Homework 2

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due date: **Wednesday, 13 February 2019** at the beginning of class, or before that by email or canvas.
Please solve **at most 3 of the 6 exercises**!

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1 Exercise 1: gcd basics

1.1 Problem

Prove the following:

(a) If a_1, a_2, b_1, b_2 are integers satisfying $a_1 \mid b_1$ and $a_2 \mid b_2$, then $\gcd(a_1, a_2) \mid \gcd(b_1, b_2)$.

(b) If a, b, c, s are integers, then $\gcd(sa, sb, sc) = |s| \gcd(a, b, c)$.

1.2 Solution

[...]

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2 Exercise 2: Products of gcds

2.1 Problem

Prove the following:

Any four integers u, v, x, y satisfy $\gcd(u, v) \gcd(x, y) = \gcd(ux, uy, vx, vy)$.

2.2 Solution

[...]

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3 Exercise 3: The gcd-lcm connection for three numbers

3.1 Problem

Let a, b, c be three integers. Prove that $\operatorname{lcm}(a, b, c) \gcd(bc, ca, ab) = |abc|$.

3.2 Solution

[...]

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4 Exercise 4: Divisibility tests for $3, 9, 11, 7$

4.1 Problem

Let n be a positive integer. Let "$d_k d_{k-1} \cdots d_0$" be the decimal representation of n; this means that $d_0, d_1, \ldots, d_k$ are digits (i.e., elements of $\{0, 1, \ldots, 9\}$) such that $n = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_0 10^0$. The digits $d_0, d_1, \ldots, d_k$ are called the *digits of* n.

(Incidentally, the quickest way to find these digits is by repeated division with remainder: To obtain the decimal representation of $n \geq 10$, you take the decimal representation of $n//10$ and append the digit $n\%10$ at the end. Thus,

$$d_0 = n\%10, \qquad d_1 = (n//10)\,\%10, \qquad d_2 = ((n//10)\,//10)\,\%10, \qquad \text{etc.}$$

But in this exercise, you can just assume that the decimal representation exists.)

(a) Prove that $3 \mid n$ if and only if $3 \mid d_k + d_{k-1} + \cdots + d_0$. (In other words, a positive integer n is divisible by 3 if and only if the sum of its digits is divisible by 3.)

(b) Prove that $9 \mid n$ if and only if $9 \mid d_k + d_{k-1} + \cdots + d_0$. (In other words, a positive integer n is divisible by 9 if and only if the sum of its digits is divisible by 9.)

(c) Prove that $11 \mid n$ if and only if $11 \mid (-1)^k d_k + (-1)^{k-1} d_{k-1} + \cdots + (-1)^0 d_0$. (In other words, a positive integer n is divisible by 11 if and only if the sum of its digits in the even positions minus the sum of its digits in the odd positions is divisible by 11.)

(d) Let $q = d_k 10^{k-1} + d_{k-1} 10^{k-2} + \cdots + d_1 10^0$. (Equivalently, $q = n//10 = \dfrac{n - d_0}{10}$; this is the number obtained from n by dropping the least significant digit.) Prove that $7 \mid n$ if and only if $7 \mid q - 2d_0$.

(This gives a recursive test for divisibility by 7.)

4.2 Solution

[...]

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5 Exercise 5: A divisibility

5.1 Problem

Let $n \in \mathbb{N}$. Prove that $7 \mid 3^{2n+1} + 2^{n+2}$.

5.2 Solution

[...]

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6 Exercise 6: A binomial coefficient sum

6.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} \binom{-2}{k} = (-1)^n \left((n+2) // 2 \right). \qquad (1)$$

6.2 Solution

[...]

References

[GrKnPa94] Ronald L. Graham, Donald E. Knuth, Oren Patashnik, *Concrete Mathematics, Second Edition*, Addison-Wesley 1994.
See `https://www-cs-faculty.stanford.edu/~knuth/gkp.html` for errata.

[Grinbe19] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 10 January 2019.
`http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf`
The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed

to match that in the citations above, see `https://github.com/darijgr/detnotes/releases/tag/2019-01-10` .