University of Minnesota, School of Mathematics

Math 4707: Combinatorics, Spring 2018
Midterm 3

Nathaniel Gorski (edited by Darij Grinberg)

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Exercise 1

Exercise 0.1. Let $D = (V, A, \psi)$ be an acyclic digraph. Then there is a list of elements $(v_1, v_2, \ldots, v_n)$ of V such that each element of V appears exactly once in the list $(v_1, v_2, \ldots, v_n)$, and whenever i and j are two elements of $[n]$, and D features an arc which starts in v_i and ends in v_j, then this implies that $i < j$.

Proof. Let $\operatorname{Anc} : V \to \{\text{subsets of } V\}$ be the function that maps each $v \in V$ to the set

$$\{w \in V \mid \text{ and there exists a walk from } w \text{ to } v\}.$$

(As we know, the existence of a walk from w to v is equivalent to the existence of a path from w to v; but we won't actually need this.)

Since V is a finite set, there exists some $n \in \mathbb{N}$ such that $|V| = n$. Consider this n. Since $|V| = n = |[n]|$, there exists a bijection $\phi : [n] \to V$. Fix this bijection ϕ.

We now define the list $(v_1, v_2, \ldots, v_n)$ to be the list of all the n elements $v \in V$ in increasing order of $|\operatorname{Anc}(v)|$, where ties are broken as follows: If $v, w \in V$ satisfy $|\operatorname{Anc}(v)| = |\operatorname{Anc}(w)|$, then v should be placed after w if $\phi(v) > \phi(w)$ (and conversely, w should be placed after v if $\phi(w) > \phi(v)$).

We will now show that this list satisfies the two requirements in the claim. First of all, it is clear that each element of V appears exactly once in this list, since this list has been constructed as a list of all elements of V in some order.

It remains to check the second requirement. In other words, it remains to show that, if i and j are two elements of $[n]$, and if D has an arc which starts at v_i and ends at v_j, then $i < j$.

Indeed, let i and j be two elements of $[n]$, and assume that D has an arc which starts at v_i and ends at v_j. We must prove $i < j$.

We will first show that $\operatorname{Anc}(v_i) \subseteq \operatorname{Anc}(v_j)$. Indeed, suppose that $w \in \operatorname{Anc}(v_i)$. Thus, there exists a walk **a** from w to v_i (by the definition of $\operatorname{Anc}(v_i)$). And because there is an arc which begins at v_i and ends at v_j, then one can add that arc to the end of the walk **a** to construct a walk from w to v_j. Hence, there exists a walk from w to v_j, so $w \in \operatorname{Anc}(v_j)$. So we have shown that $w \in \operatorname{Anc}(v_i)$ implies that $w \in \operatorname{Anc}(v_j)$. In other words, $\operatorname{Anc}(v_i) \subseteq \operatorname{Anc}(v_j)$.

We will next show that $v_j \notin \operatorname{Anc}(v_i)$. Suppose to the contrary that $v_j \in \operatorname{Anc}(v_i)$. Then there exists some walk **b** from v_j to v_i (by the definition of $\operatorname{Anc}(v_i)$). Because D is acyclic, **b** must not contain any cycles, which means that v_j does not appear in **b** except for at the very start. This means that the arc from v_i to v_j is not used in **b**, as otherwise, v_j would appear in the walk after that arc was used, which would by definition not be at the very start. Therefore, the walk constructed by adding that arc from v_i to v_j on to the end of **b** is a cycle in D (going from v_j to v_j). Thus, D has a cycle. This contradicts the assumption that D is acyclic. This contradiction reveals that $v_j \notin \operatorname{Anc}(v_i)$.

But there exists a walk from v_j to v_j (namely, the trivial walk (v_j)). Thus, $v_j \in \operatorname{Anc}(v_j)$ (by the definition of $\operatorname{Anc}(v_j)$). Contrasting this to $v_j \notin \operatorname{Anc}(v_i)$, we obtain $\operatorname{Anc}(v_i) \neq \operatorname{Anc}(v_j)$. Thus, $\operatorname{Anc}(v_i)$ is a **proper** subset of $\operatorname{Anc}(v_j)$. Hence, $|\operatorname{Anc}(v_i)| < |\operatorname{Anc}(v_j)|$. Therefore, the vertex v_i appears earlier than v_j in the list $(v_1, v_2, \ldots, v_n)$ (due to how the list was constructed). In other words, $i < j$. This concludes our proof that the second requirement holds.

Hence, the constructed list satisfies the requirements of the claim. □

Exercise 2

Exercise 0.2. Let D be an acyclic multidigraph. A vertex v of D is said to be a *sink* if there is no arc of D with source v.

If u and v are any two vertices of D, then:

- we write $u \longrightarrow v$ if and only if D has an **arc** with source u and target v;
- we write $u \overset{*}{\longrightarrow} v$ if and only if D has a **path** from u to v.

The so-called *no-watershed condition* says that for any three vertices u, v and w of D satisfying $u \longrightarrow v$ and $u \longrightarrow w$, there exists a vertex t of D such that $v \overset{*}{\longrightarrow} t$ and $w \overset{*}{\longrightarrow} t$.

If the no-watershed condition holds, then for each vertex p of D, there exists exactly one sink q of D such that $p \overset{*}{\longrightarrow} q$.

Proof. Let D be an acyclic multidigraph for which the no-watershed condition holds. Let V be the vertex set of D, and let $h : V \to \mathbb{N}$ be the function that maps each $v \in V$ to the maximum length of a path in D which begins at v.

We will first show that h is well defined. Observe that D has finitely many vertices. Also, each path in D goes through each vertex of D at most one time. Since the length of a path is equal to the number of edges taken in that path, which is equal to the number of vertices taken in that path minus one, the length of a path in D must be an integer $\leq |V| - 1$. So the set of lengths of paths which begin at a vertex $v \in V$ is some subset of $\{0, 1, \ldots, |V| - 1\}$. And since this subset is a finite nonempty set of integers (nonempty because the trivial path (v) always exists), it must have a maximum value. Hence, for all vertices $v \in V$, the number $h(v)$ is defined.

We will next show that if $u, v \in V$, and if there exists a path of nonzero length from u to v, then

$$h(v) < h(u). \tag{1}$$

[*Proof of* (1). Let $u, v \in V$, and suppose that there exists a path of nonzero length from u to v. Consider such a path $\mathbf{a}$; thus, its length is positive. By the definition of $h(v)$, we know that there is a path $\mathbf{b}$ of length $h(v)$ which begins at v. Now consider the walk $\mathbf{c}$ formed by adding the path $\mathbf{b}$ onto the end of the path $\mathbf{a}$. This new walk $\mathbf{c}$ is still a path (since otherwise, it would contain a cycle, but this would contradict the acyclicity of D), and has length $> h(v)$ (indeed, its length equals the sum of the lengths of $\mathbf{a}$ and $\mathbf{b}$, but the former length is positive and the latter is $h(v)$). Thus, $\mathbf{c}$ is a path in D which begins at u and has length $> h(v)$. Since $h(u)$ is the maximum length of a path in D which starts at u, we thus conclude that $h(u)$ is at least as large as the length of $\mathbf{c}$, which is $> h(v)$. Hence, $h(u) > h(v)$. This proves (1).]

From (1), we immediately obtain the following: If $u, v \in V$, and if there exists a path from u to v, then

$$h(v) \leq h(u). \tag{2}$$

(Indeed, if this path has nonzero length, then this inequality follows from (1), whereas otherwise it follows from $v = u$.)

The exercise claims that for each vertex v of D, there exists exactly one sink q of D such that there is a path from v to q. We will now prove this claim by strong induction on $h(v)$.

For the base case, suppose that $v \in V$ and $h(v) = 0$. Since $h(v) = 0$, there are no paths in D of nonzero length which start at v. This is possible only if there are no arcs in D which begin at v, which implies that v is a sink. Thus, v is a sink; hence, there exists only one vertex $q \in V$ for which there exists a path from v to q (namely, v itself). And since v is a sink, this means that there exists a path from v to exactly one sink (itself), and no other sinks (or even vertices for that matter). This completes the induction base.

Now, to the induction step. Let $n \in \mathbb{N}$. Assume that the claim holds for all vertices $u \in V$ such that $h(u) < n$. Consider a vertex $v \in V$ such that $h(v) = n$. We need to prove the claim for this vertex v. If $h(v) = 0$, then this has already been proven in the above induction base; thus, we assume that $h(v) > 0$. Hence, there exists a path of nonzero length which originates at v. Thus, there exists an arc with source v.

Let B be the set of the targets of all arcs with source v. Since such arcs do exist (as we have just seen), we have $B \neq \emptyset$. And also, each path which begins at v must have its second vertex be a vertex in B. And further, if $w \in B$, then there exists a path from v to w, so that $h(w) < h(v)$ (by (1)).

Now let $w_1 \in B$. Since $B \neq \emptyset$, such a w_1 must exist. And since $h(w_1) < h(v) = n$, we can apply the induction hypothesis to w_1 instead of v. We conclude that there exists exactly one sink $x \in V$ such that there is a path from w_1 to x. Consider this x. Since $w_1 \in B$, there exists a path from v to w_1, so there exists a path from v to x (via w_1).

Now let $w_2 \in B$ be arbitrary (in particular, w_2 may be equal to w_1). Since $w_1, w_2 \in B$, we have $v \longrightarrow w_1$ and $v \longrightarrow w_2$. Since the no-watershed condition holds, we conclude that there exists a vertex $t \in V$ such that there is a path from w_1 to t and there is a path from w_2 to t. Therefore, using (2), we obtain $h(t) \leq h(w_1) < h(v) = n$. So by the induction hypothesis (applied to t instead of v), there exists exactly one sink $y \in V$ such that there is a path from t to y. Consider this t. Concatenating a path from w_2 to t with a path from t to y, we obtain a walk from w_2 to y, thus a path from w_2 to y. Similarly, we find that there is a path from w_1 to y.

So y is a sink for which there exists a path from w_1 to y. But we have previously defined x to be the only such sink. Therefore, $y = x$. But recall that there is a path from w_2 to y.

In other words, there is a path from w_2 to x (since $y = x$).

We thus have shown that for each $w_2 \in B$, there is a path from w_2 to x.

Now, consider any sink z for which there is a path from v to z. This path has nonzero length (since $h(v) > 0$, so that v itself is not a sink), and thus has a second vertex. Denote this second vertex by w_2; thus, $w_2 \in B$, so that (as we have just seen) there is a path from w_2 to x. Also, from $w_2 \in B$, we obtain $h(w_2) < h(v) = n$, so that we can apply the induction hypothesis to w_2 instead of v. We thus conclude that there is exactly one sink q such that there is a path from w_2 to q. Since both x and z qualify as such q's, this entails that $z = x$. So we have proven that if z is any sink for which there is a path from v to z, then $z = x$. So there exists exactly one sink q such that there is a path from v to q, namely the sink x. This proves the claim for our vertex v. So the induction step is complete, and the claim of the exercise follows. □

Exercise 4

Part A

Definition 0.3. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. The graph $K_{n,m}$ is defined to be the simple graph with $n+m$ vertices

$$1, 2, \ldots, n, -1, -2, \ldots, -m$$

and nm edges

$$\{i, -j\} \qquad \text{for all } i \in [n] \text{ and } j \in [m].$$

(Note that $(K_{n,m}; \{1, 2, \ldots, n\}, \{-1, -2, \ldots, -m\})$ is a bipartite graph, called the *complete bipartite graph.*)

Exercise 0.4. Let $m, n \in \mathbb{N}$. Then the chromatic polynomial of $K_{n,m}$ is given by

$$\chi_{K_{n,m}} = \sum_{i=0}^{n} \operatorname{sur}(n,i) \binom{x}{i} (x-i)^m$$

Proof. Refer to the vertices $1, 2, \ldots, n$ of $K_{n,m}$ as the *positive vertices* of $K_{n,m}$, and to the vertices $-1, -2, \ldots, -m$ as the *negative vertices* of $K_{n,m}$.

Observe that for any color used in a proper coloring of $K_{n,m}$, that color can not appear on both a positive vertex and a negative vertex, since there is an edge connecting each positive vertex to each negative vertex. Hence, in a proper coloring of $K_{n,m}$, the set of colors used to color the positive vertices, and the set of colors used to color the negative vertices are disjoint.

Now, let $k \in \mathbb{N}$. Recall that the value $\chi_{K_{n,m}}(k)$ of the chromatic polynomial is equal to the number of proper k-colorings of $K_{n,m}$. We will count these k-colorings now. Let $C = [k]$; thus, a k-coloring of $K_{n,m}$ is a map from the set of vertices of $K_{n,m}$ to C. We can construct such a coloring f in the following four steps:

- Choose the number i of colors that will be used to color the positive vertices (so i will be $|f([n])|$). This is a number between 0 and n.
- Choose the set C_p of colors that will be used to color the positive vertices. This must be an i-element subset of the k-element set C. Thus, there are $\binom{k}{i}$ options here.

- Color the positive vertices with the colors from C_p, using each color at least once. This is tantamount to choosing a surjective map from the n-element set $[n]$ to the i-element set C_p (sending each positive vertex to its color); thus, there are $\operatorname{sur}(n,i)$ options for it.

- Finally, color the negative vertices. Their colors need to be chosen from the $k-i$ colors that don't belong to C_p (since the set of colors used to color the positive vertices, and the set of colors used to color the negative vertices must be disjoint in a proper k-coloring), but we don't have to use each color. Hence, this is tantamount to choosing a map from the m-element set $\{-1,-2,\ldots,-m\}$ to the $k-i$-element set $C \setminus C_p$. Thus, there are $(k-i)^m$ options at this step.

At the end of this algorithm, all vertices of $K_{n,m}$ are colored, and the resulting k-coloring is proper (because each edge connects a positive vertex with a negative vertex, and we've ensured that the latter vertex has a different color than the former). Hence, the number of all proper k-colorings of $K_{n,m}$ is $\sum_{i=0}^{n} \binom{k}{i} \operatorname{sur}(n,i)(k-i)^m$ (which we get by multiplying the numbers of options in the above algorithm). On the other hand, this is $\chi_{K_{n,m}}(k)$ (as we already showed). Comparing the two results, we find

$$\chi_{K_{n,m}}(k) = \sum_{i=0}^{n} \binom{k}{i} \operatorname{sur}(n,i)(k-i)^m .$$

Now we have proven this for each $k \in \mathbb{N}$. Thus, the two polynomials

$$\chi_{K_{n,m}}(x) \qquad \text{and} \qquad \sum_{i=0}^{n} \binom{x}{i} \operatorname{sur}(n,i)(x-i)^m$$

are equal to each other on each point $k \in \mathbb{N}$. This means that they are equal to each other on infinitely many points. Hence, they must be identical as polynomials (by the "polynomial identity trick"). In other words,

$$\chi_{K_{n,m}}(x) = \sum_{i=0}^{n} \binom{x}{i} \operatorname{sur}(n,i)(x-i)^m = \sum_{i=0}^{n} \operatorname{sur}(n,i) \binom{x}{i} (x-i)^m .$$

□

Part B

Exercise 0.5. For all $m,n \in \mathbb{N}$, it holds that

$$\sum_{i=0}^{n} \operatorname{sur}(n,i)\binom{x}{i}(x-i)^m = \sum_{i=0}^{m} \operatorname{sur}(m,i)\binom{x}{i}(x-i)^n .$$

Proof. Let $m,n \in \mathbb{N}$. We claim that the graphs $K_{n,m}$ and $K_{m,n}$ are identical up to the names of their vertices[1].

Indeed, the graph $K_{n,m}$ has vertices $1,2,\ldots,n$ and $-1,-2,\ldots,-m$, with edges $\{i,-j\}$ for $i \in [n]$ and $j \in [m]$. If one renames each vertex k as $-k$, and updates the formula for edges such that it is consistent with the new names, then the resulting graph has the

[1] That is, we can rename the vertices of $K_{n,m}$ in such a way that the resulting graph is $K_{m,n}$. In more rigorous language, we are saying that the graphs $K_{n,m}$ and $K_{m,n}$ are isomorphic.

vertices $-1, -2, \ldots, -n$ and $1, 2, \ldots, m$, with edges $\{-i, j\}$ for $i \in [n]$ and $j \in [m]$. But this is precisely the graph $K_{m,n}$. Hence, $K_{n,m}$ is equal to the graph $K_{m,n}$, except for the fact that the vertices are named differently.

And since the way the vertices of a graph are named does not in any way affect the number of proper colorings of a graph, it follows that $\chi_{K_{n,m}}(k) = \chi_{K_{m,n}}(k)$ for each $k \in \mathbb{N}$. In other words, the polynomials $\chi_{K_{n,m}}$ and $\chi_{K_{m,n}}$ are equal to each other on each point $k \in \mathbb{N}$. Hence, $\chi_{K_{n,m}} = \chi_{K_{m,n}}$.

In part **(a)**, it was shown that $\chi_{K_{n,m}} = \sum_{i=0}^{n} \operatorname{sur}(n, i) \binom{x}{i} (x - i)^m$. And swapping m and n in this formula yields $\chi_{K_{m,n}} = \sum_{i=0}^{m} \operatorname{sur}(m, i) \binom{x}{i} (x - i)^n$. Thus, the equality $\chi_{K_{n,m}} = \chi_{K_{m,n}}$ rewrites as $\sum_{i=0}^{n} \operatorname{sur}(n, i) \binom{x}{i} (x - i)^m = \sum_{i=0}^{m} \operatorname{sur}(m, i) \binom{x}{i} (x - i)^n$. □