University of Minnesota, School of Mathematics

Math 4281: Introduction to Modern Algebra, Spring 2019: Homework 1

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due date: **Friday, 8 February 2019** at the beginning of class,
or before that by email or canvas.
Please solve **at most 4 of the 6 exercises**!

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1 Exercise 1: Mutual divisibility is rare

1.1 Problem

Let a and b be two integers such that $a \mid b$ and $b \mid a$. Prove that $|a| = |b|$.

1.2 Solution

[...]

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2 Exercise 2: Congruence means equal remainders

2.1 Problem

Let n be a positive integer. Let u and v be two integers. Prove that $u \equiv v \mod n$ if and only if $u\%n = v\%n$.

2.2 Solution

[...]

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3 Exercise 3: Even and odd

3.1 Problem

Let u be an integer.

(a) Prove that u is even if and only if $u\%2 = 0$.

(b) Prove that u is odd if and only if $u\%2 = 1$.

(c) Prove that u is even if and only if $u \equiv 0 \mod 2$.

(d) Prove that u is odd if and only if $u \equiv 1 \mod 2$.

(e) Prove that u is odd if and only if $u + 1$ is even.

(f) Prove that exactly one of the two numbers u and $u + 1$ is even.

(g) Prove that $u(u+1) \equiv 0 \mod 2$.

(h) Prove that $u^2 \equiv -u \equiv u \mod 2$.

3.2 Solution

[...]

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4 Exercise 4: Factorials 102

4.1 Problem

(a) Prove that

$$\frac{1! \cdot 2! \cdot \cdots \cdot (2n)!}{n!} = 2^n \cdot \prod_{i=1}^{n} \left((2i-1)!\right)^2 \qquad \text{for each } n \in \mathbb{N}.$$

(b) Prove that

$$\sum_{k=0}^{n} \frac{1}{k! \cdot (k+2)} = 1 - \frac{1}{(n+2)!} \qquad \text{for each } n \in \mathbb{N}.$$

4.2 Solution

[...]

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5 Exercise 5: Binomial coefficients 102

5.1 Problem

Prove that

$$\frac{(ab)!}{a!\,(b!)^a} = \prod_{k=1}^{a} \binom{kb-1}{b-1}$$

for all $a \in \mathbb{N}$ and all positive integers b.

5.2 Solution

[...]

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6 Exercise 6: Binomial coefficients and coprimality

6.1 Problem

It is well-known (see, e.g., [Grinbe19, Proposition 3.20]) that $\binom{n}{k} \in \mathbb{Z}$ for all $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. (This is not at all clear from the definition of $\binom{n}{k}$; it is saying that the product of any k consecutive integers is divisible by $k!$. The case of $k = 2$ is the statement of Exercise 3 **(g)**.) Thus, we can study the divisibility of binomial coefficients by various integers. There are hundreds of theorems about this; this exercise is about one of them.

Let a and b be two coprime positive integers.

(a) Prove that $\frac{a}{a+b}\binom{a+b}{a} = \binom{a+b-1}{a-1}$ and $\frac{b}{a+b}\binom{a+b}{a} = \binom{a+b-1}{b-1}$.

(b) Prove that if $h \in \mathbb{Q}$ satisfies $ah \in \mathbb{Z}$ and $bh \in \mathbb{Z}$, then $h \in \mathbb{Z}$. (This is where the coprimality of a and b comes into play.)

(c) Prove that $a + b \mid \binom{a+b}{a}$.

(d) Find a counterexample to the claim of part **(c)** if a and b are allowed to not be coprime.

6.2 Solution

[...]

References

[GrKnPa94] Ronald L. Graham, Donald E. Knuth, Oren Patashnik, *Concrete Mathematics, Second Edition*, Addison-Wesley 1994.
See `https://www-cs-faculty.stanford.edu/~knuth/gkp.html` for errata.

[Grinbe19] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 10 January 2019.
`http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf`
The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://github.com/darijgr/detnotes/releases/tag/2019-01-10` .