University of Minnesota, School of Mathematics

Math 4707: Combinatorics, Spring 2018 Midterm 3

James Hirsch (edited by Darij Grinberg)

January 10, 2019

1 Exercise 1

1.1 Problem

Let $D = (V, A, \varphi)$ be an acyclic multidigraph. Prove that there is a list $(v_1, v_2, \ldots, v_n)$ of elements of V such that

- each element of V appears exactly once in this list $(v_1, v_2, \ldots, v_n)$;
- whenever i and j are two elements of $[n]$ such that some arc of D has source v_i and target v_j, we must have $i < j$.

1.2 Solution

Definition 1.1. Let $D = (V, A, \varphi)$ be a multidigraph. For a vertex $v \in V$, we define its *outdegree* as

$$\operatorname{outdeg}(v) = |\{a \in A \mid \operatorname{source}(a) = v\}|$$

(where $\operatorname{source}(v)$ denotes the source of v).

Lemma 1.2. *Let $D = (V, A, \varphi)$ be an acyclic multidigraph. Then, any walk in D is a path.*

Proof. Let $\mathbf{w}$ be any walk in D. We must show that $\mathbf{w}$ is a path.

Let $u_0, u_1, \ldots, u_k$ be the vertices of the walk $\mathbf{w}$, from first to last. We claim that these vertices are distinct. Indeed, assume the contrary. Thus, there exist some elements i and j of $\{0, 1, \ldots, k\}$ satisfying $i < j$ and $u_i = u_j$. In other words, there exists some $j \in \{0, 1, \ldots, k\}$ satisfying $u_j \in \{u_0, u_1, \ldots, u_{j-1}\}$. Consider the **smallest** such j. Then, there is an $i < j$ satisfying $u_j = u_i$ (since $u_j \in \{u_0, u_1, \ldots, u_{j-1}\}$). Consider this i. The vertices $u_i, u_{i+1}, \ldots, u_{j-1}$ are distinct (because of the minimality of j), but the vertex u_j equals u_i. Hence, the part of the walk $\mathbf{w}$ between u_i and u_j is a cycle of D. Hence, D

contains a cycle. But this contradicts the acyclicity of D. This contradiction shows that our assumption was wrong. Thus, the vertices $u_0, u_1, \ldots, u_k$ of the walk $\mathbf{w}$ are distinct. In other words, this walk $\mathbf{w}$ is a path. □

Lemma 1.3. *Let $D = (V, A, \varphi)$ be an acyclic multidigraph with $V \neq \varnothing$. Then, there exists a vertex $v \in V$ having* $\operatorname{outdeg}(v) = 0$.

Proof. Assume the contrary. Then, for each $v \in V$ we have $\operatorname{outdeg}(v) \neq 0$. Thus, a walk in D can be constructed by starting at an arbitrary vertex u and taking an arc leaving it (such an arc exists since $\operatorname{outdeg}(u) \neq 0$). The target of this arc also has outdegree $\neq 0$, so it has an arc leaving it, which we take. We continue taking arcs in this way until we have taken $|V|$ many arcs. The resulting walk has at least $|V|$ many arcs, and thus it has $|V| + 1$ many vertices. But this walk is a path (by Lemma 1.2), and thus its vertices are distinct. Hence, this walk has at most $|V|$ many vertices. This contradicts the fact that it has $|V| + 1$ many vertices. So, there is a contradiction and our assumption was false. □

Proposition 1.4. *Let $D = (V, A, \varphi)$ be an acyclic multidigraph. Then, there exists a list $(v_1, v_2, \ldots, v_n)$ of elements of V such that*

- *each element of V appears exactly once in this list $(v_1, v_2, \ldots, v_n)$;*
- *whenever i and j are two elements of $[n]$ such that some arc of D has source v_i and target v_j, we must have $i < j$.*

Proof by induction on $|V|$. Base case: $|V| = 0$. In this case, D is a multidigraph with no vertices, and thus the empty list () contains each element of V, and it is vacuously true that for any i and j in $[0] = \varnothing$ such that an arc of D has source v_i and target v_j, we have $i < j$. So Proposition 1.4 is proven in the case where $|V| = 0$.

Inductive Step: Fix $n \in \mathbb{N}$. Assume as the inductive hypothesis that Proposition 1.4 holds for any acyclic multidigraph having $|V| = n$. We now want to show that Proposition 1.4 holds for any acyclic multidigraph $D = (V, A, \varphi)$ having $|V| = n + 1$.

So let $D = (V, A, \varphi)$ be an acyclic multidigraph having $|V| = n + 1$. Let $u \in V$ be a vertex of D having $\operatorname{outdeg}(u) = 0$ (such a vertex exists by Lemma 1.3). Let D' be the multidigraph D with vertex u and all arcs using u removed. Removing arcs cannot create a new cycle, so D' is an acyclic multidigraph having $|V'| = n$, where $V' = V \setminus \{u\}$ is its set of vertices. By the inductive hypothesis, Proposition 1.4 holds for this multidigraph D'. Thus, there is a list $(v_1, v_2, \ldots, v_n)$ containing each element of $V' = V \setminus \{u\}$ exactly once and having the property that for any i and j in $[n]$ such that some arc of D' has source v_i and target v_j, we have $i < j$. Consider such a list. Extend it to a list $(v_1, v_2, \ldots, v_{n+1})$ of vertices of D by setting $v_{n+1} = u$. Then, the list $(v_1, v_2, \ldots, v_{n+1}) = (v_1, v_2, \ldots, v_n, u)$ contains each element of V exactly once. Furthermore, whenever i and j are two elements of $[n + 1]$ such that some arc of D has source v_i and target v_j, we must have $i < j$. (Indeed, in the case where i and j are both in $[n]$, this follows from the analogous property of the list $(v_1, v_2, \ldots, v_n)$. In the case where i is in $[n]$ and $j = n + 1$, the inequality $i < j$ is obvious. The only remaining case – the case where $i = n + 1$ – does not occur, because the vertex $v_{n+1} = u$ has outdegree 0 and thus cannot be the source of any arc of D.) Thus, the list $(v_1, v_2, \ldots, v_{n+1})$ satifies the conditions of Proposition 1.4 for our multidigraph D. This completes the inductive step. □

2 Exercise 2

2.1 Problem

Let D be an acyclic multidigraph. A vertex v of D is said to be a *sink* if there is no arc of D with source v.

If u and v are any two vertices of D, then:

- we write $u \longrightarrow v$ if and only if D has an **arc** with source u and target v;
- we write $u \overset{*}{\longrightarrow} v$ if and only if D has a **path** from u to v.

The so-called *no-watershed condition* says that for any three vertices u, v and w of D satisfying $u \longrightarrow v$ and $u \longrightarrow w$, there exists a vertex t of D such that $v \overset{*}{\longrightarrow} t$ and $w \overset{*}{\longrightarrow} t$.

Assume that the no-watershed condition holds. Prove that for each vertex p of D, there exists **exactly one** sink q of D such that $p \overset{*}{\longrightarrow} q$.

2.2 Solution

Definition 2.1. Let $D = (V, A, \varphi)$ be an acyclic multidigraph. For a vertex $p \in V$, we define its *height* $H(p)$ as the maximum number of edges in a path in D that starts at p.

Remark 2.2. The height of a vertex is an integer between 0 and $|V| - 1$, since every vertex has a path to itself (which contains 0 edges) and a path can have at most $|V| - 1$ edges (since it can contain at most $|V|$ vertices).

Remark 2.3. Let u and v be two vertices of an acyclic multidigraph D such that there is a path from u to v of nonzero length in D. Then, $H(u) > H(v)$. Indeed, a longest path in D starting at v can be concatenated with any path from u to v of nonzero length to form a longer path in D starting at u (and this concatenation is indeed a path, because of Lemma 1.2).

Proposition 2.4. *Let D be an acyclic multidigraph for which the no-watershed condition holds. Then, for each vertex p of D, there is exactly one sink q of D such that $p \overset{*}{\longrightarrow} q$.*

Proof by strong induction on $H(p)$: Base case: $H(p) = 0$.

If the height of p is 0, then the only path in D that starts at p is the "empty path", and so there is no arc of D with source p. So, in this case, p is a sink of D, and p is the unique sink of D such that there is a path from p to it. This proves Proposition 2.4 when $H(p) = 0$. This completes the induction base.

Inductive Step: Fix an integer $n \geq 1$. Assume as the inductive hypothesis that for any vertex w of D with $H(w) < n$, there is exactly one sink of D such that there is a path from w to that sink. We want to show that for a vertex p with $H(p) = n$, there is exactly one sink q of D such that $p \overset{*}{\longrightarrow} q$.

In the following, an *out-neighbor* of a vertex $x \in D$ means any vertex $y \in D$ such that D has an arc with source x and target y.

Consider any vertex p with $H(p) = n$. We have $H(p) = n \geq 1$, so there is a path in D starting at p that contains at least one edge; thus, p has an out-neighbor. Let u be any out-neighbor of p. Then, $H(p) > H(u)$ by Remark 2.3 (since there is a path from p to u of nonzero length in D), and therefore $H(u) < H(p) = n$. Hence, the inductive hypothesis shows that there is a unique sink of D such that there is a path from u to this sink. Denote this sink by q_u. Forget that we fixed u. Thus, for each out-neighbor u of p, we have defined a sink q_u of D with the property that q_u is the unique sink of D such that there is a path from u to this sink.

Let u and v be two out-neighbors of p (it is possible that $u = v$). Because there is a path from p to u of nonzero length in D, we have $H(p) > H(u)$ by Remark 2.3; similarly, $H(p) > H(v)$. Recall that q_u is the unique sink of D such that there is a path from u to q_u. Since the no-watershed condition holds (and since $p \longrightarrow u$ and $p \longrightarrow v$), there is a vertex t such that $u \overset{*}{\longrightarrow} t$ and $v \overset{*}{\longrightarrow} t$. Again by Remark 2.3, this vertex t has a smaller height than v, and so its height is smaller than $H(p) = n$. Thus, by the inductive hypothesis, t has a path in D to exactly one sink of D. Call this sink q. Because $u \overset{*}{\longrightarrow} t$ and $t \overset{*}{\longrightarrow} q$, we have $u \overset{*}{\longrightarrow} q$. This means that q is a sink of D that u has a path to in D. Since we already know that the only such sink is q_u (because this is how we defined q_u), we thus obtain $q_u = q$. Similarly, $q_v = q$. Thus, $q_u = q_v$.

Now, forget that we fixed u and v. We have shown that any two out-neighbors u and v of p satisfy $q_u = q_v$. Thus, the sink q_u corresponding to an out-neighbor u of p does not depend on u. Hence, there is a sink q of D such that each out-neighbor u of p satisfies $q_u = q$ (since we know that p has an out-neighbor). Consider this q. For each out-neighbor u of p, there is a path from u to q (since $q_u = q$). Thus, there is a walk from p to q (since there is a path from p to any of its out-neighbors and a path from any of its out-neighbors to q), therefore also a path from p to q (by Lemma 1.2). In other words, $p \overset{*}{\longrightarrow} q$.

The vertex p is not a sink (since $H(p) \geq 1$). Hence, any path from p to a sink of D must leave p, and thus must travel through some out-neighbor u of p. Hence, the sink that this path leads to must be q_u. In other words, this sink must be q (since $q_u = q$). So we have proven that if there is a path from p to a sink of D, then this sink must be q. In other words, q is the unique sink of D such that there is a path from p to this sink. Thus, there is exactly one sink r of D such that $p \overset{*}{\longrightarrow} r$ (namely, q). This completes the inductive step.

Proposition 2.4 is thus proven for any vertex p having $H(p) \geq 0$, and so it is proven for any $p \in V$. □

3 Exercise 5

Definition 3.1. Let G be a graph.

(a) An *independent set* of G means a set S of vertices of G such that no two distinct elements of S are adjacent.

(b) We let $\operatorname{ind} G$ be the number of all independent sets of G.

Definition 3.2. Let G be a graph. Let S be a set of vertices of G. Then, $G \setminus S$ will denote the graph obtained from G by removing all vertices in S (along with all edges that use these vertices).

Definition 3.3. For each $n \in \mathbb{N}$, we define the *n-th path graph* to be the simple graph

$$\begin{aligned}&(\{1, 2, \ldots, n\}, \{\{i, i+1\} \mid i \in \{1, 2, \ldots, n-1\}\})\\&= (\{1, 2, \ldots, n\}, \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\}).\end{aligned}$$

This graph is denoted by P_n.

Definition 3.4. For each integer $n > 1$, we define the *n-th cycle graph* to be the simple graph

$$\begin{aligned}&(\{1, 2, \ldots, n\}, \{\{i, i+1\} \mid i \in \{1, 2, \ldots, n-1\}\} \cup \{n, 1\})\\&= (\{1, 2, \ldots, n\}, \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}\}).\end{aligned}$$

This graph is denoted by C_n.

3.1 Problem

(a) Let v be a vertex of a graph G. Let $N(v)$ be the set of all neighbors of v. Let $N^+(v) = \{v\} \cup N(v)$. Prove that

$$\operatorname{ind} G = \operatorname{ind}(G \setminus \{v\}) + \operatorname{ind}\left(G \setminus \left(N^+(v)\right)\right).$$

(b) Compute $\operatorname{ind}(C_n)$ for each $n \geq 2$ (in terms of the Fibonacci sequence).

3.2 Solution to part a)

Proposition 3.5. *Let $G = (V, E, \varphi)$ be a graph, and let $v \in V$ be one of its vertices. Then,*

$$\operatorname{ind} G = \operatorname{ind}(G \setminus \{v\}) + \operatorname{ind}\left(G \setminus \left(N^+(v)\right)\right).$$

Proof. There are two types of independent sets of G: those that contain v and those that don't. So,

$$\begin{aligned}\operatorname{ind} G = &\left|\{\text{independent sets } S \text{ of } G \text{ with } v \notin S\}\right| \\ &+ \left|\{\text{independent sets } S \text{ of } G \text{ with } v \in S\}\right|. \qquad (1)\end{aligned}$$

There is a map

$$\begin{aligned}\{\text{independent sets } S \text{ of } G \text{ with } v \notin S\} &\to \{\text{independent sets of } G \setminus \{v\}\}, \\ S &\mapsto S\end{aligned}$$

(this is simply the identity map). (Indeed, this map is well-defined, because if S is an independent set of G with $v \notin S$, then S is a subset of V containing no vertices which are neighbors in G and satisfying $v \notin S$, so S is also a subset of $V \setminus \{v\}$ containing no vertices which are neighbors in $G \setminus \{v\}$, and therefore S is an independent set of $G \setminus \{v\}$.)
This map has an inverse map, which is

$$\begin{aligned}\{\text{independent sets of } G \setminus \{v\}\} &\to \{\text{independent sets } S \text{ of } G \text{ with } v \notin S\}, \\ S &\mapsto S.\end{aligned}$$

(This map is well-defined for similar reasons.) Thus, the map above is a bijection. Hence,

$$\begin{aligned}\left|\{\text{independent sets } S \text{ of } G \text{ with } v \notin S\}\right| &= \left|\{\text{independent sets of } G \setminus \{v\}\}\right| \\ &= \operatorname{ind}(G \setminus \{v\}). \qquad (2)\end{aligned}$$

In addition, there is a map

$$\begin{aligned}\{\text{independent sets } S \text{ of } G \text{ with } v \in S\} &\to \left\{\text{independent sets of } G \setminus \left(N^+(v)\right)\right\}, \\ S &\mapsto S \setminus \{v\}.\end{aligned}$$

(Indeed, this map is well-defined for the following reason: If S is an independent set of G with $v \in S$, then none of the neighbors of v belongs to S. Thus, $S \setminus \{v\}$ is a subset of $V \setminus (N^+(v))$. Removing an element from an independent set leaves it independent, so $S \setminus \{v\}$ is an independent set and this map is well-defined.)
This map, too, has an inverse map, which is the map

$$\begin{aligned}\left\{\text{independent sets of } G \setminus \left(N^+(v)\right)\right\} &\to \{\text{independent sets } S \text{ of } G \text{ with } v \in S\}, \\ T &\mapsto T \cup \{v\}.\end{aligned}$$

(Here is why this map is well-defined: If T is an independent set of $G \setminus (N^+(v))$, then T contains none of the neighbors of v. Thus, adding v to T keeps the set independent. Thus, $T \cup \{v\}$ is an independent set of G, and of course it satisfies $v \in T \cup \{v\}$.) Thus, the map above is a bijection. Hence,

$$\begin{aligned} |\{\text{independent sets } S \text{ of } G \text{ with } v \in S\}| &= \left|\left\{\text{independent sets of } G \setminus \left(N^+(v)\right)\right\}\right| \\ &= \operatorname{ind}\left(G \setminus \left(N^+(v)\right)\right). \end{aligned} \tag{3}$$

Now, (1) becomes

$$\begin{aligned} \operatorname{ind} G &= |\{\text{independent sets } S \text{ of } G \text{ with } v \notin S\}| + |\{\text{independent sets } S \text{ of } G \text{ with } v \in S\}| \\ &= \operatorname{ind}(G \setminus \{v\}) + \operatorname{ind}\left(G \setminus \left(N^+(v)\right)\right) \end{aligned}$$

(by (2) and (3)). This is Proposition 3.5. □

3.3 Solution to part b)

Recall that the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$ is defined recursively by

$$f_0 = 0, \qquad f_1 = 1, \qquad \text{and } f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2.$$

Lemma 3.6. *Let $n \in \mathbb{N}$. Then,*

$$\operatorname{ind}(P_n) = f_{n+2}.$$

Proof. In the n-th path graph, two vertices are neighbors if and only if they are consecutive integers. For this reason, an independent set of P_n is the same as a subset of $V = [n]$ that contains no two consecutive integers. This is what we called a lacunar subset of $[n]$. Hence,

$$\operatorname{ind}(P_n) = (\text{the number of lacunar subsets of } [n]).$$

But by Proposition 1.22 from the February 5 lecture, the number of lacunar subsets of $[n]$ is f_{n+2}. Combining the above, we obtain

$$\operatorname{ind}(P_n) = (\text{the number of lacunar subsets of } [n]) = f_{n+2}.$$

□

Proposition 3.7. *Let $n \geq 2$. Then,*

$$\operatorname{ind}(C_n) = f_{n+1} + f_{n-1}.$$

Proof. We WLOG assume that $n \geq 3$, since the case $n = 2$ can be dealt with easily by hand. Proposition 3.5 applied to $v = n$ gives

$$\operatorname{ind}(C_n) = \operatorname{ind}(C_n \setminus \{n\}) + \operatorname{ind}\left(C_n \setminus \left(N^+(n)\right)\right).$$

The graph $C_n \setminus \{n\}$ is the graph C_n with vertex n and edges $\{n-1, n\}$ and $\{n, 1\}$ removed. This is the graph P_{n-1}. By Lemma 3.6 (applied to $n-1$ instead of n), we have

$$\operatorname{ind}(P_{n-1}) = f_{n+1}.$$

In C_n, the neighbors of n are the vertices 1 and $n-1$, and so the set $N^+(n)$ is $\{1, n-1, n\}$. Thus, the graph $C_n \setminus (N^+(n))$ is C_n with the vertices 1, $n-1$, and n removed, as well as their connected edges $\{n-2, n-1\}$, $\{n-1, n\}$, $\{n, 1\}$ and $\{1, 2\}$ removed. After relabeling the remaining vertices $2, 3, \ldots, n-2$ as $1, 2, \ldots, n-3$, this graph becomes P_{n-3}. Thus, $\operatorname{ind}(C_n \setminus (N^+(n))) = \operatorname{ind}(P_{n-3})$. But by Lemma 3.6 (applied to $n-3$ instead of n), we have

$$\operatorname{ind}(P_{n-3}) = f_{n-1}.$$

So,

$$\begin{aligned}
\operatorname{ind}(C_n) &= \operatorname{ind}\left(\underbrace{C_n \setminus \{n\}}_{=P_{n-1}}\right) + \underbrace{\operatorname{ind}\left(C_n \setminus \left(N^+(n)\right)\right)}_{=\operatorname{ind}(P_{n-3})} \\
&= \underbrace{\operatorname{ind}(P_{n-1})}_{=f_{n+1}} + \underbrace{\operatorname{ind}(P_{n-3})}_{=f_{n-1}} = f_{n+1} + f_{n-1}.
\end{aligned}$$

□