Math 4707 Spring 2018 (Darij Grinberg): midterm 3
due date: Wednesday 2 May 2018 at the beginning of class, or before that by email or moodle
Please solve **at most 3** of the 5 exercises!
Collaboration is not allowed!

Contents

Please write your name on each page. Feel free to use LaTeX (here is a sample file with lots of amenities included).

Recall the following:

- If $n \in \mathbb{N}$, then $[n]$ denotes the n-element set $\{1, 2, \ldots, n\}$.
- We use the Iverson bracket notation.

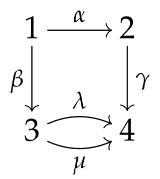
Keep in mind that **everything you claim must be proven** (unless it was stated in class or on previous homeworks/midterms/solutions), even if the exercise doesn't explicitly say so.

0.1. Ordering acyclic digraphs

See Spring 2017 Math 5707 Homework set #2 (or our class notes from April 23) for the definition of a "multidigraph". We will refer to multidigraphs simply as *digraphs*.

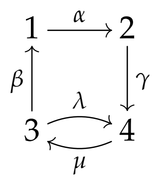
Definition 0.1. A digraph is said to be *acyclic* if it has no cycles.

For example, the digraph



(1)

is acyclic, whereas the digraph



is not acyclic (both $(1, \alpha, 2, \gamma, 4, \mu, 3, \beta, 1)$ and $(3, \lambda, 4, \mu, 3)$ are cycles of this latter digraph).

[Acyclic digraphs are often called "dags", apparently because the proper abbreviation "adgs" would be harder to pronounce.]

Exercise 1. Let $D = (V, A, \varphi)$ be an acyclic digraph. Prove that there is a list $(v_1, v_2, \ldots, v_n)$ of elements of V such that

- each element of V appears exactly once in this list $(v_1, v_2, \ldots, v_n)$;
- whenever i and j are two elements of $[n]$ such that some arc of D has source v_i and target v_j, we must have $i < j$.

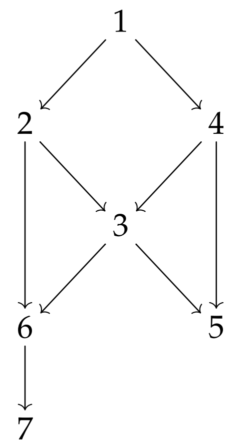
(In other words, prove that there is a list consisting of all vertices of V, which contains each of them exactly once, and with the property that the source of any arc must appear before the target of this arc in the list. For example, if D is the digraph (1), then there are two such lists: $(1, 2, 3, 4)$ and $(1, 3, 2, 4)$.)

[**Hint:** For each vertex v of D, we let $\operatorname{Anc}(v)$ be the set of all $w \in V$ such that there exists a path from w to v in D. The elements of $\operatorname{Anc}(v)$ are called the *ancestors* of v (whence the notation Anc). Now, let $(v_1, v_2, \ldots, v_n)$ be a list of all vertices of D in the order of increasing $|\operatorname{Anc}(v)|$, where ties are resolved arbitrarily. Prove that this list does the job.]

0.2. Watersheds in digraphs

A *simple digraph* means a pair (V, A), where V is a finite set, and where A is a subset of $V \times V$. We identify every simple digraph (V, A) with the multidigraph (V, A, ι), where ι is the map sending each $(u, v) \in A$ to $(u, v) \in V \times V$. Thus, simple digraphs are the same as multidigraphs whose arcs are already pairs of vertices, the first entry being the source and the second entry being the target. (So the relation between simple digraphs and multidigraphs is the same as the relation between simple graphs and multigraphs.)

Example 0.2. Consider the following simple digraph:



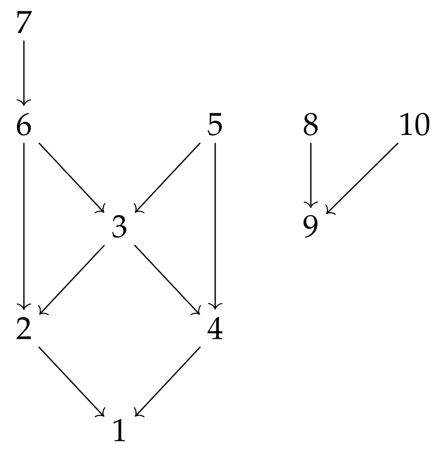
. (2)

Imagine a game chip placed initially at the vertex 1. The chip is allowed to move along the arcs of the digraph (from source to target). For example, the chip can first move along the arc $(1, 2)$ to 2, then along the arc $(2, 3)$ to 3, then along the arc $(3, 5)$ to 5. Once it arrives at 5, it can no longer move, because there are no arcs with source 5. We say that 5 is a *sink* for this reason (see Exercise 2 below for the precise definition).

Alternatively, the chip could have moved along the arc $(1, 2)$ to 2, then along the arc $(2, 6)$ to 6, then along the arc $(6, 7)$ to 7. At this point it would again be stuck, since 7 is a sink.

Thus, the chip can get stuck in **two different sinks**, depending on the path it takes. (It will always get stuck in **some** sink, because our digraph has no cycles.)

Now, consider the following simple digraph:



. (3)

This time, any chip starting at any given vertex will necessarily get stuck at **the same sink** no matter what path it takes (either the sink 1, if it started at one of the vertices $1, 2, 3, 4, 5, 6, 7$; or the sink 9, if it started at one of the vertices $8, 9, 10$). How can we show this without checking all possible paths?

One criterion, which is clearly necessary, is that there are no "watershed vertices": i.e., there is no vertex u from which the chip can take two different arcs

(u,v) and (u,w) such that v and w "never meet again" (i.e., there exists no vertex reachable both from v and from w). For example, the digraph (2) has a "watershed vertex" (namely, 3, because the arcs $(3,5)$ and $(3,6)$ lead to the vertices 5 and 6 which "never meet again").

The next exercise claims that this condition is also sufficient (as long as our digraph is acyclic). That is, if there are no "watershed vertices" and no cycles, then the sink at which a chip gets stuck is uniquely determined by the vertex it started at (rather than by the path it took).

Exercise 2. Let D be an acyclic multidigraph. A vertex v of D is said to be a *sink* if there is no arc of D with source v.

If u and v are any two vertices of D, then:

- we write $u \longrightarrow v$ if and only if D has an **arc** with source u and target v;
- we write $u \overset{*}{\longrightarrow} v$ if and only if D has a **path** from u to v.

The so-called *no-watershed condition* says that for any three vertices u, v and w of D satisfying $u \longrightarrow v$ and $u \longrightarrow w$, there exists a vertex t of D such that $v \overset{*}{\longrightarrow} t$ and $w \overset{*}{\longrightarrow} t$.

Assume that the no-watershed condition holds. Prove that for each vertex p of D, there exists **exactly one** sink q of D such that $p \overset{*}{\longrightarrow} q$.

[**Hint:** Induction on the "height" of p (that is, the length of a longest path starting at p).]

0.3. Arborescences of a wheel

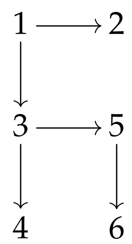
Definition 0.3. Let D be a digraph, and let u be a vertex of D.

(a) Then, D is called a *u-arborescence* if and only if for each vertex v of D, there is a **unique** walk from u to v in D.

(b) Assume that D is a simple digraph (V, A). A *u-arborescence of D* means a subset B of A such that the digraph (V, B) is a u-arborescence.

I believe that what I just called a "u-arborescence" is the same as what Vic called "arborescence with root u", except that maybe the arcs are pointing in the opposite direction.

For an example, the simple digraph

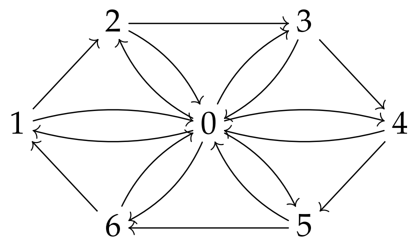


is a 1-arborescence.

Exercise 3. Let m be a positive integer. Let W_m be the simple digraph with $m+1$ vertices $0, 1, \ldots, m$ and the following $3m$ arcs:

$$\begin{aligned} &(1,2), (2,3), \ldots, (m-1,m), (m,1); \\ &(0,i) \qquad \text{for each } i \in [m]; \\ &(i,0) \qquad \text{for each } i \in [m]. \end{aligned}$$

(Visually speaking, W_m consists of a cycle that traverses m vertices $1, 2, \ldots, m$, as well as a "center vertex" 0 which is joined to each of these m vertices by one edge in each direction. For example, here is how W_6 looks like:



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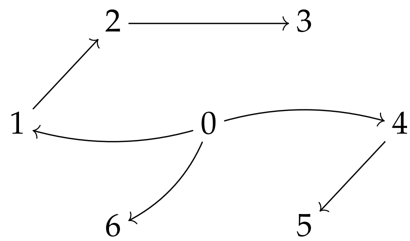
And here is a 0-arborescence of W_6:



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(a) Compute the number of 0-arborescences of W_m.

(b) Let $i \in [m]$. Compute the number of i-arborescences of W_m.

[For example, if $m = 3$, then both answers are 7, and the 0-arborescenses of W_3 are

$$\begin{aligned} &\{(0,1), (0,2), (0,3)\}, && \{(0,1), (0,2), (2,3)\}, \\ &\{(0,1), (0,3), (1,2)\}, && \{(0,1), (1,2), (2,3)\}, \\ &\{(0,2), (0,3), (3,1)\}, && \{(0,2), (2,3), (3,1)\}, \\ &\{(0,3), (1,2), (3,1)\}. && \end{aligned}$$

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0.4. Back to undirected graphs

In the following exercises, we will use the following definitions:

Definition 0.4. For each $n \in \mathbb{N}$, we define the *n-th path graph* to be the simple graph

$$\begin{aligned}&(\{1,2,\ldots,n\}, \{\{i,i+1\} \mid i \in \{1,2,\ldots,n-1\}\}) \\ &= (\{1,2,\ldots,n\}, \{\{1,2\},\{2,3\},\ldots,\{n-1,n\}\}).\end{aligned}$$

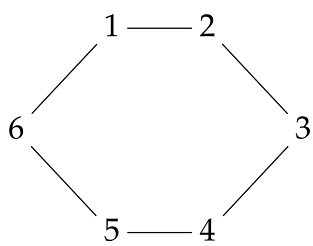
This graph is denoted by P_n. It has n vertices and $n-1$ edges (unless $n=0$, in which case it has 0 edges). Here is a drawing of P_4:

$$1 \text{ --- } 2 \text{ --- } 3 \text{ --- } 4 .$$

Definition 0.5. For each integer $n > 1$, we define the *n-th cycle graph* to be the simple graph

$$\begin{aligned}&(\{1,2,\ldots,n\}, \{\{i,i+1\} \mid i \in \{1,2,\ldots,n-1\}\} \cup \{n,1\}) \\ &= (\{1,2,\ldots,n\}, \{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\}).\end{aligned}$$

This graph is denoted by C_n. It has n vertices and $\begin{cases} n, & \text{if } n \geq 3; \\ 1, & \text{if } n = 2 \end{cases}$ edges. Here is a drawing of C_6:



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0.5. Chromatic polynomials of complete bipartite graphs

For the definition and properties of the chromatic polynomial of a simple graph, see Exercise 4 on Spring 2017 Math 5707 midterm #2. In a nutshell:

- If $G = (V,E)$ is a simple graph, then the *chromatic polynomial* χ_G of G is a polynomial in a single indeterminate x (with integer coefficients) defined by

$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} x^{\operatorname{conn}(V,F)}.$$

(Here, as usual, $\operatorname{conn} H$ denotes the number of connected components of any graph H.)

We could easily extend this definition to multigraphs, but I don't want to carry the extra notation around.

- The main property of chromatic polynomials in the following: If $G = (V, E)$ is a graph, and if $k \in \mathbb{N}$, then $\chi_G(k)$ is the number of all proper k-colorings[1] of G. Note that this property uniquely determines χ_G, because any polynomial in a single indeterminate x is uniquely determined by its values on all nonnegative integers.

- For any $n \in \mathbb{N}$, the complete graph K_n (that is, the simple graph with n vertices $1, 2, \ldots, n$ and all possible edges $\{i, j\}$ with $1 \leq i < j \leq n$) has chromatic polynomial $\chi_{K_n} = x(x-1)\cdots(x-n+1)$.

- If T is a tree with n vertices, then $\chi_T = x(x-1)^{n-1}$. Thus, in particular, for any positive integer n, the path graph P_n (see Definition 0.4) has characteristic polynomial $\chi_{P_n} = x(x-1)^{n-1}$ (since it is a tree with n vertices).

- If $n > 1$ is an integer, then the chromatic polynomial of the cycle graph C_n (see Definition 0.5) is $\chi_{C_n} = (x-1)^n + (-1)^n (x-1)$. (This is Exercise 2 **(a)** on Spring 2017 Math 5707 midterm #3.)

Definition 0.6. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. The graph $K_{n,m}$ is defined to be the simple graph with $n + m$ vertices

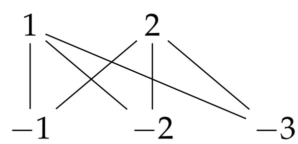
$$1, 2, \ldots, n, -1, -2, \ldots, -m$$

and nm edges

$$\{i, -j\} \qquad \text{for all } i \in [n] \text{ and } j \in [m].$$

(Note that $(K_{n,m}; \{1, 2, \ldots, n\}, \{-1, -2, \ldots, -m\})$ is a bipartite graph, called the *complete bipartite graph*.)

For example, the graph $K_{2,3}$ is



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Exercise 4. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$.

(a) Prove that the chromatic polynomial of $K_{n,m}$ is

$$\chi_{K_{n,m}} = \sum_{i=0}^{n} \operatorname{sur}(n, i) \binom{x}{i} (x-i)^m.$$

[1]Recall that a *k-coloring* of G means a map $f : V \to \{1, 2, \ldots, k\}$. (The image $f(v)$ of a vertex $v \in V$ under this map is called the *color* of v under this k-coloring f.) A k-coloring f of G is said to be *proper* if each edge $\{u, v\}$ of G satisfies $f(u) \neq f(v)$. (In other words, a k-coloring f of G is proper if and only if no two adjacent vertices share the same color.)

(Recall that $\operatorname{sur}(n,i)$ denotes the number of all surjections from $[n]$ to $[i]$.)

(b) Prove that

$$\sum_{i=0}^{n} \operatorname{sur}(n,i) \binom{x}{i} (x-i)^m = \sum_{i=0}^{m} \operatorname{sur}(m,i) \binom{x}{i} (x-i)^n .$$

Remark 0.7. Applying Exercise 4 **(b)** to $n=0$, we recover the identity

$$x^m = \sum_{i=0}^{m} \operatorname{sur}(m,i) \binom{x}{i},$$

which was Theorem 3.15 in the class of February 21.

0.6. Counting independent sets

Now let us return to independent sets of graphs.

Definition 0.8. Let G be a graph.

(a) An *independent set* of G means a set S of vertices of G such that no two distinct elements of S are adjacent.

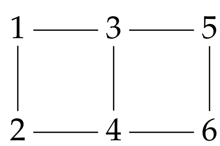
(b) We let $\operatorname{ind} G$ be the number of all independent sets of G.

There are no good formulas for $\operatorname{ind} G$ in general, but we can always try to compute it when G is a particularly simple type of graph. For example, if G is the path graph P_n for some $n \in \mathbb{N}$, then the independent sets of G are precisely the lacunar subsets of $[n]$, and thus $\operatorname{ind} G$ is the Fibonacci number f_{n+2} (by Proposition 1.22 in the February 5 class). The independent sets of the cycle graph C_n are the lacunar subsets of $[n]$ which don't contain 1 and n simultaneously (i.e., they can contain at most one of 1 and n). The following definition will help counting them:

Definition 0.9. Let G be a graph. Let S be a set of vertices of G. Then, $G \setminus S$ will denote the graph obtained from G by removing all vertices in S (along with all edges that use these vertices).

(More rigorously: If G is a simple graph (V,E), then $G \setminus S$ is the simple graph $(V \setminus S, E')$, where E' is the set of all edges $e \in E$ such that no endpoint of e belongs to S. If G is a multigraph (V,E,φ), then $G \setminus S$ is the multigraph $(V \setminus S, E', \varphi\mid_{E'})$, where E' is the set of all edges $e \in E$ such that no endpoint of e belongs to S.)

For example, if G is the graph



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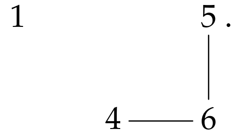
then $G \setminus \{2, 3\}$ is the graph



Exercise 5. **(a)** Let v be a vertex of a graph G. Let $N(v)$ be the set of all neighbors of v. Let $N^+(v) = \{v\} \cup N(v)$. Prove that

$$\operatorname{ind} G = \operatorname{ind}(G \setminus \{v\}) + \operatorname{ind}\left(G \setminus \left(N^+(v)\right)\right).$$

(b) Compute $\operatorname{ind}(C_n)$ for each $n \geq 2$ (in terms of the Fibonacci sequence).

Remark 0.10. **(a)** It is instructive to see what Exercise 5 **(a)** says when G is a path graph. Let $n > 1$ be an integer, and let G be the path graph P_n. Let $v \in [n]$ (so that v is a vertex of G). If $v = n$, then $G \setminus \{v\} = P_{n-1}$ and $G \setminus (N^+(v)) = P_{n-2}$ (since $N^+(v) = \{n, n-1\}$ in this case), so that Exercise 5 **(a)** yields

$$\operatorname{ind}(P_n) = \operatorname{ind}(P_{n-1}) + \operatorname{ind}(P_{n-2}).$$

This is precisely the recurrence equation of the Fibonacci numbers. Thus, we obtain a new (inductive) proof of the fact that

$$\operatorname{ind}(P_n) = f_{n+2} \qquad \text{for each } n \in \mathbb{N}. \tag{4}$$

However, we can also apply Exercise 5 **(a)** to another vertex v. Let $v \in \{2, 3, \ldots, n-1\}$. Then, $G \setminus \{v\}$ is a disconnected graph looking as follows:

$$1 \text{ --- } 2 \text{ --- } \cdots \text{ --- } v-1 \qquad v+1 \text{ --- } v+2 \text{ --- } \cdots \text{ --- } n\ .$$

We can treat this graph as a "disjoint union" of two path graphs, one of which is P_{v-1} while the other is P_{n-v} "in all but name" (its vertices are called $v+1, v+2, \ldots, n$ rather than $1, 2, \ldots, n-v$, but otherwise it is identical to P_{n-v}). To construct an independent set of $G \setminus \{v\}$, we thus just need to choose an independent set of the former path graph P_{v-1} and an independent set of the latter path grad P_{n-v} (with vertices renamed as $v+1, v+2, \ldots, n$), and take the union of these two independent sets. Hence,

$$\operatorname{ind}(G \setminus \{v\}) = \operatorname{ind}(P_{v-1}) \cdot \operatorname{ind}(P_{n-v}).$$

A similar argument shows that the graph $G \setminus (N^+(v))$ has the form

$$1 \text{ --- } 2 \text{ --- } \cdots \text{ --- } v-2 \qquad v+2 \text{ --- } v+3 \text{ --- } \cdots \text{ --- } n$$

(again, a "disjoint union" of two path graphs, which this time are P_{v-2} and P_{n-v-1}), and thus

$$\operatorname{ind}\left(G \setminus \left(N^+(v)\right)\right) = \operatorname{ind}(P_{v-2}) \cdot \operatorname{ind}(P_{n-v-1}).$$

Hence, Exercise 5 **(a)** becomes

$$\operatorname{ind} G = \underbrace{\operatorname{ind}\left(G \setminus \{v\}\right)}_{=\operatorname{ind}(P_{v-1})\cdot\operatorname{ind}(P_{n-v})} + \underbrace{\operatorname{ind}\left(G \setminus \left(N^+(v)\right)\right)}_{=\operatorname{ind}(P_{v-2})\cdot\operatorname{ind}(P_{n-v-1})}$$
$$= \operatorname{ind}(P_{v-1}) \cdot \operatorname{ind}(P_{n-v}) + \operatorname{ind}(P_{v-2}) \cdot \operatorname{ind}(P_{n-v-1}).$$

Since $G = P_n$, this rewrites as

$$\operatorname{ind}(P_n) = \operatorname{ind}(P_{v-1}) \cdot \operatorname{ind}(P_{n-v}) + \operatorname{ind}(P_{v-2}) \cdot \operatorname{ind}(P_{n-v-1}).$$

Using the equality (4), we can rewrite this as

$$f_{n+2} = f_{(v-1)+2} \cdot f_{(n-v)+2} + f_{(v-2)+2} \cdot f_{(n-v-1)+2} = f_{v+1} f_{n-v+2} + f_v f_{n-v+1}.$$

Applying this to $v = a$ and $n = a + b - 1$, we conclude that

$$f_{a+b+1} = f_{a+1} f_{b+1} + f_a f_b \qquad \text{for all } a \in \mathbb{N} \text{ and } b \in \mathbb{N}.$$

(To be more precise, we have only proven this in the case when $a > 1$ and $b > 1$; but all other cases are easy.) Thus, we have recovered the claim of Exercise 3 **(e)** on midterm #1.

(b) We can similarly count independent sets **of a given size**. For C_n, we find that the number of independent sets of C_n having size k (for a given $k \in \{0, 1, \ldots, n-1\}$) is $\dfrac{n}{n-k}\dbinom{n-k}{k}$. This can also be proven combinatorially; see [Stan11, Lemma 2.3.4].

References

[Stan11] Richard Stanley, *Enumerative Combinatorics, volume 1*, Second edition, version of 15 July 2011. Available at `http://math.mit.edu/~rstan/ec/` .