Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2019: Midterm 3 with solutions (preliminary version)

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Notations

Here is a list of notations that are used in this homework:

- We shall use the notation $[n]$ for the set $\{1, 2, \ldots, n\}$ (when $n \in \mathbb{Z}$).
- If $n \in \mathbb{N}$, then S_n denotes the set of all permutations of $[n]$.
- If $n \in \mathbb{N}$ and $\sigma \in S_n$, then:
  - the *one-line notation* $\operatorname{OLN} \sigma$ of σ is defined as the n-tuple $(\sigma(1), \sigma(2), \ldots, \sigma(n))$.
  - the *inversions* of σ are defined to be the pairs (i, j) of integers satisfying $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$.
  - the *length* $\ell(\sigma)$ of σ is defined to be the # of inversions of σ.
  - the *sign* $(-1)^\sigma$ of σ is defined to be $(-1)^{\ell(\sigma)}$.
  - we say that σ is *even* if $(-1)^\sigma = 1$ (that is, if $\ell(\sigma)$ is even).
  - we say that σ is *odd* if $(-1)^\sigma = -1$ (that is, if $\ell(\sigma)$ is odd).
  - we let $\operatorname{Fix} \sigma$ denote the set of all fixed points of σ; in other words,

$$\operatorname{Fix} \sigma = \{i \in [n] \mid \sigma(i) = i\}.$$

1 Exercise 1

1.1 Problem

Let n be an integer such that $n \geq 2$. If $w \in S_n$ is a permutation, then the *peaks* of w are defined to be the elements $i \in \{2, 3, \ldots, n-1\}$ satisfying $w(i-1) < w(i) > w(i+1)$. (For example, if $n = 7$ and if OLN $w = (4, 1, 2, 5, 3, 7, 6)$, then the peaks of w are 4 and 6. The name "peak" is explained by a look at the plot of w.)

An *n-peak set* shall mean a subset P of $\{2, 3, \ldots, n-1\}$ such that there exists a $w \in S_n$ satisfying $\{\text{peaks of } w\} = P$. (For example, the example we just gave shows that $\{4, 6\}$ is a 7-peak set.)

Find the # of all n-peak sets (for our given n).

1.2 Solution sketch (outline)

Let $(f_0, f_1, f_2, \ldots)$ denote the Fibonacci sequence (defined in [Math222, Definition 1.1.10]). Our goal is to prove that

$$(\# \text{ of } n\text{-peak sets}) = f_{n-1}.$$

Recall the notion of a lacunar set ([Math222, Definition 1.4.2]). We claim:

Observation 1: The n-peak sets are precisely the lacunar subsets of $\{2, 3, \ldots, n-1\}$.

[*Proof of Observation 1:* We must prove two facts:

- that every n-peak set is a lacunar subset of $\{2, 3, \ldots, n-1\}$, and
- that every lacunar subset of $\{2, 3, \ldots, n-1\}$ is an n-peak set.

The first of these two facts is just saying that no two consecutive integers can be peaks of the same permutation $w \in S_n$; but this is obvious (indeed, if i and $i+1$ are both peaks of a permutation $w \in S_n$, then the definition of "peak" yields $w(i-1) < w(i) > w(i+1)$ and $w(i) < w(i+1) > w(i+2)$; but these inequalities clearly contradict one another).

Thus, it remains to prove the second fact. So let L be a lacunar subset of $\{2, 3, \ldots, n-1\}$. We must prove that L is an n-peak set. In other words, we must prove that there exists a permutation $w \in S_n$ satisfying $\{\text{peaks of } w\} = L$.

We construct such a permutation $w \in S_n$ as follows:

- Set $g = |L|$.
- Let the values of w at the g elements of L be $n-g+1, n-g+2, \ldots, n$ (in increasing order).
- Let the values of w at the $n-g$ remaining elements of $[n]$ (that is, at the $n-g$ elements of $[n] \setminus L$) be $1, 2, \ldots, n-g$ (in increasing order).

We notice the following consequence of this construction: If $i \in [n]$ satisfies $i \notin L$, then

$$w(i) < w(i+1). \qquad (1)$$

[*Proof of (1):* Let $i \in [n]$ satisfy $i \notin L$. If $i+1 \notin L$, then (1) follows from the fact that the values of w at the $n-g$ elements of $[n] \setminus L$ have been chosen to be $1, 2, \ldots, n-g$ in increasing order. But if $i+1 \in L$, then (1) follows from $w(i) \leq n-g < n-g+1 \leq w(i+1)$. In either case, (1) is proved.]

For every $i \in L$, we have $w(i) \geq n-g+1$, while both $w(i-1)$ and $w(i+1)$ are $\leq n-g$ (since the lacunarity of L yields $i-1 \notin L$ and $i+1 \notin L$). Hence, each $i \in L$ satisfies $w(i-1) < w(i) > w(i+1)$. In other words, each $i \in L$ is a peak of w. In other words, $L \subseteq \{\text{peaks of } w\}$.

Conversely, if i is a peak of w, then we must have $w(i-1) < w(i) > w(i+1)$, thus in particular $w(i) > w(i+1)$, which entails that $i \in L$ (because if we had $i \notin L$, then (1) would yield $w(i) < w(i+1)$). Hence, $\{\text{peaks of } w\} \subseteq L$. Combining this with $L \subseteq \{\text{peaks of } w\}$, we obtain $\{\text{peaks of } w\} = L$.

Thus, we have found a permutation $w \in S_n$ satisfying $\{\text{peaks of } w\} = L$. This shows that L is an n-peak set. This concludes the proof of the second fact above. Hence, the proof of Observation 1 is complete.]

Now, recall the notation $[a, b]$ for the integer interval $\{a, a+1, \ldots, b\}$ whenever a and b are two integers. Thus, $\{2, 3, \ldots, n-1\} = [2, n-1] = [1+1, 1+(n-2)]$.

But [Math222, Proposition 1.4.18] (with n renamed as m) says that for any $m \in \{-1, 0, 1, \ldots\}$ and $a \in \mathbb{Z}$, we have

$$(\# \text{ of lacunar subsets of } [a+1, a+m]) = f_{m+1}. \tag{2}$$

But Observation 1 yields

$$\begin{aligned}
(\# \text{ of } n\text{-peak sets}) &= (\# \text{ of lacunar subsets of } \{2, 3, \ldots, n-1\}) \\
&= (\# \text{ of lacunar subsets of } [1+1, 1+(n-2)]) \\
&\qquad \left(\text{since } \{2, 3, \ldots, n-1\} = [1+1, 1+(n-2)]\right) \\
&= f_{(n-2)+1} \qquad (\text{by (2), applied to } m = n-2 \text{ and } a = 1) \\
&= f_{n-1}.
\end{aligned}$$

This proves our goal.

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2 Exercise 2

2.1 Problem

Let n be an integer such that $n \geq 3$. For each $k \in \mathbb{Z}$, set

$$m_k = (\# \text{ of permutations } \sigma \in S_n \text{ such that } \ell(\sigma) \equiv k \mod 3).$$

(*Example:* If $n = 3$, then m_0 counts the two permutations with one-line notations $(1, 2, 3)$ and $(3, 2, 1)$, while m_1 counts the two permutations with one-line notations $(1, 3, 2)$ and $(2, 1, 3)$, and while m_2 counts the two permutations with one-line notations $(2, 3, 1)$ and $(3, 1, 2)$.)

Prove that $m_0 = m_1 = m_2 = n!/3$.

[**Hint:** The Lehmer code (see [Grinbe15, §5.8] or [17f-hw8s, §0.4]) may be of use.]

2.2 Solution sketch (outline)

We shall first state a general rule for counting:

Proposition 2.1. *Let A and B be two sets. Let $f : A \to B$ be a bijection. Let $\mathcal{X}(b)$ be a statement for each $b \in B$. Then,*

$$(\# \text{ of } a \in A \text{ satisfying } \mathcal{X}(f(a))) = (\# \text{ of } b \in B \text{ satisfying } \mathcal{X}(b)).$$

Proof of Proposition 2.1. The map $f : A \to B$ is a bijection. Thus, it is easy to see that the map

$$\{a \in A \mid \mathcal{X}(f(a))\} \to \{b \in B \mid \mathcal{X}(b)\},$$
$$a \mapsto f(a)$$

is a bijection. Hence, the bijection principle yields

$$|\{a \in A \mid \mathcal{X}(f(a))\}| = |\{b \in B \mid \mathcal{X}(b)\}|.$$

In other words, $(\# \text{ of } a \in A \text{ satisfying } \mathcal{X}(f(a))) = (\# \text{ of } b \in B \text{ satisfying } \mathcal{X}(b))$. This proves Proposition 2.1. □

We shall next recall the basic facts about the Lehmer code:

- Whenever m is an integer, we shall use the notation $[m]_0$ for the set $\{0, 1, \ldots, m\}$.
- Let H denote the set $[n-1]_0 \times [n-2]_0 \times \cdots \times [n-n]_0$. More explicitly, this set H is

  $$H = \{(i_1, i_2, \ldots, i_n) \in \mathbb{N}^n \mid i_j \leq n - j \text{ for each } j\}.$$

  (For example, if $n = 3$, then $H = \{(2,1,0), (2,0,0), (1,1,0), (1,0,0), (0,1,0), (0,0,0)\}$.)
- If $\sigma \in S_n$ and $i \in [n]$, then $\ell_i(\sigma)$ shall denote the number of all $j \in \{i+1, i+2, \ldots, n\}$ such that $\sigma(i) > \sigma(j)$.
- For each $\sigma \in S_n$, we have

  $$\ell(\sigma) = \ell_1(\sigma) + \ell_2(\sigma) + \cdots + \ell_n(\sigma). \quad (3)$$

  (This is [Grinbe15, Proposition 5.46] or [17f-hw8s, Exercise 5 **(d)**]. The proof is almost trivial.)
- Define the map $L : S_n \to H$ by

  $$(L(\sigma) = (\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_n(\sigma)) \qquad \text{for each } \sigma \in S_n).$$

  This map $L : S_n \to H$ is well-defined and is a bijection. (This is [Grinbe15, Theorem 5.52] or [17f-hw8s, Exercise 5 **(c)**].)

  If $\sigma \in S_n$, then the n-tuple $L(\sigma) = (\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_n(\sigma))$ is called the *Lehmer code* of σ.

Thus, for each $\sigma \in S_n$, we have

$$\begin{aligned} \ell(\sigma) &= \ell_1(\sigma) + \ell_2(\sigma) + \cdots + \ell_n(\sigma) \qquad \text{(by (3))} \\ &= (\text{the sum of the entries of } L(\sigma)) \end{aligned} \tag{4}$$

(since $L(\sigma) = (\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_n(\sigma))$).

Now, we must prove that $m_0 = m_1 = m_2 = n!/3$. It clearly suffices to show that $m_k = n!/3$ for each $k \in \mathbb{Z}$.

So let us fix $k \in \mathbb{Z}$. We must show that $m_k = n!/3$.

The definition of m_k yields

$$\begin{aligned} m_k &= (\# \text{ of permutations } \sigma \in S_n \text{ such that } \ell(\sigma) \equiv k \mod 3) \\ &= (\# \text{ of } \sigma \in S_n \text{ such that } \ell(\sigma) \equiv k \mod 3) \\ &= (\# \text{ of } \sigma \in S_n \text{ such that } (\text{the sum of the entries of } L(\sigma)) \equiv k \mod 3) \\ &\qquad \text{(by (4))}. \end{aligned}$$

But Proposition 2.1 (applied to $A = S_n$, $B = H$, $f = L$ and $\mathcal{X}(b) = ((\text{the sum of the entries of } b) \equiv k \mod 3))$ yields

$$\begin{aligned} &(\# \text{ of } \sigma \in S_n \text{ such that } (\text{the sum of the entries of } L(\sigma)) \equiv k \mod 3) \\ &= (\# \text{ of } b \in H \text{ such that } (\text{the sum of the entries of } b) \equiv k \mod 3) \end{aligned}$$

(since the map $L : S_n \to H$ is a bijection). Hence,

$$\begin{aligned} m_k &= (\# \text{ of } \sigma \in S_n \text{ such that } (\text{the sum of the entries of } L(\sigma)) \equiv k \mod 3) \\ &= (\# \text{ of } b \in H \text{ such that } (\text{the sum of the entries of } b) \equiv k \mod 3) \\ &= (\# \text{ of } (i_1, i_2, \ldots, i_n) \in H \text{ such that } i_1 + i_2 + \cdots + i_n \equiv k \mod 3) \end{aligned} \tag{5}$$

(here, we have renamed the index b as $(i_1, i_2, \ldots, i_n)$).

Recall that

$$H = [n-1]_0 \times [n-2]_0 \times \cdots \times [n-n]_0 = [n-1]_0 \times [n-2]_0 \times \cdots \times [0]_0 .$$

Hence, the n-tuples $(i_1, i_2, \ldots, i_n) \in H$ are precisely the n-tuples $(i_1, i_2, \ldots, i_n)$ with

$$i_1 \in [n-1]_0, \qquad i_2 \in [n-2]_0, \qquad \ldots, \qquad i_n \in [0]_0 .$$

Thus, we can construct an n-tuple $(i_1, i_2, \ldots, i_n) \in H$ by independently choosing its n entries $i_1, i_2, \ldots, i_n$ from the sets $[n-1]_0, [n-2]_0, \ldots, [0]_0$, respectively. The total # of options for this construction is

$$\underbrace{|[n-1]_0|}_{=n} \cdot \underbrace{|[n-2]_0|}_{=n-1} \cdot \cdots \cdot \underbrace{|[0]_0|}_{=1} = n \cdot (n-1) \cdot \cdots \cdot 1 = n!.$$

Hence, the total # of n-tuples $(i_1, i_2, \ldots, i_n) \in H$ is $n!$.

Let us now use a slight variation of this idea to count the n-tuples $(i_1, i_2, \ldots, i_n) \in H$ such that $i_1 + i_2 + \cdots + i_n \equiv k \mod 3$. Indeed, we can construct any such n-tuple by the following method:

- First, we (independently) choose the $n-1$ entries $i_1, i_2, \ldots, i_{n-3}, i_{n-1}, i_n$ (that is, all entries except for i_{n-2}) from the sets $[n-1]_0, [n-2]_0, \ldots, [3]_0, [1]_0, [0]_0$, respectively. The total # of options at this step is

$$\begin{aligned} &\underbrace{|[n-1]_0|}_{=n} \cdot \underbrace{|[n-2]_0|}_{=n-1} \cdot \cdots \cdot \underbrace{|[3]_0|}_{=4} \cdot \underbrace{|[1]_0|}_{=2} \cdot \underbrace{|[0]_0|}_{=1} = n \cdot (n-1) \cdot \cdots \cdot 4 \cdot 2 \cdot 1 \\ &\qquad = \frac{1}{3} \cdot \underbrace{n \cdot (n-1) \cdot \cdots \cdot 1}_{=n!} = \frac{1}{3} n! = n!/3. \end{aligned}$$

- Finally, we choose the remaining entry $i_{n-2} \in [2]_0$ of our n-tuple. This remaining entry i_{n-2} must be chosen to belong to $[2]_0 = \{0, 1, 2\}$ and to satisfy the congruence $i_1 + i_2 + \cdots + i_n \equiv k \mod 3$. This determines it uniquely (because the congruence $i_1 + i_2 + \cdots + i_n \equiv k \mod 3$ rewrites as $i_{n-2} \equiv k - (i_1 + i_2 + \cdots + i_{n-3} + i_{n-1} + i_n) \mod 3$, which uniquely determines the equivalence class of i_{n-2} with respect to congruence modulo 3; and because the requirement that $i_{n-2} \in \{0, 1, 2\}$ leaves exactly one value for i_{n-2} for each possible equivalence class). Thus, we have only 1 choice at this step.

Thus, the total of # options to perform this construction is $n!/3 \cdot 1 = n!/3$. Hence,

$$(\# \text{ of } (i_1, i_2, \ldots, i_n) \in H \text{ such that } i_1 + i_2 + \cdots + i_n \equiv k \mod 3) = n!/3.$$

Thus, (5) becomes

$$m_k = (\# \text{ of } (i_1, i_2, \ldots, i_n) \in H \text{ such that } i_1 + i_2 + \cdots + i_n \equiv k \mod 3) = n!/3.$$

This completes our solution.

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3 Exercise 3

3.1 Problem

Let n be an integer such that $n \geq 3$. Find

$$\sum_{w \in S_n \text{ is even}} |\operatorname{Fix} w|.$$

[**Hint:** For each $i \in [n]$, compare

$$(\# \text{ of even } w \in S_n \text{ such that } w(i) = i) \text{ with } (\# \text{ of odd } w \in S_n \text{ such that } w(i) = i).$$

]

3.2 Solution

Recall the following two properties of signs of permutations:

- If i and j are two distinct elements of $[n]$, then the transposition $t_{i,j} \in S_n$ is the permutation of $[n]$ that swaps i with j while leaving all other elements of $[n]$ unchanged. The sign of this transposition is

$$(-1)^{t_{i,j}} = -1. \tag{6}$$

(This is, e.g., [Grinbe15, Exercise 5.10 **(b)**].)

- If σ and τ are two permutations in S_n, then

$$(-1)^{\sigma \circ \tau} = (-1)^{\sigma} \cdot (-1)^{\tau}. \tag{7}$$

(This is, e.g., [Grinbe15, Proposition 5.15 **(c)**].)

We also recall the following fact ([Math222, Proposition 1.6.3 **(a)**]):

Lemma 3.1. *Let S be a finite set. Let T be a subset of S. Then,*

$$|T| = \sum_{s \in S} [s \in T] .$$

Now, the key to our solution is the following fact:

Statement 1: Let $i \in [n]$. Then,

$$(\# \text{ of even permutations } w \in S_n \text{ such that } w(i) = i) = (n-1)!/2.$$

[*Proof of Statement 1:* We have $i \in [n]$, thus $|[n] \setminus \{i\}| = \underbrace{|[n]|}_{=n \geq 3} - 1 \geq 3 - 1 = 2$. Hence, the set $[n] \setminus \{i\}$ contains two distinct elements u and v. Consider these u and v. (We usually have many choices for u and v, but it suffices to pick one such choice and stick with it.)

Note that $u, v \in [n] \setminus \{i\}$; thus, both u and v are distinct from i. In other words, i equals neither u nor v.

Consider the transposition $t_{u,v} \in S_n$. This is the permutation of $[n]$ that swaps u with v while leaving all other elements of $[n]$ unchanged. Thus, $t_{u,v}$ leaves i unchanged (since i equals neither u nor v). In other words, $t_{u,v}(i) = i$.

Hence, if $w \in S_n$ is a permutation such that $w(i) = i$, then

$$(w \circ t_{u,v})(i) = w \left(\underbrace{t_{u,v}(i)}_{=i} \right) = w(i) = i. \qquad (8)$$

Furthermore, $(-1)^{t_{u,v}} = -1$ (by (6), applied to u and v instead of i and j). Thus, for any permutation $w \in S_n$, we have

$$\begin{aligned} (-1)^{w \circ t_{u,v}} &= (-1)^w \cdot \underbrace{(-1)^{t_{u,v}}}_{=-1} \qquad (\text{by (7), applied to } \sigma = w \text{ and } \tau = t_{u,v}) \\ &= -(-1)^w . \end{aligned}$$

Hence, if a permutation $w \in S_n$ is even, then the permutation $w \circ t_{u,v}$ is odd. This observation (and (8)) shows that the map

$$\{\text{even permutations } w \in S_n \mid w(i) = i\} \to \{\text{odd permutations } w \in S_n \mid w(i) = i\}, \\ w \mapsto w \circ t_{u,v}$$

is well-defined. Similarly, the map

$$\{\text{odd permutations } w \in S_n \mid w(i) = i\} \to \{\text{even permutations } w \in S_n \mid w(i) = i\}, \\ w \mapsto w \circ t_{u,v}$$

is well-defined. These two maps are easily seen to be mutually inverse (since $t_{u,v} \circ t_{u,v} = \mathrm{id}$), and thus are bijections. Hence, the bijection principle yields

$$|\{\text{even permutations } w \in S_n \mid w(i) = i\}| = |\{\text{odd permutations } w \in S_n \mid w(i) = i\}| .$$

In other words,

$$\begin{aligned} &(\# \text{ of even permutations } w \in S_n \text{ such that } w(i) = i) \\ &= (\# \text{ of odd permutations } w \in S_n \text{ such that } w(i) = i) . \end{aligned}$$

But the sum of the two sides of this equality is

$$(\# \text{ of permutations } w \in S_n \text{ such that } w(i) = i) = (n-1)!$$

(by [17f-hw7s, Lemma 0.6]). Thus, each of these two sides equals $(n-1)!/2$. Hence, in particular,

$$(\# \text{ of even permutations } w \in S_n \text{ such that } w(i) = i) = (n-1)!/2.$$

This proves Statement 1.]

Now, we proceed as in [17f-hw7s, solution to Exercise 2]: If $w \in S_n$ and $i \in [n]$, then

$$[i \in \operatorname{Fix} w] = [w(i) = i] \tag{9}$$

[1].

If $w \in S_n$, then $\operatorname{Fix} w$ is a subset of $[n]$, and therefore Lemma 3.1 (applied to $S = [n]$ and $T = \operatorname{Fix} w$) yields

$$|\operatorname{Fix} w| = \sum_{s \in [n]} [s \in \operatorname{Fix} w] = \sum_{i \in [n]} \underbrace{[i \in \operatorname{Fix} w]}_{\substack{=[w(i)=i] \\ (\text{by } (9))}} \qquad \left(\begin{array}{c}\text{here, we have renamed the} \\ \text{summation index } s \text{ as } i\end{array}\right)$$
$$= \sum_{i \in [n]} [w(i) = i]. \tag{10}$$

But if $i \in [n]$, then $\{w \in S_n \mid w \text{ is even and } w(i) = i\}$ is a subset of $\{w \in S_n \mid w \text{ is even}\}$, and therefore Lemma 3.1 (applied to $S = \{w \in S_n \mid w \text{ is even}\}$ and $T = \{w \in S_n \mid w \text{ is even and } w(i) = i\}$) yields

$$|\{w \in S_n \mid w \text{ is even and } w(i) = i\}|$$
$$= \underbrace{\sum_{s \in \{w \in S_n \mid w \text{ is even}\}}}_{= \sum_{s \in S_n \text{ is even}}} \left[\underbrace{s \in \{w \in S_n \mid w \text{ is even and } w(i) = i\}}_{\substack{\iff (s(i)=i) \\ (\text{since } s \in \{w \in S_n \mid w \text{ is even}\})}} \right]$$
$$= \sum_{s \in S_n \text{ is even}} [s(i) = i] = \sum_{w \in S_n \text{ is even}} [w(i) = i] \tag{11}$$

(here, we have renamed the summation index s as w).

Now,

$$\sum_{w \in S_n \text{ is even}} \underbrace{|\operatorname{Fix} w|}_{\substack{= \sum_{i \in [n]} [w(i)=i] \\ (\text{by } (10))}} = \underbrace{\sum_{w \in S_n \text{ is even}} \sum_{i \in [n]}}_{= \sum_{i \in [n]} \sum_{w \in S_n \text{ is even}}} [w(i) = i] = \sum_{i \in [n]} \underbrace{\sum_{w \in S_n \text{ is even}} [w(i) = i]}_{\substack{=|\{w \in S_n \mid w \text{ is even and } w(i)=i\}| \\ (\text{by } (11))}}$$
$$= \sum_{i \in [n]} \underbrace{|\{w \in S_n \mid w \text{ is even and } w(i) = i\}|}_{\substack{=(\# \text{ of even permutations } w \in S_n \text{ such that } w(i)=i) \\ =(n-1)!/2 \\ (\text{by Statement 1})}}$$
$$= \sum_{i \in [n]} (n-1)!/2 = \underbrace{|[n]|}_{=n} \cdot (n-1)!/2 = \underbrace{n \cdot (n-1)!}_{=n!}/2 = n!/2.$$

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[1] because of the equivalence $(i \in \operatorname{Fix} w) \iff (i \text{ is a fixed point of } w) \iff (w(i) = i)$

4 Exercise 4

4.1 Problem

An n-tuple $(i_1, i_2, \ldots, i_n) \in \{0,1\}^n$ (where $n \in \mathbb{N}$) will be called *upsided* if it satisfies $i_1 + i_2 + \cdots + i_p \geq p/2$ for each $p \in [n]$.

(*Example:* The 3-tuple $(1,0,1)$ is upsided (since $1 \geq 1/2$ and $1+0 \geq 2/2$ and $1+0+1 \geq 3/2$), and so is the 3-tuple $(1,1,0)$ (for similar reasons), but the 3-tuples $(1,0,0)$ and $(0,1,1)$ are not (indeed, $(1,0,0)$ is not upsided because $1+0+0 < 3/2$, whereas $(0,1,1)$ is not upsided because $0 < 1/2$). The 0-tuple $()$ is upsided (for vacuous reasons).)

For given $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, let $U(n,k)$ denote the # of upsided n-tuples $(i_1, i_2, \ldots, i_n) \in \{0,1\}^n$ satisfying $i_1 + i_2 + \cdots + i_n = k$.

(a) Prove that if $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ satisfy $k < n/2$, then $U(n,k) = 0$.

(b) Prove that if $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ satisfy $k \geq (n-1)/2$, then

$$U(n,k) = \binom{n}{k} - \binom{n}{k+1}.$$

(c) Prove that $\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor n/2 \rfloor}$ for each $n \in \mathbb{N}$.

[**Hint:** Induction can be helpful. There are many ways to solve part **(c)**, but the one using part **(b)** is perhaps the nicest.]

4.2 Solution sketch (outline)

(a) Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ satisfy $k < n/2$. Then, every upsided n-tuple $(i_1, i_2, \ldots, i_n) \in \{0,1\}^n$ satisfies $i_1 + i_2 + \cdots + i_n \neq k$ [2]. In other words, there exists no upsided n-tuple $(i_1, i_2, \ldots, i_n) \in \{0,1\}^n$ satisfying $i_1 + i_2 + \cdots + i_n = k$. Thus, $U(n,k) = 0$ (since $U(n,k)$ was defined to be the # of such n-tuples). This solves part **(a)** of the problem.

(b) We shall prove part **(b)** of the problem by induction on n.

Induction base: It is easy to see that part **(b)** of the problem holds for $n = 0$ [3]. This completes the induction base.

[2] *Proof.* Let $(i_1, i_2, \ldots, i_n) \in \{0,1\}^n$ be an upsided n-tuple. Thus, we have the inequality $i_1 + i_2 + \cdots + i_p \geq p/2$ for each $p \in [n]$ (by the definition of "upsided"). Since this inequality also holds for $p = 0$ (because $i_1 + i_2 + \cdots + i_0 = (\text{empty sum}) = 0 \geq 0/2$), we thus conclude that it holds for each $p \in \{0, 1, \ldots, n\}$. Hence, we can apply it to $p = n$. We thus obtain $i_1 + i_2 + \cdots + i_n \geq n/2 > k$ (since $k < n/2$). Thus, $i_1 + i_2 + \cdots + i_n \neq k$. Qed.

[3] *Proof.* The proof is straightforward if you take the following into account:

- There is exactly one 0-tuple $(i_1, i_2, \ldots, i_0) \in \{0,1\}^0$, namely the empty list $()$. This 0-tuple is upsided (since the condition in the definition of "upsided" is satisfied for vacuous reasons) and satisfies $i_1 + i_2 + \cdots + i_0 = (\text{empty sum}) = 0$.
- We have $\binom{0}{k} - \binom{0}{k+1} = \begin{cases} 1, & \text{if } k = 0; \\ -1, & \text{if } k = -1; \\ 0, & \text{otherwise} \end{cases}$ for each $k \in \mathbb{Z}$. If we restrict ourselves to the $k \in \mathbb{Z}$ that satisfy $k \geq (0-1)/2$, then we can simplify this to $\binom{0}{k} - \binom{0}{k+1} = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{otherwise} \end{cases}$ (since $k = -1$ would not satisfy $k \geq (0-1)/2$).

Induction step: Fix some positive integer m. Assume (as the induction hypothesis) that part **(b)** of the problem holds for $n = m - 1$. We must prove that part **(b)** of the problem holds for $n = m$.

We have assumed (as the induction hypothesis) that part **(b)** of the problem holds for $n = m - 1$. In other words, for each $k \in \mathbb{Z}$ satisfying $k \geq (m-2)/2$, we have

$$U(m-1,k) = \binom{m-1}{k} - \binom{m-1}{k+1}. \qquad (12)$$

Let $k \in \mathbb{Z}$ satisfy $k \geq (m-1)/2$. We shall prove that

$$U(m,k) = \binom{m}{k} - \binom{m}{k+1}. \qquad (13)$$

[*Proof of (13):* If $2k = m - 1$, then this is easy to see[4]. Hence, for the rest of this proof of (13), we WLOG assume that $2k \neq m - 1$. But $k \geq (m-1)/2$ and thus $2k \geq m - 1$. Combined with $2k \neq m - 1$, this yields $2k > m - 1$. Thus, $2k \geq m$ (since $2k$ and m are integers). In other words, $k \geq m/2$. Thus, $\underbrace{k}_{\geq m/2} - 1 \geq m/2 - 1 = (m-2)/2$. Also, $k \geq k - 1 \geq (m-2)/2$.

Recall that $k \geq m/2$. Hence, an m-tuple $(i_1, i_2, \ldots, i_m) \in \{0,1\}^m$ satisfying $i_1 + i_2 + \cdots + i_m = k$ is upsided if and only if the $(m-1)$-tuple $(i_1, i_2, \ldots, i_{m-1}) \in \{0,1\}^{m-1}$ is upsided[5]. Thus, the map

$$\begin{aligned} &\{\text{upsided } m\text{-tuples } (i_1, i_2, \ldots, i_m) \in \{0,1\}^m \mid i_1 + i_2 + \cdots + i_m = k \text{ and } i_m = 0\} \\ &\to \left\{\text{upsided } (m-1)\text{-tuples } (i_1, i_2, \ldots, i_{m-1}) \in \{0,1\}^{m-1} \mid i_1 + i_2 + \cdots + i_{m-1} = k\right\} \end{aligned}$$

that sends each $(i_1, i_2, \ldots, i_m)$ to $(i_1, i_2, \ldots, i_{m-1})$ is well-defined and bijective[6]. Hence, the bijection principle yields

$$\begin{aligned} &(\# \text{ of upsided } m\text{-tuples } (i_1, i_2, \ldots, i_m) \in \{0,1\}^m \\ &\qquad \text{satisfying } i_1 + i_2 + \cdots + i_m = k \text{ and } i_m = 0) \\ &= (\# \text{ of upsided } (m-1)\text{-tuples } (i_1, i_2, \ldots, i_{m-1}) \in \{0,1\}^{m-1} \\ &\qquad \text{satisfying } i_1 + i_2 + \cdots + i_{m-1} = k) \\ &= U(m-1,k) \qquad (\text{by the definition of } U(m-1,k)) \\ &= \binom{m-1}{k} - \binom{m-1}{k+1} \end{aligned} \qquad (14)$$

[4] *Proof.* Assume that $2k = m - 1$. Thus, $2k = m - 1 < m$, so that $k < m/2$. Hence, part **(a)** of this exercise (applied to $n = m$) yields $U(m,k) = 0$. But from $2k = m - 1$, we also obtain $k = m - 1 - k$ and thus $m - k = k + 1$. The symmetry of the binomial coefficients yields $\binom{m}{k} = \binom{m}{m-k} = \binom{m}{k+1}$ (since $m - k = k + 1$). Hence, $\binom{m}{k} - \binom{m}{k+1} = 0$. Comparing this with $U(m,k) = 0$, we obtain $U(m,k) = \binom{m}{k} - \binom{m}{k+1}$. Thus, (13) is proved for $k = (m-1)/2$.

[5] Indeed, the definition of "upsided" shows the following recursive criterion for upsidedness: An m-tuple $(i_1, i_2, \ldots, i_m) \in \{0,1\}^m$ is upsided if and only if the $(m-1)$-tuple $(i_1, i_2, \ldots, i_{m-1}) \in \{0,1\}^{m-1}$ is upsided and we have $i_1 + i_2 + \cdots + i_m \geq m/2$. But if our m-tuple $(i_1, i_2, \ldots, i_m) \in \{0,1\}^m$ satisfies $i_1 + i_2 + \cdots + i_m = k$, then $i_1 + i_2 + \cdots + i_m \geq m/2$ automatically holds (since $i_1 + i_2 + \cdots + i_m = k \geq m/2$). Hence, our recursive criterion simplifies to "$(i_1, i_2, \ldots, i_m)$ is upsided if and only if $(i_1, i_2, \ldots, i_{m-1})$ is upsided" in this case. Qed.

[6] The inverse map sends each $(i_1, i_2, \ldots, i_{m-1})$ to $(i_1, i_2, \ldots, i_{m-1}, 0)$.

(by (12) (since $k \geq (m-2)/2$)).

Recall again that an m-tuple $(i_1, i_2, \ldots, i_m) \in \{0,1\}^m$ satisfying $i_1 + i_2 + \cdots + i_m = k$ is upsided if and only if the $(m-1)$-tuple $(i_1, i_2, \ldots, i_{m-1}) \in \{0,1\}^{m-1}$ is upsided. Thus, the map

$$\{\text{upsided } m\text{-tuples } (i_1, i_2, \ldots, i_m) \in \{0,1\}^m \mid i_1 + i_2 + \cdots + i_m = k \text{ and } i_m = 1\}$$
$$\to \left\{\text{upsided } (m-1)\text{-tuples } (i_1, i_2, \ldots, i_{m-1}) \in \{0,1\}^{m-1} \mid i_1 + i_2 + \cdots + i_{m-1} = k-1\right\}$$

that sends each $(i_1, i_2, \ldots, i_m)$ to $(i_1, i_2, \ldots, i_{m-1})$ is well-defined and bijective[7]. Hence, the bijection principle yields

$$\begin{aligned}
&(\#\text{ of upsided } m\text{-tuples } (i_1, i_2, \ldots, i_m) \in \{0,1\}^m \\
&\qquad \text{satisfying } i_1 + i_2 + \cdots + i_m = k \text{ and } i_m = 1) \\
&= \left(\#\text{ of upsided } (m-1)\text{-tuples } (i_1, i_2, \ldots, i_{m-1}) \in \{0,1\}^{m-1} \right.\\
&\qquad \left.\text{satisfying } i_1 + i_2 + \cdots + i_{m-1} = k-1\right) \\
&= U(m-1, k-1) \qquad \text{(by the definition of } U(m-1,k-1)) \\
&= \binom{m-1}{k-1} - \binom{m-1}{(k-1)+1} \qquad (15)
\end{aligned}$$

(by (12), applied to $k-1$ instead of k (since $k-1 \geq (m-2)/2$)).

But each upsided m-tuple $(i_1, i_2, \ldots, i_m) \in \{0,1\}^m$ satisfying $i_1 + i_2 + \cdots + i_m = k$ must satisfy either $i_m = 0$ or $i_m = 1$ (but not both at the same time). Hence, the sum rule yields

$$\begin{aligned}
&(\#\text{ of upsided } m\text{-tuples } (i_1, i_2, \ldots, i_m) \in \{0,1\}^m \text{ satisfying } i_1 + i_2 + \cdots + i_m = k) \\
&= \quad (\#\text{ of upsided } m\text{-tuples } (i_1, i_2, \ldots, i_m) \in \{0,1\}^m \\
&\qquad\qquad \text{satisfying } i_1 + i_2 + \cdots + i_m = k \text{ and } i_m = 0) \\
&\qquad + (\#\text{ of upsided } m\text{-tuples } (i_1, i_2, \ldots, i_m) \in \{0,1\}^m \\
&\qquad\qquad \text{satisfying } i_1 + i_2 + \cdots + i_m = k \text{ and } i_m = 1) \\
&= \left(\binom{m-1}{k} - \binom{m-1}{k+1}\right) + \left(\binom{m-1}{k-1} - \binom{m-1}{(k-1)+1}\right) \\
&\qquad \text{(by adding together the equalities (14) and (15))} \\
&= \underbrace{\binom{m-1}{k} + \binom{m-1}{k-1}}_{\substack{=\binom{m}{k} \\ \text{(by the recurrence relation} \\ \text{of the binomial coefficients)}}} - \underbrace{\left(\binom{m-1}{k+1} + \binom{m-1}{(k-1)+1}\right)}_{\substack{=\binom{m-1}{k+1} + \binom{m-1}{k} \\ =\binom{m}{k+1} \\ \text{(by the recurrence relation} \\ \text{of the binomial coefficients)}}} \\
&= \binom{m}{k} - \binom{m}{k+1}.
\end{aligned}$$

Now, the definition of $U(m,k)$ yields

$$\begin{aligned}
&U(m,k) \\
&= (\#\text{ of upsided } m\text{-tuples } (i_1, i_2, \ldots, i_m) \in \{0,1\}^m \text{ satisfying } i_1 + i_2 + \cdots + i_m = k) \\
&= \binom{m}{k} - \binom{m}{k+1}.
\end{aligned}$$

[7]The inverse map sends each $(i_1, i_2, \ldots, i_{m-1})$ to $(i_1, i_2, \ldots, i_{m-1}, 1)$.

This proves (13).]

Forget that we fixed k. We thus have proved that (13) holds for each $k \in \mathbb{Z}$ satisfying $k \geq (m-1)/2$. In other words, part **(b)** of the problem holds for $n = m$. This completes the induction step. Thus, part **(b)** is proved by induction.

(c) Let $n \in \mathbb{N}$. We must prove that $\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor n/2 \rfloor}$.

Let us instead prove the slightly longer chain of inequalities

$$\binom{n}{-1} < \binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor n/2 \rfloor}.$$

In view of the symmetry of the binomial coefficients, this rewrites as

$$\binom{n}{n+1} < \binom{n}{n} < \binom{n}{n-1} < \cdots < \binom{n}{n - \lfloor n/2 \rfloor}.$$

This further rewrites as

$$\binom{n}{n - \lfloor n/2 \rfloor} > \binom{n}{n - \lfloor n/2 \rfloor + 1} > \cdots > \binom{n}{n+1}.$$

So we must prove this latter chain of inequalities.

In other words, we must prove that $\binom{n}{k} > \binom{n}{k+1}$ for each $k \in \{n - \lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor + 1, \ldots, n\}$.

So let $k \in \{n - \lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor + 1, \ldots, n\}$. We must prove that $\binom{n}{k} > \binom{n}{k+1}$.

From $k \in \{n - \lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor + 1, \ldots, n\}$, we obtain $k \leq n$ and $k \geq n - \underbrace{\lfloor n/2 \rfloor}_{\leq n/2} \geq n - n/2 = \underbrace{n}_{\geq n-1} /2 \geq (n-1)/2$. Hence, part **(b)** of this exercise yields

$$U(n,k) = \binom{n}{k} - \binom{n}{k+1}.$$

But $U(n,k)$ is defined as the # of upsided n-tuples $(i_1, i_2, \ldots, i_n) \in \{0,1\}^n$ satisfying $i_1 + i_2 + \cdots + i_n = k$. Since there exists at least one such n-tuple (namely, $\left(\underbrace{1,1,\ldots,1}_{k \text{ times}}, \underbrace{0,0,\ldots,0}_{n-k \text{ times}}\right)$ [8]), we thus have $U(n,k) \geq 1 > 0$. In view of $U(n,k) = \binom{n}{k} - \binom{n}{k+1}$, this rewrites as $\binom{n}{k} - \binom{n}{k+1} > 0$. In other words, $\binom{n}{k} > \binom{n}{k+1}$. This completes our solution to part **(c)**.

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[8] This n-tuple is indeed upsided, since $k \geq n/2$. (Fill in the details of this argument!)

5 Exercise 5

5.1 Problem

Let $n \in \mathbb{N}$. Recall that a *composition of* n means a tuple $(a_1, a_2, \ldots, a_k)$ of positive integers satisfying $a_1 + a_2 + \cdots + a_k = n$. Such a composition $(a_1, a_2, \ldots, a_k)$ is called *odd* if all of $a_1, a_2, \ldots, a_k$ are odd.

Let us also say that a composition $(a_1, a_2, \ldots, a_k)$ is *odd-but-one* if a_i is even for exactly one $i \in [k]$. (For example, the composition $(3, 5, 5)$ of 13 is odd; the composition $(3, 4, 1, 5)$ of 13 is odd-but-one; the composition $(6, 6, 1)$ of 13 is neither.)

Prove that

$$\begin{aligned}
&\sum_{\substack{(a_1,a_2,\ldots,a_k) \text{ is an} \\ \text{odd composition of } n}} k \\
&= (\# \text{ of odd-but-one compositions of } n+1) \\
&= \frac{(n+4) f_n + 2n f_{n-1}}{5},
\end{aligned}$$

where $(f_0, f_1, f_2, \ldots)$ is the Fibonacci sequence (defined in [Math222, Definition 1.1.10]).

5.2 Solution sketch (outline)

Forget that we fixed n. For each $n \in \mathbb{N}$, we define three rational numbers p_n, q_n and r_n by

$$\begin{aligned}
p_n &= \sum_{\substack{(a_1,a_2,\ldots,a_k) \text{ is an} \\ \text{odd composition of } n}} k \qquad \text{and} \\
q_n &= (\# \text{ of odd-but-one compositions of } n+1) \qquad \text{and} \\
r_n &= \frac{(n+4) f_n + 2n f_{n-1}}{5}.
\end{aligned}$$

Thus, the exercise demands that we prove that $p_n = q_n = r_n$ for each $n \in \mathbb{N}$. We shall achieve this by proving the following statements:

Statement 1: We have $p_n = q_n$ for each $n \in \mathbb{N}$.

Statement 2: We have $q_n = r_n$ for each $n \in \mathbb{N}$.

In the following, we shall abbreviate the word "odd-but-one" as "obo".

Let us first prove Statement 1:

[*Proof of Statement 1:* Let $n \in \mathbb{N}$. Let $k \in \mathbb{N}$. Let $p \in [k]$. If $(i_1, i_2, \ldots, i_k)$ is an obo composition of $n+1$ into k parts such that i_p is even, then all entries of $(i_1, i_2, \ldots, i_k)$ other than i_p are odd positive integers (by the definition of "obo"), and so is the number $i_p - 1$ (because if we subtract 1 from an even positive integer, then we obtain an odd positive integer). Hence, there is a bijection

$$\begin{aligned}
&\{\text{obo compositions } (i_1, i_2, \ldots, i_k) \text{ of } n+1 \text{ into } k \text{ parts such that } i_p \text{ is even}\} \\
&\to \{\text{odd compositions of } n \text{ into } k \text{ parts}\},
\end{aligned}$$

which subtracts 1 from the p-th entry of each composition (i.e., it sends each composition $(i_1, i_2, \ldots, i_k)$ to $(i_1, i_2, \ldots, i_{p-1}, i_p - 1, i_{p+1}, \ldots, i_k)$). Hence, the bijection principle yields

$$\begin{aligned}
&\left(\# \text{ of obo compositions } (i_1, i_2, \ldots, i_k) \text{ of } n+1 \text{ into } k \text{ parts such that } i_p \text{ is even}\right) \\
&= \left(\# \text{ of odd compositions of } n \text{ into } k \text{ parts}\right).
\end{aligned} \tag{16}$$

Forget that we fixed p. Thus, we have proved (16) for each $p \in [k]$.

Now, if $(i_1, i_2, \ldots, i_k)$ is an obo composition of $n+1$ into k parts, then there is **exactly one** $p \in [k]$ such that i_p is even (by the definition of "obo"). Hence, the sum rule yields

$$\begin{aligned}
&\left(\# \text{ of obo compositions } (i_1, i_2, \ldots, i_k) \text{ of } n+1 \text{ into } k \text{ parts}\right) \\
&= \sum_{p \in [k]} \underbrace{\left(\# \text{ of obo compositions } (i_1, i_2, \ldots, i_k) \text{ of } n+1 \text{ into } k \text{ parts such that } i_p \text{ is even}\right)}_{\substack{=(\# \text{ of odd compositions of } n \text{ into } k \text{ parts}) \\ (\text{by (16)})}} \\
&= \sum_{p \in [k]} \left(\# \text{ of odd compositions of } n \text{ into } k \text{ parts}\right) \\
&= \underbrace{|[k]|}_{=k} \cdot \left(\# \text{ of odd compositions of } n \text{ into } k \text{ parts}\right) \\
&= k \cdot \left(\# \text{ of odd compositions of } n \text{ into } k \text{ parts}\right).
\end{aligned} \tag{17}$$

Forget that we fixed k. We thus have proved (17) for each $k \in \mathbb{N}$.

Now, the definition of q_n yields

$$\begin{aligned}
q_n &= \left(\# \text{ of odd-but-one compositions of } n+1\right) \\
&= \left(\# \text{ of obo compositions of } n+1\right) \qquad \left(\text{since we abbreviate "odd-but-one" as "obo"}\right) \\
&= \sum_{k \in \mathbb{N}} \underbrace{\left(\# \text{ of obo compositions } (i_1, i_2, \ldots, i_k) \text{ of } n+1 \text{ into } k \text{ parts}\right)}_{\substack{=k \cdot (\# \text{ of odd compositions of } n \text{ into } k \text{ parts}) \\ (\text{by (17)})}} \\
&\qquad \left(\text{by the sum rule}\right) \\
&= \sum_{k \in \mathbb{N}} k \cdot \left(\# \text{ of odd compositions of } n \text{ into } k \text{ parts}\right).
\end{aligned}$$

On the other hand, the definition of p_n yields

$$\begin{aligned}
p_n &= \sum_{\substack{(a_1, a_2, \ldots, a_k) \text{ is an} \\ \text{odd composition of } n}} k = \sum_{k \in \mathbb{N}} \underbrace{\sum_{\substack{(a_1, a_2, \ldots, a_k) \text{ is an} \\ \text{odd composition of } n \\ \text{into } k \text{ parts}}} k}_{=(\# \text{ of odd compositions of } n \text{ into } k \text{ parts}) \cdot k} \\
&\qquad \left(\text{here, we have split the sum according to the number of entries of the composition}\right) \\
&= \sum_{k \in \mathbb{N}} \left(\# \text{ of odd compositions of } n \text{ into } k \text{ parts}\right) \cdot k \\
&= \sum_{k \in \mathbb{N}} k \cdot \left(\# \text{ of odd compositions of } n \text{ into } k \text{ parts}\right).
\end{aligned}$$

Comparing these two equalities, we find $p_n = q_n$. This proves Statement 1.]

We won't prove Statement 2 directly. Instead, we shall first derive a recursion for the q_n:

Statement 3: Let $n \geq 2$ be an integer. Then, $q_n = q_{n-1} + q_{n-2} + f_{n-1}$.

[*Proof of Statement 3:* In [19f-mt2s, solution to Exercise 6 **(b)**], we have shown that

$$(\# \text{ of odd compositions of } n) = \begin{cases} f_n, & \text{if } n > 0; \\ 1, & \text{if } n = 0. \end{cases}$$

The same argument (applied to $n-1$ instead of n) yields

$$(\# \text{ of odd compositions of } n-1) = \begin{cases} f_{n-1}, & \text{if } n-1 > 0; \\ 1, & \text{if } n-1 = 0 \end{cases} = f_{n-1} \qquad (18)$$

(since $n-1 > 0$ (because $n \geq 2$)).

Each composition of $n+1$ has at least one entry (since $n+1 \geq n \geq 2 > 0$), and thus has a well-defined last entry.

Any obo composition of $n+1$ whose last entry is 1 must have the form $(i_1, i_2, \ldots, i_k, 1)$, where $(i_1, i_2, \ldots, i_k)$ is an obo composition of n. Hence, the map

$$\{\text{obo compositions of } n+1 \text{ whose last entry is } 1\} \to \{\text{obo compositions of } n\}, \quad (i_1, i_2, \ldots, i_k, 1) \mapsto (i_1, i_2, \ldots, i_k)$$

is a bijection. Hence, the bijection principle yields

$$(\# \text{ of obo compositions of } n+1 \text{ whose last entry is } 1) = (\# \text{ of obo compositions of } n) = q_{n-1} \qquad (19)$$

(since the definition of q_{n-1} yields $q_{n-1} = (\# \text{ of obo compositions of } n)$).

Any obo composition of $n+1$ whose last entry is 2 must have the form $(i_1, i_2, \ldots, i_k, 2)$, where $(i_1, i_2, \ldots, i_k)$ is an odd composition of $n-1$. Hence, the map

$$\{\text{obo compositions of } n+1 \text{ whose last entry is } 2\} \to \{\text{odd compositions of } n-1\}, \quad (i_1, i_2, \ldots, i_k, 2) \mapsto (i_1, i_2, \ldots, i_k)$$

is a bijection. Hence, the bijection principle yields

$$(\# \text{ of obo compositions of } n+1 \text{ whose last entry is } 2) = (\# \text{ of odd compositions of } n-1) = f_{n-1} \qquad (20)$$

(by (18)).

Finally, if $(i_1, i_2, \ldots, i_k)$ is an obo composition of $n+1$ whose last entry (that is, i_k) is larger than 2, then $(i_1, i_2, \ldots, i_{k-1}, i_k - 2)$ is an obo composition of $(n+1) - 2 = n-1$ (indeed, $i_k - 2$ is a positive integer because $i_k > 2$, and furthermore this integer $i_k - 2$ has the same parity as i_k, whence the composition $(i_1, i_2, \ldots, i_{k-1}, i_k - 2)$ is obo). Thus, the map

$$\{\text{obo compositions of } n+1 \text{ whose last entry is larger than } 2\} \to \{\text{obo compositions of } n-1\}, \quad (i_1, i_2, \ldots, i_k) \mapsto (i_1, i_2, \ldots, i_{k-1}, i_k - 2)$$

(this is the map that subtracts 2 from the last entry of each composition) is a bijection. Hence, the bijection principle yields

$$\begin{aligned}&(\#\text{ of obo compositions of } n+1 \text{ whose last entry is larger than } 2)\\&= (\#\text{ of obo compositions of } n-1) = q_{n-2}\end{aligned} \tag{21}$$

(since the definition of q_{n-2} yields $q_{n-2} = (\#\text{ of obo compositions of } n-1)$).

Now, the last entry of any obo composition of $n+1$ must be either 1 or 2 or larger than 2. Hence, the sum rule yields

$$\begin{aligned}&(\#\text{ of obo compositions of } n+1)\\&= \underbrace{(\#\text{ of obo compositions of } n+1 \text{ whose last entry is } 1)}_{\substack{=q_{n-1}\\ \text{(by (19))}}}\\&\qquad + \underbrace{(\#\text{ of obo compositions of } n+1 \text{ whose last entry is } 2)}_{\substack{=f_{n-1}\\ \text{(by (20))}}}\\&\qquad + \underbrace{(\#\text{ of obo compositions of } n+1 \text{ whose last entry is larger than } 2)}_{\substack{=q_{n-2}\\ \text{(by (21))}}}\\&= q_{n-1} + f_{n-1} + q_{n-2} = q_{n-1} + q_{n-2} + f_{n-1}.\end{aligned}$$

Hence, the definition of q_n yields

$$q_n = (\#\text{ of obo compositions of } n+1) = q_{n-1} + q_{n-2} + f_{n-1}.$$

This proves Statement 3.]

Using Statement 3, we can now easily prove Statement 2 by induction:

[*Proof of Statement 2:* We proceed by strong induction on n. This is a straightforward and mostly computational argument, so we only briefly outline it.

Fix $m \in \mathbb{N}$. We assume (as induction hypothesis) that Statement 2 holds for all $n < m$. We must prove that Statement 2 holds for $n = m$. In other words, we must prove that $q_m = r_m$. This is easily verified for $m \leq 1$, so let us WLOG assume that $m > 1$. Thus, $m \geq 2$. Hence, the induction hypothesis yields that Statement 2 holds for $n = m-1$ and for $n = m-2$. In other words, we have $q_{m-1} = r_{m-1}$ and $q_{m-2} = r_{m-2}$. Hence,

$$q_{m-1} = r_{m-1} = \frac{(m+3)\, f_{m-1} + 2\,(m-1)\, f_{m-2}}{5} \qquad (\text{by the definition of } r_{m-1})$$

and

$$q_{m-2} = r_{m-2} = \frac{(m+2)\, f_{m-2} + 2\,(m-2)\, f_{m-3}}{5} \qquad (\text{by the definition of } r_{m-2})\,.$$

Now, Statement 3 (applied to m instead of n) yields

$$\begin{aligned} q_m &= \underbrace{q_{m-1}}_{=\frac{(m+3)f_{m-1}+2(m-1)f_{m-2}}{5}} + \underbrace{q_{m-2}}_{=\frac{(m+2)f_{m-2}+2(m-2)f_{m-3}}{5}} + f_{m-1} \\ &= \frac{(m+3)f_{m-1}+2(m-1)f_{m-2}}{5} + \frac{(m+2)f_{m-2}+2(m-2)f_{m-3}}{5} + f_{m-1} \\ &= \underbrace{\left(\frac{m+3}{5}+1\right)}_{=\frac{m+8}{5}} f_{m-1} + \underbrace{\left(\frac{2(m-1)}{5}+\frac{m+2}{5}\right)}_{=\frac{3m}{5}} f_{m-2} + \frac{2(m-2)}{5} f_{m-3} \\ &= \frac{m+8}{5} f_{m-1} + \frac{3m}{5} f_{m-2} + \frac{2(m-2)}{5} f_{m-3}. \end{aligned}$$

Our goal is to prove that this equals

$$r_m = \frac{(m+4)f_m + 2mf_{m-1}}{5} = \frac{m+4}{5} f_m + \frac{2m}{5} f_{m-1}.$$

Thus, it suffices to show that

$$\frac{m+8}{5} f_{m-1} + \frac{3m}{5} f_{m-2} + \frac{2(m-2)}{5} f_{m-3} = \frac{m+4}{5} f_m + \frac{2m}{5} f_{m-1}. \tag{22}$$

The easiest way to do so is to rewrite all four Fibonacci numbers appearing in this equality in terms of f_{m-1} and f_{m-2}. In fact, the recurrence equation of the Fibonacci sequence yields $f_{m-1} = f_{m-2} + f_{m-3}$; thus, $f_{m-3} = f_{m-1} - f_{m-2}$. Also, the recurrence equation of the Fibonacci sequence yields $f_m = f_{m-1} + f_{m-2}$. In light of these two equalities, we can rewrite (22) as

$$\frac{m+8}{5} f_{m-1} + \frac{3m}{5} f_{m-2} + \frac{2(m-2)}{5} (f_{m-1} - f_{m-2}) = \frac{m+4}{5} (f_{m-1} + f_{m-2}) + \frac{2m}{5} f_{m-1}.$$

But this is easily verified by direct simplification of both sides (treating f_{m-1} and f_{m-2} as two independent variables). This completes the induction step. Thus, Statement 2 is proved.]

Combining Statement 1 with Statement 2, we conclude that $p_n = q_n = r_n$ for each $n \in \mathbb{N}$. This solves the exercise.

5.3 Remark

The sequence $(p_0, p_1, p_2, \ldots) = (q_0, q_1, q_2, \ldots) = (r_0, r_1, r_2, \ldots)$ (where the notations are as in the solution above) is sequence A029907 in the OEIS. All claims of the exercise appear on the OEIS page of this sequence. In particular, part of the exercise is an observation by Joerg Arndt (May 21, 2013). A generalization of our Statement 1 is given by Jia Huang in [Huang18, Proposition 6.1].

References

[17f-hw7s] Darij Grinberg, *UMN Fall 2017 Math 4990 homework set #7 with solutions*, `http://www.cip.ifi.lmu.de/~grinberg/t/17f/hw7os.pdf`
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The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://github.com/darijgr/detnotes/releases/tag/2019-01-10` .

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