Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2019: Homework 1

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December 16, 2019

1 Exercise 1

1.1 Problem

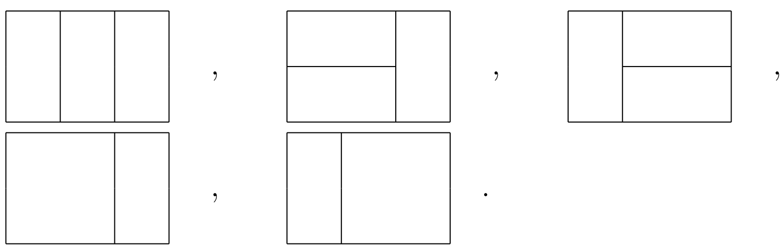
Let us define a slight variation on domino tilings. We shall use the notations of [Math222, §1.1].

A 2×2-*rectangle* will mean a set of the form $\{(i,j), (i,j+1), (i+1,j), (i+1,j+1)\}$ for some $i,j \in \mathbb{Z}$. (Visually, this is just a set of 4 mutually adjacent squares forming a 2×2-rectangle.)

A *pseudomino* will mean a set of squares that is either a domino or a 2×2-rectangle.

If S is a set of squares, then a *pseudomino tiling* of S will mean a set of disjoint pseudominos whose union is S.

For example, here are all five pseudomino tilings of the rectangle $R_{3,2}$:



For any $n \in \mathbb{N}$, we let p_n denote the number of all pseudomino tilings of the rectangle $R_{n,2}$.

[*Example:* We have $p_0 = 1$, $p_1 = 1$, $p_2 = 3$, $p_3 = 5$.]

(a) Find a recursive formula that expresses p_n in terms of p_{n-1} and p_{n-2} when $n \geq 2$.

(b) Prove that $p_n = \dfrac{(-1)^n + 2^{n+1}}{3}$ for each $n \in \mathbb{N}$.

[**Hint:** You don't need to be more detailed than in the proof of [Math222, Proposition 1.1.9].]

1.2 Solution sketch

(a) We claim that

$$p_n = p_{n-1} + 2p_{n-2} \qquad \text{for each integer } n \geq 2. \tag{1}$$

[*Proof of* (1)*:* The following proof follows closely[1] the similar proof of [Math222, Proposition 1.1.9].

Let $n \geq 2$ be an integer. Consider the last[2] column of $R_{n,2}$ (that is, the set $\{(n,1), (n,2)\}$).

In any pseudomino tiling T of $R_{n,2}$, this last column is **either** covered by 1 vertical domino, **or** covered by (parts of) 2 horizontal dominos, **or** covered by (half of) a 2×2-rectangle.

In the first of these three cases, we shall call T a *type-1 tiling*; in the second case, we shall call T a *type-2 tiling*; in the third case, we shall call T a *type-3 tiling*. Visually, these look as follows:

$$\underbrace{\begin{array}{|c|c|}\hline \text{???????????????} & \quad \\ \text{???????????????} & \quad \\ \hline \end{array}}_{\text{type-1 tiling}}, \qquad \underbrace{\begin{array}{|c|c|}\hline \text{???????????????} & \qquad \\ \cline{2-2} \text{???????????????} & \qquad \\ \hline \end{array}}_{\text{type-2 tiling}}, \qquad \underbrace{\begin{array}{|c|c|}\hline \text{???????????????} & \qquad \\ \text{???????????????} & \qquad \\ \hline \end{array}}_{\text{type-3 tiling}}$$

(where the question marks mean an unknown arrangement of pseudominos).

Let us now analyze type-1 tilings. A type-1 tiling consists of the single vertical domino $\{(n,1), (n,2)\}$ that covers its last column, and a bunch of pseudominos that cover all the remaining $n-1$ columns. This latter bunch must thus be a pseudomino tiling of $R_{n-1,2}$. Thus, a type-1 tiling consists of the single vertical domino $\{(n,1), (n,2)\}$ and an arbitrary pseudomino tiling of $R_{n-1,2}$. (Visually, this means that it looks as follows: $\begin{array}{|c|c|}\hline \text{some pseudomino} & \quad \\ \text{tiling of } R_{n-1,2} & \quad \\ \hline \end{array}$.) Hence,[3]

$$\begin{aligned}(\# \text{ of type-1 tilings}) &= (\# \text{ of pseudomino tilings of } R_{n-1,2}) &(2)\\ &= p_{n-1} &(3)\end{aligned}$$

(since p_{n-1} was defined as the # of pseudomino tilings of $R_{n-1,2}$).

Let us next analyze type-2 tilings. In a type-2 tiling, the last column is covered by (parts of) 2 horizontal dominos. These 2 dominos must extend to the left (because there is no space for them to extend to the right), and thus also cover the second-to-last column. Explicitly speaking, these 2 dominos must be $\{(n-1,1), (n,1)\}$ and $\{(n-1,2), (n,2)\}$. All the other pseudominos in the tiling must then cover the remaining $n-2$ columns, i.e.,

[1] euphemism for "is an almost verbatim copy of"

[2] i.e., easternmost

[3] When we say "type-1 tiling", we mean "type-1 tiling of $R_{n,2}$", of course. (The same will apply to "type-2 tiling" and to "type-3 tiling" later on.)

must form a pseudomino tiling of $R_{n-2,2}$. Thus, a type-2 tiling consists of the two horizontal dominos $\{(n-1,1),(n,1)\}$ and $\{(n-1,2),(n,2)\}$ and an arbitrary pseudomino tiling of $R_{n-2,2}$. (Visually, this means that it looks as follows: $\begin{array}{|c|c|}\hline \text{some pseudomino} & \qquad \\ \cline{2-2} \text{tiling of } R_{n-2,2} & \qquad \\ \hline\end{array}$.) Hence,

$$\begin{aligned}(\#\text{ of type-2 tilings}) &= (\#\text{ of pseudomino tilings of } R_{n-2,2}) &(4)\\ &= p_{n-2} &(5)\end{aligned}$$

(since p_{n-2} was defined as the # of pseudomino tilings of $R_{n-2,2}$).

Let us finally analyze type-3 tilings. In a type-3 tiling, the last column is covered by (half of) a 2×2-rectangle. This 2×2-rectangle must extend to the left (because there is no space for it to extend to the right), and thus also cover the second-to-last column. Explicitly speaking, this 2×2-rectangle must be $\{(n-1,1),(n-1,2),(n,1),(n,2)\}$. All the other pseudominos in the tiling must then cover the remaining $n-2$ columns, i.e., must form a pseudomino tiling of $R_{n-2,2}$. Thus, a type-3 tiling consists of the 2×2-rectangle $\{(n-1,1),(n-1,2),(n,1),(n,2)\}$ and an arbitrary pseudomino tiling of $R_{n-2,2}$. (Visually, this means that it looks as follows: $\begin{array}{|c|c|}\hline \text{some pseudomino} & \qquad \\ \text{tiling of } R_{n-2,2} & \qquad \\ \hline\end{array}$.) Hence,

$$\begin{aligned}(\#\text{ of type-3 tilings}) &= (\#\text{ of pseudomino tilings of } R_{n-2,2}) &(6)\\ &= p_{n-2} &(7)\end{aligned}$$

(since p_{n-2} was defined as the # of pseudomino tilings of $R_{n-2,2}$).

Now, recall that each pseudomino tiling of $R_{n,2}$ is either a type-1 tiling or a type-2 tiling or a type-3 tiling (but can never be of more than one of these types simultaneously). Hence,

$$\begin{aligned}&(\#\text{ of pseudomino tilings of } R_{n,2})\\ &= (\#\text{ of type-1 tilings}) + (\#\text{ of type-2 tilings}) + (\#\text{ of type-3 tilings})\\ &= p_{n-1} + p_{n-2} + p_{n-2} &(8)\end{aligned}$$

(by adding the equalities (3) and (5) and (7) together). Now, the definition of p_n yields

$$p_n = (\#\text{ of pseudomino tilings of } R_{n,2}) = p_{n-1} + p_{n-2} + p_{n-2} = p_{n-1} + 2p_{n-2}$$

(by (8)). This proves (1).]

(b) We shall prove the claim of part **(b)** of the exercise by strong induction on n:

Induction step: Let $m \in \mathbb{N}$. Assume (as the induction hypothesis) that part **(b)** of the exercise holds for all $n < m$. We must prove that part **(b)** of the exercise holds for $n = m$.

We have assumed that part **(b)** of the exercise holds for all $n < m$. In other words, we have

$$p_n = \frac{(-1)^n + 2^{n+1}}{3} \qquad \text{for each } n \in \mathbb{N} \text{ satisfying } n < m. \tag{9}$$

It is easy to see that $p_0 = 1$ (since the empty rectangle $R_{0,2}$ has exactly 1 pseudomino tiling – namely, the empty set) and that $p_1 = 1$ (since the rectangle $R_{1,2}$ has exactly 1 pseudomino tiling – namely, the one consisting of a single vertical domino). Comparing $p_0 = 1$ with $\frac{(-1)^0 + 2^{0+1}}{3} = \frac{1+2}{3} = 1$, we obtain $p_0 = \frac{(-1)^0 + 2^{0+1}}{3}$. Comparing $p_1 = 1$ with $\frac{(-1)^1 + 2^{1+1}}{3} = \frac{-1+4}{3} = 1$, we obtain $p_1 = \frac{(-1)^1 + 2^{1+1}}{3}$.

We must prove that part **(b)** of the exercise holds for $n = m$. In other words, we must prove that $p_m = \dfrac{(-1)^m + 2^{m+1}}{3}$. We already know that this is true for $m = 0$ (since $p_0 = \dfrac{(-1)^0 + 2^{0+1}}{3}$) and for $m = 1$ (since $p_1 = \dfrac{(-1)^1 + 2^{1+1}}{3}$). Hence, for the rest of this proof, we WLOG assume that $m \geq 2$. Thus, $m-2 \in \mathbb{N}$ and $m-2 < m$. Hence, (9) (applied to $n = m-2$) yields

$$p_{m-2} = \frac{(-1)^{m-2} + 2^{(m-2)+1}}{3} = \frac{(-1)^{m-2} + 2^{m-1}}{3}. \qquad (10)$$

Also, $m \geq 2 \geq 1$. Thus, $m-1 \in \mathbb{N}$ and $m-1 < m$. Hence, (9) (applied to $n = m-1$) yields

$$p_{m-1} = \frac{(-1)^{m-1} + 2^{(m-1)+1}}{3} = \frac{(-1)^{m-1} + 2^{m}}{3}. \qquad (11)$$

Now, (1) (applied to $n = m$) yields

$$\begin{aligned}
p_m &= p_{m-1} + 2p_{m-2} = \frac{(-1)^{m-1} + 2^m}{3} + 2 \cdot \frac{(-1)^{m-2} + 2^{m-1}}{3} \qquad \text{(by (11) and (10))} \\
&= \frac{1}{3}\left(\underbrace{(-1)^{m-1} + 2(-1)^{m-2}}_{=((-1)+2)(-1)^{m-2}} + \underbrace{2^m + 2 \cdot 2^{m-1}}_{=2^m+2^m=2\cdot 2^m=2^{m+1}} \right) = \frac{1}{3}\left(\underbrace{((-1)+2)}_{=1} \underbrace{(-1)^{m-2}}_{=(-1)^m} + 2^{m+1} \right) \\
&= \frac{1}{3}\left((-1)^m + 2^{m+1} \right) = \frac{(-1)^m + 2^{m+1}}{3}.
\end{aligned}$$

In other words, part **(b)** of the exercise holds for $n = m$. This completes the induction step. Thus, part **(b)** of the exercise is proven by induction.

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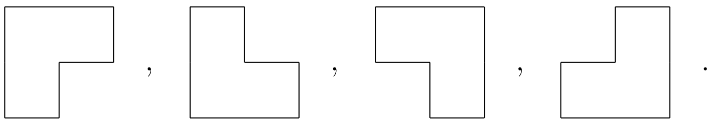
2 Exercise 2

2.1 Problem

Again, we shall use the notations of [Math222, §1.1].

An *L-tromino* will mean a set of squares that has one of the four forms



(Formally speaking, it is a set of the form $\{(i,j), (i',j), (i,j')\}$, where $i, j \in \mathbb{Z}$ and $i' \in \{i-1, i+1\}$ and $j' \in \{j-1, j+1\}$.)

If S is a set of squares, then an *L-tromino tiling* of S will mean a set of disjoint L-trominos whose union is S.

For any $n \in \mathbb{N}$, we let L_n denote the number of L-tromino tilings of the rectangle $R_{n,2}$.

We shall use the Iverson bracket notation[4].

Prove that

$$L_n = [3 \mid n] \cdot 2^{n/3} \qquad \text{for each } n \in \mathbb{N}. \qquad (12)$$

[**Hint:** Feel free to take inspiration from the solution to [18f-hw1s, Exercise 5].]

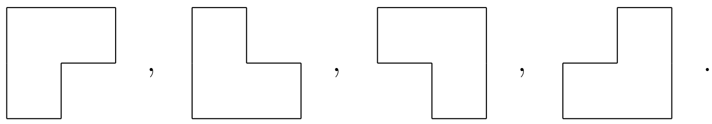
[4] This means the following:

2.2 Solution sketch

We shall first show that

$$L_n = 2L_{n-3} \qquad \text{for every integer } n \geq 3. \tag{13}$$

[*Proof of* (13)*:* The definition of L-trominos shows that there are four kinds of L-trominos:



We shall refer to these four kinds as *NW-trominos*, *SW-trominos*, *NE-trominos* and *SE-trominos*, respectively. (The names are a reference to the position in which the trominos have their "bends".)

Now, let $n \geq 3$ be an integer. We want to show that $L_n = 2L_{n-3}$.

This will be somewhat similar to [Math222, proof of Proposition 1.1.9] and to [18f-hw1s, solution to Exercise 5].

Consider the last[5] column of $R_{n,2}$ (that is, the set $\{(n,1),(n,2)\}$).

Consider any L-tromino tiling T of $R_{n,2}$. Clearly, the square $(n,1)$ must be contained in some L-tromino in T. This L-tromino cannot be a NW-tromino[6], and cannot be a SW-tromino either[7]. Hence, this L-tromino must be either a NE-tromino or a SE-tromino. In either case, this L-tromino must cover the whole last column of $R_{n,2}$ (since this is the only way to place it so that it contains $(n,1)$ but is contained in $R_{n,2}$). Of course, it is the only L-tromino in T that covers the whole last column (since the L-trominos in T are disjoint).

Now, forget that we fixed T. We thus have shown that each L-tromino tiling T of $R_{n,2}$ contains a unique L-tromino that covers the whole last column of $R_{n,2}$, and this L-tromino is either a NE-tromino or an SE-tromino. If this L-tromino is a NE-tromino, then we shall call T a *type-1 tiling*; if it instead is a SE-tromino, then we shall call T a *type-2 tiling*. Visually,

If $\mathcal{A}$ is any statement (such as "$1+1=2$" or "$1+1=1$" or "there exist infinitely many primes"), then $[\mathcal{A}]$ stands for the number

$$\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false.} \end{cases}$$

This number belongs to $\{0,1\}$, and is called the *truth value* of $\mathcal{A}$. For example,

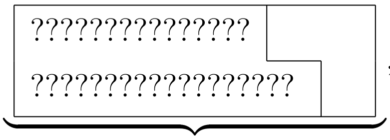
$$[1+1=2] = 1, \qquad [1+1=1] = 0, \qquad [\text{there exist infinitely many primes}] = 1.$$

[5] i.e., easternmost

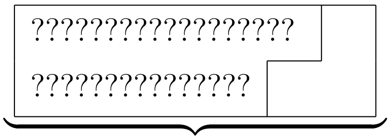
[6] since a NW-tromino that contains $(n,1)$ would also contain $(n+1,2)$, which means it would fail to be contained in $R_{n,2}$

[7] *Proof.* Assume the contrary. Thus, the square $(n,1)$ is contained in some SW-tromino $S \in T$. This SW-tromino S must be $\{(n-1,1),(n-1,2),(n,1)\}$ (since any other SW-tromino that contains $(n,1)$ would fail to be contained in $R_{n,2}$). But the square $(n,2)$ also belongs to $R_{n,2}$, and thus must be contained in some L-tromino $X \in T$. This latter L-tromino X must be distinct from S (since $(n,2) \notin \{(n-1,1),(n-1,2),(n,1)\} = S$), and thus disjoint from S. On the other hand, it must contain at least one square adjacent to $(n,2)$ (since it contains $(n,2)$, and since any L-tromino is a connected shape). But any square adjacent to $(n,2)$ either belongs to S or falls outside of $R_{n,2}$ (since $S = \{(n-1,1),(n-1,2),(n,1)\}$ contains the western and southern neighbors of $(n,2)$, whereas the eastern and northern neighbors of $(n,2)$ fall outside of $R_{n,2}$). Hence, the L-tromino X must contain at least one square that either belongs to S or falls outside of $R_{n,2}$. This contradicts the fact that X is disjoint from S and contained inside $R_{n,2}$. This contradiction shows that our assumption was false, qed.

these two types of tilings look as follows:



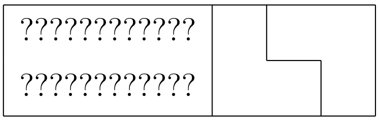
type-1 tiling



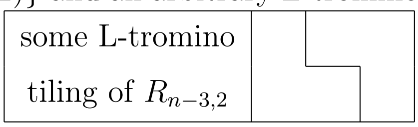
type-2 tiling

(where the question marks mean an unknown arrangement of L-trominos).

Let us now consider an arbitrary type-1 tiling T. This tiling T consists of the single NE-tromino $\{(n,1), (n,2), (n-1,2)\}$ that covers its last column, and a bunch of L-trominos that cover all the remaining $n-1$ columns except for the square $(n-1,2)$. In particular, one L-tromino of the latter bunch must contain the square $(n-1,1)$. Let us denote this L-tromino by X. Since this L-tromino X is disjoint from the L-tromino $\{(n,1), (n,2), (n-1,2)\}$ (since both of these L-trominos belong to our tiling), we thus conclude that X must be the SW-tromino $\{(n-1,1), (n-2,1), (n-2,2)\}$ (since any other L-tromino containing $(n-1,1)$ would either overlap with $\{(n,1), (n,2), (n-1,2)\}$ or fail to be contained in $R_{n,2}$). Hence, our type-1 tiling T must look as follows:



(where the question marks mean an unknown arrangement of L-trominos); in particular it contains the two L-trominos $\{(n,1), (n,2), (n-1,2)\}$ and $\{(n-1,1), (n-2,1), (n-2,2)\}$, which (in combination) cover the last 3 columns of $R_{n,2}$. The remaining L-trominos in T must cover the leftmost $n-3$ columns of $R_{n,2}$; in other words, they must form an L-tromino tiling of $R_{n-3,2}$.

Thus, we have shown that a type-1 tiling consists of the two L-trominos $\{(n,1), (n,2), (n-1,2)\}$ and $\{(n-1,1), (n-2,1), (n-2,2)\}$ and an arbitrary L-tromino tiling of $R_{n-3,2}$. (Visually, this means that it looks as follows: 

.)

Hence,[8]

$$\begin{aligned}(\text{\# of type-1 tilings}) &= (\text{\# of L-tromino tilings of } R_{n-3,2}) \\ &= L_{n-3} \qquad (14)\end{aligned}$$

(since L_{n-3} was defined as the # of L-tromino tilings of $R_{n-3,2}$).

We can similarly analyze type-2 tilings, and conclude that

$$(\text{\# of type-2 tilings}) = L_{n-3}. \qquad (15)$$

(Alternatively, we can obtain this from (14) by a symmetry argument: There is clearly a bijection from the set of all type-1 tilings to the set of all type-2 tilings[9]. This shows that (# of type-1 tilings) = (# of type-2 tilings), according to the bijection principle. Hence, (# of type-2 tilings) = (# of type-1 tilings) = L_{n-3} (by (14)).)

[8] When we say "type-1 tiling", we mean "type-1 tiling of $R_{n,2}$", of course. (The same will apply to "type-2 tiling" later on.)

[9] Namely, this bijection transforms any type-1 tiling by reflecting it across the horizontal axis of symmetry of $R_{n,2}$.

Now, recall that each L-tromino tiling of $R_{n,2}$ is either a type-1 tiling or a type-2 tiling (but never both at the same time). Hence,

$$\begin{aligned}&(\# \text{ of L-tromino tilings of } R_{n,2})\\&= (\# \text{ of type-1 tilings}) + (\# \text{ of type-2 tilings})\\&= L_{n-3} + L_{n-3}\end{aligned} \tag{16}$$

(by adding the equalities (14) and (15) together). Now, the definition of L_n yields

$$L_n = (\# \text{ of L-tromino tilings of } R_{n,2}) = L_{n-3} + L_{n-3} = 2L_{n-3}$$

(by (16)). This proves (13).]

With (13) proven, it is now a matter of straightforward induction to solve the exercise:

[*Proof of* (12)*:* We shall prove (12) by strong induction on n.

Thus, let $m \in \mathbb{N}$, and let us assume (as the induction hypothesis) that (12) holds for all $n < m$. We must prove that (12) holds for $n = m$. In other words, we must prove that $L_m = [3 \mid m] \cdot 2^{m/3}$.

The rectangle $R_{0,2}$ is the empty set, and thus has exactly one L-tromino tiling (namely, the empty set). In other words, $L_0 = 1$. Comparing this with $\underbrace{[3 \mid 0]}_{=1} \cdot \underbrace{2^{0/3}}_{=2^0=1} = 1$, we obtain $L_0 = [3 \mid 0] \cdot 2^{0/3}$. In other words, $L_m = [3 \mid m] \cdot 2^{m/3}$ holds if $m = 0$. Thus, for the rest of our proof of $L_m = [3 \mid m] \cdot 2^{m/3}$, we WLOG assume that $m \neq 0$.

The rectangle $R_{1,2}$ has 2 squares, and thus has no L-tromino tiling (since any L-tromino would take up 3 squares). In other words, $L_1 = 0$. Comparing this with $\underbrace{[3 \mid 1]}_{=0} \cdot 2^{1/3} = 0$, we obtain $L_1 = [3 \mid 1] \cdot 2^{1/3}$. In other words, $L_m = [3 \mid m] \cdot 2^{m/3}$ holds if $m = 1$. Thus, for the rest of our proof of $L_m = [3 \mid m] \cdot 2^{m/3}$, we WLOG assume that $m \neq 1$.

The rectangle $R_{2,2}$ has 4 squares, and thus has no L-tromino tiling (since any L-tromino has 3 squares, and thus any set that has an L-tromino tiling must have either 0 or 3 or 6 or more squares[10]). In other words, $L_2 = 0$. Comparing this with $\underbrace{[3 \mid 2]}_{=0} \cdot 2^{2/3} = 0$, we obtain $L_2 = [3 \mid 2] \cdot 2^{2/3}$. In other words, $L_m = [3 \mid m] \cdot 2^{m/3}$ holds if $m = 2$. Thus, for the rest of our proof of $L_m = [3 \mid m] \cdot 2^{m/3}$, we WLOG assume that $m \neq 2$.

As a consequence of our WLOG assumptions, we now have $m \neq 0$ and $m \neq 1$ and $m \neq 2$. Hence, $m \geq 3$ (since $m \in \mathbb{N}$). Thus, $m - 3 \in \mathbb{N}$ and $m - 3 < m$. Hence, (12) holds for $n = m - 3$ (due to our induction hypothesis saying that (12) holds for all $n < m$). In other words,

$$L_{m-3} = [3 \mid m-3] \cdot 2^{(m-3)/3}. \tag{17}$$

The integers $m - 3$ and m differ by 3. Thus, one of them is divisible by 3 if and only if the other is. In other words, the statements $(3 \mid m-3)$ and $(3 \mid m)$ are equivalent.

But here comes one of the most basic properties of truth values: If $\mathcal{A}$ and $\mathcal{B}$ are two equivalent statements, then $[\mathcal{A}] = [\mathcal{B}]$. (This follows easily by noticing that $\mathcal{A}$ is true if and only if $\mathcal{B}$ is true.) Applying this property to $\mathcal{A} = (3 \mid m-3)$ and $\mathcal{B} = (3 \mid m)$, we conclude that $[3 \mid m-3] = [3 \mid m]$ (since the statements $(3 \mid m-3)$ and $(3 \mid m)$ are equivalent).

Also, $(m-3)/3 = m/3 - 1$, so that $2^{(m-3)/3} = 2^{m/3-1} = 2^{m/3}/2$. Hence, (17) becomes

$$L_{m-3} = \underbrace{[3 \mid m-3]}_{=[3\mid m]} \cdot \underbrace{2^{(m-3)/3}}_{=2^{m/3}/2} = [3 \mid m] \cdot 2^{m/3}/2.$$

[10] Of course, we have used the sum rule here.

But recall that $m \geq 3$. Hence, (13) (applied to $n = m$) yields

$$L_m = 2 \underbrace{L_{m-3}}_{=[3|m]\cdot 2^{m/3}/2} = 2\,[3 \mid m] \cdot 2^{m/3}/2 = [3 \mid m] \cdot 2^{m/3}.$$

In other words, (12) holds for $n = m$. This completes the induction step. Thus, (12) is proven by induction.]

This solves the exercise.

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3 Exercise 3

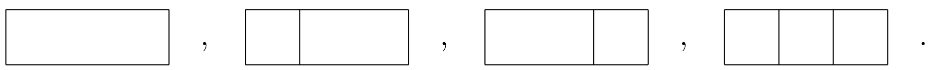
3.1 Problem

Again, we shall use the notations of [Math222, §1.1].

A *horimino* shall mean a rectangle of height 1 and positive width (i.e., formally speaking, a set of the form $\{(i,1), (i+1,1), \ldots, (j,1)\}$ for some integers $i \leq j$).

If S is a set of squares, then a *horimino tiling* of S will mean a set of disjoint horiminos whose union is S.

For example, here are all four horimino tilings of the rectangle $R_{3,1}$:



Let $n \in \mathbb{N}$. Find a simple expression for the number of all horimino tilings of the rectangle $R_{n,1}$.

[**Hint:** Make sure your answer works for $n = 0$ (you might need to handle this case separately).]

3.2 First solution sketch

We first recall the following fact:

Proposition 3.1. *Let $n \in \mathbb{N}$. Then, the number of all subsets of $\{1, 2, \ldots, n\}$ is 2^n.*

Proposition 3.1 is [19f-hw0s, Exercise 1 **(a)**], so we have no need to prove it again.

Now, let us fix $n \in \mathbb{N}$. We must compute the number of all horimino tilings of the rectangle $R_{n,1}$.

If $n = 0$, then this number is since there is exactly one horimino tiling of the rectangle $R_{0,1}$ (indeed, the rectangle $R_{0,1}$ is the empty set, so its only horimino tiling is the empty set as well). Thus, we have solved the exercise in the case $n = 0$.

Hence, we WLOG assume that $n \neq 0$ from now on. Thus, $n - 1 \in \mathbb{N}$. Thus, Proposition 3.1 (applied to $n-1$ instead of n) shows that the number of all subsets of $\{1, 2, \ldots, n-1\}$ is 2^{n-1}. In other words,

$$(\# \text{ of all subsets of } \{1, 2, \ldots, n-1\}) = 2^{n-1}. \tag{18}$$

In the following, the word "horimino tiling" shall always "horimino tiling of $R_{n,1}$".

Recall that if S is any set, then $\mathcal{P}(S)$ denotes the powerset of S (that is, the set of all subsets of S). Now, we claim that there is a bijection

$$\beta : \{\text{horimino tilings}\} \to \mathcal{P}([n-1]).$$

Let us first describe this bijection visually, before giving a formal definition.

The rectangle $R_{n,1}$ has n squares $(1,1), (2,1), \ldots, (n,1)$, separated by $n-1$ "walls". The wall between the two adjacent squares $(i,1)$ and $(i+1,1)$ will be called "wall i". Thus, when moving from the western to the eastern end of $R_{n,1}$, we have to cross the walls $1, 2, \ldots, n-1$ in this order. Hence, the set of all walls is $[n-1]$ (or at least can be identified with the set $[n-1]$).

Now, if T is a horimino tiling, and if $i \in [n-1]$, then the wall i will either **fall inside** one of the horiminos of T (that is, the squares $(i,1)$ and $(i+1,1)$ belong to one and the same horimino of T), or **separate** two horiminos of T. For example, in the following horimino tiling of $R_{7,1}$:

(where the three horiminos have widths 2, 3 and 2, from left to right), the wall 1 falls inside the leftmost horimino; the wall 2 separates this horimino from the middle horimino; the walls 3 and 4 fall inside the middle horimino; the wall 5 separates it from the rightmost horimino; and the wall 6 falls inside the rightmost horimino.

We shall say that a wall W is *visible* in a horimino tiling T if it separates two horiminos of T. Otherwise, we shall say that W is *invisible* in T.

It is intuitively obvious that any horimino tiling T is uniquely determined by the set of all walls that are visible in T (because the horiminos of T can be reconstructed from this set simply by placing a horimino into the space between any two consecutive visible walls). Conversely, for any set S of walls (i.e., for any subset S of $[n-1]$), there is exactly one horimino tiling T such that the visible walls in T will be precisely the walls in S.

Thus, we can define a bijection

$$\beta : \{\text{horimino tilings}\} \to \mathcal{P}([n-1]),$$
$$T \mapsto (\text{the set of all walls that are visible in } T).$$

As we said above, a look at the picture makes it intuitively obvious that β is a bijection; nevertheless, it is worth proving this rigorously as well. Let me give an outline of how this can be done, without going into the straightforward details. First, let us define the notions of "walls" and "visible walls" without referring to the picture. We define a *wall* simply as an element of $[n-1]$, and we say that a wall $i \in [n-1]$ is *visible* in a horimino tiling T if and only if the squares $(i,1)$ and $(i+1,1)$ belong to two different horiminos of T. (It is easy to see that this definition of "visible" matches the more geometric definition we gave above.) Thus, the map β is clearly well-defined.

In order to see that β is a bijection, we need to show that β is injective and surjective.

[*Proof of the surjectivity of β:* Let $S \in \mathcal{P}([n-1])$. We need to find a horimino tiling T such that $\beta(T) = S$.

Write the subset S of $\mathcal{P}([n-1])$ in the form $S = \{s_1, s_2, \ldots, s_k\}$ with $s_1 < s_2 < \cdots < s_k$. (Clearly, there is exactly one way to write S in this form[11].) Set $s_0 = 0$ and $s_{k+1} = n$; thus, $0 = s_0 < s_1 < s_2 \cdots < s_k < s_{k+1} = n$. Let T be the horimino tiling consisting of the following $k+1$ horiminos:

$$\begin{gathered}
\{(s_0+1,1), (s_0+2,1), \ldots, (s_1,1)\}, \\
\{(s_1+1,1), (s_1+2,1), \ldots, (s_2,1)\}, \\
\{(s_2+1,1), (s_2+2,1), \ldots, (s_3,1)\}, \\
\vdots \\
\{(s_k+1,1), (s_k+2,1), \ldots, (s_{k+1},1)\}.
\end{gathered}$$

[11]To be fully rigorous: It is [19f-hw0s, Proposition 1.3] that we are using here.

(That is, the i-th horimino is $\{(s_{i-1}+1,1),(s_{i-1}+2,1),\ldots,(s_i,1)\}$ for each $i \in [k+1]$.) It is easy to see that this T is indeed a horimino tiling (since each element of $[n]$ belongs to exactly one of the $k+1$ intervals

$$\begin{aligned}
&(s_0+1, s_0+2, \ldots, s_1), \\
&(s_1+1, s_1+2, \ldots, s_2), \\
&(s_2+1, s_2+2, \ldots, s_3), \\
&\vdots \\
&(s_k+1, s_k+2, \ldots, s_{k+1}),
\end{aligned}$$

which is because $0 = s_0 < s_1 < s_2 < \cdots < s_{k+1} = n$). It is easy to see that the walls that are visible in this tiling T are precisely the walls $s_1, s_2, \ldots, s_k$, that is, the walls in the set S (since $S = \{s_1, s_2, \ldots, s_k\}$). Hence, the definition of the map β yields $\beta(T) = S$.

Now, forget that we fixed S. Thus, for each $S \in \mathcal{P}([n-1])$, we have found a horimino tiling T such that $\beta(T) = S$. This shows that the map β is surjective.]

[*Proof of the injectivity of β:* Let T be a horimino tiling. We are going to show how T can be reconstructed from the set $\beta(T)$.

Write the tiling T in the form

$$\begin{aligned}
T = \{&\{(s_1,1),(s_1+1,1),\ldots,(t_1,1)\}, \\
&\{(s_2,1),(s_2+1,1),\ldots,(t_2,1)\}, \\
&\{(s_3,1),(s_3+1,1),\ldots,(t_3,1)\}, \\
&\vdots \\
&\{(s_p,1),(s_p+1,1),\ldots,(t_p,1)\}\}
\end{aligned} \tag{19}$$

(so that $s_1, s_2, \ldots, s_p$ are the columns in which the horiminos of T begin, and $t_1, t_2, \ldots, t_p$ are the columns in which they end). We WLOG assume that $s_1 \leq s_2 \leq \cdots \leq s_p$ (since otherwise, we just relabel the horiminos of T). Thus, it is easy to see that

$$s_1 \leq t_1 < s_2 \leq t_2 < s_3 \leq t_3 < \cdots < s_p \leq t_p \tag{20}$$

(that is, $s_i \leq t_i$ for all $i \in [p]$ and $t_i < s_{i+1}$ for all $i \in [p-1]$), because the horiminos of T must not overlap. Hence, recalling that the horiminos of T must cover the whole rectangle $R_{n,1}$, we conclude that $s_1 = 1$ and $t_p = n$ and $t_i = s_{i+1} - 1$ for each $i \in [p-1]$ (since otherwise, there would be squares that are not covered by any of the horiminos). Thus, (19) rewrites as follows:

$$\begin{aligned}
T = \{&\{(1,1),(2,1),\ldots,(s_2-1,1)\}, \\
&\{(s_2,1),(s_2+1,1),\ldots,(s_3-1,1)\}, \\
&\{(s_3,1),(s_3+1,1),\ldots,(s_4-1,1)\}, \\
&\vdots \\
&\{(s_p,1),(s_p+1,1),\ldots,(n,1)\}\}.
\end{aligned} \tag{21}$$

(If $p = 1$, then this should be understood as $T = \{\{(1,1),(2,1),\ldots,(n,1)\}\}$; this is the case when the tiling T consists of a single horimino of width n.)

From (20), we obtain $s_1 < s_2 < \cdots < s_{p+1}$, hence $s_2 < s_3 < \cdots < s_p$.

From (21), we see immediately which walls are visible in T: Namely, the walls $s_2 - 1, s_3 - 1, \ldots, s_p - 1$ (and no others) are visible in T. In other words, the set of all walls that are visible in T is $\{s_2 - 1, s_3 - 1, \ldots, s_p - 1\}$. In other words,

$$\beta(T) = \{s_2 - 1, s_3 - 1, \ldots, s_p - 1\} \tag{22}$$

(since $\beta(T)$ was defined as the set of all walls that are visible in T). Hence, the numbers $s_2 - 1, s_3 - 1, \ldots, s_p - 1$ are precisely the elements of $\beta(T)$ listed in increasing order with no repetitions (since $s_2 < s_3 < \cdots < s_p$). Thus, the set $\beta(T)$ determines the numbers $s_2 - 1, s_3 - 1, \ldots, s_p - 1$. These numbers, in turn, determine $s_2, s_3, \ldots, s_p$, and thus determine the tiling T (because (21) expresses T in terms of $s_2, s_3, \ldots, s_p$). Combining these, we conclude that the set $\beta(T)$ determines the tiling T (that is, we can uniquely reconstruct T from $\beta(T)$). In other words, if T_1 and T_2 are two horimino tilings satisfying $\beta(T_1) = \beta(T_2)$, then $T_1 = T_2$. In other words, the map β is injective.]

We have now showed that β is injective and surjective. Hence, β is bijective, i.e., a bijection.

We have now found a bijection $\beta : \{\text{horimino tilings}\} \to \mathcal{P}([n-1])$. Thus, the bijection principle shows that

$$\begin{aligned} |\{\text{horimino tilings}\}| &= |\mathcal{P}([n-1])| = (\#\text{ of subsets of } [n-1]) \\ &\qquad \big(\text{since } \mathcal{P}([n-1]) \text{ is the set of all subsets of } [n-1]\big) \\ &= (\#\text{ of all subsets of } \{1, 2, \ldots, n-1\}) \\ &\qquad \big(\text{since } [n-1] = \{1, 2, \ldots, n-1\}\big) \\ &= 2^{n-1} \qquad \big(\text{by (18)}\big). \end{aligned}$$

In other words, the number of all horimino tilings of the rectangle $R_{n,1}$ is 2^{n-1}.

Now we have solved our problem first in the case when $n = 0$, and then in the case when $n \neq 0$. We obtained the answer 1 in the former case, and the answer 2^{n-1} in the latter. We can combine these two answers into the following general formula:

$$(\#\text{ of all horimino tilings of } R_{n,1}) = \begin{cases} 2^{n-1}, & \text{if } n \neq 0; \\ 1, & \text{if } n = 0 \end{cases} \qquad \text{for all } n \in \mathbb{N}.$$

The right hand side of this formula can also be rewritten as $2^{\max\{n-1,0\}}$, where $\max S$ denotes the maximum element of a set S. (Check this!)

3.3 Second solution sketch

Forget that we fixed n. For each $n \in \mathbb{N}$, we let h_n denote the number of all horimino tilings of $R_{n,1}$. Thus, the problem asks us to find a simple expression for h_n.

We answer this as follows: We claim that

$$h_n = \begin{cases} 2^{n-1}, & \text{if } n \neq 0; \\ 1, & \text{if } n = 0 \end{cases} \qquad \text{for all } n \in \mathbb{N}. \tag{23}$$

It remains to prove this. We shall achieve this by proving a recurrence relation for the h_n first: We shall show that

$$h_n = h_0 + h_1 + \cdots + h_{n-1} \qquad \text{for all positive integers } n. \tag{24}$$

[*Proof of* (24)*:* Let n be a positive integer.

Consider the square $(n, 1)$; this is the easternmost square of $R_{n,1}$.

In any horimino tiling T of $R_{n,1}$, this square $(n, 1)$ must be covered by some horimino. We let k_T denote the size (i.e., the width) of this horimino. Clearly, this size k_T is a number in $\{1,2,\ldots,n\}$ (since any horimino that fits into $R_{n,1}$ must have size $\leq n$). Thus, we can count horimino tilings of $R_{n,1}$ according to their value of k_T:

$$\begin{aligned}&(\# \text{ of horimino tilings of } R_{n,1})\\&=\sum_{i\in\{1,2,\ldots,n\}}(\# \text{ of horimino tilings } T \text{ of } R_{n,1} \text{ satisfying } k_T=i)\,.\end{aligned} \tag{25}$$

(Strictly speaking, this is a consequence of the sum rule.)

Now, let $i\in\{1,2,\ldots,n\}$. Consider any horimino tiling T of $R_{n,1}$ satisfying $k_T=i$. Then, the square $(n,1)$ is covered by a horimino of size k_T in T (by the definition of k_T). In other words, the square $(n,1)$ is covered by a horimino of size i in T (since $k_T=i$). This horimino must end[12] in this square $(n,1)$ (since it has to be contained in $R_{n,1}$, but $(n,1)$ is the easternmost square of $R_{n,1}$), and thus must begin in the square $(n-i+1,1)$ (since it has size i). Hence, this horimino is $\{(n-i+1,1),(n-i+2,1),\ldots,(n,1)\}$, and covers the last (i.e., easternmost) i columns of $R_{n,1}$. The remaining horiminos in the tiling T must therefore cover the remaining $n-i$ columns of $R_{n,1}$; in other words, they must constitute a horimino tiling of $R_{n-i,1}$. Hence, the set

$$T\setminus\{\text{the horimino in } T \text{ that covers } (n,1)\}$$

is a horimino tiling of $R_{n-i,1}$.

Now, forget that we fixed T. We thus have shown that if T is a horimino tiling of $R_{n,1}$ satisfying $k_T=i$, then $T\setminus\{\text{the horimino in } T \text{ that covers } (n,1)\}$ is a horimino tiling of $R_{n-i,1}$. Thus, the map

$$\begin{aligned}&\{\text{horimino tilings } T \text{ of } R_{n,1} \text{ satisfying } k_T=i\}\to\{\text{horimino tilings of } R_{n-i,1}\}\,,\\&\qquad\qquad T\mapsto T\setminus\{\text{the horimino in } T \text{ that covers } (n,1)\}\end{aligned}$$

is well-defined. Furthermore, this map is a bijection[13]. Thus, the bijection principle shows that

$$\begin{aligned}&|\{\text{horimino tilings } T \text{ of } R_{n,1} \text{ satisfying } k_T=i\}|\\&=|\{\text{horimino tilings of } R_{n-i,1}\}|\,.\end{aligned} \tag{26}$$

Thus,

$$\begin{aligned}&(\# \text{ of horimino tilings } T \text{ of } R_{n,1} \text{ satisfying } k_T=i)\\&=|\{\text{horimino tilings } T \text{ of } R_{n,1} \text{ satisfying } k_T=i\}|\\&=|\{\text{horimino tilings of } R_{n-i,1}\}|\\&=(\# \text{ of horimino tilings of } R_{n-i,1})\\&=h_{n-i}\end{aligned} \tag{27}$$

(since h_{n-i} was defined to be the # of horimino tilings of $R_{n-i,1}$).

[12] We say that a horimino *ends* in its easternmost square.

[13] because the map

$$\begin{aligned}&\{\text{horimino tilings of } R_{n-i,1}\}\to\{\text{horimino tilings } T \text{ of } R_{n,1} \text{ satisfying } k_T=i\}\,,\\&\qquad Q\mapsto Q\cup\{\{(n-i+1,1),(n-i+2,1),\ldots,(n,1)\}\}\end{aligned}$$

is inverse to it

Forget that we fixed i. We thus have proved (27) for each $i \in \{1, 2, \ldots, n\}$. Now, the definition of h_n yields

$$\begin{aligned} h_n &= (\# \text{ of horimino tilings of } R_{n,1}) \\ &= \sum_{i \in \{1,2,\ldots,n\}} \underbrace{(\# \text{ of horimino tilings } T \text{ of } R_{n,1} \text{ satisfying } k_T = i)}_{\substack{= h_{n-i} \\ (\text{by } (27))}} \qquad \big(\text{by } (25)\big) \\ &= \sum_{i \in \{1,2,\ldots,n\}} h_{n-i} = h_{n-1} + h_{n-2} + \cdots + h_{n-n} = h_0 + h_1 + \cdots + h_{n-1}. \end{aligned}$$

This proves (24).]

The recurrent equation (24) is a bit complicated (it involves n terms on the right hand side), but it is already good enough to let us easily prove (23) by strong induction on n (more precisely, it reduces (23) to the identity $2^{n-1} = 1 + (2^0 + 2^1 + \cdots + 2^{n-2})$, which is well-known and itself is easy to prove by induction). Nevertheless, we can simplify our remaining work a little bit further: We can use (24) to derive a simpler recursion. Namely, we claim that

$$h_n = 2h_{n-1} \qquad \text{for all integers } n > 1. \tag{28}$$

[*Proof of* (28)*:* Let $n > 1$ be an integer. Then, $n - 1$ is a positive integer. Hence, (24) (applied to $n - 1$ instead of n) yields

$$h_{n-1} = h_0 + h_1 + \cdots + h_{(n-1)-1} = h_0 + h_1 + \cdots + h_{n-2}.$$

But n is a positive integer, too (since $n > 1$), and thus (24) yields

$$h_n = h_0 + h_1 + \cdots + h_{n-1}.$$

Subtracting the previous equality from this one, we obtain

$$h_n - h_{n-1} = (h_0 + h_1 + \cdots + h_{n-1}) - (h_0 + h_1 + \cdots + h_{n-2}) = h_{n-1}.$$

Thus, $h_n = h_{n-1} + h_{n-1} = 2h_{n-1}$. This proves (28).]

Now, proving (23) is completely straightforward:

[*Proof of* (23)*:* We shall prove (23) by induction on n:

Induction base: There is exactly 1 horimino tiling of $R_{0,1}$ (namely, the empty set). In other words, $h_0 = 1$. Thus, (23) holds for $n = 0$. This completes the induction base.

Induction step: Let m be a positive integer. Assume that (23) holds for $n = m - 1$. We must prove that (23) holds for $n = m$. In other words, we must prove that

$$h_m = \begin{cases} 2^{m-1}, & \text{if } m \neq 0; \\ 1, & \text{if } m = 0 \end{cases}.$$

This rewrites as

$$h_m = 2^{m-1}$$

(because $m \neq 0$). Thus, all we need to prove is the equality $h_m = 2^{m-1}$. This equality is easy to verify when $m = 1$, because a quick look at the rectangle $R_{1,1}$ (which has only one square, and thus only one horimino tiling) reveals that $h_1 = 1 = 2^{1-1}$. Thus, for the rest of this proof, we WLOG assume that $m \neq 1$. Hence, $m > 1$ (since m is a positive integer). Thus, $m-1 > 0$, so that $m-1 \neq 0$. But we assumed that (23) holds for $n = m-1$. Thus,

$$h_{m-1} = \begin{cases} 2^{(m-1)-1}, & \text{if } m-1 \neq 0; \\ 1, & \text{if } m-1 = 0 \end{cases} = 2^{(m-1)-1} \qquad \left(\text{since } m-1 \neq 0\right).$$

But (28) (applied to $n = m$) yields

$$h_m = 2 \underbrace{h_{m-1}}_{=2^{(m-1)-1}} = 2 \cdot 2^{(m-1)-1} = 2^{m-1}.$$

This is precisely the equality that we need to prove. Thus, we have shown that (23) holds for $n = m$. This completes the induction step, and with it our inductive proof of (23).]

3.4 Remark

There is also a third solution, similar to the second but obtaining the recurrence (28) directly (without the detour through (24)). The main idea is that (for $n > 1$) we can subdivide the horimino tilings of $R_{n,1}$ into two types: "type-1" (in which the horimino covering the square $(n, 1)$ has size 1) and "type-2" (in which the horimino covering the square $(n, 1)$ has size > 1). It is clear that the type-1 horimino tilings of $R_{n,1}$ are in bijection with the horimino tilings of $R_{n-1,1}$. But the type-2 horimino tilings of $R_{n,1}$ are also in bijection with the horimino tilings of $R_{n-1,1}$ (in fact, the bijection simply removes the square $(n, 1)$ from the horimino containing it, so that this particular horimino becomes shorter by 1). Thus, the sum rule yields (28).

---

4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$0^2 \cdot 1! + 1^2 \cdot 2! + 2^2 \cdot 3! + \cdots + (n-1)^2 \cdot n! = (n-2) \cdot (n+1)! + 2.$$

4.2 Remark

A well-known identity (see, e.g., [19s-hw0s, Exercise 2 **(b)**]) says that each $n \in \mathbb{N}$ satisfies

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1. \qquad (29)$$

The exercise above can be viewed as a variant of this identity.

4.3 Solution

We shall give two solutions, both closely resembling the two proofs of (29) given in [19s-hw0s, solution to Exercise 2 **(b)**]. The first solution is a straightforward induction argument, whereas the second is a slicker-looking application of the telescope principle (but in reality, the only difference is the presentation).

Both solutions will rely on the following fundamental fact about factorials:

$$n! = n \cdot (n-1)! \qquad \text{for each positive integer } n. \tag{30}$$

(This appears, e.g., in [19s-hw0s, Exercise 2 **(a)**]. It follows easily from the definition of factorials.)

Now we solve our exercise:

First solution (by induction): We shall prove the claim of the exercise by induction on n:

Induction base: We have

$$0^2 \cdot 1! + 1^2 \cdot 2! + 2^2 \cdot 3! + \cdots + (0-1)^2 \cdot 0! = (\text{empty sum}) = 0.$$

Comparing this with $(0-2) \cdot \underbrace{(0+1)!}_{=1!=1} + 2 = (0-2) \cdot 1 + 2 = 0$, we obtain $0^2 \cdot 1! + 1^2 \cdot 2! + 2^2 \cdot 3! + \cdots + (0-1)^2 \cdot 0! = (0-2) \cdot (0+1)! + 2$. Thus, the claim of the exercise holds for $n = 0$. This completes the induction base.

Induction step: Let $m \in \mathbb{N}$. Assume that the claim of the exercise holds for $n = m$. We must prove that the claim of the exercise holds for $n = m+1$.

We have assumed that the claim of the exercise holds for $n = m$. In other words, we have

$$0^2 \cdot 1! + 1^2 \cdot 2! + 2^2 \cdot 3! + \cdots + (m-1)^2 \cdot m! = (m-2) \cdot (m+1)! + 2.$$

Now,

$$\begin{aligned}
&0^2 \cdot 1! + 1^2 \cdot 2! + 2^2 \cdot 3! + \cdots + ((m+1)-1)^2 \cdot (m+1)! \\
&= \underbrace{0^2 \cdot 1! + 1^2 \cdot 2! + 2^2 \cdot 3! + \cdots + (m-1)^2 \cdot m!}_{=(m-2)\cdot(m+1)!+2} + \underbrace{((m+1)-1)^2}_{=m^2} \cdot (m+1)! \\
&= (m-2) \cdot (m+1)! + 2 + m^2 \cdot (m+1)! = \underbrace{\left((m-2) + m^2\right)}_{\substack{=m^2+m-2 \\ =(m-1)(m+2)}} \cdot (m+1)! + 2 \\
&= (m-1)(m+2) \cdot (m+1)! + 2.
\end{aligned} \tag{31}$$

But (30) (applied to $n = m+2$) yields

$$(m+2)! = (m+2) \cdot \left(\underbrace{(m+2)-1}_{=m+1}\right)! = (m+2) \cdot (m+1)!.$$

Hence, (31) becomes

$$\begin{aligned}
&0^2 \cdot 1! + 1^2 \cdot 2! + 2^2 \cdot 3! + \cdots + ((m+1)-1)^2 \cdot (m+1)! \\
&= \underbrace{(m-1)}_{=(m+1)-2} \underbrace{(m+2) \cdot (m+1)!}_{\substack{=(m+2)! \\ =((m+1)+1)!}} + 2 = ((m+1)-2)((m+1)+1)! + 2.
\end{aligned}$$

In other words, the claim of the exercise holds for $n = m+1$. This completes the induction step. Thus, the claim of the exercise is proven by induction.

Second solution (using telescope principle): This proof shall rely on the following fact:

Proposition 4.1. *Let $m \in \mathbb{N}$. Let $a_0, a_1, \ldots, a_m$ be $m+1$ real numbers. Then,*

$$\sum_{i=1}^{m} (a_i - a_{i-1}) = a_m - a_0.$$

Proposition 4.1 is proven (e.g.) in [19s-hw0s, proof of Proposition 2.2] and in [detnotes, proof of (16)]. It is known as the "telescope principle" since it contracts the sum $\sum_{i=1}^{m} (a_i - a_{i-1})$ to the single difference $a_m - a_0$, like folding a telescope.

Now, how can we apply Proposition 4.1 to the exercise? We have $0^2 \cdot 1! + 1^2 \cdot 2! + 2^2 \cdot 3! + \cdots + (n-1)^2 \cdot n! = \sum_{i=1}^{n} (i-1)^2 \cdot i!$. If we could write each addend $(i-1)^2 \cdot i!$ in the form $a_i - a_{i-1}$ for some $n+1$ real numbers $a_0, a_1, \ldots, a_n$, then we could use Proposition 4.1.

The tricky part is finding these a_i. Namely, set $a_i = (i-2) \cdot (i+1)!$ for each $i \in \{0, 1, \ldots, n\}$. Then, I claim that

$$(i-1)^2 \cdot i! = a_i - a_{i-1} \qquad \text{for each } i \in \{1, 2, \ldots, n\}. \tag{32}$$

The *proof of* (32) is not tricky at all: Let $i \in \{1, 2, \ldots, n\}$. Then, (30) (applied to $i+1$ instead of n) yields

$$(i+1)! = (i+1) \cdot \left(\underbrace{(i+1) - 1}_{=i} \right)! = (i+1) \cdot i!.$$

Now,

$$\begin{aligned}
& \underbrace{a_i}_{\substack{=(i-2)\cdot(i+1)! \\ \text{(by the definition of } a_i)}} - \underbrace{a_{i-1}}_{\substack{=((i-1)-2)\cdot((i-1)+1)! \\ \text{(by the definition of } a_{i-1})}} \\
&= (i-2) \cdot \underbrace{(i+1)!}_{=(i+1)\cdot i!} - \underbrace{((i-1)-2)}_{=i-3} \cdot \left(\underbrace{(i-1)+1}_{=i} \right)! = (i-2) \cdot (i+1) \cdot i! - (i-3) \cdot i! \\
&= \underbrace{((i-2) \cdot (i+1) - (i-3))}_{=(i-1)^2} \cdot i! = (i-1)^2 \cdot i!,
\end{aligned}$$

and this proves (32).

Now,

$$\begin{aligned}
& 0^2 \cdot 1! + 1^2 \cdot 2! + 2^2 \cdot 3! + \cdots + (n-1)^2 \cdot n! \\
&= \sum_{i=1}^{n} \underbrace{(i-1)^2 \cdot i!}_{\substack{=a_i - a_{i-1} \\ \text{(by (32))}}} = \sum_{i=1}^{n} (a_i - a_{i-1}) \\
&= \underbrace{a_n}_{\substack{=(n-2)\cdot(n+1)! \\ \text{(by the definition of } a_n)}} - \underbrace{a_0}_{\substack{=(0-2)\cdot(0+1)! \\ \text{(by the definition of } a_0)}} \qquad \text{(by Proposition 4.1, applied to } m = n) \\
&= (n-2) \cdot (n+1)! - (0-2) \cdot \underbrace{(0+1)!}_{=1!=1} \\
&= (n-2) \cdot (n+1)! - (0-2) \cdot 1 = (n-2) \cdot (n+1)! + 2.
\end{aligned}$$

This solves the exercise again.

4.4 Remark

The second solution above is not actually hard to find. When I said that finding the a_i is "the tricky part", I referred to the situation in which you are trying to find a formula for the sum $\sum_{i=1}^{n} (i-1)^2 \cdot i!$. But in this exercise, you are in a better situation: You know such a formula (it is stated in the exercise), and all you need is to prove it. In this situation, it is easy to reverse-engineer what the a_i should be: You want to have $a_n - a_0 = (n-2) \cdot (n+1)! + 2$, so it makes sense to set $a_i = (i-2) \cdot (i+1)! + 2 + C$ for some constant C. Choosing $C = -2$ (to cancel out the constant terms), we obtain $a_i = (i-2) \cdot (i+1)!$, which is what I used in the above solution; of course, any other value of C would have worked just as well.

Sums that can be computed using the telescope principle (Proposition 4.1) are called *telescoping sums*. See `https://brilliant.org/wiki/telescoping-series/` or `https://en.wikipedia.org/wiki/Telescoping_series` or `https://www.cut-the-knot.org/m/Algebra/TelescopingSums.shtml` for other examples of such sums. (A famous example is $\sin \alpha + \sin(2\alpha) + \sin(3\alpha) + \cdots + \sin(n\alpha)$.)

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5 Exercise 5

5.1 Problem

Let $(u_0, u_1, u_2, \ldots)$ be a sequence of real numbers such that every $n \geq 1$ satisfies

$$u_n = n u_{n-1} + (-1)^n . \tag{33}$$

Prove that $u_n = (n-1)(u_{n-1} + u_{n-2})$ for each $n \geq 2$.

5.2 Remark

This shows that the recurrence $D_n = nD_{n-1} + (-1)^n$ for the derangement numbers implies the recurrence $D_n = (n-1)(D_{n-1} + D_{n-2})$.

5.3 Solution

Let $n \geq 2$ be an integer. Thus, $n - 1 \geq 2 - 1 = 1$. Hence, (33) (applied to $n-1$ instead of n) yields

$$u_{n-1} = (n-1) \underbrace{u_{(n-1)-1}}_{=u_{n-2}} + \underbrace{(-1)^{n-1}}_{=-(-1)^n} = (n-1) u_{n-2} + (-(-1)^n) = (n-1) u_{n-2} - (-1)^n .$$

Thus,

$$u_{n-1} + (-1)^n = (n-1) u_{n-2}. \tag{34}$$

But $n \geq 2 \geq 1$. Hence, (33) yields

$$u_n = \underbrace{n}_{=(n-1)+1} u_{n-1} + (-1)^n = \underbrace{((n-1)+1) u_{n-1}}_{=(n-1)u_{n-1}+u_{n-1}} + (-1)^n = (n-1) u_{n-1} + \underbrace{u_{n-1} + (-1)^n}_{\substack{=(n-1)u_{n-2} \\ \text{(by (34))}}}$$

$$= (n-1) u_{n-1} + (n-1) u_{n-2} = (n-1)(u_{n-1} + u_{n-2}) .$$

This solves the exercise.

5.4 Remark

A more systematic (but slightly less neat) solution proceeds by applying (33) to $n-1$ and to n (as we did), and then using the resulting formulas to express both u_n and u_{n-1} in terms of u_{n-2}. Once this is done, the equality in question ($u_n = (n-1)(u_{n-1} + u_{n-2})$) can be verified by direct computation.

The claim of this exercise is due to Leonhard Euler ([Euler79, Section X]).

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6 Exercise 6

6.1 Problem

Let n and m be positive integers.

An n-tuple $(i_1, i_2, \ldots, i_n) \in \{0, 1, \ldots, m\}^n$ is said to be *even* if the sum $i_1 + i_2 + \cdots + i_n$ is even. (For example, the 4-tuple $(1, 0, 4, 1)$ is even, whereas $(1, 0, 3, 1)$ is not.)

(a) Find a formula for the number of all even n-tuples $(i_1, i_2, \ldots, i_n) \in \{0, 1, \ldots, m\}^n$ when m is odd.

(b) Find a formula for the number of all even n-tuples $(i_1, i_2, \ldots, i_n) \in \{0, 1, \ldots, m\}^n$ when m is even.

[**Hint:** Particular cases of this exercise (for $m = 1, 2, 3$) were done in [19f-hw0s, Exercise 3] and [18f-hw1s, Exercises 2 and 1]. Can you generalize some of that reasoning?]

6.2 Solution

Forget that we fixed n (but leave m fixed).

We extend our definition of even n-tuples to the case when n and m are merely nonnegative (as opposed to positive). Thus, in particular, the 0-tuple () (also known as the empty list) is even (because its sum is 0).

For every $n \in \mathbb{N}$, we let e_n denote the number of all even n-tuples $(i_1, i_2, \ldots, i_n) \in \{0, 1, \ldots, m\}^n$. (Keep in mind that m is fixed, so we don't need to mention it in our notation.) Thus, part **(a)** asks us to find a formula for e_n when m is odd, while part **(b)** asks us to do the same when m is even. We are now going to solve these two parts separately, both times using a modified version of the argument from [18f-hw1s, solution to Exercise 2]. We begin with part **(b)**:

(b) Assume that m is even. We claim that

$$e_n = \frac{(m+1)^n + 1}{2} \qquad \text{for each } n \in \mathbb{N}. \tag{35}$$

[*Proof of* (35)*:* We proceed by induction on n:

Induction base: There is only one 0-tuple $(i_1, i_2, \ldots, i_0) \in \{0, 1, \ldots, m\}^0$, namely the empty list (). This empty list is even (since the sum $i_1 + i_2 + \cdots + i_0 = (\text{empty sum}) = 0$ is even). Thus, $e_0 = 1$. Comparing this with $\frac{(m+1)^0 + 1}{2} = \frac{1+1}{2} = 1$, we conclude that $e_0 = \frac{(m+1)^0 + 1}{2}$. Hence, (35) is proven for $n = 0$. This completes the induction base.

Induction step: Let N be a positive integer. Assume that (35) holds for $n = N-1$. We must prove that (35) holds for $n = N$.

In the following, the word "k-tuple" (for k being a nonnegative integer) shall always mean "k-tuple in $\{0, 1, \ldots, m\}^k$". Thus, for each $n \in \mathbb{N}$, the number e_n is simply the number of all even n-tuples. We note that the total number of k-tuples (for a given $k \in \mathbb{N}$) is

$$\left|\{0, 1, \ldots, m\}^k\right| = |\{0, 1, \ldots, m\}|^k = (m+1)^k \qquad \left(\text{since } |\{0, 1, \ldots, m\}| = m+1\right).$$

We have assumed that (35) holds for $n = N-1$. In other words,

$$e_{N-1} = \frac{(m+1)^{N-1} + 1}{2}. \tag{36}$$

Recall that an $(N-1)$-tuple $(i_1, i_2, \ldots, i_{N-1}) \in \{0, 1, \ldots, m\}^{N-1}$ is even if and only if the sum $i_1 + i_2 + \cdots + i_{N-1}$ is even. Let us introduce the natural counterpart to this notion: An $(N-1)$-tuple $(i_1, i_2, \ldots, i_{N-1}) \in \{0, 1, \ldots, m\}^{N-1}$ is said to be *odd* if the sum $i_1 + i_2 + \cdots + i_{N-1}$ is odd.

Thus, each $(N-1)$-tuple is either even or odd, but not both at the same time. Hence,

$$\begin{aligned}
&(\text{the number of all } (N-1)\text{-tuples}) \\
&= \underbrace{(\text{the number of all even } (N-1)\text{-tuples})}_{\substack{= e_{N-1} \\ (\text{by the definition of } e_{N-1})}} + (\text{the number of all odd } (N-1)\text{-tuples}) \\
&= e_{N-1} + (\text{the number of all odd } (N-1)\text{-tuples}).
\end{aligned}$$

Thus,

$$\begin{aligned}
(\text{the number of all odd } (N-1)\text{-tuples}) &= \underbrace{(\text{the number of all } (N-1)\text{-tuples})}_{=(m+1)^{N-1}} - e_{N-1} \\
&= (m+1)^{N-1} - e_{N-1}.
\end{aligned}$$

Now, we want to count the even N-tuples $(i_1, i_2, \ldots, i_N)$. Each N-tuple $(i_1, i_2, \ldots, i_N)$ has a well-defined last entry i_N (since $N > 0$). Now, for each $j \in \{0, 1, \ldots, m\}$, we can count the even N-tuples whose last entry is j:

- Let $j \in \{0, 1, \ldots, m\}$ be even. Then, for any $(N-1)$-tuple $(i_1, i_2, \ldots, i_{N-1})$, we have the following chain of logical equivalences:

  $$\begin{aligned}
  &(\text{the } (N-1)\text{-tuple } (i_1, i_2, \ldots, i_{N-1}) \text{ is even}) \\
  \Longleftrightarrow\ &(i_1 + i_2 + \cdots + i_{N-1} \text{ is even}) \qquad (\text{by the definition of "even" for tuples}) \\
  \Longleftrightarrow\ &(i_1 + i_2 + \cdots + i_{N-1} + j \text{ is even}) \qquad (\text{since } j \text{ is even}) \\
  \Longleftrightarrow\ &(\text{the } N\text{-tuple } (i_1, i_2, \ldots, i_{N-1}, j) \text{ is even}) \\
  &\qquad (\text{by the definition of "even" for tuples}).
  \end{aligned}$$

  Hence, there is a bijection

  $$\begin{aligned}
  \{\text{even } (N-1)\text{-tuples}\} &\to \{\text{even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = j\}, \\
  (i_1, i_2, \ldots, i_{N-1}) &\mapsto (i_1, i_2, \ldots, i_{N-1}, j).
  \end{aligned}$$

[14] Hence,

$$|\{\text{even } (N-1)\text{-tuples}\}| = |\{\text{even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = j\}| .$$

In other words,

$$\begin{aligned}
&(\text{the number of all even } (N-1)\text{-tuples}) \\
&= (\text{the number of all even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = j) .
\end{aligned}$$

Hence,

$$\begin{aligned}
&(\text{the number of all even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = j) \\
&= (\text{the number of all even } (N-1)\text{-tuples}) \\
&= e_{N-1}.
\end{aligned} \tag{37}$$

Forget that we fixed j. We thus have proven (37) for each **even** $j \in \{0, 1, \ldots, m\}$.

- Let $j \in \{0, 1, \ldots, m\}$ be odd. Then, for any $(N-1)$-tuple $(i_1, i_2, \ldots, i_{N-1})$, we have the following chain of logical equivalences:

$$\begin{aligned}
&(\text{the } (N-1)\text{-tuple } (i_1, i_2, \ldots, i_{N-1}) \text{ is odd}) \\
\Longleftrightarrow\ &(i_1 + i_2 + \cdots + i_{N-1} \text{ is odd}) \qquad \left(\text{by the definition of ``odd'' for tuples}\right) \\
\Longleftrightarrow\ &(i_1 + i_2 + \cdots + i_{N-1} + j \text{ is even}) \qquad \left(\text{since } j \text{ is odd}\right) \\
\Longleftrightarrow\ &(\text{the } N\text{-tuple } (i_1, i_2, \ldots, i_{N-1}, j) \text{ is even}) \\
&\qquad \left(\text{by the definition of ``even'' for tuples}\right) .
\end{aligned}$$

Hence, there is a bijection

$$\begin{aligned}
\{\text{odd } (N-1)\text{-tuples}\} &\to \{\text{even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = j\} , \\
(i_1, i_2, \ldots, i_{N-1}) &\mapsto (i_1, i_2, \ldots, i_{N-1}, j) .
\end{aligned}$$

[15] Hence,

$$|\{\text{odd } (N-1)\text{-tuples}\}| = |\{\text{even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = j\}| .$$

In other words,

$$\begin{aligned}
&(\text{the number of all odd } (N-1)\text{-tuples}) \\
&= (\text{the number of all even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = j) .
\end{aligned}$$

Hence,

$$\begin{aligned}
&(\text{the number of all even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = j) \\
&= (\text{the number of all odd } (N-1)\text{-tuples}) \\
&= (m+1)^{N-1} - e_{N-1}.
\end{aligned} \tag{38}$$

Forget that we fixed j. We thus have proven (38) for each **odd** $j \in \{0, 1, \ldots, m\}$.

[14] We leave it to the reader to verify that this map is well-defined and is actually a bijection.
[15] We leave it to the reader to verify that this map is well-defined and is actually a bijection.

But e_N is the number of all even N-tuples. Thus,

$$\begin{aligned}
e_N &= \left(\text{the number of all even } N\text{-tuples } (i_1, i_2, \ldots, i_N)\right) \\
&= \sum_{j \in \{0,1,\ldots,m\}} \left(\text{the number of all even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = j\right) \\
&\qquad \left(\begin{array}{c}\text{because for each even } N\text{-tuple } (i_1, i_2, \ldots, i_N)\text{, there} \\ \text{exists a unique } j \in \{0,1,\ldots,m\} \text{ satisfying } i_N = j\end{array}\right) \\
&= \sum_{\substack{j \in \{0,1,\ldots,m\};\\ j \text{ is odd}}} \underbrace{\left(\text{the number of all even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = j\right)}_{\substack{=(m+1)^{N-1}-e_{N-1}\\ \text{(by (38))}}} \\
&\qquad + \sum_{\substack{j \in \{0,1,\ldots,m\};\\ j \text{ is even}}} \underbrace{\left(\text{the number of all even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = j\right)}_{\substack{=e_{N-1}\\ \text{(by (37))}}} \\
&\qquad \left(\text{since each } j \in \{0,1,\ldots,m\} \text{ is either odd or even (but not both)}\right) \\
&= \underbrace{\sum_{\substack{j \in \{0,1,\ldots,m\};\\ j \text{ is odd}}} \left((m+1)^{N-1} - e_{N-1}\right)}_{=(\text{the number of all odd } j \in \{0,1,\ldots,m\}) \cdot \left((m+1)^{N-1} - e_{N-1}\right)} \\
&\qquad + \underbrace{\sum_{\substack{j \in \{0,1,\ldots,m\};\\ j \text{ is even}}} e_{N-1}}_{=(\text{the number of all even } j \in \{0,1,\ldots,m\}) \cdot e_{N-1}} \\
&= \left(\text{the number of all odd } j \in \{0,1,\ldots,m\}\right) \cdot \left((m+1)^{N-1} - e_{N-1}\right) \\
&\qquad + \left(\text{the number of all even } j \in \{0,1,\ldots,m\}\right) \cdot e_{N-1}. \qquad (39)
\end{aligned}$$

Now, it is time to recall that m is even. Hence, there are exactly $m/2+1$ many even elements $j \in \{0,1,\ldots,m\}$ (namely, $0,2,4,\ldots,m$). In other words,

$$\left(\text{the number of all even } j \in \{0,1,\ldots,m\}\right) = m/2+1.$$

Likewise,

$$\left(\text{the number of all odd } j \in \{0,1,\ldots,m\}\right) = m/2$$

(since there are exactly $m/2$ many odd elements $j \in \{0,1,\ldots,m\}$, namely $1,3,5,\ldots,m-1$).

Hence, (39) becomes

$$\begin{aligned}
e_N &= \underbrace{(\text{the number of all odd } j \in \{0,1,\ldots,m\})}_{=m/2} \cdot \left((m+1)^{N-1} - e_{N-1}\right) \\
&\qquad + \underbrace{(\text{the number of all even } j \in \{0,1,\ldots,m\})}_{=m/2+1} \cdot e_{N-1} \\
&= (m/2) \cdot \left((m+1)^{N-1} - e_{N-1}\right) + \underbrace{(m/2+1) \cdot e_{N-1}}_{=(m/2)\cdot e_{N-1}+e_{N-1}} \\
&= \underbrace{(m/2) \cdot \left((m+1)^{N-1} - e_{N-1}\right) + (m/2) \cdot e_{N-1}}_{=(m/2)\cdot(m+1)^{N-1}} + \underbrace{e_{N-1}}_{\substack{=\frac{(m+1)^{N-1}+1}{2} \\ \text{(by (36))}}} \\
&= (m/2) \cdot (m+1)^{N-1} + \frac{(m+1)^{N-1}+1}{2} = \underbrace{\left(m/2 + \frac{1}{2}\right)}_{=\frac{1}{2}(m+1)} (m+1)^{N-1} + \frac{1}{2} \\
&= \frac{1}{2} \underbrace{(m+1)(m+1)^{N-1}}_{=(m+1)^N} + \frac{1}{2} = \frac{1}{2}(m+1)^N + \frac{1}{2} = \frac{(m+1)^N + 1}{2}.
\end{aligned}$$

In other words, (35) holds for $n = N$. This completes the induction step. Hence, (35) is proven.]

(a) Assume that m is odd. We claim that

$$e_n = \frac{(m+1)^n}{2} \qquad \text{for each positive integer } n. \tag{40}$$

(This does not hold for $n = 0$, but of course we have $e_0 = 1$ for obvious reasons.)

[*Proof of* (40)*:* Fix a positive integer N. Then, the equality (39) holds, and can be proven exactly as in our above proof of (35) (since we did not use the evenness of m in its proof).

But m is odd. Hence, there are exactly $(m+1)/2$ even elements $j \in \{0,1,\ldots,m\}$ (namely, $0,2,4,\ldots,m-1$). In other words,

$$(\text{the number of all even } j \in \{0,1,\ldots,m\}) = (m+1)/2.$$

Likewise,

$$(\text{the number of all odd } j \in \{0,1,\ldots,m\}) = (m+1)/2.$$

Hence, (39) becomes

$$\begin{aligned}
e_N &= \underbrace{(\text{the number of all odd } j \in \{0,1,\ldots,m\})}_{=(m+1)/2} \cdot \left((m+1)^{N-1} - e_{N-1}\right) \\
&\qquad + \underbrace{(\text{the number of all even } j \in \{0,1,\ldots,m\})}_{=(m+1)/2} \cdot e_{N-1} \\
&= ((m+1)/2) \cdot \left((m+1)^{N-1} - e_{N-1}\right) + ((m+1)/2) \cdot e_{N-1} \\
&= ((m+1)/2) \cdot (m+1)^{N-1} = \frac{1}{2} \underbrace{(m+1)(m+1)^{N-1}}_{=(m+1)^N} = \frac{1}{2}(m+1)^N = \frac{(m+1)^N}{2}.
\end{aligned}$$

Forget that we fixed N. Thus we have shown that $e_N = \dfrac{(m+1)^N}{2}$ for each positive integer N. Renaming N as n in this statement, we conclude that $e_n = \dfrac{(m+1)^n}{2}$ for each positive integer n. Hence, (40) is proven.]

Note that we did not have to use induction on n in the proof of (40), since the e_{N-1} on the right hand side of (39) disappeared after cancellations (and thus we did not have to know its value).

6.3 Remark

Part **(a)** can be solved in other ways, too (namely, similarly to the solutions of [19f-hw0s, Exercise 3] and [18f-hw1s, Exercise 1]); but these ways don't extend to part **(b)**.

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